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Research Article

The Neumann Problem for a Degenerate Elliptic System Near Resonance

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This paper studies the following system of degenerate equations $-\operatorname{div}(p(x)\nabla u) + q(x)u = \alpha u + \beta v + g_1(x, v) + h_1(x)$, $x \in \Omega$, $-\operatorname{div}(p(x)\nabla v) + q(x)v = \beta u + \alpha v + g_2(x, u) + h_2(x)$, $x \in \Omega$, $\partial u/\partial\nu = \partial v/\partial\nu = 0$, $x \in \partial\Omega$. Here $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, and ν is the exterior normal vector on $\partial\Omega$. The coefficient function p may vanish in $\overline{\Omega}$, $q \in L^r(\Omega)$ with $r > ns/(2s-n)$, $s > n/2$. We show that the eigenvalues of the operator $-\operatorname{div}(p(x)\nabla u) + q(x)u$ are discrete. Secondly, when the linear part is near resonance, we prove the existence of at least two different solutions for the above degenerate system, under suitable conditions on h_1, h_2, g_1, and g_2.

1. Introduction

In recent decades, many kinds of perturbed problems were studied by many scholars, such as [1–11]. Here, we want to say that the authors in [5] studied the following Dirichlet boundary problem:

$$\begin{aligned} -\Delta u &= \mu u \pm g(x, u) + h(x), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{1}$$

When the parameter μ is close to an eigenvalue of the operator $-\Delta$, they proved that problem (1) has two different solutions. Moreover, this result was extended to some equations and systems; see [6–10]. In particular, Massa and Rossato [11] studied a nondegenerate elliptic system and two solutions were obtained by using Galerkin techniques. On the other hand, we also mention that many scholars studied some elliptic equations with the Neumann or Robin boundary; see [12–17] and the references therein. Inspired by the above results, we study the following system of degenerate equations:

$$\begin{aligned} &-\operatorname{div}(p(x)\nabla u) + q(x)u \\ &\quad = \alpha u + \beta v + g_1(x, v) + h_1(x), \quad x \in \Omega, \\ &-\operatorname{div}(p(x)\nabla v) + q(x)v \\ &\quad = \beta u + \alpha v + g_2(x, u) + h_2(x), \quad x \in \Omega, \\ &\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, ν is the exterior normal vector on $\partial\Omega$, $\alpha, \beta \in \mathbb{R}$, and $h_1, h_2 \in L^2(\Omega)$. The coefficient p may vanish in $\overline{\Omega}$, $q \in L^r(\Omega)$ with $r > ns/(2s-n)$, $s > n/2$; that is, problem (2) may be degenerate; see [18]. As in [11], we will use the critical point theory and Galerkin techniques to obtain the existence of two different solutions for the above degenerate system. Now, we introduce the function set $\mathscr{F}$ which consists of functions $\omega : \Omega \to \mathbb{R}^+ \cup \{0\}$ such that

$$\omega \in L^1_{\text{loc}}(\Omega),$$
$$\omega^{-1} \in L^1_{\text{loc}}(\Omega), \tag{3}$$
$$\omega^{-s} \in L^1(\Omega) \text{ with } s > \frac{n}{2}.$$

Throughout the paper, we always assume that there exist $\omega \in \mathscr{F}$ and $\kappa \geq 1$ such that

$$\frac{\omega(x)}{\kappa} \leq p(x) \leq \kappa\omega(x), \quad \text{a.e. } x \in \Omega. \tag{4}$$

We also assume that q belongs to $L^r(\Omega)$ with $r > ns/(2s-n)$, and $g_i : \Omega \to \mathbb{R}$ is a Carathéodory mapping and satisfies the following conditions:

(g_0) For every $M \in \mathbb{R}^+$, there exists $l_M \in L^2(\Omega)$ such that, for all $|s| \leq M$

$$|g_i(x,s)| \leq l_M(x), \quad \text{a.e. } x \in \Omega. \tag{5}$$

(g_∞) $\lim_{|s|\to\infty}(g_i(x,s)/s) = 0$, uniformly in $x \in \Omega$, $i = 1, 2$.

Although the conditions (g_0) and (g_∞) were introduced in [10], it is weaker than (f_1) in [5] (or (1.2) in [11]). In fact, let $g_i(x,s) = s^{-1}\log(1+|s|)$; it is easy to see that g_i satisfies (g_∞), but the function g_i does not satisfy the condition (f_1) in [5] (or (1.2) in [11]).

In Section 2, we give some preliminary lemmas and our main results. Meanwhile, we show that the eigenvalues of the operator $-\text{div}(p(x)\nabla u) + q(x)u$ are discrete under Neumann boundary condition. In Section 3, we prove our main results through Galerkin techniques and saddle point theorem.

2. Preliminaries and Main Results

In this section, we first collect some basic facts and then give the properties of the eigenvalues of the operator $-\text{div}(p(x)\nabla u) + q(x)u$. Secondly, we define a new norm and prove it is equivalent to the usual Sobolev norm. At the end of this section, we give the main results of this paper.

Firstly, let $W^{1,2}(\Omega,\omega)$ denote the completion of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_\omega = \sqrt{\int_\Omega \left(\omega(x)|\nabla u|^2 + u^2\right)dx}, \quad \forall \omega \in \mathscr{F}. \tag{6}$$

The inner product in $W^{1,2}(\Omega,\omega)$ is denoted by

$$\langle u,v\rangle_\omega = \int_\Omega (\omega(x)\nabla u\nabla v + uv)\,dx, \quad \forall u,v \in W^{1,2}(\Omega,\omega). \tag{7}$$

From (4), we know that the spaces $W^{1,2}(\Omega,\omega)$ and $W^{1,2}(\Omega,p)$ are equivalent; see [18]. Let $r' = r/(r-1)$; from $r > ns/(2s-n)$ with $s > n/2$, one has

$$2r' < 2^*_s = \frac{2ns}{n(s+1)-2s}. \tag{8}$$

Hence, by the Sobolev embedding theorem of [18], we know that $W^{1,2}(\Omega,\omega)$ is compactly embedded in $L^{2r'}(\Omega)$. Moreover, it follows from Hölder's inequality that

$$\left|\int_\Omega q(x)u^2dx\right| \leq \|q\|_{L^r(\Omega)}\|u\|^2_{L^{2r'}(\Omega)}. \tag{9}$$

Now, we use a similar argument to that of Gasiński and Papageorgiou (see [15]). Let us study the following eigenvalue problem:

$$-\text{div}(p(x)\nabla u) + q(x)u = \mu u, \quad x \in \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega. \tag{10}$$

Firstly, from (4) and the Sobolev embedding theorem of [18], we know that

$$W^{1,2}(\Omega,p) \hookrightarrow L^{2r'}(\Omega) \hookrightarrow L^2(\Omega), \tag{11}$$

and the first embedding is compact. Then, for any $\varepsilon > 0$, we have

$$\|u\|^2_{L^{2r'}(\Omega)} \leq \varepsilon\|u\|^2_p + c_1\|u\|^2_{L^2(\Omega)}, \quad \forall u \in W^{1,2}(\Omega,p), \tag{12}$$

for some positive constant c_1; see [19].

Let us define $\sigma : W^{1,2}(\Omega,p) \times W^{1,2}(\Omega,p) \to \mathbb{R}$, $\forall u,v \in W^{1,2}(\Omega,p)$

$$\sigma(u,v) = \int_\Omega (p(x)\nabla u\nabla v + q(x)uv)\,dx. \tag{13}$$

It follows from (9) and (12) that

$$\left(1 - \varepsilon\|q\|_{L^r(\Omega)}\right)\|u\|^2_p \leq \sigma(u,u) + \left(c_1\|q\|_{L^r(\Omega)} + 1\right)\|u\|^2_{L^2(\Omega)}, \quad \forall u \in W^{1,2}(\Omega,p). \tag{14}$$

Choosing ε small enough, then from (14) one gets

$$\|u\|^2_p \leq c_2\left(\sigma(u,u) + \|u\|^2_{L^2(\Omega)}\right), \quad \forall u \in W^{1,2}(\Omega,p), \tag{15}$$

for some positive constants c_2. Hence, by Corollary 7.8 in [20], we conclude that there exists an eigenvalue sequence $\{\mu_k\}$ satisfying

$$-c_2 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \cdots \longrightarrow +\infty, \tag{16}$$

as $k \to +\infty$ and

$$\mu_1 = \inf\left\{\frac{\sigma(u,u)}{\|u\|^2_{L^2(\Omega)}} : u \in W^{1,2}(\Omega,p) \setminus \{0\}\right\}. \tag{17}$$

Let $\{\varphi_k\}$ be the corresponding eigenfunction sequence; then $\{\varphi_k\}$ is complete in $L^2(\Omega)$ and $\varphi_k \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$; see [21].

Now, let $\vartheta = \max\{-\mu_1, 0\} + 1$; since the coefficient p may vanish in $\overline{\Omega}$, we need to define a new norm:

$$\|u\|_\vartheta = \sqrt{\sigma(u,u) + \int_\Omega \vartheta u^2dx}, \quad u \in W^{1,2}(\Omega,p), \tag{18}$$

and the corresponding inner product

$$\langle u, v\rangle_{\vartheta} = \sigma(u, v) + \int_{\Omega} \vartheta u v \, dx, \quad \forall u, v \in W^{1,2}(\Omega, p). \quad (19)$$

Lemma 1. *Let $q \in L^{r}(\Omega)$ with $r > ns/(2s-n)$; then the norms $\|\cdot\|_{\vartheta}$ and $\|\cdot\|_{p}$ are equivalent.*

Proof. Firstly, it follows from (17) that

$$\chi(u) = \sigma(u, u) + \int_{\Omega} \vartheta u^2 dx \geq 0, \quad \forall u \in W^{1,2}(\Omega, p). \quad (20)$$

We prove that there exists $c_3 >$ such that

$$\chi(u) \geq c_3 \|u\|_p^2, \quad \forall u \in W^{1,2}(\Omega, p). \quad (21)$$

In fact, if (21) is false, by mean of the 2-homogeneity of χ, there exists $\{u_n\} \subset W^{1,2}(\Omega, p)$ such that $\|u_n\|_p = 1$ for all $n \geq 1$ and $\chi(u_n) \to 0^+$ as $n \to \infty$. Without loss of generality, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega, p), \\ &u_n \longrightarrow u \quad \text{strongly in } L^{r'}(\Omega). \end{aligned} \quad (22)$$

By the sequential weak lower semicontinuity of $\sigma(u, u)$ and the choice of ϑ, we know that

$$\sigma(u, u) \leq \varliminf_{n\to\infty} \sigma(u_n, u_n) \leq \mu_1 \|u\|_{L^2(\Omega)}^2. \quad (23)$$

By (17), one gets $\sigma(u, u) = \mu_1 \|u\|_{L^2(\Omega)}^2$ as well as $u = \varrho \varphi_1$ for some constant ϱ. If $\varrho = 0$, then $u_n \to 0$ in $W^{1,2}(\Omega, p)$, which contradicts $\|u_n\|_p = 1$, $\forall n \geq 1$; if $\varrho \neq 0$, by (23), one has $\sigma(\varphi_1, \varphi_1) < \mu_1 \|\varphi_1\|_{L^2(\Omega)}^2$; this is a contradiction. Hence, (21) is true.

On the flip side, from (9), we have

$$\begin{aligned} \sigma(u, u) + \int_{\Omega} \vartheta u^2 dx &\leq \int_{\Omega} \left(p(x)|\nabla u|^2 + \vartheta u^2\right) dx \\ &\quad + \|q\|_{L^r(\Omega)} \|u\|_{2r'}^2 \\ &\leq (1+\vartheta)\|u\|_p^2 + K\|q\|_{L^r(\Omega)} \|u\|_p^2 \\ &\leq \max\left\{1+\vartheta, K\|q\|_{L^r(\Omega)}\right\} \|u\|_p^2. \end{aligned} \quad (24)$$

Here K is a positive constant. Then, by (21) and (24), one gets

$$\begin{aligned} c_3 \|u\|_p^2 &\leq \sigma(u, u) + \int_{\Omega} \vartheta u^2 dx \\ &\leq \max\left\{1+\vartheta, K\|q\|_{L^r(\Omega)}\right\} \|u\|_p^2. \end{aligned} \quad (25)$$

This proved the norms $\|\cdot\|_{\vartheta}$ and $\|\cdot\|_{p}$ are equivalent. □

From now on, we always assume $W^{1,2}(\Omega, p) = (W^{1,2}(\Omega, p), \|\cdot\|_{\vartheta}, \langle\cdot,\cdot\rangle_{\vartheta})$, $\|\varphi_k\|_{\vartheta} = 1$.

Lemma 2. *Under the hypotheses of Lemma 1, the embedding $W^{1,2}(\Omega, p) \hookrightarrow L^m(\Omega)$ is continuous for $[1, 2_s^*]$, compact for $[1, 2_s^*)$.*

Proof. By Lemma 1 and the Compactness Theorem in [18], we directly conclude Lemma 2.

In addition, from Lemma 2, there exists $\ell > 0$ such that $\|u\|_{L^m(\Omega)} \leq \ell \|u\|_{\vartheta}$. For simplicity, we will assume that $\ell = 1$; that is,

$$\|u\|_{L^m(\Omega)} \leq \|u\|_{\vartheta}, \quad \forall u \in W^{1,2}(\Omega, p). \quad (26)$$

□

Now, let $G_i(x, s) = \int_0^s g_i(x, t)dt$ and $\mathfrak{F} = \{\mu_i\}_{i\in\mathbb{N}}$. For fixed $k \geq 1$, suppose that μ_k is an eigenvalue of multiplicity τ and denote by E_{μ}^k the eigenspace associated with the eigenvalue μ_k, $Z^{\pm} = \operatorname{span}\{(\varphi, \pm\varphi): \varphi \in E_{\mu}^k\}$. The main results are as follows.

Theorem 3. *Let $\mu_l \in \mathfrak{F}$ be the first value above $\alpha - \beta$ and suppose that conditions (g_0) and (g_∞) hold. Also,*

$$\lim_{|s|\to\infty} G_i(x, s) = +\infty, \quad i = 1, 2, \quad (27)$$

uniformly in $x \in \Omega$, and

$$\int_{\Omega} (h_1 \phi + h_2 \psi) dx = 0, \quad \forall (\phi, \psi) \in Z^+. \quad (28)$$

Then for any $\eta > 0$, there exists $\gamma_0 > 0$ such that $\alpha + \beta \in (\mu_k - \gamma_0, \mu_k)$; if $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta$, then problem (2) has at least two different solutions.

Theorem 4. *If we replace condition (27) of Theorem 3 with*

$$\lim_{|s|\to\infty} G_i(x, s) = -\infty, \quad i = 1, 2, \quad (29)$$

uniformly in $x \in \Omega$, then for any $\eta > 0$, there exists $\gamma_1 > 0$ such that $\alpha + \beta \in (\mu_k, \mu_k + \gamma_1)$; if $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta$, then problem (2) has at least two different solutions.

Theorem 5. *In addition to conditions (g_0), (g_∞), and (29), suppose that $\mu_l \in \mathfrak{F}$ is the first value above $\alpha + \beta$ and*

$$\int_{\Omega} (h_1 \phi + h_2 \psi) dx = 0, \quad \forall (\phi, \psi) \in Z^-. \quad (30)$$

Then for any $\eta > 0$, there exists $\gamma_2 > 0$ such that $\alpha - \beta \in (\mu_k - \gamma_2, \mu_k)$; if $\operatorname{dist}(\alpha + \beta, \mathfrak{F}) > \eta$, then problem (2) has at least two different solutions.

Theorem 6. *Let μ_l be the first eigenvalue above $\alpha + \beta$ and conditions (g_0), (g_∞), (27), and (30) hold. Then for any $\eta > 0$, there exists $\gamma_3 > 0$ such that $\alpha - \beta \in (\mu_k, \mu_k + \gamma_3)$; if $\operatorname{dist}(\alpha + \beta, \mathfrak{F}) > \eta$, then problem (2) has at least two different solutions.*

3. Proof of Main Results

In this section, we firstly prove some preliminary lemmas, and then we prove our main results through variational methods and Galerkin techniques.

For the sake of simplicity, let $DW = W^{1,2}(\Omega, p) \times W^{1,2}(\Omega, p)$ and $DL = L^2(\Omega) \times L^2(\Omega)$, with the norms

$$\|z\|_{DW} = \sqrt{\|u\|_{\vartheta}^2 + \|v\|_{\vartheta}^2}, \qquad \|z\|_{DL} = \sqrt{\|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2}, \tag{31}$$

the inner products $\langle\cdot,\cdot\rangle_{DW}$ and $\langle\cdot,\cdot\rangle_{DL}$, respectively. In addition, we will always use the notation $z = (u, v)$, $w = (\phi, \psi) \in DW$, unless otherwise specified.

Define $G: DW \to \mathbb{R}$ and $H: DW \to \mathbb{R}$, $\forall z \in DW$

$$G(z) = \int_\Omega (G_1(x, v) + G_2(x, u))\, dx, \qquad H(z) = \int_\Omega (h_1 v + h_2 u)\, dx. \tag{32}$$

For every $\varepsilon > 0$, we claim that there exist positive constants M_ε and C_h such that

$$|G(z)| \le \frac{\varepsilon}{2}\|z\|_{DW}^2 + 2M_\varepsilon \|z\|_{DW}, \qquad H(z) \le C_h \|z\|_{DW}, \tag{33}$$

$$\left|\left\langle G'(z), w\right\rangle\right| \le (\varepsilon\|z\|_{DW} + 2M_\varepsilon)\|w\|_{DW}, \qquad \left|\left\langle H'(z), w\right\rangle\right| \le C_h \|w\|_{DW}. \tag{34}$$

In fact, by means of (g_0), (g_∞), and (26), the arguments of (33) and (34) are quite similar to that of Lemma 3.1 in [11] and so is omitted.

Now, we define the functional $J(z): DW \to \mathbb{R}$, $\forall z \in DW$

$$J(z) = \frac{1}{2}B(z, z) - G(z) - H(z), \tag{35}$$

where $B: DW \times DW \to \mathbb{R}$, $\forall z, w \in DW$, given by

$$\begin{aligned} B(z, w) &= \int_\Omega p(x)(\nabla u \nabla \psi + \nabla v \nabla \phi)\, dx + \int_\Omega q(x)(u\psi + v\phi)\, dx \\ &\quad - \alpha \int_\Omega (u\psi + v\phi)\, dx - \beta \int_\Omega (u\phi + v\psi)\, dx \\ &= \langle z, \overline{w}\rangle_{DW} - (\alpha + \vartheta)\langle z, \overline{w}\rangle_{DL} - \beta\langle z, w\rangle_{DL}, \end{aligned} \tag{36}$$

where $\overline{w} = (\psi, \phi)$. By means of (g_0) and (g_∞), one has $J \in C^1(DW, \mathbb{R})$ and

$$\left\langle J'(z), w\right\rangle = B(z, w) - \left\langle G'(z), w\right\rangle - \left\langle H'(z), w\right\rangle, \tag{37}$$

which implies that the critical points of J are exactly weak solutions of problem (2).

Next, we need to consider the eigenvalue problem: $B(z, w) = \lambda\langle z, \phi\rangle_{DW}$, for $\lambda \in \mathbb{R}$; that is,

$$B(z, w) = \lambda \int_\Omega \{p(x)(\nabla u \nabla \phi + \nabla v \nabla \psi) + (q(x) + \vartheta)(u\phi + v\psi)\}\, dx. \tag{38}$$

Firstly, for any $u, v \in W^{1,2}(\Omega, p)$, one has $u = \sum_{k=1}^\infty a_i\varphi_i$ and $v = \sum_{i=1}^\infty b_i\varphi_i$ for some $a_i, b_i \in \mathbb{R}$. Now, by $\|\varphi_i\|_\vartheta = 1$ and using $(\phi, \psi) = (\varphi_i, 0)$ and $(\phi, \psi) = (0, \varphi_i)$ in (38), then a straightforward calculation shows that (38) is equivalent to

$$\begin{pmatrix} \alpha - \mu_i & \lambda(\mu_i + \vartheta) + \beta \\ \lambda(\mu_i + \vartheta) + \beta & \alpha - \mu_i \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix} = 0. \tag{39}$$

Obviously, $(a_i, b_i) \ne 0$ if and only if $(\alpha - \mu_i)^2 - \{\lambda(\vartheta + \mu_i) + \beta\}^2 = 0$. Hence, we obtain two sequences of eigenvalues

$$\lambda_{\pm i} = \frac{-\beta \pm (\mu_i - \alpha)}{\vartheta + \mu_i}, \quad \forall i \in \mathbb{N}, \tag{40}$$

and the corresponding eigenfunctions

$$\phi_{\pm i} = \frac{\sqrt{2}}{2}(\varphi_i, \pm\varphi_i), \quad \forall i \in \mathbb{N}, \tag{41}$$

are the corresponding eigenfunctions.

Let $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, for $i, j \in \mathbb{Z}_0$; a simple calculation yields

$$\begin{aligned} \|\phi_i\|_{DW} &= 1, \\ \left\langle \phi_i, \phi_j\right\rangle_{DW} &= \delta_i^j, \\ \mu_i\left\langle \phi_i, \phi_j\right\rangle_{DL} &= \delta_i^j, \\ B\left(\phi_i, \phi_j\right) &= \mu_{|i|}^{-1}\delta_i^j, \end{aligned} \tag{42}$$

where δ_i^j denotes the Kronecker symbol. Moreover, if $z = \sum_{i\in\mathbb{Z}_0} a_i\phi_i$, then

$$\|z\|_{DW} = \sum_{i\in\mathbb{Z}_0} a_i^2, \qquad B(z, z) = \sum_{i\in\mathbb{Z}_0} \lambda_i a_i^2, \qquad \|z\|_{DL} = \sum_{i\in\mathbb{Z}_0} \mu_{|i|}^{-1} a_i^2, \tag{43}$$

In addition, for every $\gamma > 0$, if $\alpha + \beta \in (\mu_k - \gamma, \mu_k)$, from (40) one gets

$$0 < \lambda_k = \frac{\mu_k - \alpha - \beta}{\vartheta + \mu_k} < \frac{\gamma}{\vartheta + \mu_k}. \tag{44}$$

Let us fix $k \ge 1$ and define

$$\begin{aligned} \mathbb{I}^0 &= \{i \in \mathbb{Z}_0 : \lambda_i = \lambda_k\}, \\ \mathbb{I}^- &= \{i \in \mathbb{Z}_0 : \lambda_i \ne \lambda_k,\ \lambda_i < 0\}, \\ \mathbb{I}^+ &= \{i \in \mathbb{Z}_0 : \lambda_i \ne \lambda_k,\ \lambda_i > 0\}, \\ Z &= \overline{\operatorname{span}\{\phi_i : i \in \mathbb{I}^0\}}, \\ V &= \overline{\operatorname{span}\{\phi_i : i \in \mathbb{I}^-\}}, \\ X &= \overline{\operatorname{span}\{\phi_i : i \in \mathbb{I}^+\}}. \end{aligned} \tag{45}$$

Meanwhile, we denote by B_V, B_{VZ}, B_X, and B_{ZX} the unitary closed balls, with respect to the norm $\|\cdot\|_{DW}$, in the spaces V, $V \oplus Z$, X, and $Z \oplus X$, respectively, and by S_V, S_{VZ}, S_X, and S_{ZX} their relative boundaries.

Lemma 7. *Suppose that g_i satisfies (g_0), (g_∞), $\alpha \pm \beta \notin \mathfrak{F}$. For fixed $k \geq 1$, let μ_k and μ_l be the first eigenvalue above $\alpha + \beta$ and $\alpha - \beta$, respectively.*

If $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta > 0$, then we have

$$J(z) \geq C^{\eta}_{\alpha+\beta}, \quad \forall z \in Z \oplus X, \qquad J(z) < C^{\eta}_{\alpha+\beta}, \quad \forall z \in \rho S_V, \ \rho \geq \rho^{\eta}_{\alpha+\beta}, \tag{46}$$

for some constants $C^{\eta}_{\alpha+\beta} \in \mathbb{R}$ and $\rho^{\eta}_{\alpha+\beta} > 0$.

Further, if condition (27) also is satisfied, then there exists a positive constant γ_0 such that, for $\alpha + \beta \in (\mu_k - \gamma_0, \mu_k)$, there exist D_η, $C^{\eta}_{\alpha+\beta} \in R$, $\rho^{\eta}_{\alpha+\beta} > R_\eta > 0$ such that (46) hold and

$$J(z) \geq D_\eta, \quad \forall z \in X, \tag{47}$$

$$J(z) < D_\eta - 1, \quad \forall z \in R_\eta S_{VZ}, \tag{48}$$

$$J(z) < D_\eta - 1, \quad \forall z \in V \text{ with } \|z\|_{DW} > R_\eta. \tag{49}$$

Here, a value D_η with index δ represents that D_η depend on δ, and other cases are similar.

Proof. Firstly, if $k \geq 2$, then $\mu_{k-1} < \alpha + \beta < \mu_k$; if $k = 1$, then $\alpha + \beta < \mu_1$.

For $k \geq 2$, if $\vartheta + \alpha + \beta \geq 0$, then the sequence $\{\lambda_i = 1 - (\vartheta + \alpha + \beta)/(\vartheta + \mu_i)\}$ is nondecreasing, which implies

$$|\lambda_i| \geq \min\left\{1 - \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_k}, -1 + \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_{k-1}}\right\} > 0, \quad \forall i \in \mathbb{N}. \tag{50}$$

If $\vartheta + \alpha + \beta < 0$, then the sequence $\{\lambda_i = 1 - (\vartheta + \alpha + \beta)/(\vartheta + \mu_i)\}$ is nonincreasing, which implies

$$\lambda_i \geq \lim_{k \to \infty}\left(1 - \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_k}\right) = 1, \quad \forall i \in \mathbb{N}. \tag{51}$$

Similarly, for $k = 1$, that is, $\alpha + \beta < \mu_1$, if $\vartheta + \alpha + \beta \geq 0$, then the sequence $\{\lambda_i = 1 - (\vartheta + \alpha + \beta)/(\vartheta + \mu_i)\}$ is also nondecreasing, which implies

$$\lambda_i \geq \lambda_1 = 1 - \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_1} > 0, \quad \forall i \in \mathbb{N}. \tag{52}$$

If $\vartheta + \alpha + \beta < 0$, then the sequence $\{\lambda_i = 1 - (\vartheta + \alpha + \beta)/(\vartheta + \mu_i)\}$ is nonincreasing, which implies $\lambda_i \geq 1$ for every $i \in \mathbb{N}$.

In a word, for fixed $k \geq 1$, there exists $P_{\alpha+\beta} > 0$ such that

$$|\lambda_i| \geq P_{\alpha+\beta}, \quad \forall i \in \mathbb{N}. \tag{53}$$

Secondly, because of the fact that μ_l is the first eigenvalue above $\alpha - \beta$ and $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta > 0$, thus $\alpha - \beta < \mu_1 - \delta$, if $l = 1$; $\mu_{l-1} + \delta < \alpha - \beta < \mu_l - \delta$, if $l \geq 2$. Proceeding as in the proof of the first step, we can also conclude that there exists $P_\eta > 0$ such that

$$|\lambda_{-i}| \geq P_\eta, \quad \forall i \in \mathbb{N}. \tag{54}$$

Let $P^{\eta}_{\alpha+\beta} = \min\{P_{\alpha+\beta}, P_\eta\} > 0$; we have by (53) and (54)

$$|\lambda_{\pm i}| \geq P^{\eta}_{\alpha+\beta}, \quad \forall i \in \mathbb{N}. \tag{55}$$

Hence, as in the proof of Lemma 4.1 in [11], we know that

$$B(z, z) \leq -P^{\eta}_{\alpha+\beta}\|z\|^2_{DW}, \quad \forall z \in V, \tag{56}$$

$$B(z, z) \geq P^{\eta}_{\alpha+\beta}\|z\|^2_{DW}, \quad \forall z \in Z \oplus X. \tag{57}$$

From (33) and (56), we get

$$J(z) \leq -\left(P^{\eta}_{\alpha+\beta} - \frac{\varepsilon}{2}\right)\|z\|^2_{DW} + (2M_\varepsilon + C_h)\|z\|_{DW}, \quad \forall z \in V. \tag{58}$$

From (33) and (57), we get

$$J(z) \geq \left(P^{\eta}_{\alpha+\beta} - \frac{\varepsilon}{2}\right)\|z\|^2_{DW} - (2M_\varepsilon + C_h)\|z\|_{DW}, \quad \forall z \in Z \oplus X. \tag{59}$$

By (58) and (59) and choosing $\varepsilon < 2P^{\eta}_{\alpha+\beta}$, we conclude that there exist $C^{\eta}_{\alpha+\beta} \in \mathbb{R}$ and $\rho^{\eta}_{\alpha+\beta} > 0$ satisfying (46).

In addition, if $\alpha + \beta$ is near enough to μ_k, in particular, if $\alpha + \beta > 0$ and $\operatorname{dist}(\alpha + \beta, \mathfrak{F} \setminus \{\mu_k\}) > d > 0$, we claim that there exists $Q^{\eta}_d > 0$ such that

$$B(z, z) \leq -Q^{\eta}_d\|z\|^2_{DW}, \quad \forall z \in V, \tag{60}$$

$$B(z, z) \geq Q^{\eta}_d\|z\|^2_{DW}, \quad \forall z \in X. \tag{61}$$

In fact, if $\alpha + \beta > 0$ and $\operatorname{dist}(\alpha + \beta, \mathfrak{F} \setminus \{\mu_k\}) > d > 0$, for $k \geq 2$, we have

$$\begin{aligned}\min_{i \in \mathbb{N} \setminus \{k, \ldots, k+\tau-1\}} |\lambda_i| &= \min\left\{1 - \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_{k+\tau}}, -1 + \frac{\vartheta + \alpha + \beta}{\vartheta + \mu_{k-1}}\right\} \\ &> \min\left\{\frac{d}{\vartheta + \mu_{k+\tau}}, \frac{d}{\vartheta + \mu_{k-1}}\right\} > 0.\end{aligned} \tag{62}$$

And, for $k = 1$, we have

$$\inf_{i \in \mathbb{N} \setminus \{1\}} |\lambda_i| = \frac{\mu_2 - \alpha - \beta}{\vartheta + \mu_2} \geq \frac{d}{\vartheta + \mu_2} > 0. \tag{63}$$

Hence, for fixed $k \geq 1$, there exists $Q^{\eta}_d > 0$ such that

$$|\lambda_i| \geq Q^{\eta}_d, \quad \forall i \in \mathbb{Z}_0 \setminus \{k, \ldots, k+\tau-1\}. \tag{64}$$

From this we easily get the estimates (60) and (61).

Next, we prove (47), (48), and (49). Let $d = (1/2)\operatorname{dist}(\mu_k, \mathfrak{F}\setminus\{\mu_k\})$ and $\gamma_0 \in (0, d)$. If $\alpha+\beta \in (\mu_k - \gamma_0, \mu_k)$, then $\operatorname{dist}(\alpha+\beta, \mathfrak{F}\setminus\{\mu_k\}) > d$.

If $z \in X$, it follows from (33) and (61) that

$$J(z) \geq \left(Q_d^{\eta} - \frac{\varepsilon}{2}\right)\|z\|_{DW}^2 - (2M_\varepsilon + C_h)\|z\|_{DW}. \quad (65)$$

By choosing $\varepsilon = Q_d^{\eta}$, then there exists $D_\eta \in \mathbb{R}$ satisfying (47).

If $z \in V$, by (33) and (60), we obtain

$$J(z) \leq -\left(Q_d^{\eta} - \frac{\varepsilon}{2}\right)\|z\|_{DW}^2 + (2M_\varepsilon + C_h)\|z\|_{DW}. \quad (66)$$

Let us choose ε small enough; then there exists $\widetilde{R} > 0$ satisfying (49) for $R_\eta > \widetilde{R}$.

Now, we prove the estimate (48). If (48) is not true, then, for any sequences $\gamma_n \to 0^+$ and $R_n > \widetilde{R}$, there exist $z_n \in R_n S_{VZ}$ and $\alpha_n, \beta_n \in \mathbb{R}$ such that

$$\begin{gathered} \alpha_n + \beta_n \in (\mu_k - \gamma_n, \mu_k), \\ \operatorname{dist}(\alpha_n - \beta_n, \mathfrak{F}\setminus\{\mu_k\}) > \eta, \\ J_n(z_n) \geq D_\eta - 1. \end{gathered} \quad (67)$$

Here B_n (or J_n) denotes the form B (or the functional J) with $\alpha = \alpha_n$ and $\beta = \beta_n$. And, $\{\lambda_i^n\}_{i\in\mathbb{Z}_0}$ denotes the eigenvalues of the bilinear form B_n.

Let $z_n = \varpi_n + \tau_n \in V \oplus Z$, and suppose $R_n \to \infty$ and $\gamma_n R_n^2 \to 0$ as $n \to \infty$. By (44), one has $0 < \lambda_k^n < \gamma_n/(\vartheta + \mu_k)$, which implies $B_n(\tau_n, \tau_n) \leq (\gamma_n/(\vartheta+\mu_k))\|\tau_n\|_{DW}^2$. So, by (60), we get

$$\begin{aligned} B_n(z_n, z_n) &\leq B_n(\varpi_n, \varpi_n) + B_n(\tau_n, \tau_n) \\ &\leq \frac{\gamma_n}{\vartheta + \mu_k}\|\tau_n\|_{DW}^2 - Q_d^{\eta}\|\varpi_n\|_{DW}^2, \end{aligned} \quad (68)$$

for all positive integers n; we get by (68)

$$\begin{aligned} D_\eta - 1 &\leq J_n(z_n) \\ &\leq \frac{\gamma_n}{\vartheta+\mu_k}\|\tau_n\|_{DW}^2 - Q_d^{\eta}\|\varpi_n\|_{DW}^2 - G(z_n) \\ &\quad - H(z_n). \end{aligned} \quad (69)$$

It follows from (33) and (69) that

$$\begin{aligned} \left(Q_d^{\eta} - \frac{\varepsilon}{2}\right)\frac{\|\varpi_n\|_{DW}^2}{R_n^2} \leq \left(\frac{\gamma_n}{\vartheta+\mu_k} + \frac{\varepsilon}{2}\right)\frac{\|\tau_n\|_{DW}^2}{R_n^2} \\ - \frac{D_\eta - 1}{R_n^2} \\ + \frac{(2M_\varepsilon + C_h)\|z\|_{DW}}{R_n^2}. \end{aligned} \quad (70)$$

We note that $\|\tau_n\|_{DW} \leq \|z_n\|_{DW} = R_n$; then from (70) we obtain

$$\limsup_{n\to\infty}\frac{\|\varpi_n\|_{DW}^2}{R_n^2} \leq \varepsilon', \quad (71)$$

where $\varepsilon' = \varepsilon/(2Q_d^{\eta} - \varepsilon)$. Note that $\dim Z < \infty$

$$\frac{\|\varpi_n\|_{DW}^2}{R_n^2} + \frac{\|\tau_n\|_{DW}^2}{R_n^2} = 1, \quad \frac{\|\tau_n\|_{DW}}{R_n} \in Z. \quad (72)$$

Then, we get by (71)

$$\begin{gathered} \lim_{n\to\infty}\frac{\|\tau_n\|_{DW}}{R_n} \geq \sqrt{1-\varepsilon'}, \\ \lim_{n\to\infty}\frac{\|\varpi_n\|_{DW}}{R_n} \leq \sqrt{\varepsilon'}. \end{gathered} \quad (73)$$

Let $\widetilde{z}_n = \widetilde{\varpi}_n + \widetilde{\tau}_n$, where $\widetilde{\varpi}_n = \varpi_n/R_n$, $\widetilde{\tau}_n = \tau_n/R_n$. Then, by (73), there exists $\widetilde{z}_0 = \widetilde{\varpi}_0 + \widetilde{\tau}_0 \in DW$ with $\|\widetilde{z}_0\|_{DW} = 1$, such that

$$\begin{gathered} \lim_{n\to\infty}\|\widetilde{\tau}_n\|_{DW} = \|\widetilde{\tau}_0\|_{DW} \geq \sqrt{1-\varepsilon'}, \\ \lim_{n\to\infty}\|\widetilde{\varpi}_n\|_{DW} = \|\widetilde{\varpi}_0\|_{DW} \leq \sqrt{\varepsilon'}. \end{gathered} \quad (74)$$

Let $z = (P_1 z, P_2 z) \in DW$. Then, we have

$$\begin{aligned} \|\widetilde{z}_n\|_{DW}^2 &= \|(P_1\widetilde{z}_n, P_2\widetilde{z}_n)\|_{DW}^2 \\ &= \|(P_1\widetilde{\varpi}_n + P_1\widetilde{\tau}_n, P_2\widetilde{\varpi}_n + P_2\widetilde{\tau}_n)\|_{DW}^2 \\ &= \|P_1\widetilde{\varpi}_n\|_{\vartheta}^2 + \|P_1\widetilde{\tau}_n\|_{\vartheta}^2 + \|P_2\widetilde{\varpi}_n\|_{\vartheta}^2 + \|P_2\widetilde{\tau}_n\|_{\vartheta}^2 \\ &= 1. \end{aligned} \quad (75)$$

From this and (74), without loss of generality, we assume

$$\begin{gathered} \|P_1\widetilde{\tau}_0\|_{\vartheta} \geq \sqrt{\frac{1-\varepsilon'}{2}}, \\ \|P_1\widetilde{\varpi}_0\|_{\vartheta} \leq \sqrt{\varepsilon'}. \end{gathered} \quad (76)$$

By $\dim Z < \infty$, one has $\|P_1\widetilde{\tau}_0\|_{L^1(\Omega)} > C\sqrt{(1-\varepsilon')/2}$ for some positive constant C. Besides, from (26) and the second inequality of (76), we obtain $\|P_1\widetilde{\varpi}_0\|_{L^1(\Omega)} < \sqrt{\varepsilon'}$. Thus, by choosing ε small enough, there exists $\zeta > 0$ such that

$$\begin{gathered} \|P_1\widetilde{\tau}_0\|_{L^1(\Omega)} > 3\zeta, \\ \|P_1\widetilde{\varpi}_0\|_{L^1(\Omega)} < \zeta. \end{gathered} \quad (77)$$

From this and (74), there exist $\eta > 0$ and $n_0 > 0$ such that

$$\left|\Omega_{P_1\widetilde{z}_n}\right| = \left|\{x \in \Omega : |P_1\widetilde{z}_n(x)| > \eta\}\right| > \eta \quad \text{for } n > n_0. \quad (78)$$

By (g_0) and (27), proceeding as in the proof of Proposition 4.6 in [11], we have

$$\lim_{R\to\infty}\inf_{n>n_0}\int_\Omega G_2(x, RP_1\widetilde{z}_n)\,dx = +\infty. \quad (79)$$

Further, because of the fact that $\int_\Omega G_1(x, RP_2\widetilde{z}_n)dx$ is bounded from below, we get

$$\lim_{n\to\infty} G(z_n) = \lim_{n\to\infty} G(R_n\widetilde{z}_n) = +\infty. \quad (80)$$

In addition, for fixed d and η, from (54) and (64), we can choose γ small enough such that $\mu_i = \mu_k$, $i = k, \ldots, k+\tau-1$; then we get $Z = Z^+$. Hence, by (28), it is easy to see that $Q_d^\eta \|\varpi_n\|_{DW}^2 \geq -\delta_2$ for some positive constant δ_2. Moreover, it follows from (69) that

$$G(z_n) \leq \frac{\gamma_n}{\vartheta + \mu_k} \|\tau_n\|_{DW}^2 - D_\eta + 1 + \delta_2. \tag{81}$$

Recall that $\|\tau_n\|_{DW} \leq R_n$ and $\gamma_n R_n^2 \to 0$ as $n \to \infty$, which contradicts with (80). Hence, there exist $R_\eta > \widetilde{R} > 0$ and $\gamma_0 \in (0, d)$ which satisfy (48).

Finally, by the process of the above proof, we easily know that all the constants of the estimates above are not contradictory; then we finished the proof of Lemma 7. □

Remark 8. In fact, from (46), we can get a solution of problem (2). And, from (47), (48), and (49), we can also get a solution of problem (2). But we do not know whether they are different. So we will use Galerkin techniques to show the existence of two different solutions.

Now, we assume that $n > k + \tau$

$$\begin{aligned} DW_n &= \operatorname{span}\{\phi_{-n}, \ldots, \phi_n\}, \\ V_n &= V \cap DW_n, \\ X_n &= X \cap DW_n. \end{aligned} \tag{82}$$

It follows from this that $Z \subset DW_n$. As before, let $B_V^n = B_1 \cap V^n$, $B_{VZ}^n = B_1 \cap (V_n \oplus Z)$, $B_{ZX}^n = B_1 \cap (Z \oplus X_n)$, S_V^n, S_{VZ}^n, and S_{ZX}^n represent their relative boundary. In addition, we know that all the estimates of Lemma 7 are true when the conditions of Theorem 3 are satisfied. Further, if $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta > 0$ and $\alpha + \beta \in (\mu_k - \gamma_0, \mu_k)$, we can also check that the similar estimates of Lemma 7, with respect to the functional J_n, hold on the spaces V_n, Z, and X_n.

Lemma 9. *Fix $n > k + \tau$, assume that $\alpha \pm \beta \notin \mathfrak{F}$, $h_1, h_2 \in L^2(\Omega)$, and g_1 and g_2 satisfy (g_0) and (g_∞). In addition, let $\{z_i\} \subset DW_n$ satisfy*

$$\left|\left\langle J'(z_i), w\right\rangle\right| < \delta_i \|w\|_{DW}, \quad \forall w = (\phi, \psi) \in DW_n, \tag{83}$$

where $\delta_i \to 0$ as $i \to \infty$. Then, there exists a subsequence $\{z_{i_k}\} \subset \{z_i\}$ such that $\{z_{i_k}\}$ converges in DW_n.

Proof. If $i \in \mathbb{Z}_0$, then $\lambda_i \neq 0$. Hence, we may divide the space DW in the two orthogonal components

$$\begin{aligned} DW^- &= \overline{\operatorname{span}\{\phi_i : i \in \mathbb{Z}_0, \lambda_i < 0\}}, \\ DW^+ &= DW \setminus DW^-. \end{aligned} \tag{84}$$

For $j > k+\tau$, let $DW_j^+ = DW^+ \cap DW_j$, $DW_j^- = DW^- \cap DW_j$. In addition, because of $\lambda_i \to \pm 1$ as $i \to \pm\infty$, then we may set $\lambda_0 = \inf\{|\lambda_i| : i \in \mathbb{Z}_0\} > 0$, and then

$$B(z, z) \leq -\lambda_0 \|z\|_{DW}^2, \quad \forall z \in DW^-, \tag{85}$$

$$B(z, z) \geq \lambda_0 \|z\|_{DW}^2, \quad \forall z \in DW^+. \tag{86}$$

Setting $z_i = z_i^- + z_i^+ \in DW^- \oplus DW^+$, by testing (83) with $w = z_i^-$ we have

$$\begin{aligned} &\left|B(z_i, z_i^-) - \left\langle G'(z_i), z_i^-\right\rangle - \left\langle H'(z_i), z_i^-\right\rangle\right| \\ &\quad \leq \delta_i \|z_i^-\|_{DW}. \end{aligned} \tag{87}$$

From this, (34), (42), and (85), we get

$$\lambda_0 \|z_i^-\|_{DW} \leq \delta_i + \varepsilon \|z_i\|_{DW} + 2M_\varepsilon + C_h. \tag{88}$$

In the same way, by testing (83) with $w = z_i^+$ we have

$$\lambda_0 \|z_i^+\|_{DW} \leq \delta_i + \varepsilon \|z_i\|_{DW} + 2M_\varepsilon + C_h. \tag{89}$$

From (88) and (89), we have

$$(\lambda_0 - 2\varepsilon) \|z_i\|_{DW} \leq 2\delta_i + 4M_\varepsilon + 2C_h. \tag{90}$$

Let us choose $\varepsilon < \lambda_0/2$; then $\{z_i\}$ is bounded in DW_n. By $\dim DW_n < \infty$, then there exists $\{z_{i_k}\} \subset \{z_i\}$ such that $\{z_{i_k}\}$ converges in DW_n. □

From the proof of Lemma 9, we can also get Lemma 10 below.

Lemma 10. *Assume that $\alpha \pm \beta \notin \mathfrak{F}$, $h_1, h_2 \in L^2(\Omega)$, and g_1 and g_2 satisfy (g_0) and (g_∞). In addition, let $\{z_i\} \subset DW_i$ satisfy*

$$\left|\left\langle J'(z_i), w\right\rangle\right| < \delta_i \|w\|_{DW} \quad \text{for every } w \in DW_i, \tag{91}$$

where $\delta_i \to 0$ as $i \to \infty$. Then $\{z_i\}$ is bounded in DW.

Clearly, by using the Saddle Point Theorem, Lemma 9, (46), (47), and (48), there exist $z_n = (u_n, v_n)$ and w_n in DW_n such that $J_n(z_n) = c$, $J_n'(z_n) = 0$, and $J_n(w_n) = s_n$, $J_n'(w_n) = 0$, where

$$\begin{aligned} c_n &= \inf_{\gamma \in \Gamma_V^n} \sup_{z \in \rho_{\alpha+\beta}^\eta B_V^n} J_n(\gamma(z)) \geq C_{\alpha+\beta}^\eta, \\ s_n &= \inf_{\gamma \in \Gamma_{VZ}^n} \sup_{w \in R_\eta B_{VZ}^n} J_n(\gamma(z)) \geq D_\eta, \\ \Gamma_V^n &= \left\{\gamma \in C\left(\rho_{\alpha+\beta}^\eta B_V^n, DW_n\right) : \gamma(z) = z, \|u\|_{DW} \right. \\ &\quad \left. = \rho_{\alpha+\beta}^\eta\right\}, \\ \Gamma_{VZ}^n &= \left\{\gamma \in C\left(R_\eta B_{VZ}^n, DW_n\right) : \gamma(w) = w, \|w\|_{DW} \right. \\ &\quad \left. = R_\eta\right\}. \end{aligned} \tag{92}$$

Lemma 11. *Assume that the conditions are the same as Lemma 7, $\alpha + \beta \in (\mu_k - \gamma_0, \mu_k)$, and $\operatorname{dist}(\alpha - \beta, \mathfrak{F}) > \eta > 0$. Then, for all $n > k + \tau$, there exists $T_\eta > 0$ such that $c_n \in [C_{\alpha+\beta}^\eta, D_\eta - 1]$ and $s_n \in [D_\eta, T_\eta]$.*

Proof. Firstly, by (92), one gets $c_n \geq D_{\alpha+\beta}$ and $s_n \geq D_\eta$. Define

$$\begin{aligned} &\gamma_0(p) \\ &\quad = \begin{cases} p, & R_\eta \leq \|p\|_{DW} \leq \rho_{\alpha+\beta}^\eta, \\ p + \sqrt{R_\eta^2 - \|p\|_{DW}^2}\,\phi_k, & \|p\|_{DW} \leq R_\eta. \end{cases} \end{aligned} \tag{93}$$

So it follows from (48) and (49) that

$$\sup_{z\in\rho^{\eta}_{\alpha+\beta}B^{n}_{V}} J_n(\gamma_0(z)) < D_\eta - 1, \tag{94}$$

which implies $c_n < D_\eta - 1$. In addition, by $\mathrm{Id}|_{R_\eta B^n_{VZ}} \in \Gamma^n_{VZ}$, one gets $s_n < \sup_{w\in R_\eta B^n_{VZ}} J(w)$.

For $z \in V \oplus Z$, it follows from $\alpha+\beta \in (\mu_k - \gamma_0, \mu_k)$ that

$$-P^{\eta}_{\alpha+\beta} < 0 < \lambda_k \le \frac{\gamma_0}{\vartheta + \mu_k}. \tag{95}$$

From this and (33), we get for any $z \in R_\eta B^n_{VZ}$

$$J(z) \le \left(\frac{\gamma_0}{\vartheta+\mu_k} + \varepsilon\right)\|z\|^2_{DW} + (2M_\varepsilon + C_h)\|z\|_{DW}. \tag{96}$$

Hence, there exists $T_\eta > 0$ such that $s_n \le T_\eta$. This finishes the proof of Lemma 11. □

Now, we start to prove our main theorems.

Proof of Theorem 3. Firstly, by Lemma 11 and without loss of generality, there exist $c \in [C^{\eta}_{\alpha+\beta}, D_\eta - 1]$ and $s \in [D_\eta, T_\eta]$ such that $c_n \to c$ and $s_n \to s$ as $n \to +\infty$.

Next, we prove that there exists $z \in DW$ such that $J(z) = c$ and $J'(z) = 0$. In fact, for all $n > k + \tau$, we have

$$J_n(z_n) = c_n, \tag{97}$$

$$\langle J'_n(z_n), w\rangle = 0, \quad \forall w \in DW_n. \tag{98}$$

Then, it follows from Lemma 10 that $\{z_n\}$ is bounded in DW. Hence, without loss of generality, there exists $z \in DW$ such that

$$\begin{aligned} z_n &\rightharpoonup z \quad \text{in } DW,\\ z_n &\longrightarrow z \quad \text{in } DL. \end{aligned} \tag{99}$$

For $\ell > k + \tau$, let $w = (\phi, 0) \in DW_\ell$ and $w = (0, \psi) \in DW_\ell$ in (98), respectively. From a direct calculation, we have for all $n > \ell$

$$\begin{aligned} \langle u_n, \psi\rangle_\vartheta &= (\alpha+\vartheta)\int_\Omega u_n\psi\,dx + \beta\int_\Omega v_n\psi\,dx \\ &\quad + \int_\Omega g_1(x, v_n)\psi\,dx + \int_\Omega h_1\psi\,dx, \\ \langle v_n, \phi\rangle_\vartheta &= (\alpha+\vartheta)\int_\Omega v_n\phi\,dx + \beta\int_\Omega u_n\phi\,dx \\ &\quad + \int_\Omega g_2(x, u_n)\phi\,dx + \int_\Omega h_2\phi\,dx. \end{aligned} \tag{100}$$

Let $n \to \infty$; we get $\langle J'(z), w\rangle = 0$ for any $w \in DW_\ell$. It follows from $DW = \overline{\cup_{\ell\in\mathbb{N}} DW_\ell}$ that $J'(z) = 0$. That is, $z = (u, v)$ is a critical point of the functional J.

Next, we prove $J(z) = c$. Firstly, let us define the orthogonal projection

$$P_n : W^{1,2}(\Omega, p) \longrightarrow \operatorname{span}\{\varphi_1, \dots, \varphi_n\}, \tag{101}$$

which implies $P_n u \to u$, $P_n v \to v$ in $W^{1,2}(\Omega, p)$. Then

$$\begin{aligned} \lim_{n\to\infty}\left(\|u_n - P_n u\|_{L^2(\Omega)} + \|v_n - P_n v\|_{L^2(\Omega)}\right) &= 0, \\ \|u_n\|_{L^2(\Omega)} + \|v_n\|_{L^2(\Omega)} &< \infty. \end{aligned} \tag{102}$$

It follows from (102) that

$$\begin{aligned} &\alpha\int_\Omega u_n(u_n - P_n u)\,dx + \beta\int_\Omega v_n(u_n - P_n u)\,dx \\ &\quad + \int_\Omega g_1(x, v_n)(u_n - P_n u)\,dx \longrightarrow 0, \\ &\alpha\int_\Omega v_n(v_n - P_n v)\,dx + \beta\int_\Omega u_n(v_n - P_n v)\,dx \\ &\quad + \int_\Omega g_2(x, u_n)(v_n - P_n v)\,dx \longrightarrow 0, \\ &\int_\Omega h_1(u_n - P_n u)\,dx + \int_\Omega h_2(v_n - P_n v)\,dx \longrightarrow 0. \end{aligned} \tag{103}$$

Let $w = (v_n - P_n v, u_n - P_n u)$ in (98); one gets

$$\langle J'_n(z_n), (v_n - P_n v, u_n - P_n u)\rangle = 0, \quad \forall n > k + \tau. \tag{104}$$

From this, (103), we obtain

$$\begin{aligned} &\int_\Omega \{p(x)\nabla u_n\nabla(u_n - P_n u) + q(x)u_n(u_n - P_n u)\}\,dx \\ &\quad + \int_\Omega \{p(x)\nabla v_n\nabla(v_n - P_n v) \\ &\quad + q(x)v_n(v_n - P_n v)\}\,dx \longrightarrow 0, \end{aligned} \tag{105}$$

which implies

$$\begin{aligned} &\int_\Omega \{p(x)(|\nabla u_n|^2 + |\nabla v_n|^2)\,dx + q(x)(u_n^2 + v_n^2)\}\,dx \\ &\quad - \int_\Omega \{p(x)(\nabla u_n\nabla u + \nabla v_n\nabla v) + q(x) \\ &\quad \cdot (u_n u + v_n v)\}\,dx + \int_\Omega p(x) \\ &\quad \cdot \{\nabla u_n\nabla(u - P_n u) + \nabla v_n\nabla(v - P_n v)\}\,dx \\ &\quad + \int_\Omega \{q(x)(u_n(u - P_n u) + v_n(v - P_n v))\}\,dx \\ &\quad \longrightarrow 0. \end{aligned} \tag{106}$$

By $\|(P_n u, P_n v) - (u, v)\|_{DW} \to 0$ and (106), one has $\|z_n\|_{DW} \to \|z\|_{DW}$. From the uniform convexity of DW, we have $\|z_n - z\|_{DW} \to 0$ as $n \to \infty$, which implies that $J(z) = c$.

In the same way, we can also prove that there exists $w \in DW$ such that $\|w_n - w\|_{DW} \to 0$, $J'(w) = 0$, and $J(w) = s$. Note that

$$c = J(z) \le D_\eta - 1 < D_\eta \le J(\psi) = s. \tag{107}$$

Hence, we get $z \ne w$, which finished the proof of Theorem 3. □

Proof of Theorem 4. Let $\widetilde{B}(z,w) = -B(z,w)$, and we use the same approach as before; we also get eigenvalues of $\widetilde{B}$:

$$\widetilde{\lambda}_{\pm i} = -\lambda_{\pm i} = -\frac{-\beta \pm (\mu_i - \alpha)}{\vartheta + \mu_i}, \quad \forall i \in \mathbb{N}, \tag{108}$$

and the same eigenfunctions

$$\phi_{\pm i} = \frac{\sqrt{2}}{2}(\varphi_i, \pm \varphi_i), \quad i \in \mathbb{N}. \tag{109}$$

In the same way, we can also define the subspaces $\widetilde{V}$, $\widetilde{Z}$, and $\widetilde{X}$. If $\alpha + \beta \in (\mu_k, \mu_k + \gamma)$ for some $\gamma > 0$, then

$$0 < \widetilde{\lambda}_k = \frac{\alpha + \beta - \mu_k}{\vartheta + \mu_k} < \frac{\gamma}{\vartheta + \mu_k}. \tag{110}$$

Next, we prove that (56), (57), (60), and (61), with respect to $\widetilde{B}$, hold on the new subspaces $\widetilde{V}$, $\widetilde{Z}$, and $\widetilde{X}$. Firstly, we prove the following claim.

Assume that μ_k and μ_l are the first eigenvalue above $\alpha+\beta$ and $\alpha-\beta$, respectively. If $\alpha \pm \beta \notin \mathfrak{F}$ and $\operatorname{dist}(\alpha-\beta, \mathfrak{F}) > \eta > 0$, then there exists $P^{\eta}_{\alpha+\beta} > 0$ such that

$$\begin{aligned} \widetilde{B}(z,z) &\leq -P^{\eta}_{\alpha+\beta}\|z\|^2_{DW}, \quad \forall z \in \widetilde{V}, \\ \widetilde{B}(z,z) &\geq P^{\eta}_{\alpha+\beta}\|z\|^2_{DW}, \quad \forall z \in \widetilde{Z} \oplus \widetilde{X}. \end{aligned} \tag{111}$$

Besides, if $\alpha+\beta$ is close to λ_k, or $\operatorname{dist}(\alpha+\beta, \mathfrak{F} \setminus \{\mu_k\}) > d > 0$, then

$$\begin{aligned} \widetilde{B}(z,z) &\leq -Q^{\eta}_{d}\|z\|^2_{DW}, \quad \forall z \in \widetilde{V}, \\ \widetilde{B}(z,z) &\geq Q^{\eta}_{d}\|z\|^2_{DW}, \quad \forall z \in \widetilde{X}, \end{aligned} \tag{112}$$

for some positive constant Q^{η}_{d}.

Actually, from (108) one can prove that an estimate like (55) holds for these new eigenvalues $\widetilde{\lambda}_{\pm i}$, and then we get (111). Moreover, when $\alpha+\beta$ is close to μ_k, we can also check that, as in the proof of (60) and (61), there exists the positive constant Q^{η}_{d} satisfying (112).

Similarly, we can also choose γ small enough such that $\widetilde{Z} = Z^{+}$. By (111) and (112), we can prove a similar result of Lemma 7. In other words, if the conditions of Theorem 4 are satisfied, we can conclude that there exist positive constants $C^{\eta}_{\alpha+\beta}$ and $\rho^{\eta}_{\alpha+\beta}$ such that the functional J satisfies (46) on the new subspaces $\widetilde{V}$, $\widetilde{Z}$, and $\widetilde{X}$. Further, there exists $\gamma_1 > 0$ such that if $\alpha+\beta \in (\lambda_k, \lambda_k+\gamma_1)$, then the functional J satisfies (46), (47), (48), and (49). Hence, we can also obtain two critical point sequences $\{\widetilde{z}_n\}$ and $\{\widetilde{\psi}_n\}$ of $J_n = J|_{DW_n}$, at critical levels $\widetilde{c}_n$, $\widetilde{s}_n$, similar to (92). Next, the remainder of the argument is similar to the proof of Theorem 3. □

Proof of Theorem 5 (or Theorem 6). In order to take advantage of the conclusion of Theorem 3 (or Theorem 4), we consider the following degenerate system:

$$\begin{aligned} &-\operatorname{div}(p(x)\nabla u) + q(x)u \\ &\quad = \alpha u + \widetilde{\beta} v + \widetilde{g}_1(x,v) + h_1(x), \quad x \in \Omega, \\ &-\operatorname{div}(p(x)\nabla v) + q(x)v \\ &\quad = \widetilde{\beta} u + \alpha v + \widetilde{g}_2(x,u) + \widetilde{h}_2(x), \quad x \in \Omega, \\ &\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, \end{aligned} \tag{113}$$

where $\widetilde{\beta} = -\beta$, $\widetilde{g}_1(x,v) = g_1(x,-v)$, $\widetilde{g}_2(x,u) = -g_2(x,u)$, and $\widetilde{h}_2 = -h_2$. Obviously, under the hypotheses of Theorem 5 (or Theorem 6), the similar hypotheses of Theorem 3 (or Theorem 4) are satisfied for problem (113). Meanwhile, if (u,v) is a solution of (113), then $(u,-v)$ is a solution of problem (2). Hence, by the proof of Theorem 3 (or Theorem 4), we know that Theorem 5 (or Theorem 6) is true. □

Competing Interests

The authors declare that they have no competing interests.

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