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Research Article

A Q-Continuum of Off-Shell Supermultiplets

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Within each supermultiplet in the standard literature, supersymmetry relates its bosonic and fermionic component fields in a fixed way, particularly to the selected supermultiplet. Herein, we describe supermultiplets wherein a *continuously variable* "tuning parameter" modifies the supersymmetry transformations, effectively parametrizing a novel "Q-continuum" of distinct finite-dimensional off-shell supermultiplets, which may be probed already with bilinear Lagrangians that couple to each other and to external magnetic fields, two or more of these continuously many supermultiplets, each "tuned" differently. The dependence on the tuning parameters cannot be removed by any field redefinition, rendering this "Q-*moduli space*" observable.

"Discreteness is the refuge of the clumsy."
Jorge Hazzan

1. Introduction, Results, and Synopsis

Supersymmetry has been studied for over forty years [1, 2], has had successful application in nuclear physics [3, 4] and critical phenomena [5, 6], and has recently found applications also in condensed matter physics: see the recent reviews [7, 8], for example. In quantum applications, the supermultiplets must be off-shell, that is, free of any (space) time-differential constraint that could play the role of the Euler-Lagrange (classical) equation of motion. The long-standing challenge of a systematic classification of off-shell supermultiplets [9, 10] has been addressed with significant success in the last decade or so; see [11–19] and references therein. One of the pivotal ideas enabling this recent development was the use of graph-theoretical methods [20–22] in assessing the structure of the supersymmetry transformations within the world-line dimensional reduction of off-shell supermultiplets, which turned out to relate the classification problem to encryption and coding theory [23–25].

Although this research program uncovered trillions of off-shell supermultiplets of world-line N-extended supersymmetry, we show herein—corroborated by concurrent research [26]—that they provide merely a discrete subset of a vast Q-*continuum*, giving rise to a Q-*moduli space*. Furthermore, we prove herein that this novel Q-continuum is physically observable.

This may well come as a surprise, since both the continuous Lie algebras and the various discrete symmetry groups familiar from physics applications all have *discrete* sequences of inequivalent unitary and linear and finite-dimensional representations. Reference [27] showed that the infinite sequence of quotient supermultiplets specified in [21] defines a similarly infinite sequence of ever larger unitary, linear, and finite-dimensional off-shell representations of

$N \geqslant 3$-extended world-line supersymmetry, and [28] finds highly nontrivial and continuously variable dynamics for the simplest of these supermultiplets, even with only bilinear Lagrangians.

Our present results, however, radically extend this line of research. By proving that each of these supermultiplets is merely a special member in a *continuum of distinct supermultiplets*, we prove that already bilinear Lagrangians of [28] can couple *continuously many* distinct supermultiplets. The coupling constants are therefore functions over this novel Q-moduli space, providing access to physically probe and observe this Q-continuum.

For simplicity and concreteness, we focus on the $N = 4$-extended world-line supersymmetry algebra without central charges:

$$\{Q_I, Q_J\} = 2i\delta_{IJ}\partial_\tau, \qquad [\partial_\tau, Q_I] = 0, \tag{1}$$

where $i\partial_\tau$ is the Hamiltonian (in the familiar $\hbar = 1 = c$ units) and $Q_1, \ldots, Q_4$ are the supercharges, four real generators of supersymmetry. For concreteness, we focus on a particular set of supermultiplets (see (2) below) which were adapted from [28] by replacing one of the component bosons with its τ-derivative and renaming the component fields. Our present results then apply equally well not only to the $N = 3$ supermultiplet of [27, 28] but also to the infinite sequence of ever larger supermultiplets constructed therein. Our present focus on world-line supersymmetry should nevertheless have implications for all supersymmetry, since, (a) by dimensional reduction, (1) is an integral part and common denominator of every supersymmetric theory, (b) it is directly relevant in diverse fields in physics, from candidates for the fundamental description of M-theory [29] to the phenomenology of topological insulators and graphene [30], and (c) it shows up in the Hilbert space of every supersymmetric quantum theory. We defer the exploration of these implications to a subsequent effort.

The paper is organized as follows. Section 2 defines an illustrative 1-parameter family of indecomposable off-shell, unitary, and finite-dimensional supermultiplets and identifies the novel Q-continuum and the corresponding Q-moduli space, $\{\alpha \in \mathbb{R}\}$. Section 3 then explicitly constructs Lagrangians that, even though being just bilinear in fields, (1) inextricably depend on the tuning parameter α, (2) pairwise couple continuously many inequivalent supermultiplets, and (3) provide for physical probing of this Q-continuum by coupling to external magnetic fields. Section 4 provides a token example of such nontrivial dynamics which essentially depend on the tuning parameter α, and our conclusions are summarized in Section 5.

2. The Q-Continuum of Off-Shell Supermultiplets

We proceed by way of a concrete example, introducing the following 1-parameter *family of variations* of the off-shell supermultiplet from [28]:

	Q_1	Q_2	Q_3	Q_4
ϕ_1	ψ_1	ψ_2	ψ_3	ψ_4
ϕ_2	$\psi_4 - \alpha\psi_6$	$\psi_3 + \alpha\psi_5$	$-\psi_2 - \alpha\psi_8$	$-\psi_1 + \alpha\psi_7$
F_3	$\dot\psi_3$	$-\dot\psi_4$	$-\dot\psi_1$	$\dot\psi_2$
F_4	$\dot\psi_2$	$-\dot\psi_1$	$\dot\psi_4$	$-\dot\psi_3$
ϕ_5	ψ_5	ψ_6	ψ_7	ψ_8
ϕ_6	ψ_8	ψ_7	$-\psi_6$	$-\psi_5$
F_7	$\dot\psi_7$	$-\dot\psi_8$	$-\dot\psi_5$	$\dot\psi_6$
F_8	$\dot\psi_6$	$-\dot\psi_5$	$\dot\psi_8$	$-\dot\psi_7$

	Q_1	Q_2	Q_3	Q_4
ψ_1	$i\dot\phi_1$	$-iF_4$	$-iF_3$	$-i\dot\phi_2 - i\alpha F_8$
ψ_2	iF_4	$i\dot\phi_1$	$-i\dot\phi_2 - i\alpha F_8$	iF_3
ψ_3	iF_3	$i\dot\phi_2 + i\alpha F_8$	$i\dot\phi_1$	$-iF_4$
ψ_4	$i\dot\phi_2 + i\alpha F_8$	$-iF_3$	iF_4	$i\dot\phi_1$
ψ_5	$i\dot\phi_5$	$-iF_8$	$-iF_7$	$-i\dot\phi_6$
ψ_6	iF_8	$i\dot\phi_5$	$-i\dot\phi_6$	iF_7
ψ_7	iF_7	$i\dot\phi_6$	$i\dot\phi_5$	$-iF_8$
ψ_8	$i\dot\phi_6$	$-iF_7$	iF_8	$i\dot\phi_5$

(2)

Omitting the fourth supersymmetry, Q_4, would result in a minimal example of this Q-continuum; see Main Theorem of [26]. The inclusion of Q_4, however, proves that our results are not an artifact of "too few supersymmetries" and also affords possible extensions to higher-dimensional spacetimes to be explored separately.

The supermultiplet (2) may be depicted (graphical depictions of supersymmetry transformation rules are a time-tested intuitive tool [31] but have been rigorously formalized only recently [21], and we adopt those conventions) in the manner of Figure 1. Component fields are depicted as nodes and the Q-transformations between them are depicted as connecting edges, variously colored to correspond to the four supercharges Q_I; these are drawn solid (dashed) to depict the positive (negative) signs in (2). This graphical rendition of the supermultiplet (2) at once reveals that the supermultiplet (2) consists of two identical submultiplets, $(\phi_1, \phi_2 \mid \psi_1, \ldots, \psi_4 \mid F_3, F_4)$ and $(\phi_5, \phi_6 \mid \psi_5, \ldots, \psi_8 \mid F_7, F_8)$, which supersymmetry connects by the one-way transformations. Such one-way transformations are exemplified by the fact that $Q_1(\phi_2)$ contains ψ_6, but $Q_1(\psi_6)$ does not contain ϕ_2; this is depicted by the tapering edges crossing the dashed vertical divider in Figure 1.

Surprisingly—and radically extending our previous work on the topic [27, 28]—we find that these one-way Q-transformations admit a *continuous* "tuning parameter." Denoting $\alpha \in \mathbb{R}$ in the tabulation (2), its value is in no way restricted by the supersymmetry algebra relations (1)! That is, the supersymmetry transformations (2) close the algebraic relations (1) on every given component field with no need

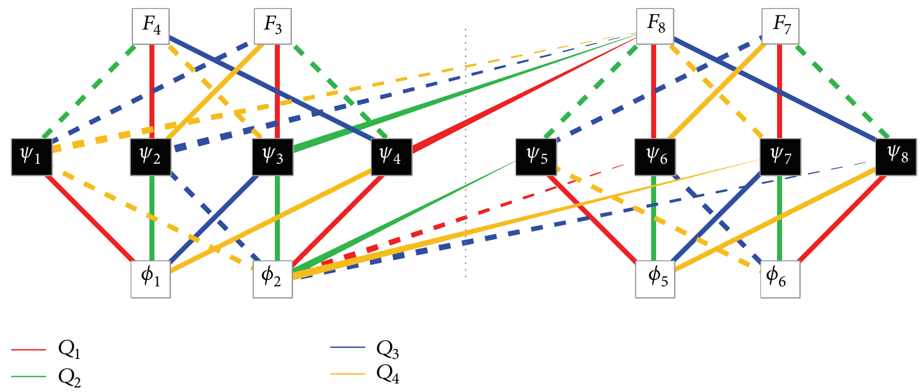

FIGURE 1: A graphical depiction of the $N = 4$ world-line supermultiplet (2).

of any τ-differential condition and for each possible value of $\alpha \in \mathbb{R}$ separately. The tabulation (2) is thus a *continuous* 1-parameter family of proper off-shell representations of $N = 4$-extended supersymmetry on the world-line. By contrast, the trillions of supermultiplets reported in [22, 25] as well as those of [27, 28] and all known supermultiplets [1, 32] form at most discrete sequences.

The supermultiplet (2) then is one of the simplest examples of the Q-continuum; see also [26]. In turn the real line $\{\alpha \in \mathbb{R}\}$ is the corresponding coarse Q-moduli space. Explicit choices of the Lagrangian will determine corresponding actions of a mapping class group, Γ, whereby $\{\alpha \in \mathbb{R}\}/\Gamma$ becomes the true (and model-dependent) Q-moduli space; see below.

The special value $\alpha = 0$ decomposes the supermultiplet (2) into two separate off-shell (2|4|2)-dimensional supermultiplets, both of which being the world-line dimensional reduction of the familiar chiral supermultiplet [1, 21]. When $\alpha \neq 0$, the off-shell supermultiplet (2) cannot be decomposed as a direct sum of two separate supermultiplets. The off-shell supermultiplets of $N = 3$-extended supersymmetry considered in [27, 28] may be similarly extended to depend on a precisely analogous tuning parameter, dialing the "magnitude" of the one-way Q_I-transformations connecting the two halves of the supermultiplet; see Figure 1. Those supermultiplets are closely related to the $\alpha = 1$ version of (2): except for some renaming of component fields, one merely needs to drop the fourth supersymmetry and replace $F_4 \mapsto \phi_4 = \int \mathrm{d}F_4$, effectively lowering the corresponding node (top, left) to the bottom level in the graph in Figure 1. This, however, obstructs dimensional extension even to just world-sheet supersymmetry [33], providing our main motivation to consider (2) instead of the slightly simpler supermultiplet of [27, 28].

Explicit attempts verify that no local component field redefinition can remove the parameter α from the supersymmetry transformations (2). As discussed subsequently, all efforts to eliminate α from the Q-transformations must involve nonlocal transformations; see also (6) below.

The table (2) thus defines a 1-parameter continuum of indecomposable off-shell, unitary, and linear representations of world-line $N = 4$-extended supersymmetry, α, parametrizing this Q-continuum and providing a coarse parametrization for the corresponding Q-moduli space.

3. Lagrangians

We now turn to show that the supersymmetry tuning parameter α does show up in the dynamics, is observable, and makes any two such supermultiplets, each with a different α-value, *usefully inequivalent* in the sense of [34]: using supermultiplets with different tuning parameter values permits constructing Lagrangians which could not be constructed without this variation.

To prove this, we construct sufficiently general Lagrangians for direct use in classical applications and in quantum models using the corresponding Hamiltonian, $H := p\cdot\dot{q}-L$, or via the partition functional $Z[\phi_*] := \int \mathbf{D}[\phi]\exp\{i\int \mathrm{d}\tau L[\phi_* + \phi, \dot{\phi}_* + \dot{\phi}, \ldots]\}$.

3.1. Simple Kinetic Terms.

3.1. Simple Kinetic Terms. Following the procedure employed in [28], we use the fact that any Lagrangian of the form

$$L := -Q_4Q_3Q_2Q_1 k(\phi, \psi, F) \tag{3}$$

is automatically supersymmetric, since its $\delta_Q := i\epsilon^I Q_I$-transformation necessarily produces a total τ-derivative. This is the direct adaptation of the construction of the so-called D-terms in standard treatments of supersymmetry [1, 2].

Dimensional analysis dictates that for kinetic-type Lagrangians we need $k(\phi, \psi, F)$ to be bilinear in the component fields $\phi_1, \phi_2, \phi_5, \phi_6$; this will produce terms of the form $\dot{\phi}_a\dot{\phi}_b$, $i\psi_\alpha\dot{\psi}_\beta$, F_AF_B, and $\dot{\phi}_aF_B$ as appropriate for kinetic terms. Table 1 lists the individually supersymmetric Lagrangian summands obtained this way after dropping total τ-derivatives. As shown, the ten bilinear functions $k(\phi) = k^{a,b}\phi_a\phi_b$ result in six linearly independent terms, so we define

$$\begin{aligned} L^{\mathrm{KE}}_{\vec{A}} := -\frac{1}{4}Q_4Q_3Q_2Q_1 \big(&A_1\phi_1{}^2 + A_2\phi_2{}^2 + A_3\phi_5{}^2 \\ &+ 2A_4\phi_1\phi_2 + 2A_5\phi_1\phi_5 + 2A_6\phi_1\phi_6\big), \end{aligned} \tag{4}$$

Table 1: Manifestly supersymmetric kinetic Lagrangian terms for the α-supermultiplet.

$\phi_i\phi_j$	$-Q^4(\phi_i\phi_j) \coloneqq -Q_4Q_3Q_2Q_1(\phi_i\phi_j)$
$(1/2)\phi_1{}^2$	$+(\dot\phi_1)^2 + (\dot\phi_2+\alpha F_8)^2 + F_3{}^2 + F_4{}^2 + i\psi_1\dot\psi_1 + i\psi_2\dot\psi_2 + i\psi_3\dot\psi_3 + i\psi_4\dot\psi_4$
$(1/2)\phi_2{}^2$	$+(\dot\phi_1-\alpha F_7)^2 + (\dot\phi_2)^2 + (F_3+\alpha\dot\phi_5)^2 + (F_4+\alpha\dot\phi_6)^2 + i(\psi_1-\alpha\psi_7)(\dot\psi_1-\alpha\dot\psi_7) + i(\psi_2+\alpha\psi_8)(\dot\psi_2+\alpha\dot\psi_8) + i(\psi_3+\alpha\psi_5)(\dot\psi_3+\alpha\dot\psi_5) + i(\psi_4-\alpha\psi_6)(\dot\psi_4-\alpha\dot\psi_6)$
$(1/2)\phi_5{}^2$	$+(\dot\phi_5)^2 + (\dot\phi_6)^2 + F_7{}^2 + F_8{}^2 + i\psi_6\dot\psi_6 + i\psi_5\dot\psi_5 + i\psi_8\dot\psi_8 + i\psi_7\dot\psi_7$
$\phi_1\phi_2{}^{\ddagger}$	$2\dot\phi_1\dot\phi_2 - 2(\dot\phi_1-\alpha F_7)(\dot\phi_2+\alpha F_8) + 2(F_3-\alpha\dot\phi_5)F_4 - 2(F_4+\alpha\dot\phi_6)F_3 + 2i\psi_1(\dot\psi_4-\alpha\dot\psi_6) + 2i\psi_2(\dot\psi_3+\alpha\dot\psi_5) - 2i\psi_3(\dot\psi_2+\alpha\dot\psi_8) - 2i\psi_4(\dot\psi_1-\alpha\dot\psi_7)$
$\phi_1\phi_5$	$+2\dot\phi_1\dot\phi_5 + 2(\dot\phi_2+\alpha F_8)\dot\phi_6 + 2F_3F_7 + 2F_4F_8 + 2i\psi_1\dot\psi_5 + 2i\psi_2\dot\psi_6 + 2i\psi_3\dot\psi_7 + 2i\psi_4\dot\psi_8$
$\phi_1\phi_6$	$+2\dot\phi_1\dot\phi_6 - 2(\dot\phi_2+\alpha F_8)\dot\phi_5 - 2F_3F_8 + 2F_4F_7 + 2i\psi_1\dot\psi_8 + 2i\psi_2\dot\psi_7 - 2i\psi_3\dot\psi_6 - 2i\psi_4\dot\psi_5$
Also, $Q^4(\phi_6{}^2) \simeq Q^4(\phi_5{}^2)$, $Q^4(\phi_6{}^2) \simeq Q^4(\phi_5{}^2)$, $Q^4(\phi_2\phi_6) \simeq Q^4(\phi_1\phi_5)$, $Q^4(\phi_5\phi_6) \simeq 0$.	

‡The $-Q^4(\phi_1\phi_2)$ entry is not simplified further to facilitate comparison with Table 2.

and we read off the actual summands from Table 1 to save space. For example,

$$\begin{aligned} L^{\mathrm{KE}}_{(1,0,1,0,0,0)} &= \frac{1}{2}\left(\dot\phi_1\right)^2 + \frac{1}{2}\left(\dot\phi_2+\alpha F_8\right)^2 + \frac{1}{2}F_3{}^2 + \frac{1}{2}F_4{}^2 \\ &+ \frac{1}{2}\left(\dot\phi_5\right)^2 + \frac{1}{2}\left(\dot\phi_6\right)^2 + \frac{1}{2}F_7{}^2 + \frac{1}{2}F_8{}^2 \\ &+ \frac{i}{2}\psi_1\dot\psi_1 + \frac{i}{2}\psi_2\dot\psi_2 + \frac{i}{2}\psi_3\dot\psi_3 + \frac{i}{2}\psi_4\dot\psi_4 \\ &+ \frac{i}{2}\psi_6\dot\psi_6 + \frac{i}{2}\psi_5\dot\psi_5 + \frac{i}{2}\psi_8\dot\psi_8 + \frac{i}{2}\psi_7\dot\psi_7 \end{aligned} \tag{5}$$

defines the "standard-looking" kinetic terms for this supermultiplet. Herein, the *local* component field redefinition

$$\left(\phi_2, F_8\right) \longmapsto \left(\widetilde\phi_2, \widetilde F_8\right), \quad \widetilde\phi_2 \coloneqq \sqrt{1+\alpha^2}\phi_2, \quad \widetilde F_8 \coloneqq \frac{F_8-\alpha\dot\phi_2}{\sqrt{1+\alpha^2}}, \tag{6}$$

would completely eliminate the appearance of the continuous tuning parameter α from the "standard-looking" Lagrangian (5) and would thus *seem* to render the supermultiplets (2) with various values of the tuning parameter α physically equivalent to each other. We note in passing that field redefinition (6) complicates the transformation table (2), the effect of which is that the partition-crossing edges in the graph in Figure 1 become regular, "two-way" edges, hiding the reducibility of the supermultiplet (2).

In fact, the α-dependence *can* be eliminated from all Lagrangians of the particular form (4) by "diagonalizing" to normal modes. To see this, note that all such Lagrangians can be written in the matrix form:

$$\begin{gathered} L^{\mathrm{KE}}_{\vec A} = \mathring\Phi^T \cdot \mathbb{K} \cdot \mathring\Phi + i\Psi^T \cdot \mathbb{M} \cdot \dot\Psi, \\ \Phi = \left(\phi_1, \phi_2, \phi_5, \phi_6 \mid F_3, F_4, F_7, F_8\right), \quad \Psi = \left(\psi_1, \dots, \psi_8\right), \quad \mathring\Phi \coloneqq \mathbb{D}\Phi, \quad \mathbb{D} \coloneqq \operatorname{diag}\left[\partial_\tau, \dots, \partial_\tau \mid 1, \dots, 1\right]. \end{gathered} \tag{7}$$

All such Lagrangians can be "diagonalized" by *local* field redefinitions $(\widetilde\Phi, \widetilde\Psi) = (\mathbb{B}\mathring\Phi, \mathbb{F}\Psi)$ defined so that $\mathbb{B}^T\mathbb{B} = \mathbb{K}$ and $\mathbb{F}^T\mathbb{F} = \mathbb{M}$. Here, $\mathbb{K}$ is manifestly symmetric and defines $\binom{8}{2} = 28$ linearly independent bosonic bilinear terms. Taking modulo total ∂_τ-derivatives, $\mathbb{M}$ defines $\binom{8}{2} = 28$ linearly independent fermionic bilinear terms. It then follows that $\mathbb{F}$ is an orthogonal basis transformation of the fermions $\psi_\alpha \to \widetilde\psi_\alpha = \Lambda_\alpha{}^\beta\psi_\beta$, and $\mathbb{B}$ is of the general form (with $\mathring{\mathbb{B}} \coloneqq \mathbb{D}^{-1}\mathbb{B}\mathbb{D}$):

$$\begin{aligned} \mathbb{B} &= \left[\begin{array}{c|c} \mathbb{A} & \mathbb{O} \\ \hline \mathbb{C} & \mathbb{E} \end{array}\right], \\ \mathring{\mathbb{B}} &= \left[\begin{array}{c|c} \mathbb{A} & \mathbb{O} \\ \hline \mathbb{C}\partial_\tau & \mathbb{E} \end{array}\right], \\ \phi_i &\longrightarrow \widetilde\phi_i = A_i{}^j\phi_j, \\ F_A &\longrightarrow \widetilde F_A = C_A{}^j\dot\phi_j + E_A{}^B F_B. \end{aligned} \tag{8}$$

This "diagonalizes" the kinetic terms (7):

$$\begin{gathered} L^{\mathrm{KE}}_{\vec A} = \mathring{\widetilde\Phi}^T \cdot \mathbb{1} \cdot \mathring{\widetilde\Phi} + i\widetilde\Psi^T \cdot \mathbb{1} \cdot \dot{\widetilde\Psi}, \\ \text{where } \mathring{\widetilde\Phi} = \mathbb{B}\mathring\Phi, \ \widetilde\Psi = \mathbb{F}\Psi. \end{gathered} \tag{9}$$

Straightforward computation then shows that

$$\begin{gathered} \text{if } \underbrace{\begin{bmatrix} \mathring\Phi' \\ \Psi' \end{bmatrix} = \begin{bmatrix} \mathbb{O} & \mathbb{L}\partial_\tau \\ \mathbb{R} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \mathring\Phi \\ \Psi \end{bmatrix}}_{\text{is a supersymmetry of (7)}} \\ \text{then } \underbrace{\begin{bmatrix} \mathring{\widetilde\Phi}' \\ \widetilde\Psi' \end{bmatrix} = \begin{bmatrix} \mathbb{B} & \mathbb{O} \\ \mathbb{O} & \mathbb{F} \end{bmatrix} \begin{bmatrix} \mathbb{O} & \mathbb{L}\partial_\tau \\ \mathbb{R} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \mathbb{B}^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{F}^{-1} \end{bmatrix} \begin{bmatrix} \mathring{\widetilde\Phi} \\ \widetilde\Psi \end{bmatrix}}_{\text{is a supersymmetry of (9)}}. \end{gathered} \tag{10}$$

Since $\det[\mathbb{B}] = \sqrt{\det[\mathbb{K}]}$ and $\det[\mathbb{F}] = \sqrt{\det[\mathbb{M}]}$, the transformation

$$\mathring{\mathbb{Q}} = \begin{bmatrix} \mathbb{O} & \mathbb{L}\partial_\tau \\ \mathbb{R} & \mathbb{O} \end{bmatrix} \longrightarrow \\ \tilde{\mathring{\mathbb{Q}}} = \begin{bmatrix} \mathbb{B} & \mathbb{O} \\ \mathbb{O} & \mathbb{F} \end{bmatrix} \begin{bmatrix} \mathbb{O} & \mathbb{L}\partial_\tau \\ \mathbb{R} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \mathbb{B}^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{F}^{-1} \end{bmatrix} \tag{11}$$

needed in (10) is well defined if and only if $\det[\mathbb{K}] \neq 0 \neq \det[\mathbb{M}]$.

Finally, since the nonzero coefficients in the final, "diagonalized" form of the kinetic Lagrangian (9) are all unity, it follows that all explicit dependence on the continuous tuning parameter α *can* be hidden from the simple kinetic Lagrangian (9) precisely if $\det[\mathbb{K}] \neq 0 \neq \det[\mathbb{M}]$.

However, the explicit dependence on the continuous tuning parameter α does not vanish from the supersymmetry transformations (2). This implies that simple kinetic Lagrangians (4) may well have a *continuum* of supersymmetries, not just a discrete number of supersymmetries as proven recently for "flat" kinetic Lagrangians [35, Appendix C.2]. Since the "kinetic-diagonalizing" field redefinition (9) complicates the supersymmetry transformation rules (2) and virtually all of the subsequent computations, we do not presume it at the outset but account for this freedom subsequently; see below.

3.2. Unhiding the Tuning Parameter. We now turn to specify some generalizations of the simple Lagrangian (4), where the explicit occurrences of α can no longer be hidden by local field redefinitions.

Mixing. Consider two separate supermultiplets of the type (2), and label their *separately variable* tuning parameters α and β, respectively:

$$(\phi_1, \phi_2, \phi_5, \phi_6 \mid \psi_1, \ldots \psi_7 \mid F_3, F_4, F_7, F_8)_\alpha, \\ (\varphi_1, \varphi_2, \varphi_5, \varphi_6 \mid \chi_1, \ldots \chi_7 \mid G_3, G_4, G_7, G_8)_\beta. \tag{12}$$

Now consider even just the bilinear coupling Lagrangians of the form

$$L^{\mathrm{KE}}_{\vec{A};\alpha,\beta} := -Q_4Q_3Q_2Q_1 \Big(\vec{A}^{(\alpha)} \cdot \vec{f}(\phi) + \vec{A}^{(\beta)} \cdot \vec{g}(\varphi) \\ + \vec{A}^{(\alpha,\beta)} \cdot \vec{h}(\phi,\varphi) \Big), \tag{13}$$

where $\vec{A}^{(\alpha)} \cdot \vec{f}(\phi)$ are the bilinear terms (4) and $\vec{A}^{(\beta)} \cdot \vec{g}(\varphi)$ terms are constructed in a precisely analogous way but for the supermultiplet $(\varphi \mid \chi \mid G)_\beta$, involving a corresponding independent set of six coefficients.

Finally, $\vec{A}^{(\alpha,\beta)} \cdot \vec{h}(\phi,\varphi)$ represents the mixing terms, constructed as a general linear combination of the fourteen analogously constructed terms, listed in Table 2, and it is the inclusion of these terms that can obstruct the hiding of the tuning parameters α and β. To see this and motivated by the analysis of (4), consider even the very simple analogue of (5):

$$L^{\mathrm{KE}}_{\mathrm{mix.}} = -\frac{1}{2} Q_4Q_3Q_2Q_1 (\phi_1\varphi_1 + \phi_5\varphi_5), \\ L^{\mathrm{KE}}_{\mathrm{mix.}} = \dot{\phi}_1\dot{\varphi}_1 + (\dot{\phi}_2 + \alpha F_8)(\dot{\varphi}_2 + \beta G_8) + F_3G_3 + F_4G_4 \\ + \dot{\phi}_5\dot{\varphi}_5 + \dot{\phi}_6\dot{\varphi}_6 + F_7G_7 + F_8G_8 + \cdots, \tag{14}$$

where the ellipses indicate that the fermionic bilinear terms were omitted. As in (6), the local field redefinition

$$\widetilde{\phi}_2 := \sqrt{1+\alpha\beta}\,\phi_2, \\ \widetilde{F}_8 := \frac{F_8 - \beta\dot{\phi}_2}{\sqrt{1+\alpha\beta}}, \\ \widetilde{\varphi}_2 := \sqrt{1+\alpha\beta}\,\varphi_2, \\ \widetilde{G}_8 := \frac{G_8 - \alpha\dot{\varphi}_2}{\sqrt{1+\alpha\beta}} \tag{15}$$

turns $L^{\mathrm{KE}}_{\mathrm{mix.}}$ into

$$L^{\mathrm{KE}}_{\mathrm{mix.}} = \dot{\phi}_1\dot{\varphi}_1 + \dot{\widetilde{\phi}}_2\dot{\widetilde{\varphi}}_2 + F_3G_3 + F_4G_4 + \dot{\phi}_5\dot{\varphi}_5 + \dot{\phi}_6\dot{\varphi}_6 \\ + F_7G_7 + \widetilde{F}_8\widetilde{G}_8 + \cdots \tag{16}$$

hiding both α and β. The transformation (15) preserves the reality of the component fields only if $\alpha\beta > -1$ and becomes undefined (diverges) when $\alpha\beta \to -1$. Also, the transformation (15) does not hide the α-dependence in (4) or the β-dependence in $-Q^4({\varphi_1}^2 + {\varphi_5}^2)$. Thus, by explicit counterexample, we have the following theorem.

Theorem 1. *There do exist Lagrangians of the generic type (13) in which no well-defined, real, local field redefinition can hide the explicit α- and β-dependence.*

The expression (13) then provides a $6 + 6 + 14 = 26$-parameter continuous family of bilinear Lagrangians for the two distinct 1-parameter families of supermultiplets: one such family for every choice of the pair $(\alpha, \beta) \in \mathbb{R}^2$! Generic choices in this 26-dimensional parameter space $\{\vec{A}^{(\alpha)}, \vec{A}^{(\beta)}, \vec{A}^{(\alpha,\beta)}\}$ define Lagrangians that depend irremovably on both tuning parameters α and β and so provide for dynamical responses that can be used to observe the values of α and β and indeed any difference between them. This then is the practical distinction between $(\phi \mid \psi \mid F)_\alpha$ and $(\varphi \mid \chi \mid G)_\beta$. For each value of the tuning parameters $(\alpha, \beta) \in \mathbb{R}^2$, they provide a distinct and *usefully inequivalent* pair of off-shell representations of $N = 4$-extended supersymmetry.

Recall that the familiar chiral and twisted-chiral superfields afford constructing σ-models with target spaces that cannot be described using only chiral superfields [36, 37]. In precise analogy, $(\phi \mid \psi \mid F)_\alpha$ and $(\varphi \mid \chi \mid G)_\beta$ *jointly* afford Lagrangians that cannot be constructed using only copies of one of the two, except that (2) presents a *continuum* of

Table 2: Fourteen bilinear "D-term"-type manifestly supersymmetric Lagrangian terms that couple the α-supermultiplet with the β-supermultiplet and showcase the appearance of these tuning parameters.

$\phi_i\varphi_j$	$-Q^4(\phi_i\varphi_j) \coloneqq -Q_4Q_3Q_2Q_1(\phi_i\varphi_j)$
$\phi_1\varphi_1$	$+2\dot{\phi}_1\dot{\varphi}_1 + 2(\dot{\phi}_2 + \alpha F_8)(\dot{\varphi}_2 + \beta G_8) + 2F_3G_3 + 2F_4G_4 + 2i\psi_1\dot{\chi}_1 + 2i\psi_2\dot{\chi}_2 + 2i\psi_3\dot{\chi}_3 + 2i\psi_4\dot{\chi}_4$
$\phi_1\varphi_2$	$+2\dot{\phi}_1\dot{\varphi}_2 - 2(\dot{\phi}_2+\alpha F_8)(\dot{\varphi}_1-\beta G_7) - 2F_3(G_4+\beta\dot{\varphi}_6) + 2F_4(G_3-\beta\dot{\varphi}_5) + 2i\psi_1(\dot{\chi}_4-\beta\dot{\chi}_6) + 2i\psi_2(\dot{\chi}_3+\beta\dot{\chi}_5) - 2i\psi_3(\dot{\chi}_2+\beta\dot{\chi}_8) - 2i\psi_4(\dot{\chi}_1-\beta\dot{\chi}_7)$
$\phi_1\varphi_5$	$+2\dot{\phi}_1\dot{\varphi}_5 + 2(\dot{\phi}_2 + \alpha F_8)\dot{\varphi}_6 + 2F_3G_7 + 2F_4G_8 + 2i\psi_1\dot{\chi}_5 + 2i\psi_2\dot{\chi}_6 + 2i\psi_3\dot{\chi}_7 + 2i\psi_4\dot{\chi}_8$
$\phi_1\varphi_6$	$+2\dot{\phi}_1\dot{\varphi}_6 - 2(\dot{\phi}_2 + \alpha F_8)\dot{\varphi}_5 - 2F_3G_8 + 2F_4G_7 + 2i\psi_1\dot{\chi}_8 + 2i\psi_2\dot{\chi}_7 - 2i\psi_3\dot{\chi}_6 - 2i\psi_4\dot{\chi}_5$
$\phi_2\varphi_1$	$+2\dot{\phi}_2\dot{\varphi}_1 - 2(\dot{\phi}_1-\alpha F_7)(\dot{\varphi}_2+\beta G_8) + 2(F_3-\alpha\dot{\phi}_5)G_4 - 2(F_4+\alpha\dot{\phi}_6)G_3 + 2i(\psi_4-\alpha\psi_6)\dot{\chi}_1 + 2i(\psi_3+\alpha\psi_5)\dot{\chi}_2 - 2i(\psi_2+\alpha\psi_8)\dot{\chi}_3 - 2i(\psi_1-\alpha\psi_7)\dot{\chi}_4$
$\phi_2\varphi_2$	$+2(\dot{\phi}_1 - \alpha F_7)(\dot{\varphi}_1 - \beta G_7) + 2\dot{\phi}_2\dot{\varphi}_2 + 2(F_3 + \alpha\dot{\phi}_5)(G_3 + \beta\dot{\varphi}_5) + 2(F_4 + \alpha\dot{\phi}_6)(G_4 + \beta\dot{\varphi}_6) + 2i(\psi_1 - \alpha\psi_7)(\dot{\chi}_1 - \beta\dot{\chi}_7) + 2i(\psi_2 + \alpha\psi_8)(\dot{\chi}_2 + \beta\dot{\chi}_8) + 2i(\psi_3 + \alpha\psi_5)(\dot{\chi}_3 + \beta\dot{\chi}_5) + 2i(\psi_4 - \alpha\psi_6)(\dot{\chi}_4 - \beta\dot{\chi}_6)$
$\phi_2\varphi_5$	$+2\dot{\phi}_2\dot{\varphi}_5 - 2(\dot{\phi}_1 - \alpha F_7)\dot{\varphi}_6 + 2(F_3 + \alpha\dot{\phi}_5)G_8 - 2(F_4 + \alpha\dot{\phi}_6)G_7 + 2i(\psi_1 - \alpha\psi_7)\dot{\chi}_8 + 2i(\psi_2 + \alpha\psi_8)\dot{\chi}_7 - 2i(\psi_3 + \alpha\psi_5)\dot{\chi}_6 - 2i(\psi_4 - \alpha\psi_6)\dot{\chi}_5$
$\phi_2\varphi_6$	$+2\dot{\phi}_2\dot{\varphi}_6 + 2(\dot{\phi}_1 - \alpha F_7)\dot{\varphi}_5 + 2(F_3 + \alpha\dot{\phi}_5)G_7 + 2(F_4 + \alpha\dot{\phi}_6)G_8 + 2i(\psi_1 - \alpha\psi_7)\dot{\chi}_5 + 2i(\psi_2 + \alpha\psi_8)\dot{\chi}_6 + 2i(\psi_3 + \alpha\psi_5)\dot{\chi}_7 + 2i(\psi_4 - \alpha\psi_6)\dot{\chi}_8$
$\phi_5\varphi_1$	$+2\dot{\phi}_5\dot{\varphi}_1 + 2\dot{\phi}_6(\dot{\varphi}_2 + \beta G_8) + 2F_7G_3 + 2F_8G_4 + 2i\psi_6\dot{\chi}_2 + 2i\psi_5\dot{\chi}_1 + 2i\psi_8\dot{\chi}_4 + 2i\psi_7\dot{\chi}_3$
$\phi_5\varphi_2$	$+2\dot{\phi}_5\dot{\varphi}_2 - 2\dot{\phi}_6(\dot{\varphi}_1 - \beta G_7) - 2F_7(G_4 + \beta\dot{\varphi}_6) + 2F_8(G_3 + \beta\dot{\varphi}_5) - 2i\psi_6(\dot{\chi}_3 - \beta\dot{\chi}_5) + 2i\psi_5(\dot{\chi}_4 - \beta\dot{\chi}_6) - 2i\psi_8(\dot{\chi}_1 - \beta\dot{\chi}_7) + 2i\psi_7(\dot{\chi}_2 - \beta\dot{\chi}_8)$
$\phi_5\varphi_5$	$+2\dot{\phi}_5\dot{\varphi}_5 + 2\dot{\phi}_6\dot{\varphi}_6 + 2F_7G_7 + 2F_8G_8 - 2i\psi_6\dot{\chi}_6 - 2i\psi_5\dot{\chi}_5 - 2i\psi_8\dot{\chi}_8 - 2i\psi_7\dot{\chi}_7$
$\phi_5\varphi_6$	$+2\dot{\phi}_5\dot{\varphi}_6 - 2\dot{\phi}_6\dot{\varphi}_5 - 2F_7G_8 + 2F_8G_7 + 2i\psi_6\dot{\chi}_7 + 2i\psi_5\dot{\chi}_8 - 2i\psi_8\dot{\chi}_5 - 2i\psi_7\dot{\chi}_6$
$\phi_6\varphi_1$	$+2\dot{\phi}_6\dot{\varphi}_1 - 2\dot{\phi}_5(\dot{\varphi}_2 + \beta G_8) + 2F_7G_4 - 2F_8G_3 - 2i\psi_6\dot{\chi}_3 - 2i\psi_5\dot{\chi}_4 + 2i\psi_8\dot{\chi}_1 + 2i\psi_7\dot{\chi}_2$
$\phi_6\varphi_2$	$+2\dot{\phi}_6\dot{\varphi}_2 + 2\dot{\phi}_5(\dot{\varphi}_1 - \beta G_7) + 2F_7(G_3 + \beta\dot{\varphi}_5) + 2F_8(G_4 + \beta\dot{\varphi}_6) + 2i\psi_6(\dot{\chi}_2 + \beta\dot{\chi}_8) + 2i\psi_5(\dot{\chi}_1 - \beta\dot{\chi}_7) + 2i\psi_8(\dot{\chi}_4 - \beta\dot{\chi}_6) + 2i\psi_7(\dot{\chi}_3 + \beta\dot{\chi}_5)$

Also, $Q^4(\phi_6\varphi_5) \simeq Q^4(\phi_5\varphi_6), Q^4(\phi_6\varphi_6) \simeq Q^4(\phi_5\varphi_5)$.

such usefully inequivalent supermultiplets, not just a discrete set of two! The most general bilinear "kinetic" Lagrangian providing for pairwise coupling of the continuously many inequivalent, off-shell supermultiplets of this type is then a double Q-moduli space *integral*:

$$L^{\mathrm{KE}}_{\mathrm{bilinear}} \coloneqq \int \mathrm{d}\alpha\mathrm{d}\beta w_{\vec{A}}(\alpha,\beta)\, L^{\mathrm{KE}}_{\vec{A};\alpha,\beta}, \tag{17}$$

where $w_{\vec{A}}(\alpha,\beta)$ is some appropriate weight function ensuring the convergence of the double integral over the coarse moduli space, $\mathbb{R}^2_{\alpha,\beta}$. Depending on the particular choice of the Lagrangian, that is, a choice of the various $\vec{A}$-parameters in (13), certain different values in the (α,β)-plane will produce physically equivalent dynamics, generating a corresponding "mapping class group," $\Gamma_{\vec{A}}$, of symmetries. In particular, Section 3.1 shows that there do exist proper local field redefinitions that can hide the 6 + 6 parameters $\vec{A}^{(\alpha)}, \vec{A}^{(\beta)}$ in (13), but it is not clear how many of the 14 parameters $\vec{A}^{(\alpha,\beta)}$—if any—can be hidden this way; a precise determination of the "mapping class group" and corresponding physical equivalences (dualities) will have to remain open for now. The weight function will have to be invariant with respect to this $\Gamma_{\vec{A}}$, seems likely to be model-dependent, and so is also beyond our present scope. Suffice it here to mention that, in familiar cases (such as the Deligne-Mumford-Teichmüller theory for Riemann surfaces, the moduli spaces of Calabi-Yau manifolds, and the (super)string landscape [38–41]), the analogue of this $\Gamma_{\vec{A}}$ is discrete and the analogue of the quotient $\mathbb{R}^2_{\alpha,\beta}/\Gamma_{\vec{A}}$ is a compact, albeit singular space.

The Lagrangians (17) depend on the continuous tuning parameters α,β differently from all previously studied supersymmetric Lagrangians. The tuning parameters α,β are not the familiar coefficients parametrizing the choice of the Lagrangian as A_i's above are. Instead, the tuning parameters α,β do parametrize the supersymmetry action within the supermultiplet (2), in a way not unlike within the formalism of "projective superspace" [42, 43]; see [44, 45] for the relation to the more general "harmonic superspace." However, in all those efforts, all supersymmetric Lagrangians are localized to special values of those parameters, whereas Lagrangians such as (13) are supersymmetric for every choice of α,β, and the integral (17) is then also supersymmetric.

Since an analogous continuous parameter $\alpha \neq 1$ may be *introduced* in the $N = 3$ supermultiplets studied in [27, 28], those specific supermultiplets are also just special members of separate Q-continua of off-shell supermultiplets, all usefully inequivalent in the same sense.

3.3. Super-Zeeman Terms. We now turn to Lagrangian terms that are still bilinear but where dimensional analysis requires an overall dimension-full parameter of the kind that may be identified as a Larmor-like frequency, coupling the supermultiplet (2) to external magnetic fields [28, 46].

In general, we seek functions $f(\phi,\psi,F)$ such that each of

$$\begin{gathered} Q_3Q_2Q_1 f(\phi,\psi,F), \\ Q_4Q_2Q_1 f(\phi,\psi,F), \\ Q_4Q_3Q_1 f(\phi,\psi,F), \\ Q_4Q_3Q_2 f(\phi,\psi,F) \end{gathered} \tag{18}$$

vanishes modulo total derivatives. Then, the six quadratic derivatives

$$\begin{gathered} Q_2Q_1 f(\phi,\psi,F), \\ Q_3Q_1 f(\phi,\psi,F), \\ Q_3Q_2 f(\phi,\psi,F), \end{gathered}$$

TABLE 3: The $Q_3Q_2Q_1$-transforms of bosonic bilinear terms, modulo total τ-derivatives.

$\phi_i\phi_j$	$-iQ_3Q_2Q_1(\phi_i\phi_j)$	
$(1/2)\phi_1\phi_1$	$+\dot{\phi}_1\psi_4 - (\dot{\phi}_2 + \alpha F_8)\psi_1 + F_3\psi_2 - F_4\psi_3$	
$(1/2)\phi_2\phi_2$	$+(\dot{\phi}_1 - \alpha F_7)(\psi_4 - \alpha\psi_6) - \dot{\phi}_2(\psi_1 - \alpha\psi_7) + (F_3 + \alpha\dot{\phi}_5)(\psi_2 + \alpha\psi_8) - (F_4 + \alpha\dot{\phi}_6)(\psi_3 + \alpha\psi_5)$	
$(1/2)\phi_5\phi_5$	$+\dot{\phi}_5\psi_8 - \dot{\phi}_6\psi_5 + F_7\psi_6 - F_8\psi_7$	subtract
$(1/2)\phi_6\phi_6$	$+\dot{\phi}_5\psi_8 - \dot{\phi}_6\psi_5 + F_7\psi_6 - F_8\psi_7$	
$\phi_1\phi_2$	$+\alpha[\dot{\phi}_1\psi_7 + (\dot{\phi}_2 + \alpha F_8)\psi_6 + F_3\psi_5 + F_4\psi_8 - \dot{\phi}_5\psi_3 - \dot{\phi}_6\psi_2 - F_7\psi_1 - F_8\psi_4]$	
$\phi_1\phi_5$	$+\dot{\phi}_1\psi_8 - (\dot{\phi}_2 + \alpha F_8)\psi_5 + F_3\psi_6 - F_4\psi_7 + \dot{\phi}_5\psi_4 - \dot{\phi}_6\psi_1 + F_7\psi_2 - F_8\psi_3$	subtract
$\phi_2\phi_6$	$+\dot{\phi}_1\psi_8 - (\dot{\phi}_2 + \alpha F_8)\psi_5 + F_3\psi_6 - F_4\psi_7 + \dot{\phi}_5\psi_4 - \dot{\phi}_6\psi_1 + F_7\psi_2 - F_8\psi_3$	
$\phi_1\phi_6$	$-\dot{\phi}_1\psi_5 - (\dot{\phi}_2 + \alpha F_8)\psi_8 + F_3\psi_7 + F_4\psi_6 + \dot{\phi}_5\psi_1 + \dot{\phi}_6\psi_4 - F_7\psi_3 - F_8\psi_2$	add
$\phi_2\phi_5$	$+\dot{\phi}_1\psi_5 + (\dot{\phi}_2 + \alpha F_8)\psi_8 - F_3\psi_7 - F_4\psi_6 - \dot{\phi}_5\psi_1 - \dot{\phi}_6\psi_4 + F_7\psi_3 + F_8\psi_2$	
$\phi_5\phi_6$	$\partial_\tau(\phi_5\psi_5 - \phi_6\psi_8) \simeq 0$	

$$Q_4Q_1 f(\phi,\psi,F),\quad Q_4Q_2 f(\phi,\psi,F),\quad Q_4Q_3 f(\phi,\psi,F) \tag{19}$$

are all manifestly supersymmetric. When applying $\delta_Q = i\epsilon \cdot Q$, Q_I from δ_Q either equals one of the two Q_I's used in the definition (19) and so produces $i\partial_\tau$ by (1) or does not and so reproduces one of the expressions (18) and again a total τ-derivative by assumption (18). Such terms remind us of the so-called F-terms in standard treatments of supersymmetry [1, 2].

We again restrict our attention to bilinear terms for simplicity, and Table 3 presents the linearly independent such terms, obtained by applying only the first batch of three supercharges. The other three expressions (18) each produce analogous results with a pattern virtually identical to the one shown in Table 3. The last-row entry, $\phi_5\phi_6$, results in a total τ-derivative all by itself, and simple row operations (indicated by braces) show that we can form three more. This means that each of the twenty-four terms

$$\begin{gathered}\frac{1}{2}Q_IQ_J\left({\phi_5}^2 - {\phi_6}^2\right),\\ Q_IQ_J\left(\phi_1\phi_5 - \phi_2\phi_6\right),\\ Q_IQ_J\left(\phi_1\phi_6 + \phi_2\phi_5\right),\\ Q_IQ_J\left(\phi_5\phi_6\right)\end{gathered} \tag{20}$$

is a supersymmetric Lagrangian contribution. This list turns out to be repetitive and contains only four linearly independent expressions, listed in Table 4. The most general super-Zeeman type Lagrangian bilinear in the component fields of the $(\phi \mid \psi \mid F)_\alpha$ supermultiplet is therefore

$$L^{\mathrm{SZ}}_{\vec{B};\alpha} := B_1Z_1 + B_2Z_2 + B_3Z_3 + B_4Z_4, \tag{21}$$

with the terms Z_i listed in Table 4. Of these, only the last term contains the expression

$$\begin{aligned} B_4Z_4 &= \cdots + \alpha B_4\phi_5\dot{\phi}_6 + \cdots \\ &\simeq \cdots + \frac{1}{2}\alpha B_4\left(\phi_5\dot{\phi}_6 - \dot{\phi}_5\phi_6\right) + \cdots \end{aligned} \tag{22}$$

which in Lagrangian physics may be interpreted as the coupling of the magnetic field B_4 to the angular momentum of rotation in the (ϕ_5,ϕ_6)-plane—if the bosons ϕ_5,ϕ_6 are interpreted as Cartesian coordinates in the target space. The elimination of the auxiliary fields F_3, F_4, F_7, F_8 (and G_3, G_4, G_7, G_8) by means of their equations of motion is expected to induce additional terms of the type (22) owing to the mixing of the auxiliary fields with the τ-derivatives of the propagating fields ϕ_i. This justifies the identification of the terms (21) and the supersymmetric version of the $\vec{B}\cdot\vec{L}$ terms exhibiting the Zeeman effect.

Summary. The four terms in Table 4 together with their $(\phi \mid \psi \mid F)_\alpha \to (\varphi \mid \chi \mid G)_\beta$ counterparts and the 26-parameter Lagrangian (13) then form the most general, 34-parameter family of bilinear Lagrangians

$$L^{\mathrm{KE}}_{\vec{A};\alpha,\beta} + L^{\mathrm{SZ}}_{\vec{B};\alpha} + L^{\mathrm{SZ}}_{\vec{B};\beta} \tag{23}$$

for two different supermultiplets from the family (2).

Many of the summands in Tables 1, 2, and 4 have negative signs and so would—if used on their own—contribute negatively to the kinetic energy, *that is*, induce nonpositivity of the kinetic energy and nonunitarity in general. However, when they are used *jointly* with the first three supersymmetric sets of kinetic terms in Table 1 (which are positive-definite), it is clear that unitarity constrains the coefficients A_i in (4) so that A_4, A_5, A_6 as well as the coefficients of the Lagrangian summands from Tables 2 and 4 should be sufficiently smaller than A_1, A_2, A_3. This is similar to the analogous case examined in [28].

Table 4: Super-Zeeman bilinear contributions, modulo total τ-derivatives.

$Z_1 := \phi_5 F_7 + \phi_6 F_8 + i\psi_6\psi_8 - i\psi_6\psi_8$
$Z_2 := \phi_5 F_8 - \phi_6 F_7 + i\psi_6\psi_5 + i\psi_8\psi_7$
$Z_3 := \phi_1 F_7 + \phi_2 F_8 + \phi_5 F_3 + \phi_6 F_4 - i\psi_1\psi_7 + i\psi_2\psi_8 + i\psi_3\psi_5 - i\psi_4\psi_6$
$Z_4 := \phi_1 F_8 - \phi_2 F_7 - \phi_6 F_3 + \phi_5 F_4 - i\psi_1\psi_6 + i\psi_2\psi_5 - i\psi_3\psi_8 + i\psi_4\psi_7 + \alpha(\phi_5\dot{\phi}_6 - i\psi_6\psi_7 - i\psi_5\psi_8)$

Requiring positivity of the kinetic energy, and unitarity more generally, restricts these parameters to an open neighborhood in this 34-dimensional parameter space. Most of the corresponding models depend explicitly on the tuning parameter $\alpha \in \mathbb{R}$ (and β for two copies of the supermultiplet (2), etc.) *besides* the dependence on the coefficients of these summands. This parameter α (and β for two copies, etc.) then provides a genuine, observable characteristic of the supermultiplet (2). We conclude that the supermultiplets (2) which differ only in a different choice of the parameter α cannot be regarded as physically equivalent in general. This dependence on the tuning parameter α becomes only more complex in the general (not just bilinear) "D-term" (3) and "F-term" (18)-(19) Lagrangian summands.

4. Sample Dynamics

To illustrate the intricate dependence on α, *that is*, on the choice from among the continuously many inequivalent supermultiplets, consider the sum of the Lagrangian summands (4) with $\vec{A}' = (a_1, 0, a_3, 0, 0, 0)$ and those in (21) with $\vec{B}' = (0, 0, 0, B_4)$, and focus only on the bosonic fields:

$$\begin{aligned} L^{\text{KE}}_{\vec{A}';\alpha} &= \frac{a_1}{2}\left[\dot{\phi}_1{}^2 + \left(\dot{\phi}_2 + \alpha F_8\right)^2 + F_3{}^2 + F_4{}^2\right] \\ &+ \frac{a_3}{2}\left[\dot{\phi}_5{}^2 + \dot{\phi}_6{}^2 + F_7{}^2 + F_8{}^2\right] + \cdots + B_4\Big[\phi_1 F_8 \\ &- \phi_2 F_7 - \phi_6 F_3 + \phi_5 F_4 + \frac{\alpha}{2}\left(\phi_5\dot{\phi}_6 - \dot{\phi}_5\phi_6\right)\Big] + \cdots, \end{aligned} \tag{24}$$

where the ellipses denote the omitted fermionic terms. The external magnetic field, B_4, is here coupled only to the angular momentum in the (ϕ_5, ϕ_6)-plane.

The Euler-Lagrange equations of motion for F_3, F_4, F_7, F_8 are of course algebraic:

$$\begin{aligned} F_3 &= \frac{B_4\phi_6}{a_1}, \\ F_4 &= -\frac{B_4\phi_5}{a_1}, \\ F_7 &= \frac{B_4\phi_2}{a_3}, \\ F_8 &= -\frac{\alpha a_1\dot{\phi}_2 + B_4\phi_1}{\alpha^2 a_1 + a_3}, \\ a_3 &\neq -\alpha^2 a_1. \end{aligned} \tag{25}$$

The special case when $a_3 = -\alpha^2 a_1$ must be treated separately. Substituting these back into the Lagrangian yields

$$\begin{aligned} L^{\text{KE}}_{\vec{A}';\alpha}\Big|_{F_A} &= \frac{a_1}{2}\dot{\phi}_1{}^2 + \frac{a_1 a_3}{2\left(a_3 + a_1\alpha^2\right)}\dot{\phi}_2{}^2 + \frac{a_3}{2}\dot{\phi}_5{}^2 \\ &+ \frac{a_3}{2}\dot{\phi}_6{}^2 - \frac{B_4{}^2}{2\left(a_3 + a_1\alpha^2\right)}\phi_1{}^2 - \frac{B_4{}^2}{2a_3}\phi_2{}^2 \\ &- \frac{B_4{}^2}{2a_1}\phi_5{}^2 - \frac{B_4{}^2}{2a_1}\phi_6{}^2 + \cdots \\ &+ \frac{a_1}{a_3 + a_1\alpha^2}\frac{\alpha}{2}B_4\left(\dot{\phi}_1\phi_2 - \phi_1\dot{\phi}_2\right) \\ &- \frac{\alpha}{2}B_4\left(\dot{\phi}_5\phi_6 - \phi_5\dot{\phi}_6\right) + \cdots \end{aligned} \tag{26}$$

which induces a coupling of the external angular momentum also to the angular momentum in the (ϕ_1, ϕ_2)-plane but with $a_1/(a_3 + a_1\alpha^2)$ times the magnitude of the interaction in the (ϕ_5, ϕ_6)-plane.

The Euler-Lagrange equations are

$$\begin{aligned} 0 &= a_1\ddot{\phi}_1 + \frac{\alpha a_1 B_4}{\alpha^2 a_1 + a_3}\dot{\phi}_2 + \frac{B_4{}^2}{\alpha^2 a_1 + a_3}\phi_1, \\ 0 &= \frac{a_1 a_3}{\alpha^2 a_1 + a_3}\ddot{\phi}_2 - \frac{\alpha a_1 B_4}{\alpha^2 a_1 + a_3}\dot{\phi}_1 + \frac{B_4{}^2}{a_3}\phi_2; \\ 0 &= a_3\ddot{\phi}_5 - \alpha B_4\dot{\phi}_6 + \frac{B_4{}^2}{a_1}\phi_5, \\ 0 &= a_3\ddot{\phi}_6 + \alpha B_4\dot{\phi}_5 + \frac{B_4{}^2}{a_1}\phi_6. \end{aligned} \tag{27}$$

The two coupled pairs of differential equations both describe a similar response to the external magnetic field B_4. A little surprisingly perhaps, the frequencies in the solutions to both pairs are (system (27) produces four decoupled 4th-order, linear equations with constant coefficients, which are easily solved using trial $e^{i\omega t}$-like functions)

$$\omega_{12\pm} = \omega_{56\pm} := \frac{\sqrt{2a_3 + a_1\alpha^2 \pm \alpha\sqrt{a_1\left(4a_3 + a_1\alpha^2\right)}}}{\sqrt{2a_1}a_3}B_4 \tag{28}$$

and clearly depend on α; note that $\omega_{ij-}(\alpha) = \omega_{ij+}(-\alpha)$. This proves that the value of the tuning parameter α, effectively limited to $\alpha \geqslant 0$, is physically observable through probing with external magnetic fields and that distinctly "tuned" supermultiplets of type (2) are observably (and so usefully) inequivalent [34]. These frequencies acquire complex or purely imaginary values for certain choices of a_1, a_3, and α, describing, respectively, attenuated/boosted oscillatory or hyperbolic motion. The frequencies are real for

$$\left((\alpha \neq 0), \left(a_1 \geqslant -\frac{4a_3}{\alpha^2}\right), (a_3 > 0)\right) \vee \left((\alpha = 0), (a_1, a_3 > 0)\right). \quad (29)$$

Furthermore, the frequencies (28) are incommensurate for most choices of a_1, a_3, and α. That is, the ratio

$$\frac{\omega_{12+}}{\omega_{12-}} = \frac{\omega_{56+}}{\omega_{56-}} = \sqrt{\frac{2a_3 + a_1\alpha^2 + \alpha\sqrt{a_1\left(4a_3 + a_1\alpha^2\right)}}{2a_3 + a_1\alpha^2 - \alpha\sqrt{a_1\left(4a_3 + a_1\alpha^2\right)}}} \quad (30)$$

is not an integer for most choices of the coefficients in the Lagrangian a_1, a_3 and the tuning parameter α; the orbits are space-filling Bowditch/Lissajous-like figures. Therefore, the dynamics of the supermultiplet (2) governed by the Lagrangian (24) exhibits nonrepetitive (pseudorandom or chaotic) oscillatory motion for most of the parameter values (29). This effect reminds us of the nonrepetitive (chaotic) dynamics also found for a similar supermultiplet of $N = 3$ supersymmetry on the world-line [28].

5. Conclusions

We have presented a 1-parameter continuous family of off-shell supermultiplets (2) of $N = 4$ world-line supersymmetry, which radically generalizes the study of the discrete sequence of off-shell supermultiplets [27, 28]. In fact, all of the qualitative conclusions from the present study of (2) apply just as well to a similar Q-continuum of $N = 3$ off-shell supermultiplets within which the supermultiplets [27, 28] are special cases.

The supermultiplet (2) exhibits an explicit, continuously variable tuning parameter, labeled α, the value of which controls the relative "magnitude" in the binomial results of applying the supercharges to the component fields. By virtue of the existence of these binomial terms, the supermultiplet (2) may be thought of as a network of Adinkras [21] connected by one-way edges, as depicted in Figure 1.

For two distinct members from this continuous family of off-shell supermultiplets, we have constructed a multiparameter family of general bilinear Lagrangians (23) which has the following characteristics:

(1) It generalizes the "standard" kinetic terms (as in (5), with $\alpha \to 0$) into a 6-parameter family of Lagrangians (4) but which all by itself may be simplified back to the "standard" form by means of local field redefinitions.

(2) It mixes two off-shell supermultiplets of the same type (2), each with a different value of the tuning parameter, given as 14-parameter linear combinations of the terms from Table 2, where the explicit dependence on the tuning parameter(s) $\alpha(\beta, \ldots)$ cannot always be eliminated.

(3) It couples such supermultiplets to external magnetic fields inducing a variant of the super-Zeeman effect, given as 4-parameter linear combinations of the terms from Table 4, where the explicit dependence on the tuning parameter(s) $\alpha(\beta, \ldots)$ cannot always be eliminated.

(4) The multidimensional parameter space of the Lagrangians (23) has at least one open neighborhood, where the kinetic energy is guaranteed to be positive, indicating unitarity of the corresponding quantum theory.

Using the constructions described in Section 3, these Lagrangians can be generalized to include (a) higher-order interaction terms and (b) couplings to additional and all differently tuned supermultiplets from the family (2).

Section 4 then demonstrates that, except for very special choices within this parameter space, the Lagrangians explicitly depend on the tuning parameter α, and also β in (13), of the supermultiplet (2), in ways that have direct dynamical consequences, and observably affect the response of these supermultiplets to probing by external magnetic fields.

Furthermore, the wealth and diversity of even just the bilinear coupling/mixing terms listed in Table 2 indicate that supermultiplets with a different choice of the tuning parameter are indeed observably different and so usefully inequivalent in the sense of [34]. The same analysis applies just as well for the infinite sequence of supermultiplets discussed in [27].

We thus have clear proof by explicit example that inequivalent off-shell supermultiplets (unitary, finite-dimensional linear representations) of world-line N-extended supersymmetry without central extensions form a physically observable continuum. We have explicitly parametrized this Q-continuum in terms of the "tuning parameter" α appearing explicitly in (2) and have shown that even simple, bilinear Lagrangians such as (23) provide pairwise coupling between *continuously* many inequivalent such off-shell supermultiplets. This in turn provides a way of physically probing the variation in the dynamics over this novel Q-moduli space.

Quite clearly, just as all bilinear Lagrangians (23) depend quadratically on the tuning parameters, general σ-model Lagrangians including but not limited to (3) will exhibit more general variation over the Q-moduli space. However, already the dynamics governed by Lagrangians (23) restricted to purely bilinear terms exhibit physically observable and highly nonlinear dependence on α—as exhibited in the frequencies (28)—albeit being derived from Lagrangian (24) that is itself only quadratic in α.

Such variations in dynamics are at the core of studies such as the Deligne-Mumford-Teichmüller theory for (super)string world-sheets, moduli spaces of Calabi-Yau compactifications, and the (super)string landscape [38–41]. We have herein shown that conceptually similar moduli spaces also emerge, in a qualitatively similar manner, in the representation theory of supersymmetry.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, *Superspace*, Benjamin/Cummings, Reading, Mass, USA, 1983.

[2] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, USA, 2nd edition, 1992.

[3] F. Iachello, "Dynamical supersymmetries in nuclei," *Physical Review Letters*, vol. 44, no. 12, pp. 772–775, 1980.

[4] R. Bijker, "Supersymmetry in nuclear physics," *Journal of Physics: Conference Series*, vol. 237, Article ID 012005, 2010.

[5] D. Friedan, Z. Qiu, and S. Shenker, "Conformal invariance, unitarity, and critical exponents in two dimensions," *Physical Review Letters*, vol. 52, no. 18, pp. 1575–1578, 1984.

[6] A. M. Gainutdinov, J. L. Jacobsen, N. Read, H. Saleur, and R. Vasseur, "Logarithmic conformal field theory: a lattice approach," *Journal of Physics A: Mathematical and Theoretical*, vol. 46, no. 49, Article ID 494012, 2013.

[7] P. K. Ghosh, "Supersymmetric many-particle quantum systems with inverse-square interactions," *Journal of Physics A. Mathematical and Theoretical*, vol. 45, no. 18, Article ID 183001, 2012.

[8] K. Hasebe and K. Totsuka, "Topological many-body states in quantum antiferromagnets via fuzzy supergeometry," *Symmetry*, vol. 5, no. 2, pp. 119–214, 2013.

[9] S. J. Gates Jr., W. D. Linch III, J. Phillips, and L. Rana, "The fundamental supersymmetry challenge remains," *Gravitation & Cosmology*, vol. 8, no. 1-2, pp. 96–100, 2002.

[10] S. J. Gates Jr., W. D. Linch III, and J. Phillips, "When superspace is not enough," http://arxiv.org/abs/hep-th/0211034.

[11] S. Bellucci, S. Krivonos, A. Marrani, and E. Orazi, "'Root' action for $N = 4$ supersymmetric mechanics theories," *Physical Review. D. Third Series*, vol. 73, no. 2, Article ID 025011, 2006.

[12] Z. Kuznetsova, M. Rojas, and F. Toppan, "Classification of irreps and invariants of the N extended supersymmetric quantum mechanics," *Journal of High Energy Physics*, vol. 3, article 098, 2006.

[13] Z. Kuznetsova and F. Toppan, "Refining the classification of the irreps of the 1D N-extended supersymmetry," *Modern Physics Letters A*, vol. 23, no. 1, pp. 37–51, 2008.

[14] E. Ivanov, O. Lechtenfeld, and A. Sutulin, "Hierarchy of N=8 mechanics models," *Nuclear Physics B*, vol. 790, no. 3, pp. 493–523, 2008.

[15] S. J. Gates Jr., J. Gonzales, B. MacGregor et al., "4D, $N = 1$ supersymmetry genomics (I)," *Journal of High Energy Physics*, vol. 12, article 009, 2009.

[16] F. Delduc and E. Ivanov, "N=4 mechanics of general (4,4,0) multiplets," *Nuclear Physics B*, vol. 855, no. 3, pp. 815–853, 2012.

[17] S. J. Gates Jr., J. Hallett, J. Parker, V. G. J. Rodgers, and K. Stiffler, "4D, $N = 1$ supersymmetry genomics (II)," *Journal of High Energy Physics*, vol. 2012, no. 6, article 71, 2012.

[18] S. J. Gates Jr., J. Hallett, T. Hübsch, and K. Stiffler, "The real anatomy of complex linear superfields," *International Journal of Modern Physics A*, vol. 27, no. 24, Article ID 1250143, 2012.

[19] I. Chappell, S. J. Gates Jr., W. D. Linch III et al., "4D, $N = 1$ supergravity genomics," *Journal of High Energy Physics*, vol. 2013, no. 10, article 004, 2013.

[20] M. Faux and S. J. Gates Jr., "Adinkras: a graphical technology for supersymmetric representation theory," *Physical Review D*, vol. 71, no. 6, Article ID 065002, 2005.

[21] C. F. Doran, M. G. Faux, S. J. Gates Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, "On graph-theoretic identifications of Adinkras, supersymmetry representations and superfields," *International Journal of Modern Physics A*, vol. 22, no. 5, pp. 869–930, 2007.

[22] C. F. Doran, M. G. Faux, S. J. Gates Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, "A superfield for every dash-chromotopology," *International Journal of Modern Physics A*, vol. 24, no. 30, pp. 5681–5695, 2009.

[23] C. F. Doran, M. G. Faux, S. J. Gates Jr. et al., "Topology types of Adinkras and the corresponding representations of N-extended supersymmetry," http://arxiv.org/abs/0806.0050.

[24] C. F. Doran, M. G. Faux, S. J. Gates Jr. et al., "Adinkras for clifford algebras, and worldline supermultiplets," http://arxiv.org/abs/0811.3410.

[25] C. F. Doran, M. G. Faux, S. J. Gates Jr. et al., "Codes and supersymmetry in one dimension," *Advances in Theoretical and Mathematical Physics*, vol. 15, no. 6, pp. 1909–1970, 2011.

[26] C. F. Doran, T. Hübsch, K. M. Iga, and G. D. Landweber, "On general off-shell representations of world line (1D) supersymmetry," *Symmetry*, vol. 6, no. 1, pp. 67–88, 2014.

[27] T. Hübsch and G. A. Katona, "On the construction and the structure of off-shell supermultiplet quotients," *International Journal of Modern Physics A*, vol. 27, no. 29, Article ID 1250173, 2012.

[28] T. Hübsch and G. A. Katona, "Golden ratio controlled chaos in supersymmetric dynamics," *International Journal of Modern Physics A*, vol. 28, no. 30, Article ID 1350156, 2013.

[29] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory: A Modern Introduction*, Cambridge University Press, 2007.

[30] C.-K. Lu and I. F. Herbut, "Supersymmetric Runge-Lenz-Pauli vector for Dirac vortex in topological insulators and graphene," *Journal of Physics A: Mathematical and Theoretical*, vol. 44, no. 29, Article ID 295003, 2011.

[31] P. Fré, "Introduction to harmonic expansions on coset manifolds and in particular on coset manifolds with Killing spinors," in *Supersymmetry and Supergravity—84 (Trieste, 1984)*, pp. 324–367, World Scientific Publishing, Singapore, 1984.

[32] I. L. Buchbinder and S. M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, Studies in High Energy Physics Cosmology and Gravitation, IOP Publishing, Bristol, England, 1998.

[33] J. Gates Jr. and T. Hübsch, "On dimensional extension of supersymmetry: from worldlines to worldsheets," *Advances in Theoretical and Mathematical Physics*, vol. 16, no. 6, pp. 1619–1667, 2012.

[34] T. Hübsch, "Linear and chiral superfields are usefully inequivalent," *Classical and Quantum Gravity*, vol. 16, no. 9, pp. L51–L54, 1999.

[35] S. J. Gates Jr., T. Hübsch, and K. Stiffler, "Adinkras and SUSY holography," *International Journal of Modern Physics A*, vol. 29, no. 7, Article ID 1450041, 14 pages, 2014.

[36] J. Gates Jr., "Superspace formulation of new nonlinear sigma models," *Nuclear Physics B*, vol. 238, no. 2, pp. 349–366, 1984.

[37] S. J. Gates Jr., C. M. Hull, and M. Roček, "Twisted multiplets and new supersymmetric non-linear σ-models," *Nuclear Physics B*, vol. 248, no. 1, pp. 157–186, 1984.

[38] J. Polchinski, *String Theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, UK, 1998.

[39] T. Hübsch, *Calabi-Yau Manifolds*, World Scientific Publishing, River Edge, NJ, USA, 2nd edition, 1994.

[40] C. Vafa, "The string landscape and the swampland," http://arxiv.org/abs/hep-th/0509212.

[41] H. Ooguri and C. Vafa, "On the geometry of the string landscape and the swampland," *Nuclear Physics. B*, vol. 766, no. 1–3, pp. 21–33, 2007.

[42] U. Lindström and M. Roček, "New hyperkähler metrics and new supermultiplets," *Communications in Mathematical Physics*, vol. 115, no. 1, pp. 21–29, 1988.

[43] U. Lindström and M. Roček, "$N = 2$ super Yang-Mills theory in projective superspace," *Communications in Mathematical Physics*, vol. 128, no. 1, pp. 191–196, 1990.

[44] S. M. Kuzenko, "Projective superspace as a double-punctured harmonic superspace," *International Journal of Modern Physics A*, vol. 14, no. 11, pp. 1737–1757, 1999.

[45] D. Jain and W. Siegel, "Deriving projective hyperspace from harmonic," *Physical Review D*, vol. 80, no. 4, Article ID 045024, 2009.

[46] C. F. Doran, M. G. Faux, S. J. Gates Jr., T. Hübsch, K. M. Iga, and G. D. Landweber, "Super-Zeeman embedding models on N-supersymmetric world-lines," *Journal of Physics A*, vol. 42, no. 6, Article ID 065402, 2009.