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Research Article

A Study on Overestimating a Given Fraction Defective by an Imperfect Inspector

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It has been believed that even an imperfect inspector with nonzero inspection errors could either overestimate or underestimate a given FD (fraction defective) with a 50 : 50 chance. What happens to the existing inspection plans, if an imperfect inspector overestimates a known FD, when it is very low? We deal with this fundamental question, by constructing four mathematical models, under the assumptions that an infinite sequence of items with a known FD is given to an imperfect inspector with nonzero inspection errors, which can be constant and/or randomly distributed with a uniform distribution. We derive four analytical formulas for computing the probability of overestimation (POE) and prove that an imperfect inspector overestimates a given FD with more than 50%, if the FD is less than a value termed as a critical FD. Our mathematical proof indicates that the POE approaches one when FD approaches zero under our assumptions. Hence, if a given FD is very low, commercial inspection plans should be revised with the POE concept in the near future, for the fairness of commercial trades.

1. Introduction

Our research started from a BLU (backlight unit) company in Korea. Inspections of a BLU, which is one of the major components attached to the back of a thin film transistor liquid crystal display unit, can be divided into several functional inspections and external appearance inspections. In the Korean BLU industry, inspection operation is usually done, like other production lines, at the end of a line, due to related costs, and the reworkability of a BLU. The inspection decision is made only by a single attribute: conforming (or C) or nonconforming (or NC). In reality, items will be misclassified, even if only a few. A C (or NC) item may be classified as NC (or C), and this probability is typically termed as type I (or type II) error. Correctly or falsely accepted items at the end of each BLU line are packaged into lots and transported to a clean storage area, where an acceptance sampling plan by attribute is performed, by a source inspector affiliated to a buyer. Even if only one NC item is found in a lot, the lot is rejected by the source inspector. The rejected items, even if there are very few NC items, must be 100% reinspected later, in another clean room (Yang and Cho [1]).

As the fraction defective (FD) of BLU items waiting for source inspections has been gradually lowered to either the thousands or hundreds PPM level, most of the quality control managers have continually raised the possibility that the FD judged by a source inspector has "always" been overestimated, because his inspection could not be perfect, but also his inspection severity would be advantageous to his company. In addition, various questions about unknown dependencies between FD and inspection errors have been raised. In fact, the FD judged by an inspector should be underestimated or overestimated with a 50 : 50 chance, regardless of a given FD and/or inspections errors. In order to verify their presumptions, after having formed lots with a very low FD comprised of hundreds of BLU items randomly mixed up with C and NC items, they carried out significant field experiments controlled by an expert, where the lots were tested by an inspector, with type I error 0.86% and type II error 4.50%, as estimated. They concluded that the possibility of overestimation by the source inspector seemed to be at least significantly larger than 50% but they could not prove it mathematically and that there might exist some relationships between overestimation, very low FD, and inspection errors. From the above facts, many basic questions may be raised, but above all, here we are interested only in the following fundamental and theoretical questions:

> *"Does an imperfect inspector overestimate a given fraction defective, when it is very low?"*

In other words, "Could the FDA (fraction defective after inspection) always be larger than the FDB (fraction defective before inspection), if the FDB is very or extremely low?" In order to answer the above question, we need to find a way to compute the probability of overestimation (POE), when an FD and inspection errors are given.

As far as we know, there have been no papers directly related to our problem. However, some studies dealing with nonzero inspection errors have appeared in the literature, since the 1970s. Collins et al. [2] considered the effects of inspection error on the probability of a lot of acceptance, average outgoing quality, and average total inspection, under both replacement and nonreplacement assumptions, and suggested that an acceptance sampling plan may be designed, based on inspection error. Dorris and Foote [3] surveyed the state of knowledge on inspection and measurement errors and suggested future lines of investigation about inspection errors. Raz and Thomas [4] presented a branch-and-bound method for determining an optimum sequencing inspection plan, for a group of inspectors operating at different skill and cost levels. Tang [5] provided a rule for determining the optimal sequence of multiple quality characteristics, for minimizing the cost of inspection within each inspection stage. Lee [6] developed the stop rule, for seeking the optimal number of inspection stages. Sylla and Drury [7] dealt with the apparent fraction nonconforming $q_e = (1-q)\alpha + q(1-\beta)$, where q is an FD, α is the probability of rejecting a C item, and β is the probability of accepting an NC item. They found n', the sample size and c', the cut-off value for single sampling by attributes, considering fraction defective, types I and II errors, and error-related payoffs, and proposed the concept of liability which is an inspector's ability to respond to information, like payoffs, fraction defectives, and discriminability between noise and signal distributions. Burk et al. [8] derived a relation $q = (q_e - \alpha)(1-\alpha-\beta)^{-1}$ and showed a table about the relation. They noted that as the type I error approaches q_e, q approaches zero and that for very good process, q_e is actually a type I error. They suggested a procedure for estimating the types I and II errors and gave an industrial example. These were shown as major variables, in the last contribution related to our current research.

Several models that are partially related to our problem have appeared in the literature. In order to attain a prespecified quality rate at the end of an assembly line, Yang [9] suggested a K-stage inspection-rework (K-IR) system, which was composed of a series of K stages, each of which included an inspection process and a rework process. He suspected the effectiveness of the K-IR system and proved mathematically that FDA is always larger than FDB if FDB is less than a value that depends on a FD of rework and inspection errors. Based on his assumptions, he suggested a necessary condition for inspection effectiveness that the sum of two errors must be less than one. However, the necessary condition is so rough that it cannot be used practically.

In this paper, we deal with the above fundamental question. In Section 2, we describe our problem in detail. Assuming that an imperfect inspector classifies an infinite sequence of items with a known FD and that each inspection error of type I or type II is either a constant or a uniform random variable on an interval, we provide four mathematical models: Model I (C, C) with both type I error and type II error being constant, Model II (R, C) with type I error and type II error being random and constant, respectively, Model III (C, R) with type I error and type II error being constant and random, respectively, and Model IV (R, R) with type I error and type II error being random and random, respectively. In Sections 3 through 6, for each model, we derive formulas for computing the probability of overestimation (POE) and a critical FD satisfying POE = 50%. In Section 7, in order to extract some relation between the results of the previous sections, we make a reasonable assumption and prove a theorem that answers our question.

2. Problem Statement

Suppose that an imperfect inspector with nonzero inspection errors classifies one-by-one an infinite sequence of items with a known fraction defective q, but which is unknown to the inspector. Then the sequence can be considered as an infinite Bernoulli process $\{X_i,\ i = 1, 2, 3, \ldots\}$ such that, for each i, the value of X_i is either zero, representing a C item, or one, representing an NC item; for all values of i, the probability that $X_i = 1$, $\Pr\{X_i = 1\}$, is the same number q. Let A_i be the probability (the type I error) that the ith C item is misclassified as NC and falsely rejected by the inspector; let B_i be the probability (the type II error) that the ith NC item is misclassified as C and falsely accepted by the inspector. Let Y_i be zero, if the ith item is judged as a C item by the inspector, and one, otherwise. That is, $A_i = \Pr\{Y_i = 1 \mid X_i = 0\}$ and $B_i = \Pr\{Y_i = 0 \mid X_i = 1\}$. Then, $\Pr\{Y_i = 1\}$ and $E[Y_i]$ can be obtained as

$$E[Y_i] = \Pr\{Y_i = 1\} = \sum_{k=0}^{1} \Pr\{Y_i = 1 \mid X_i = k\} \Pr\{X_i = k\} = A_i(1-q) + (1-B_i)q. \tag{1}$$

Since the expected number of rejected items after the nth inspections is $\sum_{i=1}^{n} E[Y_i]$, the FD of an infinite number of items judged by the inspector, denoted by Q, becomes $\lim_{n\to\infty}(1/n)\sum_{i=1}^{n} E[Y_i]$ and can be reduced to

$$Q = q + \lim_{n\to\infty} \frac{1}{n} \left\{(1-q)\sum_{i=1}^{n} A_i - q\sum_{i=1}^{n} B_i\right\}. \tag{2}$$

The above equation implies that the value of Q becomes q if all A_i and B_i are zeros since the value of the limit term

Table 1: Four types of POE and CFD analysis.

Models	Model I (C, C)	Model II (R, C)	Model III (C, R)	Model IV (R, R)
Input				
Known FD	Constant, q	Constant, q	Constant, q	constant, q
Type I error	Constant, α	Random variable, $A \sim f_A(a)$	Constant, α	Random variable, $A \sim f_A(a)$
Type II error	Constant, β	Constant, β	Random variable, $B \sim f_B(b)$	Random variable, $B \sim f_B(b)$
Output				
POE	$\mathrm{POE}_{\mathrm{cc}}(q)$	$\mathrm{POE}_{\mathrm{rc}}(q)$	$\mathrm{POE}_{\mathrm{cr}}(q)$	$\mathrm{POE}_{\mathrm{rr}}(q)$
CFD	$\mathrm{CFD}_{\mathrm{cc}}$	$\mathrm{CFD}_{\mathrm{rc}}$	$\mathrm{CFD}_{\mathrm{cr}}$	$\mathrm{CFD}_{\mathrm{rr}}$

becomes zero. We assume that two types of inspection errors are nonzero and less than or equal to one unless specially mentioned. That is, $0 < A_i,\ B_i \le 1$ for all i. Hence, the value of the limit terms can be positive, zero, or negative, corresponding to overestimation, correct estimation, or underestimation, respectively. Either overestimation or underestimation does not raise any problems by themselves, as long as their probabilities are exactly the same. In fact, we are likely to believe that all inspectors are expected to either overestimate or underestimate a given FD with a 50 : 50 chance. Otherwise, either buyer or supplier must face economic loss due to an unfair inspection game. However, unfortunately, it turns out in this paper that the 50 : 50 chance is not always true and that it depends upon q and $\{(A_i, B_i),\ i = 1, 2, 3, \ldots\}$. Let POE be the probability of overestimation by the inspector, which can be reduced to

$$\begin{aligned}\mathrm{POE} &= \Pr\{Q > q\} \\ &= \Pr\left\{\lim_{n\to\infty}\frac{1}{n}\left\{(1-q)\sum_{i=1}^{n}A_i - q\sum_{i=1}^{n}B_i\right\} > 0\right\}.\end{aligned} \tag{3}$$

If q is an input variable, POE can be expressed as POE(q). If there exists a unique FD q^* such that $\mathrm{POE}(q^*) = 0.5$, then q^* is termed as "the critical fraction defective" or CFD. This definition implies that $E[Q \mid q = q^*] = q^*$; that is, Q is an conditional unbiased estimator of CFD when Q is a random variable. In the case that such a CFD does not exist, q will be called as CFD only if an inspector estimate q correctly, that is, $Q = q$, and POE(q) is defined to be 0.5.

Since inspection error A_i as well as B_i can be assumed to be either a constant or a random variable, we need four kinds of analyses of POE and CFD, as shown in Table 1. Note that the subscript "c" (or "r") under the right sides of POE and CFD in the table represents that the type I or type II error is assumed to be constant (or random). In order to obtain some fundamental properties, we assume that each random variable follows a uniform distribution with an interval on (0, an upper value]. That is, $f_A(a) = (1/\alpha_u)I_{(0,\alpha_u]}(a)$, and $f_B(b) = (1/\beta_u)I_{(0,\beta_u]}(b)$ where $I_{(0,x_u]}(x)$ is an indicator function with one for $0 < x \le x_u$, and zero otherwise. This uniformity assumption with zero/an upper value may be justified, since it has been hoped that inspection errors would become smaller and smaller, and there has been almost no information on the distribution of inspection error, up to now. It is well known that uniform distribution gives maximum uncertainty. Our problem can be summarized as follows: derive both POE(q) and CFD for each model and answer the fundamental question:

> *"Does an imperfect inspector overestimate a given FD when it is very low?"*

Throughout this paper, the first and second derivatives of a function $f(x)$ will be expressed as $f'(x)$ and $f''(x)$, respectively, and the expectation of a random variable X will be denoted by $E[X]$.

3. Analysis of Model I (C, C)

Suppose that $A_i = \alpha$ and $B_i = \beta$, for all i, and both α and β are real-valued constants. Then, since $\Pr\{Y_i = 1 \mid X_i = 0\} = \alpha$ and $\Pr\{Y_i = 1 \mid X_i = 1\} = 1 - \beta$, the process $\{Y_i,\ i = 1, 2, 3, \ldots\}$ becomes an infinite Bernoulli process with $\Pr\{Y_i = 1\}$ being the same number $\{\alpha(1-q) + (1-\beta)q\}$. The following proposition indicates that $\mathrm{POE}_{\mathrm{cc}}(q)$ depends on q and ρ_{cc} where $\rho_{\mathrm{cc}} = \beta/\alpha$ and that $\mathrm{CFD}_{\mathrm{cc}}$ depends only on ρ_{cc}, not q. Let $q_{\mathrm{cc}} = 1/(1 + \rho_{\mathrm{cc}})$ and assume that ρ_{cc} is a rational number.

Proposition 1. *Under the assumption that $A_i = \alpha$ and $B_i = \beta$, for all i, and both α and β are real-valued constants with $0 < \alpha,\ \beta \le 1$,*

(1)

$$POE_{cc}(q) = \begin{cases} 1, & for\ 0 \le q < q_{cc}, \\ 0.5, & for\ q = q_{cc}, \\ 0, & for\ q_{cc} < q \le 1, \end{cases} \tag{4}$$

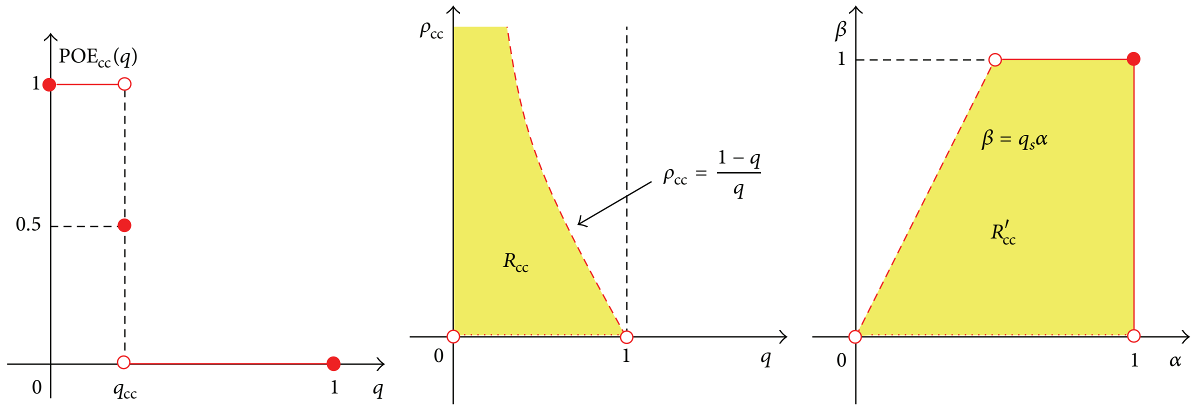
(2) $CFD_{cc} = q_{cc}$,

(3) *an inspector with (α, β) always overestimates q for $0 \le q < q_{cc}$, estimates q correctly for $q = q_{cc}$ with $POE_{cc}(q) = 0.5$, and always underestimates q for $q_{cc} < q \le 1$.*

Proof. Equations (2) and (3) can be derived, respectively, as

$$\begin{aligned}Q_{\mathrm{cc}} &= q + (\alpha + \beta)(q_{\mathrm{cc}} - q), \\ \mathrm{POE}_{\mathrm{cc}}(q) &= \Pr\{Q_{\mathrm{cc}} > q\} = \Pr\{(\alpha + \beta)(q_{\mathrm{cc}} - q) > 0\}.\end{aligned} \tag{5}$$

For $0 \le q < q_{\mathrm{cc}}$, since the inequality $(\alpha + \beta)(q_{\mathrm{cc}} - q) > 0$ is "always" true, (i.e., an inspector with ρ_{cc} "always" overestimates q), $\mathrm{POE}_{\mathrm{cc}}(q)$ can be defined to be one. For $q_{\mathrm{cc}} < q \le 1$,



(a) The graph of POE$_{cc}(q)$

(b) The region of overestimation in (q, ρ_{cc}) plane

(c) The region of overestimation in (α, β) plane when $q_s > \beta/\alpha$

FIGURE 1: The graph of POE$_{cc}(q)$ and the overestimation region in two different planes.

since an inspector with ρ_{cc} "always" underestimates q, POE$_{cc}(q)$ can be defined to be zero. In the case of $q = q_{cc}$, since $Q_{cc} = q$, that is, an inspector estimate q correctly, by our definition in Section 2, it follows that CFD$_{cc}$ $= q_{cc}$ and POE$_{cc}(q_{cc}) = 0.5$. However, q is a rational number (note that q can be expressed as $\lim_{n\to\infty}(1/n)\sum_{i=1}^{n} X_i$, which can be expressed as the fraction m/n of two integers, with $n > 0$), while q_{cc} can be an irrational number depending on the values of α and β. Hence, q cannot be equal to q_{cc} if ρ_{cc} (or q_{cc}) is an irrational number. Since we assume that ρ_{cc} (or q_{cc}) is a rational number, we have, $Q_{cc} = q$ if and only if $q = q_{cc}$ and Proposition 1-(3) holds true. □

From Proposition 1, POE$_{cc}(q)$ can be drawn as shown in Figure 1(a). Suppose that both q and ρ_{cc} are input variables. Then, since an inspector with ρ_{cc} overestimates if and only if $0 \le q < q_{cc} = 1/(1+\rho_{cc})$, every point (q, ρ_{cc}) satisfying $q(1+\rho_{cc}) < 1$ gives overestimation. For $q \ne 0$, every point (q, ρ_{cc}) satisfying $0 < \rho_{cc} < (1-q)/q$ gives overestimation. For $q = 0$, every point $(0, \rho_{cc})$ for $\rho_{cc} > 0$ gives overestimation. Let R_{cc} be the overestimation region as shown in Figure 1(b). Then, an inspector with ρ_{cc} overestimates q if $(q, \rho_{cc}) \in R_{cc}$, estimates q correctly if $\rho_{cc} = (1-q)/q$, and underestimates q otherwise. Note that the point $(0, 0)$ cannot be included in R_{cc} since $\rho_{cc} \ne 0$ due to the assumption that $\alpha \ne 0$.

Suppose that both α and β are input variables and that q is given as a constant. Then, since an inspector with ρ_{cc} overestimates if and only if $q < q_{cc} = \alpha/(\alpha+\beta)$, every point (α, β) satisfying $q(\alpha+\beta) < \alpha$ gives overestimation. For $q \ne 0$, every point (α, β) satisfying $\beta < q_s\alpha$ gives overestimation where $q_s = (1-q)/q$. For $q = 0$, every point (α, β) satisfying $\alpha > 0$ gives overestimation. Let R'_{cc} be the overestimation region bounded by $0 < \beta < q_s\alpha$, $0 < \alpha \le 1$, and $0 < \beta \le 1$ as shown in Figure 1(c) where $q_s > \beta/\alpha$. That is, an inspector with (α, β) overestimates q if $(\alpha, \beta) \in R'_{cc}$, estimates q correctly if $\beta = q_s\alpha$, and underestimates q otherwise.

If $q = 0$, then $Q_{cc} = \alpha$ from (2) and the inspector estimates correctly if and only if $\alpha = 0$. Hence, POE$_{cc}(0)$ has two values either 0.5 (when $\alpha = 0$) or one (when $0 < \alpha \le 1$) depending on α. If $q = 1$, then $Q_{cc} = 1 - \beta$ from (2), and the inspector estimates correctly if and only if $\beta = 0$. Hence, POE$_{cc}(1)$ has two values either 0.5 (when $\beta = 0$) or zero (when $0 < \beta \le 1$) depending on β. These special cases of $(q, \alpha) = (0, 0)$ and $(q, \beta) = (1, 1)$ will be discussed only if necessary.

Since CFD$_{cc}$ depends only on ρ_{cc}, CFD$_{cc}$ remains the same even if both α and β are multiplied by the same number. For example, the CFD$_{cc}$ of $(\alpha, \beta) = (1\,\text{ppm}, 3\,\text{ppm})$ is 0.25, the same as that of $(\alpha, \beta) = (10\%, 30\%)$. Typical CFD$_{cc}$ values for $0 \le \alpha, \beta \le 1$ are summarized in Table 2. As shown in the table, the value of CFD$_{cc}$ is 50% on the diagonal, more than 50% above the diagonal, and less than 50% below the diagonal. It can be proved that (1) CFD$_{cc}$ is zero if and only if $\alpha = 0$ and is one if and only if $\beta = 0$ except the case that $(\alpha, \beta) = (0, 0)$, and (2) for a given β, CFD$_{cc}$ increases strictly and forms a concave shape as α increases, while for a given α, CFD$_{cc}$ decreases strictly and forms a convex shape as β increases.

Example 2. Suppose that $q = 5\%$ and in order to estimate the FD correctly, we must select one inspector among three inspectors with (α_i, β_i) for $i = 1, 2, 3$ as shown in Table 3. In order to reflect actual inspection errors of a back-light unit manufacturer, the value of α_i is set to be smaller than that of β_i. At first glance, the inspector with either $(\alpha_1, \beta_1) = (0.1\%, 2\%)$ or $(\alpha_2, \beta_2) = (0.2\%, 2.5\%)$ is more likely to be selected than the inspector with $(\alpha_3, \beta_3) = (0.3\%, 5.7\%)$ since $\alpha_1 < \alpha_2 < \alpha_3$ and $\beta_1 < \beta_2 < \beta_3$. However, it is a wrong decision since inspector 1 underestimates q as 4.995% and inspector 2 overestimates q as 5.065%, while the right decision is to select the worst inspector 3 since only inspector 3 can estimate q correctly. This decision seems to be very strange at first time but gives a new concept to quality control managers, who could utilize intentionally this perspective in special situations.

4. Analysis of Model II (R, C)

Suppose that $A_i = A$ and $B_i = \beta$ for all i where β is a constant with $0 < \beta \le 1$ and A is a random variable distributed with

TABLE 2: Critical fraction defective for $0 \le \alpha,\ \beta \le 1$.

β	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	—	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	0.00	0.50	0.67	0.75	0.80	0.83	0.86	0.88	0.89	0.90	0.91
0.2	0.00	0.33	0.50	0.60	0.67	0.71	0.75	0.78	0.80	0.82	0.83
0.3	0.00	0.25	0.40	0.50	0.57	0.63	0.67	0.70	0.73	0.75	0.77
0.4	0.00	0.20	0.33	0.43	0.50	0.56	0.60	0.64	0.67	0.69	0.71
0.5	0.00	0.17	0.29	0.38	0.44	0.50	0.55	0.58	0.62	0.64	0.67
0.6	0.00	0.14	0.25	0.33	0.40	0.45	0.50	0.54	0.57	0.60	0.63
0.7	0.00	0.13	0.22	0.30	0.36	0.42	0.46	0.50	0.53	0.56	0.59
0.8	0.00	0.11	0.20	0.27	0.33	0.38	0.43	0.47	0.50	0.53	0.56
0.9	0.00	0.10	0.18	0.25	0.31	0.36	0.40	0.44	0.47	0.50	0.53
1	0.00	0.09	0.17	0.23	0.29	0.33	0.38	0.41	0.44	0.47	0.50

TABLE 3: $\mathrm{POE}_{\mathrm{cc}}(q)$, $\mathrm{CFD}_{\mathrm{cc}}$, and related results for $n = 10{,}000$, $q = 5\%$, and (α_j, β_j) for $j = 1, 2, 3$.

	Inspector 1	Inspector 2	Inspector 3
α_j	0.1%	0.2%	0.3%
β_j	2.0%	2.5%	5.7%
ρ_{cc}	20.0	12.5	19.0
Q_{cc}	4.995%	5.065%	5.000%
$\mathrm{POE}_{\mathrm{cc}}(q)$	0	1	0.5
$\mathrm{CFD}_{\mathrm{cc}}(=q_{\mathrm{cc}})$	4.7619%	7.4074%	5.0000%

the probability density function $f_A(a)$ for $0 < a \le \alpha_u$ and $0 < \alpha_u \le 1$. Then, since $\Pr\{Y_i = 1 \mid X_i = 0\} = A$ and $\Pr\{Y_i = 0 \mid X_i = 1\} = \beta$, the process $\{Y_i, i = 1, 2, 3, \ldots\}$ becomes an infinite "random-Bernoulli" process with $\Pr\{Y_i = 1\}$ being not a constant but the random variable $\{(1-q)A + (1-\beta)q\}$. From (2), Q_{rc} can be obtained as $(1-q)A + (1-\beta)q$. From (3), we have

$$\mathrm{POE}_{\mathrm{rc}}(q) = \Pr\{Q_{\mathrm{rc}} > q\} = \Pr\{(1-q)A > \beta q\}. \quad (6)$$

For $q \ne 1$, (6) can be further reduced to

$$\mathrm{POE}_{\mathrm{rc}}(q) = \Pr\left\{A > \frac{\beta q}{1-q}\right\} = \begin{cases} 1, & \text{for } \dfrac{\beta q}{1-q} = 0, \\ 1 - F_A\left(\dfrac{\beta q}{1-q}\right), & \text{for } 0 < \dfrac{\beta q}{1-q} \le \alpha_u, \\ 0, & \text{for } \alpha_u < \dfrac{\beta q}{1-q}, \end{cases} \quad (7)$$

where $F_A(q) = \int_0^q f_A(a)da$. For $q = 1$, $\mathrm{POE}_{\mathrm{rc}}(1)$ becomes zero since $Q_{\mathrm{rc}} = (1-\beta)$ is always smaller than $q = 1$. If $q = 1$ and $\beta = 0$, then since $Q_{\mathrm{rc}} = q = 1$, every inspector with $\beta = 0$ always estimates correctly regardless of any distribution of $f_A(a)$, and by our definition of POE, $\mathrm{POE}_{\mathrm{rc}}(1) = 0.5$. Hence, $\mathrm{POE}_{\mathrm{rc}}(1)$ has two values either 0.5 (when $\beta = 0$) or zero (when $0 < \beta \le 1$) depending on β. Suppose that $f_A(a) = (1/\alpha_u)I_{(0,\alpha_u]}(a)$. Then, we have the following proposition indicating that $\mathrm{POE}_{\mathrm{rc}}(q)$ can be expressed as a function of two input variables ρ_{rc}, and q where $\rho_{\mathrm{rc}} = 2\beta/\alpha_u$.

Proposition 3. *Under the assumptions that $A_i = A$ and $B_i = \beta$ for all i, where β is a constant with $0 < \beta \le 1$ and A is a random variable distributed with $f_A(a) = (1/\alpha_u)I_{(0,\alpha_u]}(a)$ for $0 < a \le \alpha_u$ and $0 < \alpha_u \le 1$,*

(1)
$$POE_{rc}(q) = \begin{cases} 1 - \dfrac{\rho_{rc} q}{2(1-q)}, & \text{for } 0 \le q \le q_{rc}, \\ 0, & \text{for } q_{rc} \le q \le 1, \end{cases} \quad (8)$$

where $q_{rc} = 2/(2 + \rho_{rc})$,

(2) *$POE_{rc}(q)$ is a strictly decreasing concave function of q for $0 \le q \le q_{rc}$,*

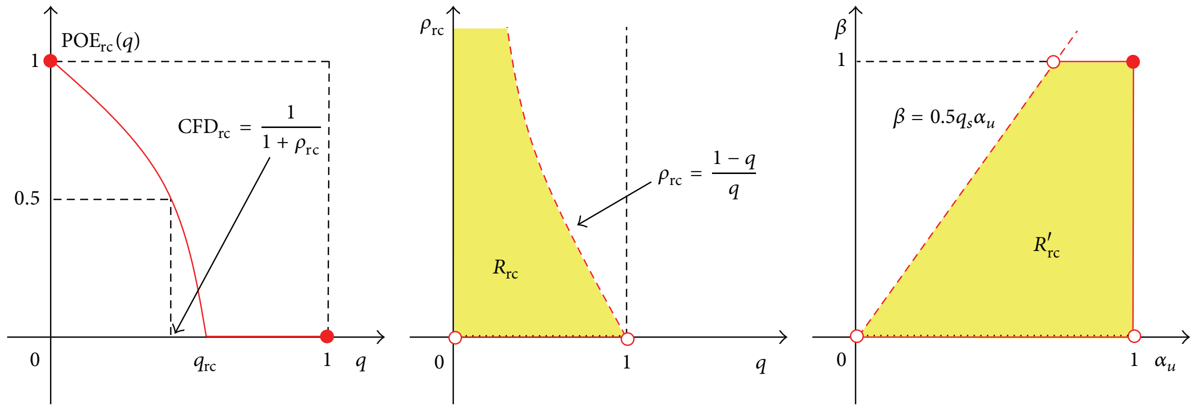
(3) *$CFD_{rc} = 1/(1 + \rho_{rc})$,*

(4) *the inspector with (α_u, β) overestimates q with $POE > 0.5$ for $0 \le q < CFD_{rc}$, estimates q correctly with $POE = 0.5$ for $q = CFD_{rc}$, and underestimates q with $POE < 0.5$ for $CFD_{rc} < q \le 1$.*

Proof. From (7), since $F_A(\beta q/(1-q)) = \beta q/\alpha_u(1-q)$ for $0 \le q \le q_{\mathrm{rc}}$, $\mathrm{POE}_{\mathrm{rc}}(q)$ can be derived as above. Since $\mathrm{POE}'_{\mathrm{rc}}(q) = -\rho_{\mathrm{rc}}/2(1-q)^2 < 0$ and $\mathrm{POE}''_{\mathrm{rc}}(q) = -\rho_{\mathrm{rc}}/(1-q)^3 < 0$, $\mathrm{POE}_{\mathrm{rc}}(q)$ is a strictly decreasing concave function of q for $0 \le q \le q_{\mathrm{rc}}$. Since $0 \le \mathrm{POE}_{\mathrm{rc}}(q) \le 1$ for $0 \le q \le q_{\mathrm{rc}}$, solving $\mathrm{POE}_{\mathrm{rc}}(q^*) = 0.5$ for $0 \le q^* \le q_{\mathrm{rc}}$ gives $\mathrm{CFD}_{\mathrm{rc}} = 1/(1 + \rho_{\mathrm{rc}})$. Hence, (4) holds true. □

Note that $\mathrm{POE}_{\mathrm{rc}}(q)$ and $\mathrm{CFD}_{\mathrm{rc}}$ for any density function $f_A(a)$ can be generally derived as shown in Proposition A.1 in Appendix. Also note that $E[Q_{\mathrm{rc}} \mid q = \mathrm{CFD}_{\mathrm{rc}}] = (1 - \mathrm{CFD}_{\mathrm{rc}})E[A] + (1-\beta)\mathrm{CFD}_{\mathrm{rc}} = \mathrm{CFD}_{\mathrm{rc}}$ where $E[A] = \alpha_u/2$.

A representative graph of $\mathrm{POE}_{\mathrm{rc}}(q)$ is shown in Figure 2(a). Suppose that both q and ρ_{rc} are input variables. Then, solving $\mathrm{POE}_{\mathrm{rc}}(q) = 1 - \rho_{\mathrm{rc}}q/2(1-q) > 0.5$ for $0 \le q \le q_{\mathrm{rc}}$ gives $0 < \rho_{\mathrm{rc}} < (1-q)/q$ and Proposition 3 implies that every point (q, ρ_{rc}) in the shaded region R_{rc} as shown in Figure 2(b) gives overestimation with POE > 0.5, where R_{rc} can be represented by $\{(q, \rho_{\mathrm{rc}}) \mid 0 < \rho_{\mathrm{rc}} < (1-q)/q,\ 0 \le q < 1\}$. That is, an



(a) The graph of $\mathrm{POE}_{\mathrm{rc}}(q)$

(b) The region of overestimation in (q, ρ_{rc}) plane

(c) The region of overestimation in (α_u, β) plane given q when $0.5q_s > \beta/\alpha_u$

FIGURE 2: The graph of $\mathrm{POE}_{\mathrm{rc}}(q)$ and the overestimation region in two different planes.

TABLE 4: $\mathrm{POE}_{\mathrm{rc}}(q)$ and $\mathrm{CFD}_{\mathrm{rc}}$ for $q = 5\%$ and different inspectors.

	Inspector 1	Inspector 2	Inspector 3
α_u (%)	0.2	0.4	0.6
β (%)	2.0	2.5	5.7
$\rho_{\mathrm{rc}} = 2\beta/\alpha_u$	20.0	12.5	19.0
$\mathrm{CFD}_{\mathrm{rc}} = q_{\mathrm{rc}}$	4.76%	7.41%	5.00%
$\mathrm{POE}_{\mathrm{rc}}(q)$	47.37%	67.11%	50.00%

inspector with ρ_{rc} overestimates q if $(q, \rho_{\mathrm{rc}}) \in R_{\mathrm{rc}}$, estimates q with POE $= 0.5$ if $\rho_{\mathrm{rc}} = (1-q)/q$, and underestimates q otherwise. Note that if $\beta = 0$, then every point $(q, 0)$ for $0 < q < 1$ on the line $\rho_{\mathrm{rc}} = 0$ gives overestimation with POE > 0.5 since $Q_{\mathrm{rc}} = (1-q)A + q$ is always greater than q.

Suppose that both α_u and β are input variables and q is given as a constant. Then, solving $\mathrm{POE}_{\mathrm{rc}}(q) = 1 - \beta q/\alpha_u(1-q) > 0.5$ for $0 \le q \le q_{\mathrm{rc}}$ gives $0 < \beta < 0.5q_s\alpha_u$ and Proposition 3 implies that every point (α_u, β) in the shaded region R'_{rc} as shown in Figure 2(c) gives overestimation with POE > 0.5 where $R'_{\mathrm{rc}} = \{(\alpha_u, \beta) \mid 0 < \beta < 0.5q_s\alpha_u,\ 0 < \alpha_u \le 1,\ 0 < \beta \le 1\}$. That is, an inspector with (α_u, β) overestimates with POE > 0.5 if $(\alpha_u, \beta) \in R'_{\mathrm{rc}}$, estimates q with POE $= 0.5$ if $\beta = 0.5q_s\alpha_u$, and underestimates q with POE < 0.5 otherwise.

Example 4. For $q = 5\%$ and three inspectors given in Table 4, $\mathrm{POE}_{\mathrm{rc}}(q)$ and $\mathrm{CFD}_{\mathrm{rc}}$ for each inspector can be computed and summarized in the table. If we would like to select the inspector satisfying $\mathrm{POE}_{\mathrm{rc}}(q) = 50\%$, then inspector 3 will be selected again.

5. Analysis of Model III (C, R)

Suppose that $A_i = \alpha$ and $B_i = B$ for all i where α is a constant with $0 < \alpha \le 1$ and B is a random variable distributed with the probability density function $f_B(b)$ for $0 < b \le \beta_u$ and $0 < \beta_u \le 1$. Then, since $\Pr\{Y_i = 1 \mid X_i = 0\} = \alpha$, $\Pr\{Y_i = 0 \mid X_i = 1\} = B$, the process $\{Y_i, i = 1, 2, 3, \ldots\}$ becomes an infinite random-Bernoulli process with $\Pr\{Y_i = 1\}$ being the random variable $\{\alpha(1-q) + (1-B)q\}$. From (2), Q_{cr} can be obtained as $\alpha(1-q) + (1-B)q$. From (3), we have

$$\mathrm{POE}_{\mathrm{cr}}(q) = \Pr\{Q_{\mathrm{cr}} > q\} = \Pr\{qB < \alpha(1-q)\}. \tag{9}$$

For $q \ne 0$, (9) can be further reduced to

$$\begin{aligned}\mathrm{POE}_{\mathrm{cr}}(q) &= \Pr\left\{B < \frac{\alpha(1-q)}{q}\right\} \\ &= \begin{cases} 1, & \text{for } \dfrac{\alpha(1-q)}{q} \ge \beta_u, \\ F_B\left(\dfrac{\alpha(1-q)}{q}\right), & \text{for } \dfrac{\alpha(1-q)}{q} < \beta_u, \end{cases}\end{aligned} \tag{10}$$

where $F_B(q) = \int_0^q f_B(b)db$.

For $q = 0$, $\mathrm{POE}_{\mathrm{cr}}(0)$ becomes one since $Q_{\mathrm{cr}} = \alpha$ is always greater than $q = 0$ from (9). If $q = \alpha = 0$, then since $Q_{\mathrm{cr}}(= 0)$ is always equal to $q(= 0)$, every inspector with $\alpha = 0$ always estimates correctly regardless of any distribution of $f_B(b)$, and, by our definition of POE, $\mathrm{POE}_{\mathrm{cr}}(0)$ is not one but 50% in this case. This result implies that $\mathrm{POE}_{\mathrm{cr}}(0)$ can be either one or 50% depending on the value of α. Since we assume that $0 < \alpha \le 1$, we have $\mathrm{POE}_{\mathrm{cr}}(0) = 1$. Suppose that $f_B(b) = (1/\beta_u)I_{(0,\beta_u]}(b)$. Then, we have the following proposition indicating that $\mathrm{POE}_{\mathrm{cr}}(q)$ can be expressed as a function of two input variables ρ_{cr} and q where $\rho_{\mathrm{cr}} = \beta_u/2\alpha$.

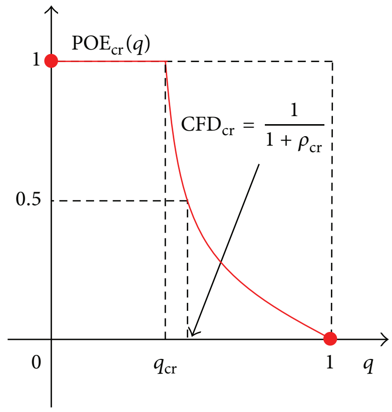
Proposition 5. *Under the assumption that $A_i = \alpha$ and $B_i = B$ for all i where α is a constant with $0 < \alpha \le 1$ and B is a random variable distributed with $f_B(b)$ for $0 < b \le \beta_u$ and $0 < \beta_u \le 1$,*

(1)

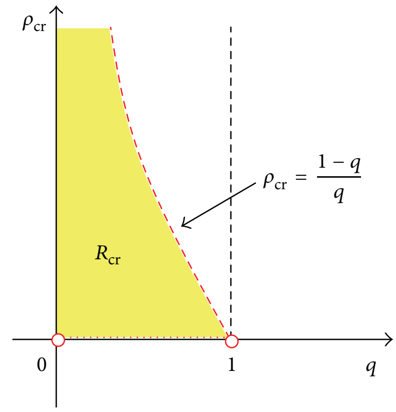
$$POE_{cr}(q) = \begin{cases} 1, & \textit{for } 0 \le q \le q_{cr}, \\ \dfrac{(1-q)}{2\rho_{cr}q}, & \textit{for } q_{cr} \le q \le 1, \end{cases} \tag{11}$$

where $q_{cr} = 1/(1+2\rho_{cr})$,

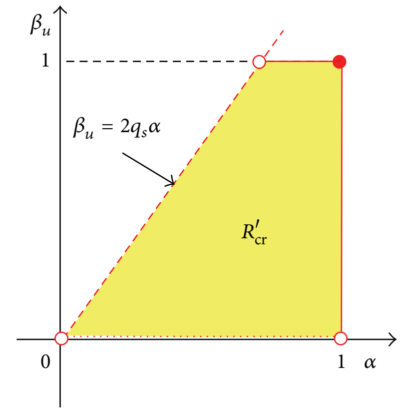
(2) *$POE_{cr}(q)$ is a strictly decreasing convex function of q for $q_{cr} \le q \le 1$,*



(a) The graph of $\mathrm{POE}_{cr}(q)$



(b) The region of overestimation in (q, ρ_{cr}) plane



(c) The region of overestimation in (α, β_u) plane given q when $2q_s > \beta_u/\alpha$

FIGURE 3: The graph of $\mathrm{POE}_{cr}(q)$ and the overestimation region in two different planes.

(3) $CFD_{cr} = 1/(1+\rho_{cr})$,

(4) *the inspector with* (α, β_u) *overestimates* q *with POE* > 0.5 *for* $0 \le q < CFD_{cr}$, *estimates* q *with POE* = 0.5 *for* $q = CFD_{cr}$, *and underestimates* q *with POE* < 0.5 *for* $CFD_{cr} < q \le 1$.

Proof. From (10), since $F_B(\alpha(1-q)/q) = \alpha(1-q)/\beta_u q$ for $q_{cr} \le q \le 1$, $\mathrm{POE}_{cr}(q)$ can be derived as above. Since $\mathrm{POE}'_{cr}(q) = -1/2\rho_{cr}q^2 < 0$ and $\mathrm{POE}''_{cr}(q) = 1/\rho_{cr}q^3 > 0$ for $q_{cr} \le q \le 1$, $\mathrm{POE}_{cr}(q)$ is a strictly decreasing convex function of q for $q_{cr} \le q \le 1$. Since $0 \le \mathrm{POE}_{cr}(q) \le 1$ for $q_{cr} \le q \le 1$, solving $\mathrm{POE}_{cr}(q^*) = 0.5$ for $q_{cr} \le q^* \le 1$ gives $\mathrm{CFD}_{cr} = 1/(1+\rho_{cr})$. Hence, (4) holds true. □

Note that generally $\mathrm{POE}_{cr}(q)$ and CFD_{cr} for any density function $f_B(b)$ can be derived as shown in Proposition A.2 in Appendix. Also note that $E[Q_{cr} \mid q = \mathrm{CFD}_{cr}] = (1 - \mathrm{CFD}_{cr})\alpha + (1 - E[B])\mathrm{CFD}_{cr} = \mathrm{CFD}_{cr}$ where $E[B] = \beta_u/2$.

A representative graph of $\mathrm{POE}_{cr}(q)$ is shown in Figure 3(a). Suppose that both q and ρ_{cr} are input variables. Then, solving $\mathrm{POE}_{cr}(q) = (1-q)/2\rho_{cr}q > 0.5$ for $q_{cr} \le q \le 1$ gives $0 < \rho_{cr} < (1-q)/q$ and Proposition 5 implies that every point (q, ρ_{cr}) in the shaded region R_{cr} as shown in Figure 3(b) gives overestimation with POE > 0.5 where $R_{cr} = \{(q, \rho_{cr})\ 0 < \rho_{cr} < (1-q)/q,\ 0 \le q < 1\}$. That is, an inspector with ρ_{cr} overestimates q if $(q, \rho_{cr}) \in R_{cr}$, estimates q with POE = 0.5 if $\rho_{cr} = (1-q)/q$, and underestimates q with POE < 0.5 otherwise. Note that if $\alpha = 0$, then every point $(q, 0)$ for $0 < q < 1$ gives underestimation since $\Pr\{Q_{cr} > q\} = \Pr\{B < 0\} = 0$.

Suppose that both α and β_u are input variables and q is given as a constant. Then, solving $\mathrm{POE}_{cr}(q) = (1-q)\alpha/q\beta_u > 0.5$ for $q_{cr} \le q \le 1$ gives $0 < \beta_u < 2q_s\alpha$ and Proposition 5 implies that every point (α, β_u) in the shaded region R'_{cr} as shown in Figure 1(c) which gives overestimation with POE > 0.5 where $R'_{cr} = \{(\alpha, \beta_u) \mid 0 < \beta_u < 2q_s\alpha,\ 0 < \alpha \le 1,\ 0 < \beta_u \le 1\}$. That is, an inspector with (α, β_u) overestimates with POE > 0.5 if $(\alpha, \beta_u) \in R'_{cr}$, estimates q with POE = 0.5 if $\beta = 2q_s\alpha_u$, and underestimates q with POE < 0.5 otherwise.

TABLE 5: $\mathrm{POE}_{cr}(q)$ and CFD for q = 5% and different inspectors.

	Inspector 1	Inspector 2	Inspector 3
α (%)	0.1	0.2	0.3
β_u (%)	4.0	5.0	11.4
$\rho_{cr} = \beta_u/2\alpha$	20.0	12.5	19.0
$\mathrm{CFD}_{cr} = q_{cr}$	4.76%	7.41%	5.00%
$\mathrm{POE}_{cr}(q)$	47.50%	76.00%	50.00%

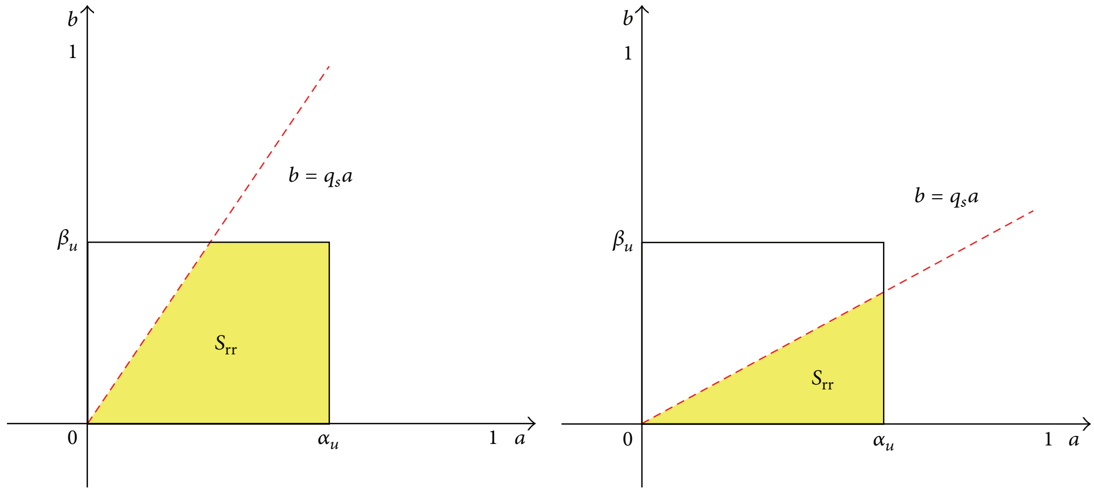
Example 6. For q = 5% and three inspectors given in Table 5, $\mathrm{POE}_{cr}(q)$ and CFD_{cr} for each inspector can be computed and summarized in the table. If we would like to select the inspector satisfying $\mathrm{POE}_{cr}(q)$ = 50%, then inspector 3 will be selected again.

6. Analysis of Model IV (R, R)

Suppose that $A_i = A$ and $B_i = B$ for all i where A and B are random variables distributed with the probability density functions $f_A(a)$ and $f_B(b)$, respectively, for $0 < a \le \alpha_u$, $0 < b \le \beta_u$ and $0 < \alpha_u, \beta_u \le 1$. Then, since $\Pr\{Y_i = 1 \mid X_i = 0\} = A$ and $\Pr\{Y_i = 0 \mid X_i = 1\} = B$, the process $\{Y_i, i = 1, 2, 3, \ldots\}$ becomes an infinite random-Bernoulli process with $\Pr\{Y_i = 1\}$ being the random variable $\{A(1-q) + (1-B)q\}$. From (2), Q_{rr} can be obtained as $A(1-q) + (1-B)q$. From (3), POE can be expressed as

$$\mathrm{POE}_{rr}(q) = \Pr\{Q_{rr} > q\} = \Pr\{(1-q)A > qB\}. \quad (12)$$

If $q = 1$, then $\mathrm{POE}_{rr}(q) = 0$ since $\mathrm{POE}_{rr}(q) = \Pr\{B < 0\} = 0$; that is, even perfect inspector always underestimates q regardless of any distribution of $(f_A(a), f_B(b))$. On the other hand, if $q = 0$, then $\mathrm{POE}_{rr}(0) = 1$ since $\mathrm{POE}_{rr}(0) = \Pr\{A > 0\} = 1$; that is, even perfect inspector always overestimates q



(a) For $q_s \geq \rho_{\mathrm{rr}}$ (or $0 < q \leq q_{\mathrm{rr}}$) (b) For $0 < q_s \leq \rho_{\mathrm{rr}}$ (or $q_{\mathrm{rr}} \leq q < 1$)

FIGURE 4: The region of S_{rr} depending on q_s and ρ_{rr}.

regardless of any distribution of $(f_A(a), f_B(b))$. For $0 < q < 1$, (12) can be reduced to

$$\begin{aligned} \mathrm{POE}_{\mathrm{rr}}(q) &= \int_0^{\beta_u} \Pr\{(1-q)A > qB \mid B = b\} f_B(b)\, db \\ &= \int_0^{\beta_u} \Pr\left\{\frac{qb}{1-q} < A \leq \alpha_u\right\} f_B(b)\, db \\ &= \int_0^{\beta_u} \left\{\int_{qb/(1-q)}^{\alpha_u} f_A(a)\, da\right\} f_B(b)\, db \\ &= \iint_{(a,b)\in S_{\mathrm{rr}}} f_A(a) f_B(b)\, da\, db, \end{aligned} \tag{13}$$

where $S_{\mathrm{rr}} = \{(a,b) \mid b < q_s a,\ 0 \leq a \leq \alpha_u,\ 0 \leq b \leq \beta_u, \text{ and } 0 < \alpha_u, \beta_u \leq 1\}$.

Suppose that $f_A(a) = (1/\alpha_u) I_{(0,\alpha_u]}(a)$ and $f_B(b) = (1/\beta_u) I_{(0,\beta_u]}(b)$. Then, we have the following proposition indicating that $\mathrm{POE}_{\mathrm{rr}}(q)$ can be expressed as a function of two input variables ρ_{rr} and q where $\rho_{\mathrm{rr}} = \beta_u/\alpha_u$.

Proposition 7. *Under the assumption that $A_i = A$ and $B_i = B$ for all i where A and B are random variables distributed with $f_A(a) = (1/\alpha_u) I_{(0,\alpha_u]}(a)$ and $f_B(b) = (1/\beta_u) I_{(0,\beta_u]}(b)$, respectively, for $0 < a \leq \alpha_u$, $0 < b \leq \beta_u$, and $0 < \alpha_u, \beta_u \leq 1$,*

(1)

$$POE_{rr}(q) = \begin{cases} POE_{rr1}(q) = 1 - \dfrac{\rho_{rr} q}{2(1-q)}, \\ \qquad \text{for } 0 \leq q \leq q_{rr} \ \ (\text{or } q_s \geq \rho_{rr}), \\ POE_{rr2}(q) = \dfrac{1-q}{2\rho_{rr} q}, \\ \qquad \text{for } q_{rr} \leq q \leq 1 \ \ (\text{or } 0 \leq q_s \leq \rho_{rr}), \end{cases} \tag{14}$$

where $q_{rr} = 1/(1+\rho_{rr})$,

(2) *$POE_{rr1}(q)$ is a strictly decreasing concave function of q with $POE_{rr1}(0) = 1$, and $POE_{rr2}(q)$ is a strictly decreasing convex function of q with $POE_{rr2}(1) = 0$,*

(3) *$CFD_{rr} = 1/(1+\rho_{rr})$,*

(4) *the inspector with ρ_{rr} overestimates q with $POE > 0.5$ for $0 \leq q < CFD_{rr}$, estimates q with $POE = 0.5$ for $q = CFD_{rr}$, and underestimates q with $POE < 0.5$ for $CFD_{rr} < q \leq 1$.*

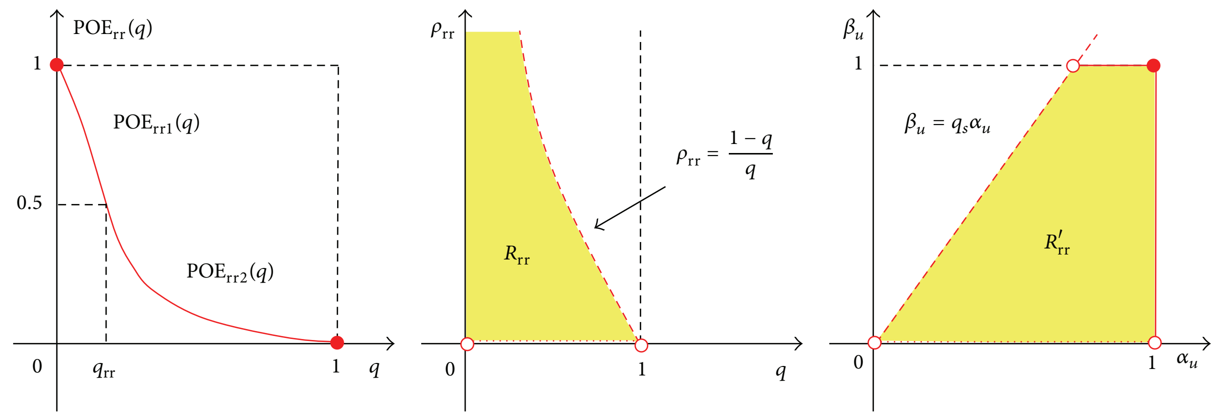
Proof. It is proved from (12) that $\mathrm{POE}_{\mathrm{rr}}(0) = 1$ and $\mathrm{POE}_{\mathrm{rr}}(1) = 0$. For $0 < q < 1$, since the shape of S_{rr} depends upon q_s and ρ_{rr}, for $q_s \geq \rho_{\mathrm{rr}}$ as shown in Figure 4(a), (13) can be reduced to

$$\mathrm{POE}_{\mathrm{rr1}}(q) = \frac{1}{\alpha_u \beta_u} \int_0^{\beta_u} \int_{qb/(1-q)}^{\alpha_u} da\, db = 1 - \frac{\rho_{\mathrm{rr}} q}{2(1-q)}; \tag{15}$$

and for $0 < q_s \leq \rho_{\mathrm{rr}}$ as shown in Figure 4(b), (13) can be reduced to

$$\mathrm{POE}_{\mathrm{rr2}}(q) = \frac{1}{\alpha_u \beta_u} \int_0^{((1-q)/q)\alpha_u} \int_{qb/(1-q)}^{\alpha_u} da\, db = \frac{1-q}{2\rho_{\mathrm{rr}} q}. \tag{16}$$

Equation (15) holds when $q = 0$ (equivalently, $q_s = \infty$) since replacing q in (15) with zero gives $\mathrm{POE}_{\mathrm{rr1}}(0) = 1$ and (16) holds when $q = 1$ (equivalently, $q_s = 0$) since replacing q in (16) with one gives $\mathrm{POE}_{\mathrm{rr2}}(1) = 0$. For $0 \leq q \leq q_{\mathrm{rr}}$, since $\mathrm{POE}'_{\mathrm{rr1}}(q) = -\rho_{\mathrm{rr}}/2(1-q)^2 < 0$ and $\mathrm{POE}''_{\mathrm{rr1}}(q) = -\rho_{\mathrm{rr}}/(1-q)^3 < 0$, $\mathrm{POE}_{\mathrm{rr1}}(q)$ is a strictly decreasing concave function of q. For $q_{\mathrm{rr}} \leq q \leq 1$, since $\mathrm{POE}'_{\mathrm{rr2}}(q) = -1/2\rho_{\mathrm{rr}} q^2 < 0$ and $\mathrm{POE}''_{\mathrm{rr2}}(q) = 1/\rho_{\mathrm{rr}} q^3 > 0$, $\mathrm{POE}_{\mathrm{rr2}}(q)$ is a strictly decreasing convex function of q. Note that $\mathrm{POE}_{\mathrm{rr}}(q)$ is differentiable at $q = q_{\mathrm{rr}}$. Hence, $\mathrm{POE}_{\mathrm{rr}}(q)$ is a strictly decreasing function of q with $\mathrm{POE}_{\mathrm{rr}}(0) = \mathrm{POE}_{\mathrm{rr1}}(0) = 1$ and $\mathrm{POE}_{\mathrm{rr}}(1) = \mathrm{POE}_{\mathrm{rr2}}(1) = 0$. Since $\mathrm{POE}_{\mathrm{rr1}}(q_{\mathrm{rr}}) = \mathrm{POE}_{\mathrm{rr2}}(q_{\mathrm{rr}}) = 0.5$, $\mathrm{CFD}_{\mathrm{rr}}$ is q_{rr}. Since $\mathrm{POE}_{\mathrm{rr1}}(q) > 0.5$ for $0 \leq q < \mathrm{CFD}_{\mathrm{rr}}$ and $\mathrm{POE}_{\mathrm{rr2}}(q) < 0.5$ for $\mathrm{CFD}_{\mathrm{rr}} < q \leq 1$, (4) holds true. □



(a) The graph of $\mathrm{POE}_{\mathrm{rr}}(q)$

(b) The region of overestimation in (q, ρ_{rr}) plane

(c) The region of overestimation in (α_u, β_u) plane for given q when $q_s > \beta_u/\alpha_u$

FIGURE 5: The graph of $\mathrm{POE}_{\mathrm{cr}}(q)$ and the overestimation region in two different planes.

Since A and B are uniform random variables, respectively, (15) and (16) can be derived by a different method as follows. Equation (13) can be expressed as

$$\mathrm{POE}_{\mathrm{rr}}(q) = \iint_{(a,b)\in S_{\mathrm{rr}}} f_A(a)\, f_B(b)\, da\, db = \frac{1}{\alpha_u \beta_u} \iint_{(a,b)\in S_{\mathrm{rr}}} da\, db. \quad (17)$$

Since the value of the integral is equivalent to the size of the area S_{rr} as shown in Figure 4, $\mathrm{POE}_{\mathrm{rr}}(q)$ can be interpreted as the ratio of the size of S_{rr} to the size of the rectangle $\alpha_u\beta_u$. The shape of S_{rr} varies depending on the straight line $b = q_s a$, which cuts the rectangle into two parts, that is, an upper part and a lower part as shown in Figure 4. The lower part of the rectangle under the line corresponds to S_{rr}, a right-angled triangle when $0 < q_s \le \rho_{\mathrm{rr}}$, and a trapezoid when $q_s > \rho_{\mathrm{rr}}$. Since the size of the trapezoid and the triangle can be derived as $(2q_s\alpha_u - \beta_u)\beta_u/2q_s$ and $q_s\alpha_u^2/2$, respectively, $\mathrm{POE}_{\mathrm{rr}}(q)$ can be easily obtained.

It seems to be mathematically hard to derive $\mathrm{POE}_{\mathrm{rr}}(q)$ and $\mathrm{CFD}_{\mathrm{rr}}$ generally for any density functions. From Propositions A.1 and A.2 in Appendix, our conjecture is that $\mathrm{CFD}_{\mathrm{rr}} = F_A^{-1}(0.5)/(F_A^{-1}(0.5)+F_B^{-1}(0.5))$. Note that $E[Q_{\mathrm{rr}} \mid q = \mathrm{CFD}_{\mathrm{rr}}] = (1-\mathrm{CFD}_{\mathrm{rr}})E[A] + (1-E[B])\mathrm{CFD}_{\mathrm{rr}} = \mathrm{CFD}_{\mathrm{rr}}$.

From Proposition 7, a graph of $\mathrm{POE}_{\mathrm{rr}}(q)$ can be drawn as shown in Figure 5(a). Suppose that q and ρ_{rr} are input variables. Then, since $\mathrm{POE}_{\mathrm{rr}1}(q) > 0.5$, solving $\mathrm{POE}_{\mathrm{rr}1}(q) = 1 - \rho_{\mathrm{rr}}q/2(1-q) > 0.5$ for $0 \le q \le q_{\mathrm{rr}} = 1/(1+\rho_{\mathrm{rr}})$ gives $0 < \rho_{\mathrm{rr}} < (1-q)/q$ and Proposition 7 implies that every point (q, ρ_{rr}) in the shaded region R_{rr} as shown in Figure 5(b) gives overestimation with POE > 0.5 where $R_{\mathrm{rr}} = \{(q, \rho_{\mathrm{rr}}) \mid 0 < \rho_{\mathrm{rr}} < (1-q)/q,\ 0 \le q < 1\}$. That is, an inspector with ρ_{rr} overestimates q with POE > 0.5 if $(q, \rho_{\mathrm{rr}}) \in R_{\mathrm{rr}}$, estimates q with POE $= 0.5$ if $\rho_{\mathrm{rr}} = q_s$, and underestimates q with POE < 0.5 otherwise.

TABLE 6: $\mathrm{POE}_{\mathrm{rr}}(q)$ and CFD for $q = 5\%$ and different inspectors.

	Inspector 1	Inspector 2	Inspector 3
α_u (%)	0.2	0.4	0.6
β_u (%)	4.0	5.0	11.4
$\rho_{\mathrm{rr}} = \beta_u/\alpha_u$	20.0	12.5	19.0
$\mathrm{CFD}_{\mathrm{rr}} = q_{\mathrm{rr}}$	4.76%	7.41%	5.00%
$\mathrm{POE}_{\mathrm{rr}}(q)$	47.50%	67.11%	50.00%

Suppose that q is a given constant and both α_u and β_u are input variables. Then, similarly solving $\mathrm{POE}_{\mathrm{rr}1}(q) = 1 - (q/2(1-q))\cdot(\beta_u/\alpha_u) > 0.5$ for $0 \le q \le q_{\mathrm{rr}} = \alpha_u/(\alpha_u+\beta_u)$ gives $0 < \beta_u < ((1-q)/q)\alpha_u$ and Proposition 7 implies that every point (α_u, β_u) in the shaded region R'_{rr} as shown in Figure 5(c) gives overestimation POE > 0.5 where $R'_{\mathrm{rr}} = \{(\alpha_u, \beta_u) \mid 0 < \beta_u < q_s\alpha_u,\ 0 < \alpha_u \le 1,\ 0 < \beta_u \le 1\}$. That is, an inspector with (α_u, β_u) overestimates q with POE > 0.5 if $(\alpha_u, \beta_u) \in R'_{\mathrm{rr}}$, estimates q with POE $= 0.5$ if $\beta_u = q_s\alpha_u$, and underestimates q with POE < 0.5 otherwise.

Example 8. For $q = 5\%$ and three inspectors given in Table 6, $\mathrm{POE}_{\mathrm{rr}}(q)$ and $\mathrm{CFD}_{\mathrm{rr}}$ for each inspector can be computed and summarized in the table. If we would like to select the inspector satisfying $\mathrm{POE}_{\mathrm{rr}}(q) = 50\%$, then inspector 3 will be selected again as before.

7. Summary

In order to find relations between four propositions, the constants α and β can be assumed to be $E[A]$ and $E[B]$, respectively. This assumption may be justified since a constant can be interpreted as a representative value. Since A and B are uniform random variables on $(0, \alpha_u]$ and $(0, \beta_u]$, respectively, we have, $\alpha = E[A] = \alpha_u/2$ and $\beta = E[B] = \beta_u/2$, respectively. Thus, the following theorem holds true.

Theorem 9. *Assuming an infinite sequence of items with a known FD q and nonzero inspection errors with $\alpha = E[A] = \alpha_u/2$ and $\beta = E[B] = \beta_u/2$,*

(1) *an imperfect inspector with nonzero inspection errors $(E[A], E[B])$ has his/her own POE curve and CFD,*

(2) *POE is a function of two variables q and ρ, denoted by $POE(q, \rho)$,*

(3) *POE is a decreasing function of q with $POE(0, \rho) = 1$ and $POE(1, \rho) = 0$,*

(4) *POE is a decreasing function of ρ with $POE(q, 0) = 1$ and $POE(q, 1) = 0$,*

(5) *there always exists a unique $CFD = 1/(1 + \rho)$, which depends only on inspection errors and not q,*

(6) *the inspector overestimates q with $POE > 0.5$ for $0 \le q < CFD$, estimates q with $POE = 0.5$ for $q = CFD$, and underestimates q with $POE < 0.5$ for $CFD < q \le 1$.*

Proof. If $\alpha = E[A] = \alpha_u/2$ and $\beta = E[B] = \beta_u/2$, then we have $\rho_{cc} = \rho_{rc} = \rho_{cr} = \rho_{rr} = E[B]/E[A]$ and $\mathrm{CFD}_{cc} = \mathrm{CFD}_{rc} = \mathrm{CFD}_{cr} = \mathrm{CFD}_{rr} = E[A]/(E[A] + E[B]) = 1/(1 + \rho)$, where $\rho = E[B]/E[A]$. Let $\mathrm{CFD} = 1/(1 + \rho)$. Then, from the results of four propositions, the theorem except (4) holds true and those results are summarized in Table 7. By using the similar method used for proofs of propositions, (4) can be proved. □

Since we have

$$
\begin{aligned}
q_{cc} &= \frac{E[A]}{E[A] + E[B]} = \frac{1}{1+\rho},\\
q_{rc} &= \frac{2E[A]}{2E[A] + E[B]} = \frac{2}{2+\rho},\\
q_{cr} &= \frac{E[A]}{E[A] + 2E[B]} = \frac{1}{1+2\rho},\\
q_{rr} &= \frac{E[A]}{E[A] + E[B]} = \frac{1}{1+\rho},\\
\mathrm{POE}_{rc}(q) &= 1 - \frac{\rho q}{2(1-q)} \quad \text{for } 0 \le q \le \frac{2}{2+\rho},\\
\mathrm{POE}_{cr}(q) &= \frac{1-q}{2\rho q} \quad \text{for } \frac{1}{1+2\rho} \le q \le 1,\\
\mathrm{POE}_{rr}(q) &= \begin{cases} 1 - \dfrac{\rho q}{2(1-q)}, & \text{for } 0 \le q \le \dfrac{1}{1+\rho},\\ \dfrac{1-q}{2\rho q}, & \text{for } \dfrac{1}{1+\rho} \le q \le 1, \end{cases}
\end{aligned}
\tag{18}
$$

it can be observed in Table 7 that $\mathrm{POE}_{rr}(q)$ has the same form as $\mathrm{POE}_{rc}(q)$ for $0 \le q \le 1/(1+\rho)$ and $\mathrm{POE}_{cr}(q)$ for $1/(1+\rho) \le q \le 1$ even though their related domains are not the same. Now, based on our theorem, our answer to the fundamental question "Does an imperfect inspector overestimate a known fraction defective when it is very low?" could be "certainly yes at least under our assumptions" since POE $\approx$ 1 when FD is very low.

8. Conclusion

Overestimation by an inspector may be explained not only by inspection errors but also by other factors such as psychological aspects of inspectors, incentive plans for inspectors, workload, conflicts among inspectors, and so on. However, our results and concepts are based on four assumptions: the assumption of an infinite sequence of items, the assumption of a fixed known FD, the assumption of nonzero inspection errors, and the assumption of a uniform distribution. Further research may be concentrated on relaxing these assumptions. We may obtain slightly different results, by assuming a finite sequence of items; or by assuming that FD is not a fixed constant, but a random variable; or by assuming other distributions, such as a skewed triangular, a truncated normal, or an empirical distribution for a lower/upper limit interval. However, our strong conjecture is that our theorem will still be true, regardless of any distribution, and even a finite number of items, as long as FD is a constant. Since our mathematical models do not consider any related costs, a cost-based optimization model with POE could be constructed, to determine a trade-off point between buyers and sellers.

If we consider the fair trade between a seller and a buyer, and the trend that FD's of manufacturers have been continuously approaching zero, Theorem 9 implies that either the ratio of type I error to type II error must go to infinity, or the type I error must be zero in order for CFD to approach zero, and that all commercial inspection plans should be revised with the concept of POE in the near future, for the fairness of commercial trades, since the smaller (up to several hundreds ppm level) the FD of items sold by sellers is, the more their unfair loss is forced to be. We hope that the concept of POE should become one of the major criteria in the future. Our methodology used in this paper could, with slight modification, be applied and extended to the existing sampling plans.

Appendix

Proposition A.1. *Under the assumption that type I error is distributed with $f_A(a)$ and that type II error is given as a constant β where $0 < a \le \alpha_u$, $0 < \alpha_u \le 1$, and $0 < \beta \le 1$,*

(1)

$$
POE_{rc}(q) = \begin{cases} 1 - F_A\left(\dfrac{\beta q}{1-q}\right), & \text{for } 0 \le q \le q_{rc},\\ 0, & \text{for } q_{rc} \le q \le 1, \end{cases}
\tag{A.1}
$$

(2) *$POE_{rc}(q)$ is a strictly decreasing function of q for $0 \le q \le q_{rc}$,*

(3) *$CFD_{rc} = F_A^{-1}(0.5)/(F_A^{-1}(0.5) + \beta)$,*

(4) *the inspector with ρ_{rc} overestimates q with $POE > 0.5$ for $0 \le q < CFD_{rc}$, estimates q with $POE = 0.5$ for $q = CFD_{rc}$, and underestimates q with $POE < 0.5$ for $CFD_{rc} < q \le 1$.*

Table 7: Four types of POE and CFD analysis under the assumption of $\alpha = E[A] = \alpha_u/2$ and $\beta = E[B] = \beta_u/2$.

Models	Model I (C, C)	Model II (R, C)	Model III (C, R)	Model IV (R, R)
FD	Constant, q	Constant, q	Constant, q	Constant, q
Type I error	Constant, α	Uniform random variable on $(0, \alpha_u]$	Constant, α	Uniform random variable on $(0, \alpha_u]$
Type II error	Constant, β	Constant, β	Uniform random variable on $(0, \beta_u]$	Uniform random variable on $(0, \beta_u]$
ρ	$\rho_{\mathrm{cc}} = \dfrac{\beta}{\alpha} = \rho$	$\rho_{\mathrm{rc}} = \dfrac{2\beta}{\alpha_u} = \rho$	$\rho_{\mathrm{cr}} = \dfrac{\beta_u}{2\alpha} = \rho$	$\rho_{\mathrm{rr}} = \dfrac{\beta_u}{\alpha_u} = \rho$
POE	$\mathrm{POE}_{\mathrm{cc}}(q,\rho) =$ $1 \text{ for } 0 \le q < \dfrac{1}{1+\rho}$ $0.5 \text{ for } q = \dfrac{1}{1+\rho}$ $0 \text{ for } \dfrac{1}{1+\rho} < q \le 1$	$\mathrm{POE}_{\mathrm{rc}}(q,\rho) =$ $1 - \dfrac{\rho q}{2(1-q)} \text{ for } 0 \le q \le \dfrac{2}{2+\rho}$ $0 \text{ for } \dfrac{2}{2+\rho} \le q \le 1$	$\mathrm{POE}_{\mathrm{cr}}(q,\rho) =$ $0 \text{ for } 0 \le q \le \dfrac{1}{1+2\rho}$ $\dfrac{(1-q)}{2\rho q} \text{ for } \dfrac{1}{1+2\rho} \le q \le 1$	$\mathrm{POE}_{\mathrm{rr}}(q,\rho) =$ $1 - \dfrac{\rho q}{2(1-q)} \text{ for } 0 \le q \le \dfrac{1}{1+\rho}$ $\dfrac{1-q}{2\rho q} \text{ for } \dfrac{1}{1+\rho} < q \le 1$
CFD	$\mathrm{CFD}_{\mathrm{cc}} = \dfrac{\alpha}{\alpha+\beta} = \dfrac{1}{1+\rho}$	$\mathrm{CFD}_{\mathrm{rc}} = \dfrac{0.5\alpha_u}{0.5\alpha_u+\beta} = \dfrac{1}{1+\rho}$	$\mathrm{CFD}_{\mathrm{cr}} = \dfrac{\alpha}{\alpha+0.5\beta_u} = \dfrac{1}{1+\rho}$	$\mathrm{CFD}_{\mathrm{rr}} = \dfrac{\alpha_u}{\alpha_u+\beta_u} = \dfrac{1}{1+\rho}$

Proof. Replacing q in (7) with zero and q_{rc} gives $\mathrm{POE}_{\mathrm{rc}}(0) = \int_0^{\alpha_u} f_A(a)da = 1$ and $\mathrm{POE}_{\mathrm{rc}}(q_{\mathrm{rc}}) = 1$, respectively. Thus, (1) holds true for $q = 0$ and $q = q_{\mathrm{rc}}$. Since $\mathrm{POE}'_{\mathrm{rc}}(q) = -(\partial/\partial q)F_A(\beta q/(1-q)) = -(\beta/(1-q)^2)f_A(\beta q/(1-q)) < 0$, $\mathrm{POE}_{\mathrm{rc}}(q)$ is a strictly decreasing function of q for $0 \le q \le q_{\mathrm{rc}}$. Since $0 \le \mathrm{POE}_{\mathrm{rc}}(q) \le 1$ for $0 \le q \le q_{\mathrm{rc}}$, solving $\mathrm{POE}_{\mathrm{rc}}(q^*) = 0.5$ for $0 \le q^* < q_{\mathrm{rc}}$ gives $\mathrm{CFD}_{\mathrm{rc}} = F_A^{-1}(0.5)/(F_A^{-1}(0.5) + \beta)$. Hence, (3) and (4) hold true. □

Proposition A.2. *Under the assumption that type I error is given as a constant α and that type II error is distributed with $f_B(b)$ where $0 < b \le \beta_u$, $0 < \beta_u \le 1$, and $0 < \alpha \le 1$,*

(1)

$$POE_{cr}(q) = \begin{cases} 1, & for 0 \le q \le q_{cr}, \\ F_B\left(\dfrac{\alpha(1-q)}{q}\right), & for q_{cr} \le q \le 1 \end{cases} \quad \text{(A.2)}$$

(2) *$POE_{cr}(q)$ is a strictly decreasing function of q for $q_{cr} \le q \le 1$,*

(3) *$CFD_{cr} = \alpha/(\alpha + F_B^{-1}(0.5))$,*

(4) *the inspector with ρ_{cr} overestimates q with $POE > 0.5$ for $0 \le q < CFD_{cr}$, estimates q with $POE = 0.5$ for $q = CFD_{cr}$, and underestimates q with $POE < 0.5$ for $CFD_{cr} < q \le 1$.*

Proof. Since $\mathrm{POE}_{\mathrm{cr}}(0) = 1$ for $\alpha \ne 0$, (1) holds true for $q = 0$. Since $\mathrm{POE}'_{\mathrm{cr}}(q) = (\partial/\partial q)F_B(\alpha(1-q)/q) = -(\alpha/q^2)f_B(\alpha(1-q)/q) < 0$, $\mathrm{POE}_{\mathrm{cr}}(q)$ is a strictly decreasing function of q for $q_{\mathrm{cr}} \le q \le 1$. Since $0 \le \mathrm{POE}_{\mathrm{cr}}(q) \le 1$ for $q_{\mathrm{cr}} \le q \le 1$, solving $\mathrm{POE}_{\mathrm{cr}}(q^*) = 0.5$ for $q_{\mathrm{cr}} \le q^* \le 1$ gives $\mathrm{CFD}_{\mathrm{cr}} = \alpha/(\alpha + F_B^{-1}(0.5))$. Hence, (3) and (4) hold true. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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