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Research Article

Existence and Numerical Solution of the Volterra Fractional Integral Equations of the Second Kind

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This work presents the possible generalization of the Volterra integral equation second kind to the concept of fractional integral. Using the Picard method, we present the existence and the uniqueness of the solution of the generalized integral equation. The numerical solution is obtained via the Simpson 3/8 rule method. The convergence of this scheme is presented together with numerical results.

1. Introduction

The integral equations form an important part of applied mathematics, with links with many theoretical fields, especially with practical fields [1–3]. The Volterra integral [1–3] equations were introduced by Vito Volterra and then studied by Traian Lalescu in his 1908 thesis. Volterra integral equations find application in demography, the study of viscoelastic materials, and in insurance mathematics through the renewal equation. Fredholm equations [4] arise naturally in the theory of signal processing, most notably as the famous spectral concentration problem popularized by David Slepian [4]. They also commonly arise in linear forward modeling and inverse problems. Throughout the last decade, physicists and mathematicians have paid attention to the concept of fractional calculus [5–9]. Actually, real problems in scientific fields such as groundwater problems, physics, mechanics, chemistry, and biology are described by partial differential equations or integral equations. Many scholars have shown with great success the applications of fractional calculus to groundwater pollution and groundwater flow problems [5–9], acoustic wave problems [10], and others [11–14]. There are also several iteration methods for solving fractional integral equations like homotopy decomposition method [15–17], variational iteration method [18–20], Adomian decomposition method [21, 22], and others [23, 24]. But in this work, we will make use of the numerical method called the Simpson 3/8 rule. The general equation under analysis here is given as

$$F(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [K(t,\tau) F(\tau)]\, d\tau, \qquad 0 < \tau \le t \le T; \quad \alpha \ge 0. \tag{1}$$

Here,

$$\begin{aligned} F(t) &= [f_1(t), f_2(t), f_3(t), f_4(t), \dots f_n(t)]^T, \\ G(t) &= [g_1(t), g_2(t), g_3(t), g_4(t), \dots g_n(t)]^T, \\ K&(t,\tau,F(\tau)) \\ &= \begin{pmatrix} K_1([t,\tau,f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau), \dots f_n(\tau)]) \\ K_2([t,\tau,f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau), \dots f_n(\tau)]) \\ \vdots \\ K_n([t,\tau,f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau), \dots f_n(\tau)]) \end{pmatrix}. \end{aligned} \tag{2}$$

For the rest of this paper, we assume that $a < \tau \le t \le T < \infty$. In this paper, system (1) can be linear or nonlinear.

2. Basic Information about the Fractional Calculus

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space $C_{-\mu}$, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$, such that $f(x) = x^p h(x)$, where $h(x) \in C[0,\infty)$, and it is said to be in space C_μ^m if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_{-}\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x). \tag{3}$$

Properties of the operator can be found in [25–29]; we mention only the following.

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$,

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \qquad J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$
$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{4}$$

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\begin{aligned} {}_a^cD_x^\alpha f(x) &= J^{m-\alpha} D^m f(x) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^m(t)\,dt, \\ &\qquad m-1 < \alpha \leq m,\ m \in \mathbb{N}, \\ &\qquad x > 0,\ f \in C_{-1}^\mu. \end{aligned} \tag{5}$$

Also, we need here two of its basic properties.

Lemma 3. *If $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and $f \in C_\mu^m$, $\mu \geq -1$, then*

$$ {}_a^cD_x^\alpha J^\alpha f(x) = f(x),$$
$$J^\alpha D_0^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{6}$$

3. Existence and Uniqueness Analysis

The analysis of the existence and the uniqueness analysis are important aspects that must be investigated before the presentation of the solution. One of the most common techniques used to achieve this is the fixed point theorem technique. To prove the existence and uniqueness of the solution of the system (1), we make use of the method of successive approximation, also called the Picard method [30]. This consists of simple iterations. Before we start this proof, we will assume the following.

First, making use of the vector norm, we assume that

$$\|F(t)\| = \max_{1\leq i\leq n} |f_i(t)|,$$
$$\|K(s,t)\| = \max_{1\leq i\leq n} \sum_k^n |k_{i,j}(s,t)|. \tag{7}$$

In this method, we assume that the following iteration can be used to provide a series solution of the problem under investigation:

$$F_n(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [K(t,\tau) F_{n-1}(\tau)]\,d\tau. \tag{8}$$

It also assumes that the initial component of the series solution is given as

$$F_0(t) = G(t). \tag{9}$$

Let us in addition put the difference between the consecutive components as

$$\delta_n(t) = F_n(t) - F_{n-1}(t). \tag{10}$$

It is very easy to see that

$$F_n(t) = \sum_{i=0}^n \delta_i(t). \tag{11}$$

3.1. Existence and Uniqueness of the Linear Volterra Fractional Integral Equations of the Second Kind

Theorem 4 (see [31]). *Under the conditions that the vector functions $G(t)$ and $K(s,t)$ are continuous $0 \leq a < \tau \leq t \leq T < \infty$, then, the system of Volterra fractional integral equations of the second kind (1) has a unique continuous solution for $0 \leq a < t \leq T < \infty$.*

The proof is similar to the one in [31].

Theorem 5. *Assuming that the system (1) has a unique solution, say $F(t)$ in $a < t \leq T$, such that $K(t,\tau)F(\tau)$ is absolutely fractionally integrable, and if in addition*

$$\|G(t)\| < g(t), \qquad \|K(t,s)\| < k(t,s), \tag{12}$$

providing that k and g are continuous functions, then it is possible to find a function, say $f(t)$, such that

$$\|F(t)\| < f(t), \tag{13}$$

where $f(t)$ is the continuous function solutions of

$$f(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [K(t,\tau) f(\tau)]\,d\tau. \tag{14}$$

Proof. From (1), applying the vector norm on both sides, we obtain the following:

$$\|F(t)\| = \left\| G(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{1-\alpha} [K(t,\tau) F(\tau)]\,d\tau \right\|. \tag{15}$$

Now, making use of the inequality triangular, we obtain the following:

$$\|F(t)\| \le \|G(t)\| + \left\| \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [K(t,\tau) F(\tau)]\, d\tau \right\|,$$

$$\|F(t)\| \le \|G(t)\| + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [\|K(t,\tau)\| \|F(\tau)\|]\, d\tau. \tag{16}$$

Thus, making use of the hypothesis, we obtain

$$\|F(t)\| < g(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [k(t,\tau) \|F(\tau)\|]\, d\tau. \tag{17}$$

If now the difference between (14) and (17) gives

$$f(t) - \|F(t)\| > \frac{1}{\Gamma(\alpha)} \times \int_a^t (t-\tau)^{\alpha-1} \times \left[k(t,\tau) \left(f(t) - \|F(\tau)\|\right)\right] d\tau, \tag{18}$$

since $f(a) - \|F(a)\| > 0$ and also $k(t,\tau)$ is a continuous positive function, it is then true to conclude that

$$f(t) - \|F(t)\| > 0, \quad \text{for } t \le T, \tag{19}$$

which concludes the proof. □

Theorem 6. *Under the condition that $G(t)$, $K(t,s)$, $\Delta G(t)$, and $\Delta K(t,s)$ are smooth functions and bounded one has*

$$\begin{aligned} &\|K(t,\tau)\| \le k, \qquad \|\Delta K(t,\tau)\| \le \Delta k, \\ &\|G(t)\| \le g, \qquad \|\Delta G(t)\| \le \Delta g. \end{aligned} \tag{20}$$

Let $F_{exact}(t)$ be the exact solution of

$$F_{exact}(t) = G(t) + \Delta G(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} d\tau \times \left[(K(t,\tau) - \Delta K(t,\tau)) \ F_{exact}(\tau)\right]; \tag{21}$$

then,

$$\begin{aligned} &\left\|F_{exact}(t) - F(t)\right\| \\ &\quad \le \left\{\Delta g + \Delta k t \left(g + \Delta g\right) E_\alpha\left((k+\Delta k) t\right)\right\} \times E_\alpha\left(k^{-\alpha} t\right) \\ &\quad = O\left(\Delta g\right) + O\left(\Delta k\right), \end{aligned} \tag{22}$$

with F_{exact} is the solution of system (1).

Proof. Since $G(t)$, $K(t,s)$ are smooth functions and bounded, using Theorem 4, there exists a positive smooth function $f(t)$ such that

$$f(t) = g(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} [k f(\tau)]\, d\tau. \tag{23}$$

For simplicity, we chose $a = 0$; then,

$$f(t) = g(t) + \frac{k}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [f(\tau)]\, d\tau. \tag{24}$$

Using the methodology of the homotopy decomposition method, we arrive at the following exact solution:

$$f(t) = g E_\alpha\left(k^{-\alpha} t\right), \tag{25}$$

where

$$E_\alpha\left(k^{-\alpha} t\right) = \sum_{n=0}^{\infty} \frac{k^n t^{n\alpha}}{\Gamma[1 + n\alpha]} \tag{26}$$

known as the Mittag-Leffler function. Therefore,

$$\|F(t)\| < g E_\alpha\left(k^{-\alpha} t\right). \tag{27}$$

With the above in hand, it is very easy to show that

$$\left\|F_{\text{exact}}(t)\right\| \le \left(g + \Delta g\right) E_\alpha\left((k + \Delta k)^{-\alpha} t\right). \tag{28}$$

Since $F_{\text{exact}}(t)$ is the approximate solution of system (1), then it follows that

$$F_{\text{exact}}(t) \approx G(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[K(t,\tau) F_{\text{exact}}(\tau)\right] d\tau. \tag{29}$$

So the error in the approximation can be represented as

$$R(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \times \left[K(t,\tau) F_{\text{exact}}(\tau)\right] d\tau - F_{\text{exact}}(t). \tag{30}$$

Now replacing $F_{\text{exact}}(t)$ as in (21), we obtain

$$R(t) = -\Delta G(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[\Delta K(t,\tau) F_{\text{exact}}(\tau)\right] d\tau. \tag{31}$$

Then, the difference between the exact solution and the approximate solution can be obtained as

$$\begin{aligned} &F(t) - F_{\text{exact}}(t) \\ &\quad = R(t) + \int_0^T \frac{1}{\Gamma(\alpha)} \times (t-\tau)^{\alpha-1} [\Delta K(t,\tau) \times \left[F(t) - F_{\text{exact}}(\tau)\right]]\, d\tau \end{aligned} \tag{32}$$

so that

$$\begin{aligned} &\left\|F(t) - F_{\text{exact}}(t)\right\| \\ &< \Delta g + \Delta k t \ \max\left(\|F(t)\|\right) \\ &\le \left\{\Delta g + \Delta k t \left(g + \Delta g\right) \times E_\alpha\left((k+\Delta k) t\right)\right\} E_\alpha\left(k^{-\alpha} t\right) \\ &= O\left(\Delta g\right) + O\left(\Delta k\right), \end{aligned} \tag{33}$$

which completes the proof. □

3.2. Existence and Uniqueness of the Nonlinear Volterra Fractional Integral Equations of the Second Kind. In this case, the nonlinear Volterra fractional integral equations of the second kind considered here are

$$F(t) = G(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}[K(t,\tau,F(\tau))]\,d\tau, \qquad 0<\tau\le t\le T;\ \alpha\ge 0. \tag{34}$$

$F(t)$, $G(t)$, and $K(t,\tau,F(\tau))$ have the same form as in the previous subsection; also the norm used in the previous section is maintained.

In analogy with what was done in Section 3.1, we define the iteration formula as

$$F_n(t) = G(t) + \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}[K(t,\tau,F(\tau))]\,d\tau, \tag{35}$$

with initial component

$$F_0(t) = G(t). \tag{36}$$

Similarly, the difference between the consecutive terms is given as

$$F_n(t) - F_{n-1}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\left[K(t,\tau,F_{n-1}(\tau)) - K(t,\tau,F_{n-2}(\tau))\right]d\tau. \tag{37}$$

We will perhaps recall that in case the kernel K satisfies the Lipchitz conditions, we have the following inequality:

$$\left\|K(t,\tau,F_1) - K(t,\tau,F_2)\right\| \le H\left\|F_1 - F_2\right\|, \tag{38}$$

with of course H being a real positive number not depending on the parameters t, τ, F_1, and F_2.

As in Section 3.1, we put

$$\delta_n(t) = F_n(t) - F_{n-1}(t). \tag{39}$$

Again, we have that

$$F_n(t) = \sum_{i=0}^{n}\delta_i(t). \tag{40}$$

Then,

$$\left\|\delta_n(t)\right\| = \left\|\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\left[K(t,\tau,F_{n-1}(\tau)) - K(t,\tau,F_{n-2}(\tau))\right]d\tau\right\|$$

$$\left\|\delta_n(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \times \left\|K(t,\tau,F_{n-1}(\tau) - K(t,\tau,F_{n-2}(\tau)\right\|d\tau. \tag{41}$$

In case the Lipchitz condition is satisfied by the kernel, we have the following inequality:

$$\left\|\delta_n(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} \times \left\|K(t,\tau,F_{n-1}(\tau))\,d\tau, -K(t,\tau,F_{n-2}(\tau))\right\| \tag{42}$$

$$\left\|\delta_n(t)\right\| \le \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\left\|\delta_{n-1}\right\|d\tau. \tag{43}$$

We will then present the following theorem.

Theorem 7. *Under the conditions that $G(t)$, $K(t,\tau,F)$ are continuous in $0<\tau\le t\le T<\infty$, $-\infty<F<\infty$ and the kernel satisfies the Lipchitz condition, that is, $\|K(t,\tau,F_1) - K(t,\tau,F_2)\| \le H\|F_1-F_2\|$, then,* (34) *has a unique solution.*

Proof. From (43), it follows that

$$\left\|\delta_n(t)\right\| \le \max_{0<t\le T}\left\|G(t)\right\|\frac{(H^{-\alpha}t)^{n\alpha}}{\Gamma(1+n\alpha)}; \tag{44}$$

therefore,

$$F(t) = \sum_{i=0}^{n}\delta_n(t) \tag{45}$$

exists and is a continuous function. However, to prove that the above function is the solution of the system of the nonlinear Volterra fractional integral equations (34) of the second kind, we let

$$F(t) = F_n(t) - P_n(t). \tag{46}$$

Now, using (36), we have the following equation:

$$F(t) - F_n(t) = G(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}K(t,\tau,F(\tau) - P_n(\tau))\,d\tau. \tag{47}$$

It follows that

$$F(t) - G(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}K(t,\tau,F(\tau))\,d\tau = P_n(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \times K(t,\tau,F(\tau) - P_n(\tau) - K(t,\tau,F(\tau))\,d\tau). \tag{48}$$

Now, applying the norm and Lipchitz condition, we arrive at the following inequality:

$$\left\|F(t) - G(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}K(t,\tau,F(\tau))\,d\tau\right\| \le \left\|P_n(t)\right\| + Ht\left\|P_{n-1}(t)\right\|. \tag{49}$$

Applying the limit on both sides of the above inequality when n tends to infinity, the right-hand side tends to zero; then, $F(t)$ in (45) satisfies

$$F(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} K(t,\tau,F(\tau))\, d\tau, \quad (50)$$

and indeed it is the solution of (34).

We will now present the uniqueness of this solution. To achieve this, we assume that (34) has another solution, say $F_1(t)$; then,

$$\begin{aligned} F(t) - F_1(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \\ &\quad \times [K(t,\tau,F(\tau)) - K(t,\tau,F_1(\tau))]\, d\tau. \end{aligned} \quad (51)$$

Applying the norm and making use of the Lipchitz condition of the kernel, we arrive at

$$\|F(t) - F_1(t)\| \leq H \int_0^t (t-\tau)^{\alpha-1} \|F(\tau) - F_1(\tau)\|\, d\tau, \quad (52)$$

but $\|F(t) - F_1(t)\| \leq D$; then,

$$\|F(t) - F_1(t)\| \leq D \frac{(H^{-\alpha} t)^{n\alpha}}{\Gamma(1+n\alpha)}, \quad (53)$$

for any n; then,

$$F(t) = F_1(t). \quad (54)$$

□

4. Numerical Method to Solve the Volterra Fractional Integral Equations

We consider the general form of the Volterra fractional integral equation as

$$\begin{aligned} F(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \\ \times [K(t,\tau,F(\tau))]\, d\tau, \\ 0 < \tau \leq t \leq T; \quad \alpha \geq 0, \end{aligned} \quad (55)$$

where $G(t)$ is a known function, $F(t)$ is an unknown function to be determined, and the kernel is $K(t, y, f)$.

In numerical analysis, Simpson's rule is a method for numerical integration, the numerical approximation of definite integrals [32]. Simpson's rule also corresponds to the 3-point Newton-Cotes quadrature rule. The method is credited to the mathematician Thomas Simpson (1710–1761) of Leicestershire, England. Simpson's rule is a staple of scientific data analysis and engineering. It is widely used, for example, by naval architects, to numerically integrate hull offsets and cross-sectional areas to determine volumes and centroids of ships or lifeboats [33].

4.1. Application of the Simpson 3/8 to Volterra Fractional Integral Equation. To use Simpson's rule here, we let $0 = a < t_1 < t_2 < t_3 \cdots < t_n$ be a possible division of $[0, b]$, with step size $x_i = ik$ for $i = 0, 1, \ldots N$. We construct a block by block method that is the system of Volterra fractional integral equation (1) for $q > 1$ simultaneous equations is then a set of q simultaneous value of the function F. Without loss of generality, we consider $q = 6$.

For the rest of the paper,

$$(t-\tau)^{\alpha-1} K[t, \tau\ F(\tau)] = K_\alpha(t, \tau, F(\tau)) \quad (56)$$

will be called the fractional kernel. Having the fractional kernel in hand, system (1) can be rewritten as follows:

$$F(t) = G(t) + \frac{1}{\Gamma(\alpha)} \int_0^t [K_\alpha(t,\tau,F(\tau))]\, d\tau. \quad (57)$$

Now, if we set $t = t_{3n+1}$ in the previous equation, we obtain

$$\begin{aligned} F_{1,3n+1}(t) = g_1(t_{3n+1}) &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_{3n+1}} K_{\alpha(1,1)}[t_{3n+1}, \tau, f_1(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_{3n}} K_{\alpha(1,2)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{3n}}^{t_{3n+1}} K_{\alpha(1,1)}[t_{3n+1}, \tau, f_1(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{3n}}^{t_{3n+1}} K_{\alpha(1,2)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau, \end{aligned} \quad (58)$$

$$\begin{aligned} F_{2,3n+1}(t) = g_2(t_{3n+1}) &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_{3n+1}} K_{\alpha(2,1)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^{t_{3n}} K_{\alpha(2,2)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{3n}}^{t_{3n+1}} K_{\alpha(2,1)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_{3n}}^{t_{3n+1}} K_{\alpha(2,2)}[t_{3n+1}, \tau, f_2(\tau)]\, d\tau. \end{aligned} \quad (59)$$

From here if one integrates over the interval $[0, t_{3n}]$, we can apply Simpson's 3/8 rule, and also by integrating over $[t_{3n}, t_{3n+1}]$, one can calculate it by using a cubic interpolation. Then, we can have the following:

$$\begin{aligned} &F_{1,3n+1}(t_{3n+1}) \\ &= g_1(t_{3n+1}) \end{aligned}$$

$$+\frac{3h}{8\Gamma(\alpha)}\left(K_{\alpha(1,1)}\left(t_{3n+1},t_0,F_{1,0}\right)\right.$$
$$+3K_{\alpha(1,1)}\left(t_{3n+1},t_1,F_{1,1}\right)$$
$$+3K_{\alpha(1,1)}\left(t_{3n+1},t_2,F_{1,2}\right)$$
$$+2K_{\alpha(1,1)}\left(t_{3n+1},t_3,F_{1,3}\right)+\cdots$$
$$\left.+K_{\alpha(1,1)}\left(t_{3n+1},t_{3n},F_{1,3n}\right)\right)$$
$$+\frac{3h}{8\Gamma(\alpha)}\left(K_{\alpha(2,1)}\left(t_{3n+1},t_0,F_{2,0}\right)\right.$$
$$+3K_{\alpha(2,1)}\left(t_{3n+1},t_1,F_{2,1}\right)$$
$$+3K_{\alpha(2,1)}\left(t_{3n+1},t_2,F_{2,2}\right)$$
$$+2K_{\alpha(2,1)}\left(t_{3n+1},t_3,F_{2,3}\right)+\cdots$$
$$\left.+K_{\alpha(2,1)}\left(t_{3n+1},t_{3n},F_{2,3n}\right)\right)$$
$$+\frac{h}{8\Gamma(\alpha)}\left(K_{\alpha(1,1)}\left(t_{3n+1},t_{3n},F_{1,3n}\right)+3K_{\alpha(1,1)}\right.$$
$$\times\left(t_{3n+1},t_{3n+1/3},\frac{40}{81}F_{1,3n}\right.$$
$$\left.-\frac{20}{27}F_{1,3n+1}-\frac{8}{27}F_{1,3n+1}+\frac{5}{81}F_{1,3n+3}\right)$$
$$+3K_{\alpha(1,1)}$$
$$\times\left(t_{3n+1},t_{3n+2/3},\frac{14}{81}F_{1,3n+1}\right.$$
$$\left.-\frac{7}{27}F_{1,3n+2}+\frac{4}{81}F_{1,3n+3}\right)$$
$$\left.+K_{\alpha(1,1)}\left(t_{3n+1},t_{3n+1},F_{1,3n}\right)\right)$$
$$+\frac{h}{8\Gamma(\alpha)}\left(K_{\alpha(1,2)}\left(t_{3n+1},t_{3n},F_{2,3n}\right)\right.$$
$$+3K_{\alpha(1,2)}\left(t_{3n+1},t_{3n+1/3},\frac{40}{81}F_{2,3n}\right.$$
$$-\frac{20}{27}F_{2,3n+1}-\frac{8}{27}F_{2,3n+2}$$
$$\left.+\frac{5}{81}F_{2,3n+3}\right)$$
$$+3K_{\alpha(1,2)}\left(t_{3n+1},t_{3n+2/3},\frac{14}{81}F_{2,3n+1}\right.$$
$$\left.-\frac{7}{27}F_{2,3n+2}+\frac{4}{81}F_{2,3n+3}\right)$$
$$\left.+K_{\alpha(1,2)}\left(t_{3n+1},t_{3n+1},F_{2,3n}\right)\right). \tag{60}$$

In a similar way, if one let $x=x_{3n+2}$, we get the following:

$$F_{1,3n+2}(t)=g_1\left(t_{3n+2}\right)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_{3n+2}}K_{\alpha(1,1)}\times\left[t_{3n+2},\tau,f_1(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_{3n}}K_{\alpha(1,2)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{3n}}^{t_{3n+2}}K_{\alpha(1,1)}\times\left[t_{3n+1},\tau,f_1(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{3n}}^{t_{3n+2}}K_{\alpha(1,2)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau,$$
$$F_{2,3n+2}(t)=g_2\left(t_{3n+1}\right)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_{3n+2}}K_{\alpha(2,1)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^{t_{3n}}K_{\alpha(2,2)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{3n}}^{t_{3n+2}}K_{\alpha(2,1)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau$$
$$+\frac{1}{\Gamma(\alpha)}\int_{t_{3n}}^{t_{3n+2}}K_{\alpha(2,2)}\times\left[t_{3n+2},\tau,f_2(\tau)\right]d\tau. \tag{61}$$

From here if one integrates over the interval $[0,t_{3n}]$, we can apply Simpson's 3/8 rule, and also by integrating over $[t_{3n},t_{3n+2}]$, one can calculate it by using a cubic interpolation. Then, we can have the following:

$$F_{1,3n+2}\left(t_{3n+2}\right)$$
$$=g_1\left(t_{3n+2}\right)$$
$$+\frac{3h}{8\Gamma(\alpha)}\left(K_{\alpha(1,1)}\left(t_{3n+2},t_0,F_{1,0}\right)\right.$$
$$+3K_{\alpha(1,1)}\left(t_{3n+1},t_1,F_{1,1}\right)$$
$$+3K_{\alpha(1,1)}\left(t_{3n+2},t_2,F_{1,2}\right)$$

$$+ 2K_{\alpha(1,1)}\left(t_{3n+2}, t_3, F_{1,3}\right) + \cdots$$
$$+ K_{\alpha(1,1)}\left(t_{3n+2}, t_{3n}, F_{1,3n}\right))$$
$$+ \frac{3h}{8\Gamma(\alpha)}\left(K_{\alpha(2,1)}\left(t_{3n+2}, t_0, F_{2,0}\right)\right.$$
$$+ 3K_{\alpha(2,1)}\left(t_{3n+2}, t_1, F_{2,1}\right)$$
$$+ 3K_{\alpha(2,1)}\left(t_{3n+2}, t_2, F_{2,2}\right)$$
$$+ 2K_{\alpha(2,1)}\left(t_{3n+2}, t_3, F_{2,3}\right) + \cdots$$
$$\left.+ K_{\alpha(2,1)}\left(t_{3n+1}, t_{3n}, F_{2,3n}\right)\right)$$
$$+ \frac{h}{4\Gamma(\alpha)}\left(K_{\alpha(1,1)}\left(t_{3n+2}, t_{3n}, F_{1,3n}\right) + 3K_{\alpha(1,1)}\right.$$
$$\times\left(t_{3n+2}, t_{3n+2/3}, \frac{14}{81}F_{1,3n} + \frac{28}{27}F_{1,3n+2}\right.$$
$$\left. - \frac{7}{27}F_{1,3n+2} + \frac{4}{81}F_{1,3n+3},\right)$$
$$+ 3K_{\alpha(1,1)}$$
$$\times\left(t_{3n+1}, t_{3n+2/3}, -\frac{5}{81}F_{1,3n} + \frac{20}{27}F_{1,3n+1}\right.$$
$$\left. + \frac{10}{27}F_{1,3n+2} - \frac{4}{81}F_{1,3n+3}\right)$$
$$\left. + K_{\alpha(1,1)}\left(t_{3n+2}, t_{3n+1}, F_{1,3n}\right)\right)$$
$$+ \frac{h}{8\Gamma(\alpha)}\left(K_{\alpha(1,2)}\left(t_{3n+2}, t_{3n}, F_{2,3n}\right) + 3K_{\alpha(1,2)}\right.$$
$$\times\left(t_{3n+1}, t_{3n+2/3}, \frac{40}{81}F_{2,3n} - \frac{20}{27}F_{2,3n+2}\right.$$
$$\left. - \frac{8}{27}F_{2,3n+2} + \frac{5}{81}F_{2,3n+3}\right)$$
$$+ 3K_{\alpha(1,2)}$$
$$\times\left(t_{3n+1}, t_{3n+2/3}, \frac{14}{81}F_{2,3n+1}\right.$$
$$\left. - \frac{7}{27}F_{2,3n+2} + \frac{4}{81}F_{2,3n+3}\right)$$
$$\left. + K_{\alpha(1,2)}\left(t_{3n+2}, t_{3n+2}, F_{2,3n+3}\right)\right). \tag{62}$$

Using the similar formulas, we can obtain $F_{1,3n+3}$ and $F_{2,3n+3}$. Now, from (58) to (62), we formed a system of six equations with normally six unknowns for $n = 1, 2, \ldots$.

In particular, we do have six simultaneous equations for each step. Our next concern in this work is to show the convergence analysis of Simpson's 3/8 rule for solving the Volterra fractional integral equations, and this will be presented in the next section.

5. Convergence Analysis of Simpson's 3/8 Rule for Solving the Volterra Fractional Integral Equations

This section is devoted to the discussion underpinning the convergence of the well-known Simpson's 3/8 rule to approximate the Volterra fractional equation of second kind. There are also other numerical methods to deal with these equations [32, 34–36]. To achieve this, and without loss of generality, we assume that the error in approximating the solution of the Volterra fractional equation of second kind via Simpson's 3/8 rule is $R_{1,3n+1}$ for the first approximation in (60); the rest can be obtained similarly; then,

$$\left|R_{1,3n+1}\right|$$
$$= \left|F_{1,3n+1} - f_1\left(x_{3n+1}\right)\right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left| h\sum_{k=0}^{3m}\psi_k K_{\alpha(1,1)}\right.$$
$$\times\left(t_{3n+1}, t_k, F_{1,k}\right)$$
$$+ h\sum_{k=0}^{3m}\psi_k K_{\alpha(1,2)}$$
$$\times\left(t_{3n+1}, t_k, F_{2,k}\right)$$
$$+ \frac{h}{8\Gamma(\alpha)}$$
$$\times\left(K_{\alpha(1,1)}\left(t_{3n+1}, t_{3n}, F_{1,3n}\right) + 3K_{\alpha(1,1)}\right.$$
$$\times\left(t_{3n+1}, t_{3n+1/3}, \frac{40}{81}F_{1,3n}\right.$$
$$- \frac{20}{27}F_{1,3n+1} - \frac{8}{27}F_{1,3n+2}$$
$$\left. + \frac{5}{81}F_{1,3n+3}\right)$$
$$+ 3K_{\alpha(1,1)}$$
$$\times\left(t_{3n+1}, t_{3n+2/3}, \frac{14}{81}F_{1,3n+1}\right.$$
$$\left. - \frac{7}{27}F_{1,3n+2} + \frac{4}{81}F_{1,3n+3}\right)$$
$$\left. + K_{\alpha(1,1)}\left(t_{3n+1}, t_{3n+1}, F_{1,3n}\right)\right)$$
$$+ \frac{h}{8\Gamma(\alpha)}$$
$$\times\left(K_{\alpha(1,2)}\left(t_{3n+1}, t_{3n}, F_{2,3n}\right) + 3K_{\alpha(1,2)}\right.$$

$$\times \left(t_{3n+1}, t_{3n+1/3}, \frac{40}{81}F_{2,3n} - \frac{20}{27}F_{2,3n+1} - \frac{8}{27}F_{2,3n+2} + \frac{5}{81}F_{2,3n+3}\right)$$
$$+ 3K_{\alpha(1,2)} \times \left(t_{3n+1}, t_{3n+2/3}, \frac{14}{81}F_{2,3n+1} - \frac{7}{27}F_{2,3n+2} + \frac{4}{81}F_{2,3n+3}\right)$$
$$+ \frac{h}{8\Gamma(\alpha)} \times \Bigg(K_{\alpha(1,2)}\left(t_{3n+2}, t_{3n}, F_{2,3n}\right) + 3K_{\alpha(1,2)} \times \left(t_{3n+1}, t_{3n+2/3}, \frac{40}{81}F_{2,3n} - \frac{20}{27}F_{2,3n+2} - \frac{8}{27}F_{2,3n+2} + \frac{5}{81}F_{2,3n+3}\right)$$
$$+ 3K_{\alpha(1,2)} \times \left(t_{3n+1}, t_{3n+2/3}, \frac{14}{81}F_{2,3n+1} - \frac{7}{27}F_{2,3n+2} + \frac{4}{81}F_{2,3n+3}\right) + K_{\alpha(1,2)}\left(t_{3n+2}, t_{3n+2}, F_{2,3n+3}\right)\Bigg)$$
$$+ \frac{h}{8}K_{1,2} \times \left(x_{3n+1}, x_{3n+1}, F_{2,3n+1}\right) - \sum_{l=1}^{2}\int_{0}^{3n+1} K_{1,l}\left(t_{3n+1}, \tau, f_l(\tau)\right) d\tau \Bigg|. \quad (63)$$

Now, employing the Lipchitz condition for the fractional kernel function, we can arrive at the following:

$$\left|R_{1,3n+1}\right| \le \frac{h}{\Gamma(\alpha)}a_1 \sum_{k=0}^{3n}\left|R_{1,k}\right| + \frac{h}{\Gamma(\alpha)}a_2 \times \sum_{k=0}^{3n}\left|R_{2,k}\right| + \frac{h}{\Gamma(\alpha)}a_3\left|R_{1,3n+1}\right| + \frac{h}{\Gamma(\alpha)}a_4\left|R_{2,3n+1}\right| + \frac{h}{\Gamma(\alpha)}a_5\left|R_{1,3n+2}\right|$$
$$+ \frac{h}{\Gamma(\alpha)}a_7\left|R_{1,3n+3}\right| + \frac{h}{\Gamma(\alpha)}a_8\left|R_{2,3n+3}\right| + \frac{1}{\Gamma(\alpha)}\left(\left|w_{1,3n+1}\right| + \left|w_{2,3n+1}\right| + \left|w_{1,3n+2}\right| + \left|w_{2,3n+2}\right|\right). \quad (64)$$

Here, it is important to recall that $w_{k,3n+1}$, $w_{k,3n+2}$ $(k = 1, 2)$ are the errors of integration rule. In addition, without loss of generality, we assume that

$$\left\|R_{k,i}\right\|_\infty = \max_{k=1,2}\max_{i=3n+1,3n+2,3n+3}\left|R_{k,i}\right| = \left|R_{1,3n+1}\right|; \quad (65)$$

thus, by letting $A = \max_m[|w_{1,m}|, |w_{2,m}|]$, consequently

$$\left\|R_{k,i}\right\|_\infty \le \frac{h}{\Gamma(\alpha)}a\sum_{n=0}^{3n}\left(\left|R_{1,n}\right|, \left|R_{2,n}\right|\right) + \frac{6h}{\Gamma(\alpha)}a'\left\|R_{k,i}\right\|_\infty + 4A. \quad (66)$$

Now by rearranging, we obtain the following inequality:

$$\left\|R_{k,i}\right\|_\infty \le \frac{(h/\Gamma(\alpha))\,a}{1-(6h/\Gamma(\alpha))\,a'} \times \sum_{n=0}^{3n}\left(\left|R_{1,n}\right|, \left|R_{2,n}\right|\right) + \frac{4A}{1-(6h/\Gamma(\alpha))\,a'}. \quad (67)$$

However, making use of the so-called Gronwall inequality, we arrive at

$$\left\|R_{k,i}\right\|_\infty \le \frac{4A}{1-(6h/\Gamma(\alpha))\,a'}\operatorname{Exp}\left[\frac{ha}{1-(6h/\Gamma(\alpha))\,a'}x_n\right]. \quad (68)$$

For the fractional kernel function K_α and F with at least fourth-order derivatives, we have $A = O(h^4)$ and then $\|R_{k,i}\|_\infty = O(h^4)$. Therefore, we can state the following theorem.

Theorem 8. *Simpson's* 3/8 *rule for solving the Volterra fractional integral equations of second kind is convergent and its order of convergence is at least four.*

6. Numerical Solutions

In this section, we present some numerical examples of solutions of the Volterra fractional integral equations via the so-called Simpson's 3/8 rule.

Example 9. Let us consider the following Volterra fractional integral equation for which the order is half:

$$f(x) = 2\sqrt{x} - \int_0^x \frac{f(t)}{\sqrt{x-t}}dx = x, \qquad 0 \le t < x < 1. \quad (69)$$

The exact solution of this equation is given as

$$f(x) = 1 - \exp[\pi x]\,\mathrm{er}\ f\left(\sqrt{x\pi}\right). \quad (70)$$

Table 1: Numerical errors corresponding to the value of h.

x	h				
	0.001	0.01	0.02 0.03	0.04	0.05
	Error (1) $\times 10^{-5}$	Error (2) $\times 10^{-5}$	Error (3) $\times 10^{-5}$	Error (5) $\times 10^{-5}$	Error (6) $\times 10^{-5}$
0	0.01	0.02	0.03	0.04	0.05
0.1	0.00216833	0.00433666	0.00650499	0.00867332	0.0108417
0.2	0.003822808	0.00765615	0.0114842	0.0153123	0.0191404
0.3	0.0113064	0.0226128	0.339192	0.0452256	0.0565321
0.4	0.0211693	0.0423387	0.063508	0.0846773	0.105847
0.5	0.0344335	0.0688669	0.1033	0.137734	0.172167
0.6	0.0524239	0.104848	0.157272	0.209695	0.262119
0.7	0.0769263	0.157272	0.230779	0.307705	0.384631
0.8	0.110371	0.220743	0.331114	0.441485	0.551856
0.9	0.156078	0.312156	0.468234	0.624312	0.780391
1	0.218586	0.437173	0.655759	0.874345	1.09293

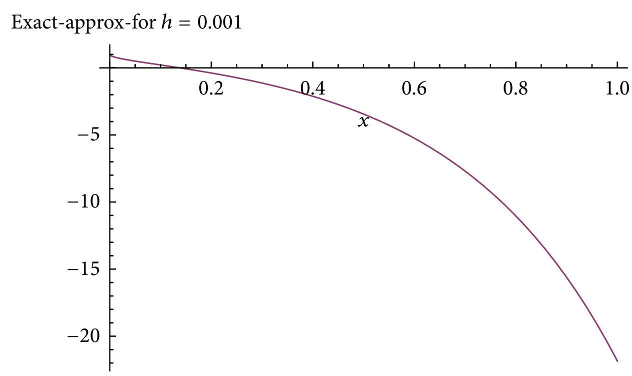

Figure 1: Comparison of exact solution and approximate solution for $h = 0.001$.

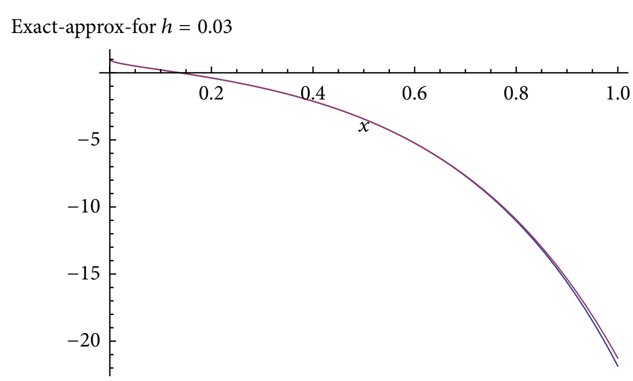

Figure 2: Comparison of exact solution and the approximate solution for $h = 0.03$.

Using the Simpson 3/8, we obtained the following numerical values indicated in Table 1.

The approximate solutions have been depicted in Figures 1, 2, and 3. Figure 1 shows the comparison of the exact and approximate solutions for $h = 0.001$, Figure 2 shows the comparison for $h = 0.03$, and Figure 3 shows the comparison for $h = 0.04$. The numerical solution shows that the method is very efficient and accurate.

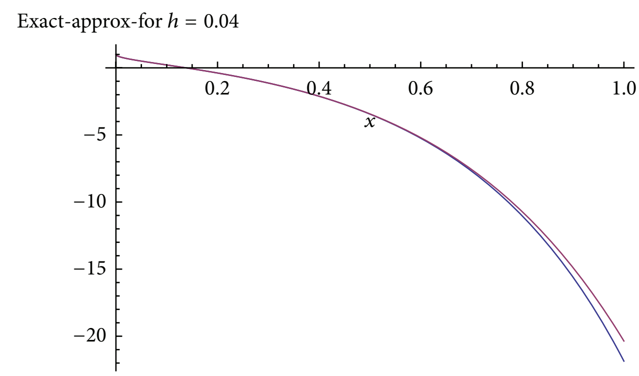

Figure 3: Comparison of the exact and approximate solution for $h = 0.04$.

7. Conclusion

The existence and the uniqueness of the Volterra fractional integral equations second kind were examined in this work. The numerical method called the Simpson 3/8 rule method was used to present the numerical solution of these equations. We presented the convergence analysis of this numerical scheme.

Conflict of Interests

The authors declare no conflict of interests.

Author's Contribution

Abdon Atangana wrote the first draft and Necdet Bildik corrected the revised form; the both authors read the revised and submitted the paper.

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