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On orthogonal polynomials and quadrature rules related to the second kind of beta distribution

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Abstract
We consider a finite class of weighted quadratures with the weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$, which is valid only for finite values of n (the number of nodes). This means that classical Gauss-Jacobi quadrature rules cannot be considered for this class, because some restrictions such as $\{\max n\} \le a+b-1/2$, $a<1/2$, $b>0$ and $(-1)^{2a}=1$ must be satisfied for its orthogonality relation. Some analytic examples are given in this sense.

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1 Introduction

The differential equation

$$x^2\left(px^2+q\right)\Phi_n''(x)+x\left(rx^2+s\right)\Phi_n'(x) - \left(n\left(r+(n-1)p\right)x^2+\left(1-(-1)^n\right)s/2\right)\Phi_n(x)=0 \tag{1}$$

was introduced in [1], and it was established that the symmetric polynomials

$$\Phi_n(x)=S_n\left(\begin{array}{cc|c} r & s & \\ p & q & x\end{array}\right) = \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{(2i+(-1)^{n+1}+2[n/2])p+r}{(2i+(-1)^{n+1}+2)q+s}\right)x^{n-2k} \tag{2}$$

are a basis solution of it. If this equation is written in a self-adjoint form, then the first-order equation

$$x\frac{d}{dx}\left(\left(px^2+q\right)W(x)\right)=\left(rx^2+s\right)W(x) \tag{3}$$

would appear. The solution of equation (3) is known as an analogue of Pearson distributions family and can be indicated as

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}\,dx\right). \tag{4}$$

There are four main sub-classes of distributions family (4) (and consequently, sub-solutions of equation (3)) whose explicit probability density functions are, respectively, as follows:

$$\begin{aligned} &K_1 W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right) \\ &\quad = \frac{\Gamma(a+b+3/2)}{\Gamma(a+1/2)\Gamma(b+1)} x^{2a}\left(1-x^2\right)^b; \quad -1 \le x \le 1; a+1/2>0; b+1>0, \end{aligned} \tag{5}$$

$$K_2 W\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right) = \frac{1}{\Gamma(a+1/2)} x^{2a} e^{-x^2}; \quad -\infty < x < \infty; a+1/2>0, \tag{6}$$

$$\begin{aligned} &K_3 W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) \\ &\quad = \frac{\Gamma(b)}{\Gamma(b+a-1/2)\Gamma(-a+1/2)} \frac{x^{-2a}}{(1+x^2)^b}; \quad -\infty < x < \infty; b+a>1/2; a<1/2; b>0, \end{aligned} \tag{7}$$

$$K_4 W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right) = \frac{1}{\Gamma(a-1/2)} x^{-2a} e^{-\frac{1}{x^2}}; \quad -\infty < x < \infty; a>1/2, \tag{8}$$

where K_i; $i = 1,2,3,4$ play the normalizing constant role.

Clearly, the value of distribution vanishes at $x = 0$ in each of the above mentioned four cases, *i.e.*, $W(p,q,r,s;0) = 0$ for $s \neq 0$.

As a special case of (4), let us consider the values $p = 1$, $q = 1$, $r = -2a-2b+2$ and $s = -2a$ corresponding to distribution (7) and replace them in equation (1) to get

$$\begin{aligned} &x^2\left(x^2+1\right)\Phi_n''(x) - 2x\left((a+b-1)x^2+a\right)\Phi_n'(x) \\ &\quad + \left(n\left(2a+2b-(n+1)\right)x^2 + \left(1-(-1)^n\right)a\right)\Phi_n(x) = 0. \end{aligned} \tag{9}$$

By solving equation (9), the polynomial solution of monic type is derived

$$\begin{aligned} &\bar{S}_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) \\ &\quad = \prod_{i=0}^{[n/2]-1} \frac{2i+(-1)^{n+1}+2-2a}{2i+2[n/2]+(-1)^{n+1}+2-2a-2b} \\ &\qquad \times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{2i+2[n/2]+(-1)^{n+1}+2-2a-2b}{2i+(-1)^{n+1}+2-2a}\right) x^{n-2k}. \end{aligned} \tag{10}$$

According to [1], these polynomials are finitely orthogonal with respect to the second kind of beta weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$ if and only if $\{\max n\} \le a+b-1/2$, *i.e.*, we have

$$\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} \bar{S}_n \left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix} \middle| x \right) \bar{S}_m \left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix} \middle| x \right) dx$$
$$= \left((-1)^n \prod_{i=1}^{n} \frac{(i-(1-(-1)^i)a)(i-(1-(-1)^i)a-2b)}{(2i-2a-2b+1)(2i-2a-2b-1)} \right)$$
$$\times \frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)} \delta_{n,m}, \tag{11}$$

if $m,n = 0,1,\dots,N \leq a+b-1/2$, where $N = \max\{m,n\}$, $\delta_{n,m} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m), \end{cases}$ $a < 1/2$, $b > 0$ and $(-1)^{2a} = 1$. Moreover, they satisfy a three-term recurrence relation

$$\bar{S}_{n+1}(x) = x\bar{S}_n(x) + \frac{(n-(1-(-1)^n)a)(n-(1-(-1)^n)a-2b)}{(2n-2a-2b+1)(2n-2a-2b-1)} \bar{S}_{n-1}(x),$$
$$\text{with } \bar{S}_0(x) = 1, \bar{S}_1(x) = x, n \in \mathbf{N}. \tag{12}$$

The orthogonality property (11) shows that the polynomials $\bar{S}_n(1,1,-2a-2b+2,-2a;x)$ are a suitable tool to finitely approximate the functions that satisfy the Dirichlet conditions [2–5].

For example, if $N = \{\max n\} = 3$, $a+b \geq 7/2$, $a < 1/2$, $b > 0$ and $(-1)^{2a} = 1$ in (10), then the arbitrary function $f(x)$ can be approximated as

$$f(x) \cong \sum_{m=0}^{3} B_m \bar{S}_m(1,1,-2a-2b+2,-2a;x), \tag{13}$$

where

$$B_m = \left((-1)^m \prod_{i=1}^{m} \frac{(i-(1-(-1)^i)a)(i-(1-(-1)^i)a-2b)}{(2i-2a-2b+1)(2i-2a-2b-1)} \right) \frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}$$
$$\times \int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} \bar{S}_m \left(\begin{matrix} -2a-2b+2 & -2a \\ 1 & 1 \end{matrix} \middle| x \right) f(x)\, dx. \tag{14}$$

This means that the finite set $\{\bar{S}_i(1,1,-2a-2b+2,-2a;x)\}_{i=0}^3$ is a basis space for all polynomials of degree at most three, *i.e.*, for $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, the approximation (13) is exact. This matter helps us to state the application of symmetric orthogonal polynomials (10) in weighted quadrature rules [6–9].

2 Application of $\bar{S}_n(1,1,-2a-2b+2,-2a;x)$ in quadrature rules

Consider the general form of a weighted quadrature

$$\int_{\alpha}^{\beta} w(x) f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i) + \sum_{k=1}^{m} v_k f(z_k) + R_{n,m}[f], \tag{15}$$

where $w(x)$ is a positive function on $[\alpha,\beta]$; $\{w_i\}_{i=1}^n$, $\{v_k\}_{k=1}^m$ are unknown coefficients; $\{x_i\}_{i=1}^n$ are unknown nodes; $\{z_k\}_{k=1}^m$ are pre-determined nodes [7, 8]; and finally, the residue $R_{n,m}[f]$

is determined (see, *e.g.*, [8]) by

$$R_{n,m}[f] = \frac{f^{(2n+m)}(\xi)}{(2n+m)!}\int_\alpha^\beta w(x)\prod_{k=1}^{m}(x-z_k)\prod_{i=1}^{n}(x-x_i)^2\,dx; \quad \alpha<\xi<\beta. \tag{16}$$

It can be shown in (15) that $R_{n,m}[f] = 0$ for any linear combination of the sequence $\{1, x, \ldots, x^{2n+m+1}\}$ if and only if $\{x_i\}_{i=1}^n$ are the roots of orthogonal polynomials of degree n with respect to the weight function $w(x)$, and $\{z_k\}_{k=1}^m$ belong to $[\alpha,\beta]$; see [7] for more details. Also, it is proved that to derive $\{w_i\}_{i=1}^n$ in (15), when $m = 0$, it is not required to solve the following linear system of order $n \times n$:

$$\sum_{i=1}^{n} w_i x_i^j = \int_\alpha^\beta w(x)x^j\,dx \quad \text{for } j = 0,1,\ldots,2n-1. \tag{17}$$

Rather, one can directly use the relation

$$\frac{1}{w_i} = \hat{P}_0^2(x_i) + \hat{P}_1^2(x_i) + \cdots + \hat{P}_{n-1}^2(x_i) \quad \text{for } i = 1,2,\ldots,n, \tag{18}$$

in which $\hat{P}_i(x)$ is the orthonormal polynomial of $P_i(x)$, *i.e.*,

$$\hat{P}_i(x) = \left(\int_\alpha^\beta w(x)P_i^2(x)\,dx\right)^{-1/2} P_i(x). \tag{19}$$

Now, by noting that the symmetric polynomials (10) are finitely orthogonal with respect to the weight function $W(x,a,b) = x^{-2a}(1+x^2)^{-b}$ on the real line, we consider the following finite class of quadrature rules:

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} f(x)\,dx \\ &\quad= \sum_{j=1}^{n} w_j f(x_j) + \frac{f^{(2n)}(\xi)}{(2n)!}\int_{-\infty}^{\infty}\frac{x^{-2a}}{(1+x^2)^b}\prod_{j=1}^{n}(x-x_j)^2\,dx, \quad \xi\in\mathbf{R},\end{aligned} \tag{20}$$

where x_j are the roots of polynomials $\bar{S}_n(1,1,-2a-2b+2,-2a;x)$ and w_j are calculated by

$$\frac{1}{w_j} = \sum_{i=0}^{n-1}\left(\bar{S}_i^*(1,1,-2a-2b+2,-2a;x_j)\right)^2 \quad \text{for } j = 0,1,2,\ldots,n. \tag{21}$$

2.1 An important remark

The change of variable $x = t^{-1/2}(1-t)^{1/2}$ in the left-hand side of (20) first changes the interval $(-\infty,\infty)$ to $[0,1]$ such that we have

$$\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} f(x)\,dx = \int_0^1 t^{a+b-\frac{3}{2}}(1-t)^{-a-\frac{1}{2}} f\left(\sqrt{\frac{1}{t}-1}\right)dt. \tag{22}$$

As the right-hand integral of (22) shows, the shifted Jacobi weight function $(1-x)^u x^v$ has appeared for $u = -a-1/2$ and $v = a+b-3/2$. Hence, the shifted Gauss-Jacobi quadrature rule [6, 9] with the special parameters $u = -a - 1/2$ and $v = a + b - 3/2$ can also be applied for estimating (22). This procedure eventually changes (20) into the form

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} f(x)\,dx \\ &\quad = \sum_{j=1}^{n} w_j^{(-a-\frac{1}{2},a+b-\frac{3}{2})} f\left(\frac{1}{\sqrt{x_j^{(-a-1/2,a+b-3/2)}}}\right) + R_n\left[f\left(\sqrt{\frac{1}{x}-1}\right)\right],\end{aligned} \tag{23}$$

where $x_j^{(-a-1/2,a+b-3/2)}$ are the zeros of shifted Jacobi polynomials $P_{n,+}^{(-a-1/2,a+b-3/2)}(x)$ on $[0,1]$. But, there is the main problem for the formula (23). From (16), it is generally known that the residue of quadrature rules depends on $f^{(2n)}(\xi)$; $\alpha < \xi < \beta$. Therefore, by noting (23), we should have

$$\frac{d^{2n} f(\sqrt{x^{-1}-1})}{dx^{2n}} = \sum_{i=0}^{2n} \varphi_i(x) f^{(i)}\big(\sqrt{x^{-1}-1}\big), \tag{24}$$

where φ_i are real functions to be computed and $f^{(i)}$, $i = 0,1,2,\dots,2n$ are the successive derivatives of the function f. On the other hand, the function f cannot be in the form of an arbitrary polynomial in order that the right-hand side of (24) becomes zero. In other words, the formula (23) cannot be exact for all elements of the basis $f(x) = x^j$; $j = 0,1,2,\dots,2n-1$. This is the main disadvantage of using (23), which shows the importance of the polynomials (10) in estimating a class of weighted quadrature rules [10]. The following examples clarify this remark.

Example 1 Consider the two-point quadrature formula

$$\int_{-\infty}^{\infty} x^{-2a}\big(1+x^2\big)^{-b} f(x)\,dx \cong w_1 f(x_1) + w_2 f(x_2), \tag{25}$$

in which $a + b \geq 5/2$, $a < 1/2$, $b > 0$ and $(-1)^{2a} = 1$. According to the explained comments, (25) must be exact for all elements of the basis $f(x) = \{1, x, x^2, x^3\}$ if and only if x_1, x_2 are two roots of $\bar{S}_2(1,1,-2a-2b+2,-2a;x)$. As a particular sample, let us take $a = 0$ and $b = 3$. Then (25) is reduced to

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} f(x)\,dx \cong w_1 f\left(\frac{\sqrt{3}}{3}\right) + w_2 f\left(-\frac{\sqrt{3}}{3}\right), \tag{26}$$

in which $\sqrt{3}/3$ and $-\sqrt{3}/3$ are zeros of $\bar{S}_2(1,1,-4,0;x)$ and w_1, w_2 are computed by solving the linear system

$$w_1 + w_2 = \int_{-\infty}^{\infty} \big(1+x^2\big)^{-3}\,dx = \frac{3}{8}\pi, \qquad \frac{\sqrt{3}}{3}(w_1 - w_2) = \int_{-\infty}^{\infty} x\big(1+x^2\big)^{-3}\,dx = 0. \tag{27}$$

After deriving w_1, w_2 in (27), the complete form of (26) would be

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} f(x)\,dx = \frac{3\pi}{16}\left(f\left(\frac{\sqrt{3}}{3}\right) + f\left(-\frac{\sqrt{3}}{3}\right)\right) + R_2[f], \tag{28}$$

where

$$R_2[f] = \frac{f^{(4)}(\xi)}{4!} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} \left(\bar{S}_2 \left(\begin{matrix} -4 & 0 \\ 1 & 1 \end{matrix} \middle| x \right) \right)^2 dx = \frac{\pi}{72} f^{(4)}(\xi), \quad \xi \in \mathbf{R}. \tag{29}$$

Relation (28) shows that it is exact for any arbitrary polynomial of degree at most three.

Example 2 To have a three-point formula of type (20), first we should note that the conditions $a + b \geq 7/2$, $a < 1/2$, $b > 0$ and $(-1)^{2a} = 1$ must be satisfied. For instance, if $a = -1$ and $b = 5$, then after some computations, the related formula takes the form

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^5} f(x)\, dx = \frac{\pi}{1{,}280} \left(9f\left(\sqrt{\frac{5}{3}}\right) + 32f(0) + 9f\left(-\sqrt{\frac{5}{3}}\right) \right) + R_3[f], \tag{30}$$

where

$$R_3[f] = \frac{f^{(6)}(\xi)}{6!} \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^5} \left(\bar{S}_3 \left(\begin{matrix} -6 & 2 \\ 1 & 1 \end{matrix} \middle| x \right) \right)^2 dx = \frac{5\pi}{3{,}456} f^{(6)}(\xi), \quad \xi \in \mathbf{R}, \tag{31}$$

and $x_1 = \sqrt{5/3}$, $x_2 = 0$, $x_3 = -\sqrt{5/3}$ are the roots of $\bar{S}_3(1, 1, -6, 2; x) = x^3 - (5/3)x$.

Example 3 To derive a four-point formula of type (20), first the conditions $a + b \geq 9/2$, $a < 1/2$, $b > 0$ and $(-1)^{2a} = 1$ must be satisfied. For example, if $a = 0$ and $b = 6$, then

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^6} f(x)\, dx \\ &\quad = \frac{7\pi}{6{,}144}(54 - 11\sqrt{21}) \left(f\left(\sqrt{\frac{21+4\sqrt{21}}{35}}\right) + f\left(-\sqrt{\frac{21+4\sqrt{21}}{35}}\right) \right) \\ &\qquad + \frac{7\pi}{6{,}144}(54 + 11\sqrt{21}) \left(f\left(\sqrt{\frac{21-4\sqrt{21}}{35}}\right) + f\left(-\sqrt{\frac{21-4\sqrt{21}}{35}}\right) \right) + R_4[f], \end{aligned} \tag{32}$$

where

$$\begin{aligned} R_4[f] &= \frac{f^{(8)}(\xi)}{8!} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^6} \left(\bar{S}_4 \left(\begin{matrix} -10 & 0 \\ 1 & 1 \end{matrix} \middle| x \right) \right)^2 dx \\ &= \frac{\pi}{2{,}822{,}400} f^{(8)}(\xi), \quad \xi \in \mathbf{R}. \end{aligned} \tag{33}$$

This formula is exact for all elements of the basis $f(x) = x^j$; $j = 0, 1, 2, \ldots, 7$ and its nodes are the roots of $\bar{S}_4(1, 0, -8, 2; x) = x^4 - (6/5)x^2 + 3/35$.

Tables 1-3 show some numerical examples related to three given examples.

Table 1 Numerical results for two-point formula (28)

$f(x)$	Approximate value (2-point)	Exact value	Error
$\cos x^2$	1.113251175	1.041656130	0.071595045
$\exp(-x^2/2)$	0.997237788	1.037543288	0.040305500
$\exp(-\cos x)$	0.509660126	0.519034734	0.009374608
$\sqrt{1+x^2}$	1.360349524	1.333333333	0.027016191

Table 2 Numerical results for three-point formula (30)

$f(x)$	Approximate value (3-point)	Exact value	Error
$\cos x^2$	0.0743108795	0.09326578594	0.01895490641
$\exp(-x^2/2)$	0.09773977703	0.09545329274	0.00228738430
$\exp(-\cos x)$	0.06241097330	0.06149960816	0.00091136514
$\sqrt{1+x^2}$	0.15068324430	0.15238095240	0.00169770810

Table 3 Numerical results for four-point formula (32)

$f(x)$	Approximate value (4-point)	Exact value	Error
$\cos x^2$	0.7563575358	0.7567616833	0.0004041475
$\exp(-x^2/2)$	0.7341056789	0.7341611797	0.0000555010
$\exp(-\cos x)$	0.3013485879	0.3013339743	0.0000146136
$\sqrt{1+x^2}$	0.8128655892	0.8126984127	0.0001671765

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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