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Research Article

A Integrable Generalized Super-NLS-mKdV Hierarchy, Its Self-Consistent Sources, and Conservation Laws

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A generalized super-NLS-mKdV hierarchy is proposed related to Lie superalgebra $B(0,1)$; the resulting supersoliton hierarchy is put into super bi-Hamiltonian form with the aid of supertrace identity. Then, the super-NLS-mKdV hierarchy with self-consistent sources is set up. Finally, the infinitely many conservation laws of integrable super-NLS-mKdV hierarchy are presented.

1. Introduction

The superintegrable systems have aroused strong interest in recent years; many experts and scholars do research on the field and obtain lots of results [1, 2]. In [3], the supertrace identity and the proof of its constant γ are given by Ma et al. As an application, super-Dirac hierarchy and super-AKNS hierarchy and its super-Hamiltonian structures have been furnished. Then, like the super-C-KdV hierarchy, the super-Tu hierarchy, the multicomponent super-Yang hierarchy, and so on were proposed [4–9]. In [10], the binary nonlinearization and Bargmann symmetry constraints of the super-Dirac hierarchy were given.

Soliton equations with self-consistent sources have important applications in soliton theory. They are often used to describe interactions between different solitary waves, and they can provide variety of dynamics of physical models; some important results have been got by some scholars [11–18]. Conservation laws play an important role in mathematical physics. Since Miura et al. discovery of conservation laws for KdV equation in 1968 [19], lots of methods have been presented to find them [20–23].

In this work, a generalized super-NLS-mKdV hierarchy is constructed. Then, we present the super bi-Hamiltonian form for the generalized super-NLS-mKdV hierarchy with the help of the supertrace identity. In Section 3, we consider the generalized super-NLS-mKdV hierarchy with self-consistent sources based on the theory of self-consistent sources. Finally, the conservation laws of the generalized super-NLS-mKdV hierarchy are given.

2. The Generalized Superequation Hierarchy

Based on the Lie superalgebra $B(0,1)$

$$
\begin{gathered}
e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
e_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \\
e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};
\end{gathered} \tag{1}
$$

that is, along with the communicative operation

$$\begin{aligned}
&[e_1, e_2] = 2e_3,\\
&[e_1, e_3] = 2e_2,\\
&[e_1, e_4] = [e_2, e_5] = [e_3, e_5] = e_4,\\
&[e_4, e_2] = [e_5, e_1] = [e_3, e_4] = e_5,\\
&[e_2, e_3] = 2e_1,\\
&[e_4, e_4]_+ = -e_2 - e_3,\\
&[e_4, e_5]_+ = e_1,\\
&[e_5, e_5]_+ = e_3 - e_2.
\end{aligned} \tag{2}$$

To set up the generalized super-NLS-mKdV hierarchy, the spectral problem is given as follows:

$$\begin{aligned}
\varphi_x &= U\varphi,\\
\varphi_t &= V\varphi,
\end{aligned} \tag{3}$$

with

$$\begin{aligned}
U &= \begin{pmatrix} \lambda + w & u_1 + u_2 & u_3 \\ u_1 - u_2 & -\lambda - w & u_4 \\ u_4 & -u_3 & 0 \end{pmatrix},\\
V &= \begin{pmatrix} A & B + C & \sigma \\ B - C & -A & \rho \\ \rho & -\sigma & 0 \end{pmatrix},
\end{aligned} \tag{4}$$

where $w = \varepsilon((1/2)u_1^2 - (1/2)u_2^2 + u_3u_4)$ with ε being an arbitrary even constant, λ is the spectral parameter, u_1 and u_2 are even potentials, and u_3 and u_4 are odd potentials. Note that spectral problem (3) with $\varepsilon = 0$ is reduced to super-NLS-mKdV hierarchy case [24].

Setting

$$\begin{aligned}
A &= \sum_{m\geq 0} a_m \lambda^{-m},\\
B &= \sum_{m\geq 0} b_m \lambda^{-m},\\
C &= \sum_{m\geq 0} c_m \lambda^{-m},\\
\sigma &= \sum_{m\geq 0} \sigma_m \lambda^{-m},\\
\rho &= \sum_{m\geq 0} \rho_m \lambda^{-m},
\end{aligned} \tag{5}$$

solving the equation $V_x = [U, V]$, we have

$$\begin{aligned}
a_{mx} &= 2u_2 b_m - 2u_1 c_m + u_3 \rho_m + u_4 \sigma_m,\\
b_{mx} &= 2c_{m+1} - 2u_2 a_m - u_3 \sigma_m + u_4 \rho_m + 2w c_m,\\
c_{mx} &= 2b_{m+1} - 2u_1 a_m - u_3 \sigma_m - u_4 \rho_m + 2w b_m,\\
\sigma_{mx} &= \sigma_{m+1} + w\sigma_m + (u_1 + u_2)\rho_m - u_3 a_m - u_4 b_m\\
&\quad - u_4 c_m,\\
\rho_{mx} &= -\rho_{m+1} - w\rho_m + u_1 \sigma_m - u_2 \sigma_m - u_3 b_m + u_3 c_m\\
&\quad + u_4 a_m,
\end{aligned} \tag{6}$$

and, from the above recursion relationship, we can get the recursion operator L which meets the following:

$$\begin{pmatrix} b_{m+1} \\ -c_{m+1} \\ \rho_{m+1} \\ -\sigma_{m+1} \end{pmatrix} = L \begin{pmatrix} b_m \\ -c_m \\ \rho_m \\ -\sigma_m \end{pmatrix}, \quad m \geqslant 0, \tag{7}$$

where the recursive operator L is given as follows:

$$L = \begin{pmatrix}
-w + 2u_1\partial^{-1}u_2 & -\frac{1}{2}\partial + 2u_1\partial^{-1}u_1 & \frac{1}{2}u_4 + u_1\partial^{-1}u_3 & -\frac{1}{2}u_3 - u_1\partial^{-1}u_4 \\
-\frac{1}{2}\partial - 2u_2\partial^{-1}u_2 & -w + 2u_2\partial^{-1}u_1 & \frac{1}{2}u_4 - u_2\partial^{-1}u_3 & \frac{1}{2}u_3 + u_2\partial^{-1}u_4 \\
-u_3 + 2u_4\partial^{-1}u_2 & -u_3 + 2u_4\partial^{-1}u_1 & -\partial - w + u_4\partial^{-1}u_3 & -u_1 + u_2 - u_4\partial^{-1}u_4 \\
-u_4 - 2u_3\partial^{-1}u_2 & u_4 - 2u_3\partial^{-1}u_1 & u_1 + u_2 - u_3\partial^{-1}u_3 & \partial - w + u_3\partial^{-1}u_4
\end{pmatrix}. \tag{8}$$

Choosing the initial data

$$\begin{aligned}
a_0 &= 1,\\
b_0 &= c_0 = \sigma_0 = \rho_0 = 0,
\end{aligned} \tag{9}$$

from the recursion relations in (6), we can obtain

$$\begin{aligned}
a_1 &= 0,\\
b_1 &= u_1,\\
c_1 &= u_2,\\
\sigma_1 &= u_3,\\
\rho_1 &= u_4,\\
a_2 &= -\frac{1}{2}u_1^2 + \frac{1}{2}u_2^2 - u_3u_4,\\
b_2 &= \frac{1}{2}u_{2x} - wu_1,
\end{aligned}$$

$$\begin{aligned}
c_2 &= \frac{1}{2}u_{1x} - wu_2, \\
\sigma_2 &= u_{3x} - wu_3, \\
\rho_2 &= -u_{4x} - wu_4, \\
a_3 &= \frac{1}{2}\left(u_2u_{1x} - u_1u_{2x}\right) + u_3u_{4x} + u_4u_{3x} \\
&\quad + w\left(u_1^2 - u_2^2\right) + 2u_3u_4, \\
b_3 &= \frac{1}{4}u_{1xx} - \frac{1}{2}w_xu_2 + \frac{1}{2}u_1u_2^2 - \frac{1}{2}u_1^3 + \frac{1}{2}u_3u_{3x} \\
&\quad - \frac{1}{2}u_4u_{4x} - u_1u_3u_4 - wu_{2x} + w^2u_1, \\
c_3 &= \frac{1}{4}u_{2xx} - \frac{1}{2}w_xu_1 + \frac{1}{2}u_2^3 - \frac{1}{2}u_2u_1^2 + \frac{1}{2}u_3u_{3x} \\
&\quad + \frac{1}{2}u_4u_{4x} - u_2u_3u_4 - wu_{1x} + w^2u_2, \\
\sigma_3 &= u_{3xx} + \frac{1}{2}u_3u_2^2 - \frac{1}{2}u_3u_1^2 + \frac{1}{2}u_4u_{1x} + \frac{1}{2}u_4u_{2x} \\
&\quad + u_1u_{4x} + u_2u_{4x} - w_xu_3 + w^2u_3 - 2wu_{3x}, \\
\rho_3 &= u_{4xx} + \frac{1}{2}u_4u_2^2 - \frac{1}{2}u_4u_1^2 + \frac{1}{2}u_3u_{1x} - \frac{1}{2}u_3u_{2x} \\
&\quad + u_1u_{3x} - u_2u_{3x} + w_xu_4 + w^2u_4 + 2wu_{4x}.
\end{aligned} \tag{10}$$

Then we consider the auxiliary spectral problem

$$\varphi_{t_n} = V^{(n)}\varphi, \tag{11}$$

with

$$V^{(n)} = \sum_{m=0}^{n}\begin{pmatrix} a_m & b_m + c_m & \sigma_m \\ b_m - c_m & -a_m & \rho_m \\ \rho_m & -\sigma_m & 0 \end{pmatrix}\lambda^{n-m} + \Delta_n, \quad n \geq 0. \tag{12}$$

Suppose

$$\Delta_n = \begin{pmatrix} a & b+c & e \\ b-c & -a & f \\ f & -e & 0 \end{pmatrix}, \tag{13}$$

substituting $V^{(n)}$ into the zero curvature equation

$$U_{t_n} - V_x^{(n)} + \left[U, V^{(n)}\right] = 0, \tag{14}$$

where $n \geq 0$. Making use of (6), we have

$$\begin{aligned}
w_{t_n} &= a_x, \\
&b = c = e = f = 0, \\
u_{1t_n} &= b_{nx} - 2wc_n + 2u_2a_n + u_3\sigma_n - u_4\rho_n + b_x - 2wc \\
&\quad + u_3e - u_4f + 2u_2a = 2c_{n+1} + 2u_2a, \\
u_{2t_n} &= c_{nx} - 2wb_n + 2u_1a + u_3\sigma_n + u_4\rho_n + c_x + u_3e \\
&\quad + u_4f + 2u_1a - 2wb = 2b_{n+1} + 2u_1a, \\
u_{3t_n} &= \sigma_{nx} - w\sigma_n - u_1\rho_n - u_2\rho_n + u_3a_n + u_4b_n + u_4c_n \\
&\quad + e_x - we - u_1f - u_2f + u_3a + u_4b + u_4c \\
&= \sigma_{n+1} + u_3a, \\
u_{4t_n} &= \rho_{nx} + w\rho_n - u_1\sigma_n + u_2\sigma_n + u_3b_n - u_3c_n - u_4a_n \\
&\quad - u_4a + f_x + wf - u_1e + u_2e + u_3b - u_3c \\
&= -\rho_{n+1} - u_4a,
\end{aligned} \tag{15}$$

which guarantees that the following identity holds true:

$$\begin{aligned}
&\left(\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2 + u_4u_3\right)_{t_n} \\
&\quad = 2u_2b_{n+1} - 2u_1c_{n+1} - \rho_{n+1}u_3 + u_4\sigma_{n+1} = a_{n+1x}.
\end{aligned} \tag{16}$$

Choosing $a = -\varepsilon a_{n+1}$, we arrive at the following generalized super-NLS-mKdV hierarchy:

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{t_n} = \begin{pmatrix} 2c_{n+1} - 2\varepsilon u_2a_{n+1} \\ 2b_{n+1} - 2\varepsilon u_1a_{n+1} \\ \sigma_{n+1} - \varepsilon u_3a_{n+1} \\ -\rho_{n+1} + \varepsilon u_4a_{n+1} \end{pmatrix}, \tag{17}$$

where $n \geq 0$. The case of (17) with $\varepsilon = 0$ is exactly the standard supersoliton hierarchy [24].

When $n = 1$ in (17), the flow is trivial. Taking $n = 2$, we can obtain second-order generalized super-NLS-mKdV equations

$$\begin{aligned}
u_{1t_2} &= \frac{1}{2}u_{2xx} + u_2^3 - u_2u_1^2 + u_3u_{3x} + u_4u_{4x} - 2u_2u_3u_4 \\
&\quad + \varepsilon\left(2u_1u_2u_{2x} - 2u_1^2u_{1x} + u_{4x}u_3u_1 - u_{3x}u_4u_1\right. \\
&\quad - 2u_3u_4u_{1x} - 2u_2u_3u_{4x} - 2u_2u_4u_{3x} - 2u_2u_1^2u_3u_4 \\
&\quad \left.+ 2u_2^3u_3u_4\right) + 2\varepsilon^2u_2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)^2 \\
&\quad + 2\varepsilon^2u_2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_2^2 - u_1^2\right), \\
u_{2t_2} &= \frac{1}{2}u_{1xx} - u_1^3 + u_1u_2^2 - 2u_1u_3u_4 + u_3u_{3x} - u_4u_{4x} \\
&\quad + \varepsilon\left(2u_2^2u_{2x} - 2u_1u_2u_{1x} - u_{3x}u_4u_2 + u_{4x}u_3u_2\right.
\end{aligned}$$

$$- 2u_3u_4u_{2x} - 2u_1u_3u_{4x} - 2u_1u_4u_{3x}\Big) - 2\varepsilon^2 u_1 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_1^2 - u_2^2\right) + 2\varepsilon^2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)^2 u_1 - 2\varepsilon^2 u_1u_3\left(u_1^2 - u_2^2\right)u_4,$$

$$u_{3t_2} = u_{3xx} + \frac{1}{2}u_3u_2^2 - \frac{1}{2}u_3u_1^2 + \frac{1}{2}u_4u_{1x} + \frac{1}{2}u_4u_{2x} + u_1u_{4x} + u_2u_{4x} + \varepsilon\Big(\frac{1}{2}u_3u_1u_{2x} - \frac{1}{2}u_3u_2u_{1x} + u_2^2u_{3x} - u_1^2u_{3x} + u_2u_{2x}u_3 - u_1u_{1x}u_3 - 2u_3u_4u_{3x}\Big) + \varepsilon^2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)^2 u_3 - \varepsilon^2 u_3\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_1^2 - u_2^2\right),$$

$$u_{4t_2} = -u_{4xx} - \frac{1}{2}u_4u_2^2 + \frac{1}{2}u_4u_1^2 + \frac{1}{2}u_3u_{2x} - \frac{1}{2}u_3u_{1x} - u_1u_{3x} + u_2u_{3x} + \varepsilon\Big(\frac{1}{2}u_4u_2u_{1x} - \frac{1}{2}u_4u_1u_{2x} - u_1u_{1x}u_4 + u_2u_{2x}u_4 - u_1^2u_{4x} + u_2^2u_{4x} - 2u_3u_4u_{4x}\Big) - \varepsilon^2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)^2 u_4 - \varepsilon^2 u_4\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_1^2 - u_2^2\right). \tag{18}$$

From (3) and (11), one infers the following:

$$V^{(2)} = \begin{pmatrix} V_{11}^{(2)} & V_{12}^{(2)} & V_{13}^{(2)} \\ V_{21}^{(2)} & -V_{11}^{(2)} & V_{23}^{(2)} \\ V_{23}^{(2)} & -V_{13}^{(2)} & 0 \end{pmatrix}, \tag{19}$$

with

$$V_{11}^{(2)} = \lambda^2 + \frac{1}{2}u_2^2 - \frac{1}{2}u_1^2 - u_3u_4 - \varepsilon\left(\frac{1}{2}u_2u_{1x} - \frac{1}{2}u_1u_{2x} + u_4u_{3x} + u_3u_{4x}\right) - 2\varepsilon^2\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)^2,$$

$$V_{12}^{(2)} = \left(u_1 + u_2\right)\lambda + \frac{1}{2}\left(u_1 + u_2\right)_x - \varepsilon\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_1 + u_2\right),$$

$$V_{13}^{(2)} = u_3\lambda + u_{3x} - \varepsilon\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)u_3,$$

$$V_{21}^{(2)} = \left(u_1 - u_2\right)\lambda + \frac{1}{2}\left(u_2 - u_1\right)_x + \varepsilon\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\left(u_2 - u_1\right),$$

$$V_{23}^{(2)} = u_4\lambda - u_{4x} - \varepsilon\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)u_4. \tag{20}$$

In what follows, we shall set up super-Hamiltonian structures for the generalized super-NLS-mKdV hierarchy.

Through calculations, we obtain

$$\begin{aligned} \operatorname{Str}\left(V\frac{\partial U}{\partial \lambda}\right) &= 2A, \\ \operatorname{Str}\left(\frac{\partial U}{\partial u_1}V\right) &= 2\left(B + \varepsilon u_1 A\right), \\ \operatorname{Str}\left(\frac{\partial U}{\partial u_2}V\right) &= -2\left(C + \varepsilon u_2 A\right), \\ \operatorname{Str}\left(\frac{\partial U}{\partial u_3}V\right) &= 2\left(\rho + \varepsilon u_4 A\right), \\ \operatorname{Str}\left(\frac{\partial U}{\partial u_4}V\right) &= -2\left(\sigma + \varepsilon u_3 A\right). \end{aligned} \tag{21}$$

Substituting the above results into the supertrace identity [3] and balancing the coefficients of λ^{n+2}, we obtain

$$\frac{\delta}{\delta u}\int a_{n+2}dx = \left(\gamma - n - 1\right)\begin{pmatrix} b_{n+1} + \varepsilon u_1 a_{n+1} \\ -c_{n+1} - \varepsilon u_2 a_{n+1} \\ \rho_{n+1} + \varepsilon u_4 a_{n+1} \\ -\sigma_{n+1} - \varepsilon u_3 a_{n+1} \end{pmatrix}, \quad n \geq 0. \tag{22}$$

Thus, we have

$$H_{n+1} = \int -\frac{a_{n+2}}{n+1}dx,$$

$$\frac{\delta H_{n+1}}{\delta u} = \begin{pmatrix} b_{n+1} + \varepsilon u_1 a_{n+1} \\ -c_{n+1} - \varepsilon u_2 a_{n+1} \\ \rho_{n+1} + \varepsilon u_4 a_{n+1} \\ -\sigma_{n+1} - \varepsilon u_3 a_{n+1} \end{pmatrix}, \quad n \geqslant 0. \tag{23}$$

Moreover, it is easy to find that

$$\begin{pmatrix} b_{n+1} \\ -c_{n+1} \\ \rho_{n+1} \\ -\sigma_{n+1} \end{pmatrix} = R_1 \begin{pmatrix} b_{n+1} + \varepsilon u_1 a_{n+1} \\ -c_{n+1} - \varepsilon u_2 a_{n+1} \\ \rho_{n+1} + \varepsilon u_4 a_{n+1} \\ -\sigma_{n+1} - \varepsilon u_3 a_{n+1} \end{pmatrix}, \quad n \geq 0, \tag{24}$$

where R_1 is given by

$$R_1 = \begin{pmatrix} 1-2\varepsilon u_1\partial^{-1}u_2 & -2\varepsilon u_1\partial^{-1}u_1 & \varepsilon u_1\partial^{-1}u_3 & \varepsilon u_1\partial^{-1}u_4 \\ 2\varepsilon u_2\partial^{-1}u_2 & 1+2\varepsilon u_2\partial^{-1}u_1 & \varepsilon u_2\partial^{-1}u_3 & -\varepsilon u_2\partial^{-1}u_4 \\ -2\varepsilon u_4\partial^{-1}u_2 & -2\varepsilon u_4\partial^{-1}u_1 & 1-\varepsilon u_4\partial^{-1}u_3 & \varepsilon u_4\partial^{-1}u_4 \\ 2\varepsilon u_3\partial^{-1}u_2 & 2\varepsilon u_3\partial^{-1}u_1 & \varepsilon u_3\partial^{-1}u_3 & 1-\varepsilon u_3\partial^{-1}u_4 \end{pmatrix}. \quad (25)$$

Therefore, superintegrable hierarchy (17) possesses the following form:

$$u_{t_n} = R_2\begin{pmatrix} b_{n+1} \\ -c_{n+1} \\ \rho_{n+1} \\ -\sigma_{n+1} \end{pmatrix} = R_2R_1\begin{pmatrix} b_{n+1}+\varepsilon u_1a_{n+1} \\ -c_{n+1}-\varepsilon u_2a_{n+1} \\ \rho_{n+1}+\varepsilon u_4a_{n+1} \\ -\sigma_{n+1}-\varepsilon u_3a_{n+1} \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \quad n\ge 0, \quad (26)$$

where

$$R_2 = \begin{pmatrix} -4\varepsilon u_2\partial^{-1}u_2 & -2-4\varepsilon u_2\partial^{-1}u_1 & -2\varepsilon u_2\partial^{-1}u_3 & 2\varepsilon u_2\partial^{-1}u_4 \\ 2-4\varepsilon u_1\partial^{-1}u_2 & -4\varepsilon u_1\partial^{-1}u_1 & -2\varepsilon u_1\partial^{-1}u_3 & 2\varepsilon u_1\partial^{-1}u_4 \\ -4\varepsilon u_3\partial^{-1}u_2 & -4\varepsilon u_3\partial^{-1}u_1 & -2\varepsilon u_3\partial^{-1}u_3 & -1+2\varepsilon u_3\partial^{-1}u_4 \\ 2\varepsilon u_4\partial^{-1}u_2 & 2\varepsilon u_4\partial^{-1}u_1 & -1+\varepsilon u_4\partial^{-1}u_3 & -\varepsilon u_4\partial^{-1}u_4 \end{pmatrix}, \quad (27)$$

and super-Hamiltonian operator J is given by

$$J = R_2R_1 = \begin{pmatrix} -8\varepsilon u_2\partial^{-1}u_2 & -2-8\varepsilon u_2\partial^{-1}u_1 & -4\varepsilon u_2\partial^{-1}u_3 & 4\varepsilon u_2\partial^{-1}u_4 \\ 2-8\varepsilon u_1\partial^{-1}u_2 & -8\varepsilon u_1\partial^{-1}u_1 & -4\varepsilon u_1\partial^{-1}u_3 & 4\varepsilon u_1\partial^{-1}u_4 \\ -6\varepsilon u_3\partial^{-1}u_2 & -6\varepsilon u_3\partial^{-1}u_1 & -3\varepsilon u_3\partial^{-1}u_3 & -1+3\varepsilon u_3\partial^{-1}u_4 \\ 4\varepsilon u_4\partial^{-1}u_2 & 4\varepsilon u_4\partial^{-1}u_1 & -1+2\varepsilon u_4\partial^{-1}u_3 & -2\varepsilon u_4\partial^{-1}u_4 \end{pmatrix}. \quad (28)$$

In addition, generalized super-NLS-mKdV hierarchy (17) also possesses the following super-Hamiltonian form:

$$u_{t_n} = R_2L\begin{pmatrix} b_n \\ -c_n \\ \rho_n \\ -\sigma_n \end{pmatrix} = R_2LR_1\begin{pmatrix} b_n+\varepsilon u_1a_n \\ -c_n-\varepsilon u_2a_n \\ \rho_n+\varepsilon u_4a_n \\ -\sigma_n-\varepsilon u_3a_n \end{pmatrix} = M\frac{\delta H_n}{\delta u}, \quad n\ge 0, \quad (29)$$

where $M = R_2LR_1 = (M_{ij})_{4\times 4}$ is the second super-Hamiltonian operator.

3. Self-Consistent Sources

Consider the linear system

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}, \qquad \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_t = V\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}. \quad (30)$$

From the result in [25], we set

$$\frac{\delta\lambda_j}{\delta u_i} = \frac{1}{3}\operatorname{Str}\left(\Psi_j\frac{\partial U(u,\lambda_j)}{\delta u_i}\right), \quad i = 1,2,\ldots,4, \quad (31)$$

with Str on behalf of the supertrace, and

$$\Psi_j = \begin{pmatrix} \varphi_{1j}\varphi_{2j} & -\varphi_{1j}^2 & \varphi_{1j}\varphi_{3j} \\ \varphi_{2j}^2 & -\varphi_{1j}\varphi_{2j} & \varphi_{2j}\varphi_{3j} \\ \varphi_{2j}\varphi_{3j} & -\varphi_{1j}\varphi_{3j} & 0 \end{pmatrix}, \quad j = 1, 2, \ldots, N. \quad (32)$$

From system (30), we get $\delta\lambda_j/\delta u$ as follows:

$$\sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u_i} = \sum_{j=1}^{N} \begin{pmatrix} \mathrm{Str}\left(\Psi_j \frac{\delta U}{\delta u_1}\right) \\ \mathrm{Str}\left(\Psi_j \frac{\delta U}{\delta u_2}\right) \\ \mathrm{Str}\left(\Psi_j \frac{\delta U}{\delta u_3}\right) \\ \mathrm{Str}\left(\Psi_j \frac{\delta U}{\delta u_4}\right) \end{pmatrix} = \begin{pmatrix} 2\varepsilon u_1 \langle \Phi_1, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \\ -2\varepsilon u_2 \langle \Phi_1, \Phi_2 \rangle + \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \\ 2\varepsilon u_4 \langle \Phi_1, \Phi_2 \rangle - 2 \langle \Phi_2, \Phi_3 \rangle \\ 2\varepsilon u_3 \langle \Phi_1, \Phi_2 \rangle - 2 \langle \Phi_1, \Phi_3 \rangle \end{pmatrix}, \quad (33)$$

where $\Phi_i = (\varphi_{i1}, \ldots, \varphi_{iN})^T$, $i = 1, 2, 3$.

So, we obtain the self-consistent sources of generalized super-NLS-mKdV hierarchy (17):

$$u_{t_n} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_{t_n} = J \frac{\delta H_{n+1}}{\delta u_i} + J \sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u_i} = J \begin{pmatrix} b_{n+1} + \varepsilon u_1 a_{n+1} \\ -c_{n+1} - \varepsilon u_2 a_{n+1} \\ \rho_{n+1} + \varepsilon u_4 a_{n+1} \\ -\sigma_{n+1} - \varepsilon u_3 a_{n+1} \end{pmatrix} + J \begin{pmatrix} 2\varepsilon u_1 \langle \Phi_1, \Phi_2 \rangle - \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \\ -2\varepsilon u_2 \langle \Phi_1, \Phi_2 \rangle + \langle \Phi_1, \Phi_1 \rangle + \langle \Phi_2, \Phi_2 \rangle \\ 2\varepsilon u_4 \langle \Phi_1, \Phi_2 \rangle - 2 \langle \Phi_2, \Phi_3 \rangle \\ 2\varepsilon u_3 \langle \Phi_1, \Phi_2 \rangle - 2 \langle \Phi_1, \Phi_3 \rangle \end{pmatrix}. \quad (34)$$

For $n = 2$, we get supersoliton equation with self-consistent sources as follows:

$$\begin{aligned}
u_{1t_2} &= \frac{1}{2} u_{2xx} + u_2^3 - u_2 u_1^2 + u_3 u_{3x} + u_4 u_{4x} - 2u_2 u_3 u_4 \\
&\quad + \varepsilon \left(2u_1 u_2 u_{2x} - 2u_1^2 u_{1x} + u_{4x} u_3 u_1 - u_{3x} u_4 u_1 \right. \\
&\quad - 2u_3 u_4 u_{1x} - 2u_2 u_3 u_{4x} - 2u_2 u_4 u_{3x} - 2u_2 u_1^2 u_3 u_4 \\
&\quad \left. + 2u_2^3 u_3 u_4\right) + 2\varepsilon^2 u_2 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)^2 \\
&\quad + 2\varepsilon^2 u_2 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)\left(u_2^2 - u_1^2\right) \\
&\quad + 2\varepsilon u_1 \sum_{j=1}^{N} \varphi_{1j}\varphi_{2j} - \sum_{j=1}^{N} \left(\varphi_{1j}^2 - \varphi_{2j}^2\right), \\
u_{2t_2} &= \frac{1}{2} u_{1xx} - u_1^3 + u_1 u_2^2 - 2u_1 u_3 u_4 + u_3 u_{3x} - u_4 u_{4x} \\
&\quad + \varepsilon \left(2u_2^2 u_{2x} - 2u_1 u_2 u_{1x} - u_{3x} u_4 u_2 + u_{4x} u_3 u_2 \right. \\
&\quad \left. - 2u_3 u_4 u_{2x} - 2u_1 u_3 u_{4x} - 2u_1 u_4 u_{3x}\right) \\
&\quad - 2\varepsilon^2 u_1 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)\left(u_1^2 - u_2^2\right) \\
&\quad + 2\varepsilon^2 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)^2 u_1 - 2\varepsilon^2 u_1 u_3 \left(u_1^2\right. \\
&\quad \left. - u_2^2\right) u_4 - 2\varepsilon u_2 \sum_{j=1}^{N} \varphi_{1j}\varphi_{2j} + \sum_{j=1}^{N} \left(\varphi_{1j}^2 + \varphi_{2j}^2\right), \\
u_{3t_2} &= u_{3xx} + \frac{1}{2} u_3 u_2^2 - \frac{1}{2} u_3 u_1^2 + \frac{1}{2} u_4 u_{1x} + \frac{1}{2} u_4 u_{2x} \\
&\quad + u_1 u_{4x} + u_2 u_{4x} + \varepsilon \left(\frac{1}{2} u_3 u_1 u_{2x} - \frac{1}{2} u_3 u_2 u_{1x} \right. \\
&\quad + u_2^2 u_{3x} - u_1^2 u_{3x} + u_2 u_{2x} u_3 - u_1 u_{1x} u_3 \\
&\quad \left. - 2u_3 u_4 u_{3x}\right) + \varepsilon^2 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)^2 u_3 \\
&\quad - \varepsilon^2 u_3 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)\left(u_1^2 - u_2^2\right) \\
&\quad + 2\varepsilon u_4 \sum_{j=1}^{N} \varphi_{1j}\varphi_{2j} - \sum_{j=1}^{N} \varphi_{2j}\varphi_{3j}, \\
u_{4t_2} &= -u_{4xx} - \frac{1}{2} u_4 u_2^2 + \frac{1}{2} u_4 u_1^2 + \frac{1}{2} u_3 u_{2x} - \frac{1}{2} u_3 u_{1x} \\
&\quad - u_1 u_{3x} + u_2 u_{3x} + \varepsilon \left(\frac{1}{2} u_4 u_2 u_{1x} - \frac{1}{2} u_4 u_1 u_{2x} \right. \\
&\quad - u_1 u_{1x} u_4 + u_2 u_{2x} u_4 - u_1^2 u_{4x} + u_2^2 u_{4x} \\
&\quad \left. - 2u_3 u_4 u_{4x}\right) - \varepsilon^2 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)^2 u_4 \\
&\quad - \varepsilon^2 u_4 \left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 u_4\right)\left(u_1^2 - u_2^2\right) \\
&\quad + 2\varepsilon u_3 \sum_{j=1}^{N} \varphi_{1j}\varphi_{2j} - 2\sum_{j=1}^{N} \varphi_{1j}\varphi_{3j}, \\
\varphi_{1jx} &= (\lambda + w)\varphi_{1j} + (u_1 + u_2)\varphi_{2j} + u_3 \varphi_{3j}, \\
\varphi_{2jx} &= (u_1 - u_2)\varphi_{1j} - (\lambda + w)\varphi_{2j} + u_4 \varphi_{3j}, \\
\varphi_{3jx} &= u_4 \varphi_{1j} - u_3 \varphi_{2j}, \\
&\qquad j = 1, \ldots, N.
\end{aligned} \quad (35)$$

4. Conservation Laws

In the following, we shall derive the conservation laws of supersoliton hierarchy. Introducing the variables

$$K = \frac{\varphi_2}{\varphi_1}, \qquad G = \frac{\varphi_3}{\varphi_1}, \tag{36}$$

then we obtain

$$\begin{aligned} K_x &= u_1 - u_2 - 2(\lambda + w) K + u_4 G - (u_1 + u_2) K^2 \\ &\quad - u_3 K G, \\ G_x &= u_4 - u_3 K - (\lambda + w) G - (u_1 + u_2) G K - u_3 G^2. \end{aligned} \tag{37}$$

Next, we expand K and G as series of the spectral parameter λ

$$K = \sum_{j=1}^{\infty} k_j \lambda^j, \qquad G = \sum_{j=1}^{\infty} g_j \lambda^j. \tag{38}$$

Substituting (38) into (37), we raise the recursion formulas for k_j and g_j:

$$\begin{aligned} k_{j+1} &= -\frac{1}{2} k_{jx} - w k_j + \frac{1}{2} u_4 g_j - \frac{1}{2}(u_1 + u_2) \sum_{l=1}^{j-1} k_l k_{j-l} \\ &\quad - \frac{1}{2} u_3 \sum_{l=1}^{j-1} k_l g_{j-l}, \\ g_{j+1} &= -g_{jx} - u_3 k_j - w g_j - (u_1 + u_2) \sum_{l=1}^{j-1} g_l k_{j-l} \\ &\quad - u_3 \sum_{l=1}^{j-1} g_l g_{j-l}, \quad j \geq 2. \end{aligned} \tag{39}$$

We write the first few terms of k_j and g_j:

$$\begin{aligned} k_1 &= \frac{1}{2}(u_1 - u_2), \\ g_1 &= u_4, \\ k_2 &= -\frac{1}{4}(u_1 - u_2) x \\ &\quad - \frac{1}{2} \varepsilon (u_1 - u_2) \left(\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + u_3 u_4 \right), \\ g_2 &= -u_{4x} - \frac{1}{2}(u_1 - u_2)(u_3 + \varepsilon u_1 + \varepsilon u_2), \\ k_3 &= \frac{1}{8}(u_1 - u_2)_{xx} - \frac{1}{8}(u_1 + u_2)(u_1 - u_2)^2 \\ &\quad + \frac{1}{4} w_x (u_1 - u_2) + \frac{1}{2}(u_1 - u_2)_x w \\ &\quad + \frac{1}{2}(u_1 - u_2) w^2 - \frac{1}{2} u_4 u_{4x}, \\ g_3 &= u_{4xx} + \frac{1}{2}(u_1 - u_2) u_{3x} - \frac{1}{2}(u_1^2 - u_2^2) u_4 \\ &\quad + \frac{3}{4}(u_1 - u_2)_x u_3 + w_x u_4 + 2 w u_{4x} \\ &\quad + w (u_1 - u_2) u_3 + w^2 u_4, \ldots. \end{aligned} \tag{40}$$

Note that

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\lambda + w + (u_1 + u_2) K + u_3 G \right] \\ &\quad = \frac{\partial}{\partial x} \left[A + (B + C) K + \sigma G \right], \end{aligned} \tag{41}$$

setting $\delta = \lambda + w + (u_1 + u_2)K + u_3 G, \theta = A + (B + C)K + \sigma G$, which admitted that the conservation laws is $\delta_t = \theta_x$. For (19), one infers that

$$\begin{aligned} A &= \lambda^2 + \frac{1}{2} u_2^2 - \frac{1}{2} u_1^2 - u_3 u_4, \\ B &= u_1 \lambda + \frac{1}{2} u_{2x} - \varepsilon u_1 \left(\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + u_3 u_4 \right), \\ C &= u_2 \lambda + \frac{1}{2} u_{1x} - \varepsilon u_2 \left(\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + u_3 u_4 \right), \\ \sigma &= u_3 \lambda + u_{3x} - \frac{1}{2} \varepsilon u_3 (u_1^2 - u_2^2). \end{aligned} \tag{42}$$

Expanding δ and θ as

$$\begin{aligned} \delta &= \lambda + w + \sum_{j=1}^{\infty} \delta_j \lambda^{-j}, \\ \theta &= \lambda^2 + \frac{1}{2} u_2^2 - \frac{1}{2} u_1^2 - u_3 u_4 + \sum_{j=1}^{\infty} \theta_j \lambda^{-j}, \end{aligned} \tag{43}$$

conserved densities and currents are the coefficients δ_j, θ_j, respectively. The first two conserved densities and currents are read:

$$\begin{aligned} \delta_1 &= \frac{1}{2}(u_1^2 - u_2^2) + u_3 u_4, \\ \theta_1 &= -\frac{1}{4}(u_1 + u_2)(u_1 - u_2) x + \frac{1}{4}(u_1 - u_2)(u_1 + u_2) \\ &\quad \cdot x - \frac{1}{4} \varepsilon (u_1^2 - u_2^2) \left(\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + u_3 u_4 \right) - \frac{1}{2} (u_1^2 \end{aligned}$$

$$- u_2^2\right)\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right) - \frac{1}{2}\varepsilon\left(u_1^2 - u_2^2\right)u_3 - \frac{1}{2}$$
$$\cdot \varepsilon\left(u_1^2 - u_2^2\right)u_3u_4 - u_3u_{4x} + u_{3x}u_4,$$
$$\delta_2 = -\frac{1}{4}\left(u_1 + u_2\right)\left(u_2 - u_1\right)_x - \frac{1}{2}\varepsilon\left(u_1^2 - u_2^2\right)\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3 + u_3u_4\right) - u_3u_{4x},$$
$$\theta_2 = \left(u_1 + u_2\right)k_3 - \frac{1}{4}\left[\left(u_1 + u_2\right)_x - \varepsilon\left(u_1 + u_2\right)\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\right]$$
$$\cdot\left[\frac{1}{2}\left(u_1 - u_2\right)x + \left(u_1 - u_2\right)\times\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\right] + u_3g_3 - \left[u_{3x} - \frac{1}{2}\varepsilon\left(u_1^2 - u_2^2\right)u_3\right]\left[u_{4x} + \frac{1}{2}\left(u_1 - u_2\right)\left(u_3 + \varepsilon u_1 + \varepsilon u_2\right)\right], \tag{44}$$

where k_3 and g_3 are given by (40). The recursion relationship for δ_j and θ_j is as follows:

$$\begin{aligned}\delta_j &= \left(u_1 + u_2\right)k_j + u_3g_j,\\ \theta_j &= \left(u_1 + u_2\right)k_{j+1} + \frac{1}{2}\left[\left(u_1 + u_2\right)_x - \varepsilon\left(u_1 + u_2\right)\left(\frac{1}{2}u_1^2 - \frac{1}{2}u_2^2 + u_3u_4\right)\right]k_j + u_3g_{j+1}\\ &\quad + \left[u_{3x} - \frac{1}{2}\varepsilon\left(u_1^2 - u_2^2\right)u_3\right]g_j,\end{aligned} \tag{45}$$

where k_j and g_j can be recursively calculated from (39). We can display the first two conservation laws of (18) as

$$\begin{aligned}\delta_{1t} &= \theta_{1x},\\ \delta_{2t} &= \theta_{2x},\end{aligned} \tag{46}$$

where $\delta_1, \theta_1, \delta_2$, and θ_2 are defined in (44). Then we can obtain the infinitely many conservation laws of (17) from (37)–(46).

5. Conclusions

In this work, we construct the generalized super-NLS-mKdV hierarchy with bi-Hamiltonian forms with the help of variational identity. Self-consistent sources and conservation laws are also set up. In [26–29], the nonlinearization of AKNS hierarchy and binary nonlinearization of super-AKNS hierarchy were given. Can we do the binary nonlinearization for hierarchy (17)? The question may be investigated in further work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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