Research Article

Real Zeros of a Class of Hyperbolic Polynomials with Random Coefficients

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We have proved here that the expected number of real zeros of a random hyperbolic polynomial of the form $y = P_n(t) = \sqrt{\binom{n}{1}}a_1 \cosh t + \sqrt{\binom{n}{2}}a_2 \cosh 2t + \cdots + \sqrt{\binom{n}{n}}a_n \cosh nt$, where $a_1, \ldots, a_n$ is a sequence of standard Gaussian random variables, is $\sqrt{n}/2 + o_p(1)$. It is shown that the asymptotic value of expected number of times the polynomial crosses the level $y = K$ is also $\sqrt{n}/2$ as long as K does not exceed $\sqrt{2^n e^{\mu(n)}}$, where $\mu(n) = o(n)$. The number of oscillations of $P_n(t)$ about $y = K$ will be less than $\sqrt{n}/2$ asymptotically only if $K = \sqrt{2^n e^{\mu(n)}}$, where $\mu(n) = O(n)$ or $n^{-1}\mu(n) \to \infty$. In the former case the number of oscillations continues to be a fraction of $\sqrt{n}$ and decreases with the increase in value of $\mu(n)$. In the latter case, the number of oscillations reduces to $o_p(\sqrt{n})$ and almost no trace of the curve is expected to be present above the level $y = K$ if $\mu(n)/(n \log n) \to \infty$.

1. Introduction

Let (Ω, A, Pr) be a fixed probability space and let $\{a_k(\omega)\}_{k=1}^{j=n}$ be a sequence of independent random variables defined on Ω. The sum $a_0(\omega)f_1(t) + a_2(\omega)f_2(t) + \cdots + a_n(\omega)f_n(t)$ is traditionally known as a random algebraic polynomial if $f_i(t) = t^i$, a random trigonometric polynomial if $f_i(t) = \cos(it)$ or $\sin(it)$, and a random hyperbolic polynomial if $f_i(t) = \cosh(it)$ or $\sinh(it)$. One can have useful information about the behaviour of these ensembles of polynomials if the average number of times these polynomials oscillate about the line $y = K$ is known. The reader is referred to the book by Farahmand [1] where an exhaustive account of progress made in study of random polynomials has been presented. It is to be noted that there is significantly more published literature on random algebraic and random trigonometric polynomials than that of random hyperbolic polynomials. Let

$$y = Q_n(t) = \sum_{k=1}^{n} a_k(\omega) \cosh kt, \qquad (1)$$

where $a_k(\omega)$ are normally distributed random variables with mean zero and variance one. One knows that Das [2] first calculated the expected number of real zeros of $Q_n(t)$. Farahmand [3] calculated the asymptotic estimate of oscillations of $Q_n(t)$ about $y = K$ if $K = o(\sqrt{n})$. Some of the other works in this direction are due to Mahanti [4–6]. Wilkins [7] determined real zeros of $Q_n(t)$ when $\mathrm{var}(a_k(\omega)) = k^p$, $p \geq 0$. We observe that the asymptotic value of the oscillations of random hyperbolic polynomials is $(2/\pi) \log n$ in each of these cases.

One is tempted to ask whether $Q_n(t)$ has more than $(2/\pi) \log n$ oscillations under certain conditions. In this context, we are reminded of a recent work of Edelman and Kostlan [8] where it has been found out that the expected number of real zeros of random algebraic polynomials increases significantly if the variance of the coefficients changes from unity to $\sqrt{\binom{n}{k}}$. Therefore, we examine what effect this new assumption on variance of the coefficients has on number of

oscillations of $Q_n(t)$. In other words, we calculate the number of oscillations of the polynomial

$$
y = P_n(t) = \sum_{k=1}^{n} a_k(\omega) \sqrt{\binom{n}{k}} \cosh kt, \tag{2}
$$

where $a_k(\omega)$ are normally distributed random variables with mean zero and variance one.

In Theorem 1 we have shown that the number of real zeros of $P_n(t)$ is substantially larger than that of $Q_n(t)$. Moreover, there is a significant difference in the way real zeros of $P_n(t)$ and $Q_n(t)$ lie on the t-axis. Most of the real zeros of $Q_n(t)$ are confined to the interval $[-1, 1]$ and there are negligible numbers of them if $|t| > 1$ (see [4]). But there are large numbers of real zeros of $P_n(t)$ outside $[-1, 1]$. This phenomenon can be deduced from the formula given in (26) and Lemma 10. In fact, the number of real zeros of $P_n(t)$ in the region $|t| > \alpha$, $\alpha > 1/\sqrt{n}$, is dependent on α. The real zeros decrease in number with increase in α in the region $|t| > \alpha$ and there are negligible numbers of them only when $\alpha \to \infty$.

Let $\mu(n)$ be any function of n such that $\mu(n)/n \to 0$. In Theorem 2 we have shown that the number of oscillations of $P_n(t)$ about the line $y = K$, where $K \leq \sqrt{2^n e^{\mu(n)}}$, is equal to its axis crossings. Thus, for all these values of K one can say that most of the oscillations of $P_n(t)$ that cross the t-axis reach up to the level $y = K$. If $\mu(n) = O(n)$ or $n^{-1}\mu(n) \to \infty$, we have proved in the theorem that the asymptotic value of number of oscillations of $P_n(t)$ about $y = K = \sqrt{2^n e^{\mu(n)}}$ is less than $\sqrt{n}/2$. If $\mu(n) = O(n)$ the number of oscillations continues to be a fraction of $\sqrt{n}$ and decreases with the increase in value of $\mu(n)$. If $n^{-1}\mu(n) \to \infty$ the number of oscillations is reduced to $o_p(\sqrt{n})$. Inequality (35) together with Lemma 14 provides a glimpse of the manner in which the number of oscillations decreases with increase in value of K. Inequality (35) also shows that there is hardly any trace of the curve $P_n(t)$ above the level $y = K = \sqrt{2^n e^{\mu(n)}}$ if $\mu(n)/(n \log n) \to \infty$.

Let the coefficients $\{a_k(\omega)\}_{k=1}^{k=n}$ of $P_n(t)$ be standard normal random variables. The expected number of oscillations of $P_n(t)$ about the line $y = K$, $t \in (\alpha, \beta)$, has been denoted by us as $\mathrm{EN}_{n,K}(\alpha, \beta)$ in the following two theorems.

Theorem 1. *For sufficiently large n,*

$$
\mathrm{EN}_{n,0}(-\infty, \infty) = \frac{\sqrt{n}}{2} + O_p\left(n^{-1/2} \log n\right). \tag{3}
$$

Theorem 2. *For sufficiently large n,*

   (i) $\mathrm{EN}_{n,K}(-\infty, \infty) = \sqrt{n}/2 + O_p(n^{-1/2} \log n)$ if $K^2/2^n \to o$,

   (ii) $\mathrm{EN}_{n,K}(-\infty, \infty) = \sqrt{n}/2 + (2/\sqrt{\pi})\mathrm{erf}(K/\sqrt{2^n}) + O_p(n^{-1/2} \log n)$ if $K^2 = O(2^n)$,

   (iii) $\mathrm{EN}_{n,K}(-\infty, \infty) = \sqrt{n}/2 - n^{-1/2}\mu(n)/(2\pi) + 2/\sqrt{\pi} + O_p(n^{-1/2} \log n)$ if $\mu(n)/n \to 0$, $\mu(n) \to \infty$, and $K = \sqrt{2^n e^{\mu(n)}}$,

   (iv) $\mathrm{EN}_{n,K}(-\infty, \infty) = (\sqrt{n}/\pi)\sin^{-1}(e^{-n^{-1}\mu(n)}/\sqrt{2}) + 2/\sqrt{\pi} + O_p(n^{-1/2} \log n)$ if $\mu(n) = O(n)$ and $K = \sqrt{2^n e^{\mu(n)}}$,

   (v) $\mathrm{EN}_{n,K}(-\infty, \infty) = \sqrt{n}e^{-n^{-1}\mu(n)}/(2\sqrt{2}) + 2/\sqrt{\pi} + O_p(n^{-1/2} \log n)$ if $\mu(n)/n \to \infty$ and $K = \sqrt{2^n e^{\mu(n)}}$.

Two more differences in behaviour of $P_n(t)$ and $Q_n(t)$ are noteworthy. In what follows we will find out that most of the axis crossings of $P_n(t)$ reach the level $\sqrt{2^n e^{\mu(n)}}$. However, the branches of $Q_n(t)$ that cross the axis do not travel beyond $y = K$, where $K = O(\sqrt{n})$ (see Mahanti and Sahoo [6]). Almost all of the polynomial $Q_n(t)$ lie below the level K, where $n^{-1}\log(K/\sqrt{n}) \to \infty$ (Mahanti and Sahoo [6]). However, a large part of $P_n(t)$ stretches above this level.

2. Formula for the Proof of the Theorems

The proof of the theorem is based on the formula for expected number of level crossings given by Crammer and Leadbetter [9, page 285]. Using it for $P_n(t) - K = 0$ in the interval (α, β) we can show that

$$
\mathrm{EN}_{n,K}(\alpha, \beta) = \int_{\alpha}^{\beta} \frac{\sqrt{A_n C_n - B_n^2}}{A_n} \phi\left(-\frac{K}{\sqrt{A_n}}\right) \tag{4}
$$
$$
\cdot \left[2\phi(\eta) + \eta\left\{2\Phi(\eta) - 1\right\}\right] dt,
$$

where $A_n = \mathrm{var}\{P_n(t) - K\}$, $B_n = \mathrm{cov}[\{P_n(t) - K\}, P_n'(t)]$, $C_n = \mathrm{var}\{P_n'(t)\}$, $\Phi(t) = (1/\sqrt{2\pi})\int_{-\infty}^{t} \exp(-x^2/2)dx$, $\phi(t) = (1/\sqrt{2\pi})\exp(-x^2/2)$, and $\eta = -B_n K/\sqrt{A_n}\sqrt{A_n C_n - B_n^2}$. Since the coefficients of $P_n(t)$ are independent and $\mathrm{var}(a_j) = \sqrt{\binom{n}{k}}$, it is easy to derive using little algebra that

$$
A_n = \sum_{k=1}^{n} \sqrt{\binom{n}{k}} \cosh^2 kt = 2^{n-1}\left(\cosh nt \cosh^n t + 1\right), \tag{5}
$$

$$
B_n = \frac{1}{2}\sum_{k=1}^{n} k\sqrt{\binom{n}{k}} \sinh 2kt = 2^{n-2}n\sinh(n+1)t \tag{6}
$$
$$
\cdot \cosh^{n-1} t,
$$

$$
C_n = \sum_{k=1}^{n} k^2 \sqrt{\binom{n}{k}} \sinh^2 k
$$
$$
= 2^{n-3}\left[\left(n^2\cosh(n+2)t + n\cosh nt\right)\cosh^{n-2}t \right.
$$
$$
\left. - n^2 - n\right]. \tag{7}
$$

Let

$$
D_n = A_n C_n - B_n^2 = 2^{2n-4}\left(n^2\left(\cosh^{2n}t - 1\right) \right.
$$
$$
+ \left(n^2 - n\right)\cosh^{n-2}t \cosh nt \sinh^2 t
$$
$$
+ 2n^2\cosh^{n-1}t \sinh nt \sinh t
$$
$$
\left. + n\left(\cosh^{2n-2}t \cosh^2 nt - 1\right)\right). \tag{8}
$$

Formula (4) now can be written as

$$\mathrm{EN}_{n,K}\left(\alpha,\beta\right) = I_1\left(\alpha,\beta\right) + I_2\left(\alpha,\beta\right), \qquad (9)$$

where

$$I_1\left(\alpha,\beta\right) = \frac{1}{\pi}\int_\alpha^\beta \phi_n\left(t\right)\exp\left\{-\frac{K^2 C_n}{2D_n}\right\}dt,$$

$$I_2\left(\alpha,\beta\right)$$
$$= \int_\alpha^\beta \frac{\sqrt{2}KB_n}{\pi A_n^{3/2}}\exp\left\{-\frac{K^2}{2A_n}\right\}\mathrm{erf}\left(\frac{KB_n}{\sqrt{2A_nD_n}}\right)dt, \qquad (10)$$

$$\phi_n\left(t\right) = \frac{\sqrt{D_n}}{A_n}.$$

As a special case, we can also obtain the famous Kac-Rice formula [10] for expected number of zeros of $P_n(t)$ by putting $K = 0$ in (9). Thus we have

$$\mathrm{EN}_{n,0}\left(\alpha,\beta\right) = \frac{1}{\pi}\int_\alpha^\beta \phi_n\left(t\right)dt. \qquad (11)$$

3. Preliminary Analysis

To evaluate the integrals in (9) and (11) we need to find out the dominant terms of A_n, B_n, and C_n. The inequalities mentioned in Lemmas 4–7 will be helpful for the purpose. We first mention the following form of L'Hôpital's Rule which we use to derive some of the inequalities.

Lemma 3 (the monotonic form of L'Hôpital's Rule [11]). *For $-\infty < a < b < \infty$, let $g, h : [a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b) with $h'(t) \ne 0$ on (a, b). If $g'(t)/h'(t)$ is increasing or decreasing on (a, b), then so are $(g(t) - g(a))/(h(t) - h(a))$ and $(g(t) - g(b))/(h(t) - h(b))$. If $g'(t)/h'(t)$ is strictly monotone, then the monotonicity of the above two quotients is also strict.*

Lemma 4. *If $0 \le t < \pi/2$, then $e^{nt^2/2 - nt^4/8} \le (\cosh t)^n \le e^{nt^2/2}$. In particular in the interval $(0, n^{-1/2})$*

$$1 + \frac{n\sinh^2 t}{2} + \frac{n(n-2)\sinh^4 t}{8} < (\cosh t)^n$$
$$< 1 + \frac{n\sinh^2 t}{2} + \frac{n^2\sinh^4 t}{4}. \qquad (12)$$

Thus $(\cosh t)^n = 1 + O(nt^2)$ if $t = o(n^{-1/2})$.

Proof. The series representation of $\tanh t$ is given by [12, page 42] $\tanh t = \sum_{k=1}^\infty s_k t^{2k-1}$, where $s_k = 2^{2k}(4^k - 1)B_{2k}/(2k)!$. Using the fact that $\zeta(2k) = 2^{2k-1}\pi^{2k}|B_{2k}|/(2k)!$ [12, page 1038], we have $|s_k|t^{2k-1} = \pi^{-2k}(2t)^{2k-1}(1 - 2^{-2k})\zeta(2k)$.

Now, $(1 - 2^{-n})\zeta(n) = \sum_{j=0}^\infty(2j+1)^{-n}$ is a monotonically decreasing function of n. Therefore, the power series of $\tanh t$ converges absolutely and uniformly for $0 \le t < \pi/2$. Integrating term by term we have the alternative series

$\log\cosh t = \sum_{k=1}^\infty (2k)^{-1}s_k t^{2k}$ for $0 \le t < \pi/2$. Since the coefficients of t^{2k} are monotonically decreasing, we find that the first part of the lemma is true.

Let $h_1(t) = r_1(t)/r_2(t)$, where $r_1(t) = (\cosh t)^n - 1$ and $r_2(t) = n\sinh^2 t/2 + n^2\sinh^4 t/4$, with $r_1(0) = r_2(0) = 0$. Then

$$\left(\frac{r_1'(t)}{r_2'(t)}\right)' = \left[(n-2)\left(n^2 - 4n\right)(\cosh t)^{n-3}\sinh t\right.$$
$$\left.\cdot\left\{\sinh^2 t - \frac{(n+2)}{(n^2 - 4n)}\right\}\right]\left\{2\left(1 + n\sinh^2 t\right)^2\right\}^{-1}. \qquad (13)$$

It is easy to verify that $t + t^3/5 < (n+2)/(n^2 - 4n)$ if $t \le n^{-1/2}$. Using the series representation of $\sinh t$ we find that

$$t + \frac{t^3}{5} - \sinh t = \frac{t^3}{30} - \sum_{k=2}^\infty \frac{t^{2k+1}}{(2k+1)!}$$
$$\ge t^3\left(\frac{1}{30} - \sum_{k=2}^\infty \frac{1}{(2k+1)!}\right) \ge 0. \qquad (14)$$

Therefore $\sinh^2 t - (n+2)/(n^2 - 4n) < 0$ if $t \in (0, n^{-1/2})$. As a consequence, $r_1'(t)/r_2'(t)$ is strictly decreasing in the interval $(0, n^{-1/2})$. By Lemma 3, $h_1(t)$ is decreasing in $(0, n^{-1/2})$ and by L'Hôpital's Rule $h_1(0+) = 1$. Therefore, $r_1(t) < r_2(t)$. Let $h_2(t) = r_3(t)/r_4(t)$, where $r_3(t) = (\cosh t)^n - 1 - (n\sinh^2 t)/2$ and $r_4(t) = (n^2\sinh^4 t)/4$ with $r_3(0) = r_4(0) = 0$. Then $r_3'(t)/r_4'(t) = (\cosh^{n-2} t - 1)(n\sinh^2 t)^{-1} = r_5(t)/r_6(t)$, where $r_5(t) = (\cosh t)^{n-2} - 1$ and $r_6(t) = n\sinh^2 t$ with $r_5(0) = r_6(0) = 0$. Now $r_5'(t)/r_6'(t) = (n-2)(\cosh t)^{n-4}/(2n)$, which is increasing. So, by Lemma 3, $h_2(t)$ is strictly increasing. By L'Hôpital's Rule we find that $h_2(0+) = (n-2)/(2n)$. Therefore,

$$1 + \frac{n\sinh^2 t}{2} + \frac{n(n-2)\sinh^4 t}{8} < (\cosh t)^n. \qquad (15)$$
□

Lemma 5. *If $t \to 0$, then $\pi/4 + t/2 - t^3/12 < \tan^{-1}e^t < \pi/4 + t/2 - t^3/12 + t^5/48$.*

Proof. Let $r_7(t) = \tanh^{-1}e^t - \pi/4$, $r_8(t) = t/2 - t^3/12$, and $r_9(t) = t/2 - t^3/12 + t^5/48$. Observe that $r_7(0) = r_8(0) = r_9(0) = 0$. Let $h_3(t) = r_7(t)/r_8(t)$ and $h_4(t) = r_7(t)/r_9(t)$. Then $r_7'(t)/r_8'(t) = \sec ht$ is a monotonically decreasing and $r_7'(t)/r_9'(t) = 2\sec ht/(2 - t^2)$ is monotonically increasing function of t. By L'Hôpital Rule $h_3(0+) = h_4(0+) = 1$. Therefore, by Lemma 3, $r_7(t) < r_8(t)$ and $r_9(t) < r_7(t)$ for $t > 0$. □

Lemma 6. *Let $m \ge 0$, $p \ge 1$. Then $\sinh^p t(\sinh nt)^{-1}(\cosh t)^{-m}$ is a monotonically decreasing function of t in $(0, \infty)$ if $p = 1$ and is a monotonically decreasing function of t in $(p/n, \infty)$ if $p \ge 2$ and $n > \sqrt{p}\{3(1 - \tanh p)\}^{-1/2}$.*

Proof. Since

$$
\left(\sinh^p t \, (\sinh nt)^{-1} \, (\coth t)^{-m}\right)' = \sinh^{p-1} t
$$

$$
\cdot (\cosh t)^{m+1} \cosh nt \tag{16}
$$

$$
\cdot \left(p \tanh nt - n \tanh t - m \tanh nt \tanh^2 t\right),
$$

to prove the lemma we show that $p \tanh nt - n \tanh t < 0$ under the conditions mentioned in the lemma. We first note that $(\tanh nt - n \tanh t)' = n(\sec h^2 nt - \sec h^2 t) < 0$. Therefore, if $p = 1$ and $t > 0$, $p \tanh nt - n \tanh t < 0$ since its value is zero at $t = 0$.

Now let $p \geq 2$. Let $q(t) = r_1(t)/r_2(t)$, where $r_1(t) = \tanh t$ and $r_2(t) = t$ with $r_1(0) = 0$, $r_2(0) = 0$. As $r_1'(t)/r_2'(t) = \sec h^2 t$ is a decreasing function of t, it follows from Lemma 3 that $t^{-1} \tanh t$ is monotonically decreasing. Therefore, for a fixed p, $(p/n)^{-1}\tanh(p/n)$ increases with n. As $lt_{n \to \infty}(p/n)^{-1} \tanh(p/n) = 1$ and $\tanh p < 1$, there exists an integer n_0 such that, for $n > n_0$, $(p/n)^{-1} \tanh(p/n) > \tanh p$. In other words, for $n > n_0$, $p \tanh nt - n \tanh t < 0$ at

$t = p/n$. Since $(\tanh nt - n \tanh t)' < 0$, we conclude that $p \tanh nt - n \tanh t < 0$ if $t \geq p/n$ and $n > n_0$. We can find out n_0 in the following manner.

We note that $t^{-1} \tanh t - 1 + t^2/3 > 0$ as $(\tanh t - t + t^3/3)' = t^2 \sec h^2 t (\cosh^2 t - t^{-2} \sinh^2 t) > 0$ and $\tanh t - t + t^3/3$ is zero at $t = 0$. So $(p/n)^{-1}\tanh(p/n) > \tanh p$ if $1 - p^2/(3n^2) > \tanh p$; that is, $n > n_0 = \sqrt{p}\{3(1 - \tanh p)\}^{-1/2}$. $\square$

Lemma 7. *If $0 \leq x \leq 1/\sqrt{2}$, then*

$$
\frac{(\pi/2) \, x \sqrt{1 - x^2}}{\left(1 + 2^{-1}\left(1 - 2x^2\right)\right)^{1.27\ldots}} < \sin^{-1} x
$$

$$
< \frac{(\pi/2) \, x \sqrt{1 - x^2}}{\left(1 + 2^{-1}\left(1 - 2x^2\right)\right)^{1.1137\ldots}}. \tag{17}
$$

Proof. Let $x = \sin\theta$ and $\alpha = 2\theta$. Let $r_{10}(\theta) = \log(1 + 2^{-1} \cos\alpha)$ and $r_{11}(\theta) = \log(\pi \sin\alpha/(2\alpha))$. We observe that $r_{10}(\pi/2) = r_{11}(\pi/2) = 0$. After differentiation we have

$$
\left(\frac{r_{11}'(\theta)}{r_{10}'(\theta)}\right)' = \left(\frac{(2 + \cos\alpha)\left(\alpha^{-1} - \cot\alpha\right)}{\sin\alpha}\right)' = \frac{2\left[(2 + \cos\alpha)\left(\cos ec^2\alpha - \alpha^{-2}\right)\sin\alpha + \left(\cot\alpha - \alpha^{-1}\right)(1 + 2\cos\alpha)\right]}{\sin^2\alpha}. \tag{18}
$$

We now use the following power series representation (see [12]) convergent for $|t| < \pi^2$, $t^{-1} - \cot t = \sum_{k=1}^{\infty}(2^{2k}|B_{2k}|/(2k)!)t^{2k-1}$, where B_{2k} is the Bernoulli number of degree $2k$.

Then,

$$
\sin^2\alpha \left(\frac{r_{11}'(\theta)}{r_{10}'(\theta)}\right)' = 2 \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} (\alpha)^{2k-2} \tag{19}
$$

$$
\cdot ((2k-1)(2 + \cos\alpha)\sin\alpha - \alpha(1 + 2\cos\alpha)).
$$

We observe that $(2 + \cos\alpha)\sin 2\theta - \alpha(1 + 2\cos\alpha)$ is nonnegative in $(0, \pi/4)$ since it vanishes at zero and its derivative, that is, $2(\alpha \sin\alpha - \sin^2\alpha)$ is positive.

It follows from (19) that $(r_{11}'(\theta)/r_{10}'(\theta))$ is a nondecreasing function in $(0, \pi/4)$ since

$$
(2k-1)(2 + \cos\alpha)\sin\alpha - \alpha(1 + 2\cos\alpha)
$$

$$
\geq (2 + \cos\alpha)\sin 2\theta - \alpha(1 + 2\cos\alpha). \tag{20}
$$

$lt_{\theta \to 0} r_{11}(\theta)/r_{10}(\theta) = \ln(\pi/2)/\ln(3/2) = 1.1137\ldots$ and by L'Hôpital's Rule $lt_{\theta \to \pi/4} r_{11}(\theta)/r_{10}(\theta) = 4/\pi = 1.27\ldots$, we obtain the proof of Lemma 7 using Lemma 3. $\square$

Lemma 8. *Consider*

$$
b_0 - b_1 < \frac{\sqrt{\pi}}{2} \int_0^v e^{-q^2 x^2} \operatorname{erf}(x) \, dx < b_0, \tag{21}
$$

where

$$
b_k = \frac{\left(1 - e^{-(qv)^2} \sum_{j=0}^{k}\left((qv)^{2j}/j!\right)\right)}{\left(2(2k+1)q^{2k+2}\right)}, \quad q > 1. \tag{22}
$$

Proof. Consider the following power series representation of $\operatorname{erf}(x)$ [12, eq. 3.321.1] $\operatorname{erf}(x) = (2/\sqrt{\pi}) \sum_{k=0}^{k=\infty}(-1)^k x^{2k+1}/ (k!(2k + 1))$. The power series converges absolutely and uniformly for $x \in R$. Hence [12, page 346]

$$
\frac{\sqrt{\pi}}{2} \int_0^v e^{-q^2 x^2} \operatorname{erf}(x) \, dx
$$

$$
= \sum_{k=0}^{k=\infty} (-1)^k \frac{\int_0^v x^{2k+1} e^{-q^2 x^2} \, dx}{k!(2k+1)} \tag{23}
$$

$$
= \sum_{k=0}^{k=\infty} (-1)^k \frac{\gamma\left(k + 1, q^2 v^2\right)}{2q^{2k+2} k!(2k+1)} = \sum_{k=0}^{k=\infty} (-1)^k b_k,
$$

where $b_k = (1 - e^{-(qv)^2} \sum_{j=0}^{k}((qv)^{2j}/j!))/(2(2k+1)q^{2k+2})$. It is easy to see that b_k is nonincreasing and $b_k \to 0$ as $k \to \infty$. Thus, the statement of the lemma is true. $\square$

Lemma 9. *Let $u_n = o(n^{-1/2})$; then*

$$
\int_0^{u_n} \phi_n(t) \, dt = O\left(u_n \sqrt{n}\right). \tag{24}
$$

Proof. By Lemma 4, D_n defined in (8) can be written as

$$D_n = 2^{2n-4} n \sinh^2 nt$$

$$\cdot \left\{ (n-1) \sinh^2 t \cos ech^2 nt \left(\cosh^2 nt - \cosh nt \right) \right. \tag{25}$$

$$\left. + (n \sinh t \cos echnt + 1)^2 \right\} \left(1 + O(nt)^2 \right).$$

By Lemma 6 we find that $(n \sinh t \cos echnt + 1)^2 = O(1)$ in $(0,1)$ and $(\sinh t \cos echnt)^2 \cosh^2 nt = O(n^{-1})$ if $nt = O(1)$. Also, $\sinh^2 t \coth^2 nt = O(t^2)$ if $nt \to \infty$ and $t \in (0,1)$. So, $(2^{2n-4} n \sinh^2 nt)^{-1} D_n \sim C_1$, $1 \le C_1 \le 2$. Also using Lemma 4 and (5) we find that $A_n/2^{n-1} \sim \cosh nt + 1$. By (11), we see that Lemma 9 is valid. $\square$

Lemma 10. *If $\beta > \alpha \ge 2n^{-1} \log n$, then*

$$\int_\alpha^\beta \phi_n(t) \, dt = \sqrt{n} \left(\tan^{-1} e^\beta - \tan^{-1} e^\alpha \right) \tag{26}$$

$$+ O\left(n^{3/2} \sinh \alpha \exp(-n\alpha) \right).$$

Proof. Using Lemmas 6 and 4 we obtain from (8) that $D_n = 2^{2n-4} n \cosh^{2n-2} t \cosh^2 nt (1 + O(n \sinh u \exp(-nu)))$ and $A_n = 2^{n-1} \cosh nt \cosh^n t (1 + O(\exp(-nu)))$. The conclusion of the lemma follows now from (11). $\square$

4. Proof of Theorem 1

If u_n is taken as $2n^{-1} \log n$ we can derive from Lemmas 9 and 10 the following relations:

$$\mathrm{EN}_{n,0}\left(0, 2n^{-1} \log n \right) = O_p\left(n^{-1/2} \log n \right),$$

$$\mathrm{EN}_{n,0}\left(2n^{-1} \log n, \infty \right) \tag{27}$$

$$= \frac{\sqrt{n}}{2} - \frac{\sqrt{n} \tan^{-1} e^{2n^{-1} \log n}}{\pi} + O_p\left(n^{-1/2} \right).$$

Since the integrand in (11) is an even function of t, we have

$$\mathrm{EN}_{n,0}(-\infty, \infty) = 2\mathrm{EN}_{n,0}(0, \infty). \tag{28}$$

Using Lemma 5 to approximate $\tan^{-1} e^{2n^{-1} \log n}$, we see that Theorem 1 is true.

5. Proof of Theorem 2

To determine $\mathrm{EN}_{n,K}(-\infty, \infty)$ we need to calculate $I_1(-\infty, \infty)$ and $I_2(-\infty, \infty)$ for different ranges of values of K as both quantities depend on magnitude of K. In Lemma 14 we have calculated $I_2(-\infty, \infty)$. Value of $I_1(-\infty, \infty)$ for three different ranges of value of K has been calculated in Lemmas 11–13. The relations (32), (34), and (35) and Lemma 14 establish Theorem 2. Note that we only need to calculate $I_1(0, \infty)$ and $I_2(0, \infty)$ since $I_1(\alpha, \beta)$ and $I_2(\alpha, \beta)$ are even functions of t.

Since $I_1(\alpha, \beta) \le \mathrm{EN}_{n,0}(\alpha, \beta)$, by Lemma 9 we have

$$I_1\left(0, 2n^{-1} \log n \right) = O_p\left(n^{-1/2} \log n \right). \tag{29}$$

In order to calculate $I_1(2n^{-1} \log n, \infty)$, we need the dominant term of C_n/D_n, which can be calculated with the help of Lemmas 4 and 6 as

$$\frac{C_n}{D_n} = n \left(2^{n-1} \cosh nt \cosh^n t \right)^{-1} \left(1 + O(n^{-1}) \right). \tag{30}$$

For brevity, we have written $K' = K^2/2^n$ in the following lemmas.

Lemma 11. *As long as K' does not exceed $e^{\mu(n)}$, where $\mu(n) = o(\sqrt{n})$,*

$$I_1(0, \infty) = \frac{\sqrt{n}}{4} + O_p\left(n^{-1/2} \log n \right). \tag{31}$$

Proof. Let $K' \le n^\alpha$, $\alpha > 0$. Then by Lemma 9 we find that $I_1(0, n^{-1}(\alpha + 2) \log n) = O_p(n^{-1/2} \log n)$. Using (30) and Lemma 10, we obtain that $I_1(n^{-1}(\alpha + 2) \log n, \infty) = \sqrt{n}/4 + O_p(n^{-1/2} \log n)$. Hence $I_1(0, \infty) = \sqrt{n}/4 + O_p(n^{-1/2} \log n)$.

Now let $K' \ge n^\alpha$, $\alpha \ge 2$, and $n^{-1/2} \log(K') = o(1)$. It follows from (30) that the maximum value of $e^{-K' C_n D_n^{-1}}$ in $(2n^{-1} \log n, n^{-1} \log(nK'^2/\log n))$ is $O(n^{-1})$.

Hence $I_1((2n^{-1} \log n, n^{-1} \log(nK'^2/\log n))) = O_p(n^{-1/2})$. As the integrand of $I_1(a, b)$ is bounded, we have $I_1(n^{-1} \log(nK'/\log n), n^{-1} \log(K'^2 n^2)) = O_p(n^{-1/2})$. By Lemma 10 and (30) we find that $I_1(n^{-1} \log(K'n^2), \infty) = \sqrt{n}/4 + O_p(n^{-1/2})$. Taking into account (29) we see that (31) is true. $\square$

Lemma 12. *Let $n^{-1/2} \log K' = \mu(n)$, where either $\mu(n) = O(1)$ or $\mu(n) \to \infty$, but $\mu(n) = o(\sqrt{n})$; then*

$$I_1(0, \infty) = \frac{\sqrt{n}}{2} - \frac{\mu(n)}{2\pi} + O_p\left(n^{-1/2} \log n \right). \tag{32}$$

Proof. By Lemma 10 and (30) we find that the following relations are true:

$$I_1\left(2n^{-1} \log n, (2n)^{-1} \log n + \mu(n) n^{-1/2} - \frac{\mu^2(n)}{2} \right)$$

$$= O_p\left(\sqrt{n} e^{-\sqrt{n}} \right),$$

$$I_1\left(2n^{-1} \log n + \mu(n) n^{-1/2} - \frac{\mu^2(n)}{2}, \infty \right) = \frac{\sqrt{n}}{4}$$

$$- \frac{\mu(n)}{\pi} + O_p\left(n^{-1/2} \right), \tag{33}$$

$$I_1\left((2n)^{-1} \log n + \mu(n) n^{-1/2} - \frac{\mu^2(n)}{2}, 2n^{-1} \log n \right.$$

$$\left. + \mu(n) n^{-1/2} - \frac{\mu^2(n)}{2} \right) = O_p\left(n^{-1/2} \log n \right).$$

These relations and (29) yield (32). $\square$

Lemma 13. *Let* $\log K' = n\mu(n)$, *where* $\mu(n) = O(1)$ *or* $\mu(n) \to \infty$. *Then*

$$I_1(0, \infty) = \left(\frac{2\sqrt{n}}{\pi}\right) \sin^{-1}\left(\frac{e^{-\mu(n)}}{\sqrt{2}}\right) \\ + O_p\left(n^{-1/2}\log n\right). \tag{34}$$

For large values of n

$$\frac{\sqrt{n}x\sqrt{1-x^2}}{4\left(1+\sqrt{1-x^2+x^4}\right)^{1.27\ldots}} < I_1(0,\infty) \\ < \frac{\sqrt{n}x\sqrt{1-x^2}}{4\left(1+\sqrt{1-x^2+x^4}\right)^{1.1137\ldots}}, \tag{35}$$

where $x = e^{-\mu(n)}/\sqrt{2}$.

Moreover, $I_1(0,\infty) = O_p(1)$ *if* $\mu(n) = O(n\log n)$ *and* $I_1(0,\infty) = o_p(1)$ *if* $\mu(n)/(n\log n) \to \infty$.

Proof. Let t_1 and t_2 be the points in R where $(\exp(2t)+1)^n/2^n$ assumes the values $(n/\log n)\exp(n\mu(n))$ and $n^2\exp(n\mu(n))$, respectively. It follows from (30) and Lemma 10 that

$$I_1\left(2n^{-1}\log n, t_1\right) = O_p\left(n^{-1/2}\right). \tag{36}$$

Let $z = \ln((e^{2t}+1)/2)$ and $s = 2e^z$.

Then $\int \sec ht\, dt = \int (dz/\sqrt{2e^z-1}) = \int (ds/s\sqrt{s-1}) = -2\sin^{-1}(s^{-1/2})$. Therefore, by (30) and the definition of $I_1(\alpha, \beta)$, we have $I_1(t_1, t_2) = O_p(n^{-1/2}\log n)$ and $I_1(t_2, \infty) = \sqrt{n}\sin^{-1}(e^{-\mu(n)}/\sqrt{2}) + O_p(n^{-1/2})$. We obtain (34) if (29) is also considered. Inequality (35) follows immediately from Lemma 7. From this inequality the other two estimates of $I_1(0,\infty)$ valid for $\mu(n) = O_p(n\log n)$ and $\mu(n)$ such that $\mu(n)/(n\log n) \to \infty$ follow. $\qquad\square$

Lemma 14. *The dependence of* $I_2(-\infty, \infty)$ *on* K *is given by the following relations:*

   (i) *if* $K^2 = o(2^n)$, *then* $I_2(-\infty, \infty) = o_p(1)$,

   (ii) *if* $K^2 = O(2^n)$, *then* $I_2(-\infty, \infty) = (2/\sqrt{\pi})\mathrm{erf}(u) + O_p(e^{-n})$,

   (iii) *if* $K^2/2^n \to \infty$, *then* $I_2(-\infty, \infty) = (2/\sqrt{\pi}) + O_p(e^{-K^2/2^n})$,

   where $u = \sqrt{K'}$.

Proof. Using Lemma 4 we find that in $(0, u_n)$

$$D_n = 2^{2n-4}n\sinh^2 nt\,(1 + 2n\sinh t\cos echnt + o\,(1)), \\ B_n = 2^{n-2}n\sinh^2 nt\,(1 + o\,(1)). \tag{37}$$

Hence, $B_n/\sqrt{D_n} \sim \sqrt{n}/c$, where $1 \le c \le \sqrt{3}$.

On the other hand, using Lemma 6 and the definitions of B_n and D_n in (u_n, ∞), where $u_n > 2n^{-1}\log n$, we have $B_n/\sqrt{D_n} \sim \sqrt{n}e^t$.

Let $s = K/\sqrt{2A_n}$. It is not difficult to see that $(K/s)^{1/n} \sim e^t$ in the interval (u_n, ∞)

$$I_2(-\infty, \infty) = 2I_2(0, \infty) \\ = \left(\frac{4}{\pi}\right)\int_0^u \exp\left(-s^2\right)\mathrm{erf}\,(sp)\,ds \\ = \left(\frac{2}{\sqrt{\pi}}\right)\mathrm{erf}\,(up)\,\mathrm{erf}\,(u) \\ - 4p\int_0^u \exp\left(-p^2s^2\right)\mathrm{erf}\,(s)\,ds, \tag{38}$$

where $\sqrt{n}/\sqrt{3} \le p \le \sqrt{n}e^{\mu(n)}$.

Clearly $I_2(-\infty, \infty) = o_p(1)$ if $K^2 = o(2^n)$. The other two parts of the lemma are found to be true by virtue of Lemma 8. $\qquad\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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