

## Research Article

# Bifurcation Analysis and Spatiotemporal Patterns of Nonlinear Oscillations in a Ring Lattice of Identical Neurons with Delayed Coupling

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We investigate the dynamics of a delayed neural network model consisting of  $n$  identical neurons. We first analyze stability of the zero solution and then study the effect of time delay on the dynamics of the system. We also investigate the steady state bifurcations and their stability. The direction and stability of the Hopf bifurcation and the pitchfork bifurcation are analyzed by using the derived normal forms on center manifolds. Then, the spatiotemporal patterns of bifurcating periodic solutions are investigated by using the symmetric bifurcation theory, Lie group theory and  $S^1$ -equivariant degree theory. Finally, two neural network models with four or seven neurons are used to verify our theoretical results.

## 1. Introduction

Growing biological evidence suggests that propagation delay in axons and dendrites may play a key role in the processing of sensory information by the brain. For instance, spatiotemporal patterns of synchronous oscillations in  $\gamma$  and  $\theta$  frequency bands emerge in the hippocampus when a rat goes through previously visited places. These oscillations and their interaction form a coding scheme that is used to readout from long-time memory (see, e.g., [1]). Although there are many experimental results pointing to the  $\gamma$  phase-locked firing of pyramidal cells (see, e.g., [2]), the mechanisms underlying this phenomenon are fairly unknown. The hippocampus has a laminar stratified structure, where each pyramidal cell receives and integrates a large amount of spikes arriving at different time instants to different parts of the cell. Then their active integration by the cell, to produce an output spike, depends on the relative time delays of the incoming spikes. Thus the cell output is conditioned by delays occurring in incoming electric wave, which justifies the need of mathematical modeling of these complex phenomena [3].

On the other hand, artificial neural networks including delays have been known to be useful for mimicking various neuroprocesses like in the image processing (see, e.g., [4]). Their further expansion and exploitation are limited by the added complexity of the mathematical analysis brought by the delay (ordinary or partial differential equations become functional differential equations thus implying an infinite number of degrees of freedom). There are, however, some advantages. For instance, a harmonic oscillator augmented with time delay shows stable “robust” oscillations [5]. The “fragile” character of linear oscillations and linear waves when unfolded in space precludes their utility in reliably carrying information. An interesting point, however, is that a harmonic oscillator augmented with white noise permits transferring the latter into a colored noise thus creating a memory-like stochastic process and so a kind of delay process. In [6] this idea has been used to model quasiharmonic oscillations observed in inferior olive. The spiking propagation process in the brain comes indeed from robust oscillations as known by many authors. This fact permits modeling the dynamics using harmonic oscillators with appropriate nonlinearity which is like in the case of a van der Pol-Bonhöffer oscillator and

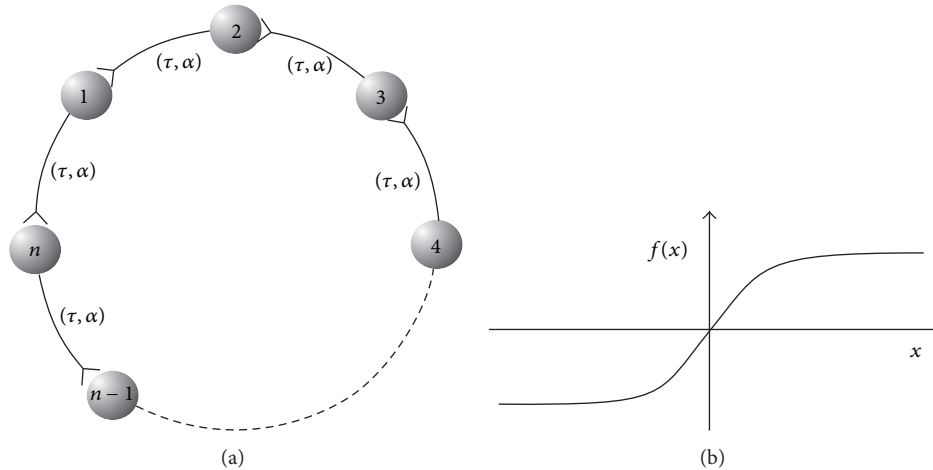


FIGURE 1: Lattice ring. (a) Network architecture with  $n$  units unidirectionally coupled. Each unit receives a delayed output from the preceding unit. (b) Transfer (synaptic) function.

the Fitzhugh-Nagumo excitability dynamics underlying the Hodgkin-Huxley equation. An alternative to van der Pol's idea was earlier suggested by Lord Ragleigh who also proposed augmenting the harmonic oscillator with suitable active friction thus allowing maintained oscillations. In both cases we have systems where there is an appropriate input-output energy balance and thus driven-dissipative systems.

Focusing on robots or neuron inspired artifacts, the use of ring lattice models has shown potential in different applications, for example, from electron transport to the dynamical decentralized gait control in robotics [7]. In the latter case movement of each robot leg is associated (coupled) with the behavior of the corresponding unit in the ring. Then standing waves excited in the ring can be used to reproduce the leg movement without the need of a computer, unlike what happens in nature, which uses central pattern generators (CPG) designing artificial locomotion for this purpose [8]. It has been shown in [9] for a hexapod robot that a ring of six units is able to reproduce metachronal, caterpillar, and tripod gaits. However, ripple gait, widely employed by insects for medium speed movement, is difficult to mimic. We foresee that the use of delays may help in CPGs to control the robot walking behavior.

Since Marcus and Westervelt [10] incorporated a single time delay into the connection term of Hopfield's model and observed sustained oscillations resulting from this time delay, there has been a growing interest to the dynamical properties of neural networks when the time delay cannot be neglected. It is known that delay can significantly alter the dynamics of neural networks [10–12], for example, leading to instability of a delay-free stable solution [13]. There have been lots of researches on the neural networks with delay (see, e.g., [14–18] and references therein).

A ring with identical elements often has a kind of topological symmetry. Considering these symmetries helps in the analysis of wave solutions in a ring lattice. Golubitsky et al. [19] have shown that systems with symmetry can exhibit different oscillatory patterns, which are predictable

based on the theory of equivariant bifurcations. Later, in a series of papers, Wu and coworkers [20–22] have extended the theory of equivariant Hopf bifurcations to functional differential equations. These theoretical advances have led to a vast literature on the mechanisms of spatiotemporal activity in neural networks with symmetry and delays (see, e.g., [22–35]). The majority of these studies have focused on a ring structure with bidirectional couplings between the neighboring elements. From the view point of group theory, the symmetry in a ring with such bidirectional coupling if there is a reflection symmetry is given by a dihedral group  $D_n$  (the symmetry of a regular  $n$ -gon:  $n$  is the number of units). Another type of symmetry in a ring is a cyclic group  $Z_n$  (the symmetry of a direct  $n$ -gon) if one direction is preferred; that is, the coupling goes in one direction. The latter type of symmetry is more natural for the neural networks. In 1994, Baldi and Atiya [11] proposed a neural network consisting of  $n$  neurons connected in a ring by unidirectional couplings with delays (Figure 1(a)):

$$\frac{dx_i}{dt} = -x_i(t) + \alpha_{i+1} f_{i+1}(x_{i+1}(t - \tau_{i+1})), \quad i \pmod{n}, \quad (1)$$

where  $x_i$  is the state of the  $i$ th unit (related to the membrane potential),  $f$  is the transfer function, which describes chemical interaction between pre- and postsynaptic neurons, and  $\alpha$  accounts for the coupling strength. For positive or negative  $\alpha$  we can speak of *excitatory* or *inhibitory* coupling, respectively. The nonnegative constants  $\tau_i$  represent the transmission delays between neighboring neurons.

Baldi and Atiya [11] investigated the effects of delays on the ring dynamics, in particular on the oscillatory properties, showing numerically that when the ring has *odd* number of units and the coupling is *inhibitory* ( $\alpha < 0$ ), the origin (the zero solution) is the only fixed point, and if it is unstable, the dynamics of the network always converges to a stable limit cycle. They also argued that, for *even*  $n$  and  $\alpha < 0$ , there are no stable limit cycles. But a strict mathematical proof is still lacking. Model (1) has been studied by several researchers.

In [36, 37], the authors studied a ring with two and three units and obtained sufficient conditions for the asymptotic stability of the zero equilibrium and for the existence of Hopf bifurcations. Later Wei and Li investigated the global existence of multiple periodic solutions in a ring of three units [38]. The stability analysis and local Hopf bifurcation of the zero solution in a ring of four units have been given in [39]. In the above-mentioned studies the authors considered nonidentical elements, where the theory of equivariant bifurcations is not applicable. Besides, the properties of nonzero equilibria have not been addressed. To fill this gap Guo [40] studied a ring (1) consisting of four identical neurons coupled with the same delay. He provided conditions on the linear stability of zero solution, spatiotemporal patterns of nonlinear oscillations, albeit restricting consideration to the case of *excitatory* coupling ( $\alpha > 1$ ). The direction and stability of bifurcating periodic solutions and steady state bifurcations were not considered. Thus there is no result on the case of the unidirectional ring consisting of arbitrary number of units. In particular, little is known about the patterns of nonlinear oscillations and their stability even in a ring of more than four identical neurons. The steady state bifurcation patterns are also lacking. This paper is a generalization of the previous investigation for the case of the unidirectional ring consisting of two or three neurons with discrete delay in [37]. We obtain not only the stability and delay-induced Hopf bifurcation but also the spatiotemporal patterns of periodic oscillations. We would also like to mention that the rings of neural networks with unidirectional coupling and distributed delays have recently been considered in [41–43]. The paper also can be considered the complement of the results in [41–43] for the case of distributed delays.

In this paper, we consider model (1) consisting of  $n$  (arbitrary) identical neurons; that is,  $\alpha_i \equiv \alpha$ ,  $f_i \equiv f$  coupled with the same delay  $\tau_i \equiv \tau$ . We assume that the synaptic coupling is described by a sufficiently smooth sigmoidal function (Figure 1(b)). A widely used example in the literature is  $f(x) = \tanh(x)$ . However, here we only assume  $f(0) = 0$ ,  $f \in C^1(\mathbb{R})$  for the stability analysis, and we require  $f \in C^3(\mathbb{R})$ ,  $f'(0)f'''(0) \neq 0$ ,  $f(0) = f''(0) = 0$  for the bifurcation analysis. We investigate the global and local stability, Hopf bifurcations, pitchfork bifurcations, and spatiotemporal patterns of bifurcating periodic solutions. We also derive normal forms on center manifolds and determine direction and stability of the Hopf bifurcations.

## 2. Background

For convenience, we recall results for the model (1) for the case of a ring composed of  $n$  identical neurons

$$\frac{dx_i}{dt} = -x_i(t) + \alpha f(x_{i+1}(t - \tau)), \quad i \pmod n. \quad (2)$$

System (2) admits the zero as an equilibrium solution  $x_0 = (0, 0, \dots, 0)$ , which we will refer to as the rest state.

Every initial state  $\varphi \in C$  uniquely defines a solution  $x(\varphi, t)$  of system (2) for all  $t \geq -\tau$ . Due to the uniqueness of the Cauchy initial value problem of system (2), every initial state

$\varphi$  satisfying  $\varphi_1 = \varphi_2 = \dots = \varphi_n$  (called synchronous phase point) gives a synchronous solution  $x(\varphi, t)$ , that is,  $x_1(\varphi, t) = x_2(\varphi, t) = \dots = x_n(\varphi, t)$ , and the solution  $x_i(\varphi, t)$  of system (2) can be characterized by the scalar delay differential equation

$$\frac{dx}{dt} = -x(t) + \alpha f(x(t - \tau)). \quad (3)$$

System (3) has been studied by many researchers. For example, Mallet-Paret and Nussbaum [44] have obtained some results on global continuation and asymptotical behavior of periodic solutions, and Krisztin et al. [45] and Krisztin and Walther [46] give a complete description of the global attractor of (3) as a three-dimensional smooth solid spindle when  $\tau$  is in a certain range. A few results from Mallet-Paret and Nussbaum [44] are needed for our subsequent work.

**Lemma 1.** *Assume that  $f$  is a smooth sigmoid-like function,  $x_*$  is a steady state of (3), and  $\tau_{0,j}$  is defined by (13) with  $j = 0, 1, \dots$*

- (i) *If  $|\alpha f'(x_*)| < 1$ , then  $x_*$  is asymptotically stable for any  $\tau \geq 0$ .*
- (ii) *If  $\alpha f'(x_*) < -1$ , then  $x_*$  is asymptotically stable for  $\tau \in [0, \tau_{0,0})$  and unstable for  $\tau > \tau_{0,0}$ .*
- (iii) *If  $\alpha f'(x_*) > 1$ , then  $x_*$  is unstable for any  $\tau \geq 0$ .*
- (iv) *If  $|\alpha f'(x_*)| > 1$ , then (3) undergoes a Hopf bifurcation at  $x_*$  when  $\tau = \tau_{0,j}$ .*
- (v) *If  $|\alpha f'(x_*)| > 1$ , then (3) has at least  $(j + 1)$  periodic solution for  $\alpha < -1$  with  $\tau > \tau_{0,j}$  or  $\alpha > 1$  with  $\tau > \tau_{0,j+1}$ .*

## 3. Stability and Hopf Bifurcations of the Rest State

In this section, we assume

$$(H1) \quad f \in C^1(\mathbb{R}), \quad f(0) = 0, \quad f'(0) = 1.$$

Let us first study the global stability of the rest solution  $x_0$ .

**Lemma 2.** *If  $|\alpha| < 1$ , then the solution  $x_0$  of system (2) is globally asymptotically stable for any  $\tau \geq 0$ .*

The proof is given in Appendix A.

Lemma 2 shows that the dynamics of the network (2) is simple when  $|\alpha| < 1$ . Any perturbation of the network state decays in time. Therefore, in the remainder of this paper, we will investigate the dynamics of (2) with  $|\alpha| \geq 1$ .

The linearization of system (2) around the rest state  $x_0$  is given by

$$\frac{dx_i}{dt} = -x_i(t) + \alpha x_{i+1}(t - \tau), \quad i \pmod n. \quad (4)$$

The characteristic matrix associated with system (4) is

$$M_n(\tau, \lambda) = (\lambda + 1)I - \alpha e^{-\lambda\tau}M, \quad (5)$$

where  $I$  is  $n \times n$  identity matrix and

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}. \tag{6}$$

Let  $\chi = e^{(2\pi/n)i}$ ,  $v_k = (1, \chi^k, \chi^{2k}, \dots, \chi^{(n-1)k})^T$ ,  $k = 0, 1, \dots, n-1$ . Then, we get

$$M_n(\tau, \lambda) v_k = (\lambda + 1 - \alpha \chi^k e^{-\lambda \tau}) v_k. \tag{7}$$

It follows that the characteristic equation is

$$\det M_n(\tau, \lambda) = \prod_{k=0}^{n-1} \Delta_k(\tau, \lambda) = 0, \tag{8}$$

where

$$\Delta_k(\tau, \lambda) = \lambda + 1 - \alpha e^{(2k\pi/n)i} e^{-\lambda \tau}. \tag{9}$$

An equilibrium solution of a delay differential equation is locally asymptotically stable if all roots of the corresponding characteristic equation have negative real parts and unstable if at least one root has positive real part (see, e.g., [47]). Thus, in order to investigate the stability of the rest state, it is necessary to study the distribution of roots of the characteristic equation (8) and when they cross the imaginary axis.

It is straightforward to see that if  $|\alpha| > 1$ , then  $\lambda = 0$  is not a root of  $\Delta_k(\tau, \lambda) = 0$  for any  $k \in \{0, 1, 2, \dots, n-1\}$  and  $\tau \geq 0$ . Therefore, in what follows we assume that two complex roots cross the imaginary axis at some positive critical values of time delay; that is,  $\lambda(\tau) = iv$ . Then from (8) and (9) it follows that  $v$  satisfies

$$iv + 1 - \alpha e^{(2k\pi/n - v\tau)i} = 0, \tag{10}$$

which gives

$$\begin{aligned} 1 &= \alpha \cos\left(\frac{2k\pi}{n} - v\tau\right), \\ v &= \alpha \sin\left(\frac{2k\pi}{n} - v\tau\right). \end{aligned} \tag{11}$$

Then for  $|\alpha| > 1$ ,  $v = \pm\omega_0$  with  $\omega_0 = \sqrt{\alpha^2 - 1}$ . Let us first consider  $\alpha < -1$ . Set

$$\begin{aligned} \tilde{\tau}_{k,j}^+ &= \frac{1}{\omega_0} \left\{ \frac{2k\pi}{n} + \arccos\frac{1}{\alpha} + (j-1)2\pi \right\}, \\ \tilde{\tau}_{k,j}^- &= \frac{1}{\omega_0} \left\{ -\frac{2k\pi}{n} + \arccos\frac{1}{\alpha} + 2j\pi \right\}, \end{aligned} \tag{12}$$

where  $k = 0, 1, \dots, n-1$  and  $j \in \{0, 1, \dots\}$ . It is easy to verify that  $\Delta_k(\tau, \lambda)$  has a purely imaginary root  $\omega_0 i$  for  $\tau = \tilde{\tau}_{k,j}^+$  and  $-\omega_0 i$  for  $\tau = \tilde{\tau}_{k,j}^-$ . Since  $\tilde{\tau}_{k,j}^- = \tilde{\tau}_{n-k,j}^+$ , one can conclude that the

characteristic equation (8) with  $\alpha < -1$  has a pair of simple purely imaginary roots  $\pm i\omega_0$  at  $\tau = \tilde{\tau}_{k,j}^-$ . Proceeding for the case  $\alpha > 1$  exactly as done above we can get similar results. In fact, we can set

$$\tau_{k,j} = \begin{cases} \frac{1}{\omega_0} \left\{ -\frac{2k\pi}{n} + \arccos\frac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha < -1, \\ \frac{1}{\omega_0} \left\{ \frac{2k\pi}{n} - \arccos\frac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha > 1, \end{cases} \tag{13}$$

with  $k = 0, 1, \dots, n-1$  and  $j \in \{0, 1, \dots\}$  such that  $\tau_{k,j} > 0$ . Then the characteristic equation (8) has a pair of simple purely imaginary roots  $\pm i\omega_0$  at  $\tau = \tau_{k,j}$ .

Differentiating  $\Delta_k(\tau, \lambda)$  with respect to  $\lambda$  we obtain that

$$\begin{aligned} \left. \frac{\partial}{\partial \lambda} \Delta_k(\tau, \lambda) \right|_{\lambda=i\omega_0, \tau=\tau_{k,j}} &= 1 + \tau_{k,j} \alpha e^{(2k\pi/n)i} e^{-\lambda \tau} \Big|_{\lambda=i\omega_0, \tau=\tau_{k,j}} \\ &= 1 + \tau_{k,j} (1 + i\omega_0) \neq 0. \end{aligned} \tag{14}$$

Thus, the implicit function theorem implies that there exist  $\delta > 0$  and a smooth curve  $\lambda : (\tau_{k,j} - \delta, \tau_{k,j} + \delta) \rightarrow \mathbb{C}$  such that  $\Delta_k(\tau, \lambda(\tau)) = 0$  and  $\lambda(\tau_{k,j}) = i\omega_0$ . Differentiating  $\Delta_k(\tau, \lambda)$  with respect to  $\tau$  after some algebraic calculus we get

$$\operatorname{Re} \left\{ \frac{d\lambda}{d\tau} \right\}_{\tau=\tau_{k,j}} = \frac{\omega_0^2}{\tau_{k,j}^2 \omega_0^2 + (1 + \tau_{k,j})^2} > 0. \tag{15}$$

Now we can state the following results about the distribution of roots of (8).

**Lemma 3.** Assume that  $\tau_{k,j}$  is defined by (13) with  $k = 0, 1, \dots, n-1$  and  $j \in \{0, 1, \dots\}$  such that  $\tau_{k,j} > 0$ .

- (I) If  $n$  is even, then when  $|\alpha| > 1$  the characteristic equation (8) has at least one root with positive real part for all  $\tau \geq 0$ .
- (II) If  $n$  is odd, then
  - (i) when  $\alpha < \sec((n-1)\pi/n)$  or  $\alpha > 1$  the characteristic equation (8) has at least one root with positive real part for all  $\tau \geq 0$ ;
  - (ii) when  $\sec((n-1)\pi/n) < \alpha < -1$  all roots of the characteristic equation (8) have negative real parts for  $\tau \in [0, \tau_{(n-1)/2,0})$  and all roots of the characteristic equation (8) except  $\pm i\omega_0$  have negative real parts at  $\tau = \tau_{(n-1)/2,0}$ , but the characteristic equation (8) has at least two roots with positive real parts for  $\tau > \tau_{(n-1)/2,0}$ .
- (III) If  $|\alpha| > 1$ , then when  $\tau = \tau_{k,j}$ , the characteristic equation (8) has a pair of simple purely imaginary roots  $\pm i\omega_0$ .

The proof is given in Appendix B.

From Lemma 3, the transversality condition (15), and the standard Hopf bifurcation theorem of delay differential equations (see, e.g., [47]), we can state the following theorem.



**Theorem 4.** Assume that  $|\alpha| > 1$  and  $\tau_{k,j}$  is defined by (13).

(I) If  $n$  is even, then the rest state  $x_0$  of system (2) is unstable for all  $\tau \geq 0$ .

(II) If  $n$  is odd, then

(i) when  $\alpha < \sec((n-1)\pi/n)$  or  $\alpha > 1$  the rest state  $x_0$  of system (2) is unstable for all  $\tau \geq 0$ ;

(ii) when  $\sec((n-1)\pi/n) < \alpha < -1$  the rest state  $x_0$  of system (2) is asymptotically stable for  $\tau \in [0, \tau_{(n-1)/2,0})$  and unstable for  $\tau > \tau_{(n-1)/2,0}$ .

(III) The system (2) undergoes a Hopf bifurcation at  $\tau = \tau_{k,j}$ . That is, there exists a unique branch of periodic solutions  $x^{(k,j)}(t, \tau)$  with period  $p^{(k,j)}(\tau)$ , and  $x^{(k,j)}(t, \tau) \rightarrow 0, p^{(k,j)}(\tau) \rightarrow 2\pi/\omega_0$  as  $\tau \rightarrow \tau_{k,j}$ .

**Theorem 5.** (I) Assume that  $\alpha = 1$ . Then for all  $\tau \geq 0$  the characteristic equation (8) has one simple root  $\lambda = 0$ , and other roots have negative real parts.

(II) Assuming that  $\alpha = -1$  and  $n$  is even, then, for all  $\tau \geq 0$ , the characteristic equation (8) has one simple root  $\lambda = 0$ , and other roots have negative real parts.

The proof is given in Appendix C.

#### 4. Stability and Steady State Bifurcations

In this section, we study the properties of the equilibrium of the system (2) and assume

$$(H2) f \in C^2(\mathbb{R}), f(0) = 0, f'(0) = 1, f'(x) > 0 \quad \forall x \in \mathbb{R}, xf''(x) < 0 \quad \forall x \neq 0, \lim_{x \rightarrow \pm\infty} |f(x)| < +\infty.$$

From system (2), we immediately have the following lemma.

**Lemma 6.** If  $(x_1, x_2, \dots, x_n)$  is an equilibrium of system (2) and there exists one component  $x_i = 0$ , then it must be the zero equilibrium  $x_0 = (0, 0, \dots, 0)$ .

**Theorem 7.** Assuming that  $\alpha > 1$ , every equilibrium of system (2) must be synchronous and system (2) has exactly three synchronous equilibria: the zero equilibrium  $x_0$ , the negative equilibrium  $x_- = (u_-, u_-, \dots, u_-)$ , and the positive equilibrium  $x_+ = (u_+, u_+, \dots, u_+)$ , where  $u_{\pm}$  are nonzero solutions of the equation  $u = \alpha f(u)$ . Moreover, these two nonzero equilibria  $x_-$  and  $x_+$  are both asymptotically stable for all  $\tau \geq 0$ .

The proof is given in Appendix D.

**Theorem 8.** Assuming that  $n$  is odd and  $\alpha \leq 1$ , system (2) has just one zero equilibrium  $x_0$ .

The proof is given in Appendix E.

**Theorem 9.** If  $n$  is even and  $\alpha < -1$ , then system (2) has exactly three equilibria:  $x_0, x_1^* = (u_+, u_-, \dots, u_+, u_-), x_2^* = (u_-, u_+, \dots, u_-, u_+)$ , where  $u_{\pm}$  are the nonzero solutions of  $u = \alpha f(\alpha f(u))$ . Moreover, these two nonzero equilibria are both asymptotically stable for all  $\tau \geq 0$ .

The proof is given in Appendix F. From Theorems 7, 8, and 9, we have the following.

**Theorem 10.** System (2) undergoes a pitchfork bifurcation at  $|\alpha| = 1$  when  $n$  is even and at  $\alpha = 1$  when  $n$  is odd.

*Remark 11.* From Theorems 7, 8, and 9, we also know that the size of the network does not affect the position of the steady states (see Figures 2 and 3 obtained from numerical simulations) and their stability.

#### 5. Center Manifold Reduction and Normal Forms

In this section, we will apply the method of Faria and Magalhães [48, 49] to obtain normal forms on center manifold. After that, we will study the properties of Hopf and steady state bifurcations. Without loss of generality, we assume

$$(H3) f \in C^3(\mathbb{R}), f(0) = f''(0) = 0, f'(0) = 1, f'''(0) \neq 0.$$

Firstly, we introduce a new time scale  $t \mapsto t/\tau$  to normalize the delay and give universalities to our analysis. Then system (2) can be written as

$$\dot{z}(t) = F(z_t, \tau), \tag{16}$$

in the phase space  $C = C([-1, 0], \mathbb{R}^n)$ . For  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in C$ , we have

$$(F(\varphi, \tau))_i = -\tau\varphi_i(0) + \tau\alpha f(\varphi_{i+1}(-1)), \quad i \pmod n. \tag{17}$$

Under the assumption (H3), we can expand  $f$  in the Taylor series

$$f(z) = z + \gamma z^3 + \text{h.o.t.}, \tag{18}$$

where  $\gamma = (1/3!)f'''(0)$  and h.o.t. stands for the higher order terms.

*5.1. Properties of Hopf Bifurcations:*  $|\alpha| > 1$ . From Lemma 3 and Theorem 4, we know that for  $|\alpha| > 1$  the characteristic equation (8) has a pair of simple purely imaginary roots  $\pm i\omega_0$  and system (2) undergoes a Hopf bifurcation at the critical delay values  $\tau_{k,j}$ . In this subsection, we will determine the direction and stability of Hopf bifurcations by calculating explicitly the normal form of system (2) on the associated two-dimensional (2D) center manifold. For a Hopf bifurcation, the normal form on the center manifold is given in polar coordinates  $\rho, \xi$  by

$$\dot{\rho} = K_1\mu\rho + K_2\rho^3 + O(\mu^2\rho + |(\rho, \mu)|^4), \tag{19}$$

$$\dot{\xi} = -\omega_* + O(|(\rho, \mu)|).$$

Here,  $\mu = \tau - \tau_{k,j}$  is the bifurcation parameter,  $\omega_* = \omega_0\tau_{k,j}$ , and the coefficients  $K_1$  and  $K_2$  are real numbers. The qualitative behavior of the asymptotic solutions of (16) (or equivalently of (2)) is the same as the behavior of solutions

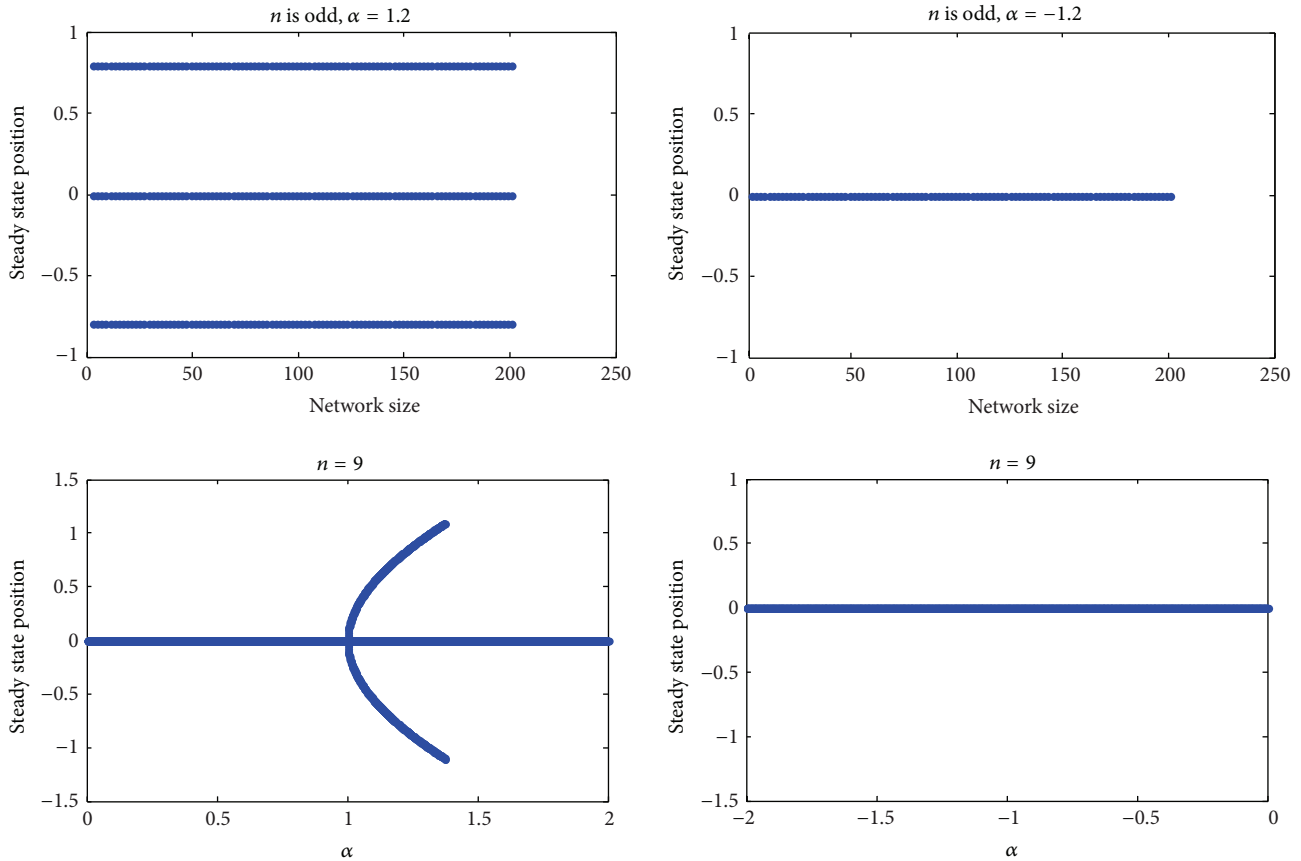


FIGURE 2: Steady state pitchfork bifurcation of system (2) for  $n$  odd.

of (19), which, in turn, only depends on the signs of the two coefficients  $K_1$  and  $K_2$ . The sign of the product  $K_1K_2$  determines the direction of the bifurcation (supercritical if  $K_1K_2 < 0$  and subcritical if  $K_1K_2 > 0$ ), and the sign of  $K_2$  determines the stability of the nontrivial periodic orbits (stable if  $K_2 < 0$  and unstable if  $K_2 > 0$ ) (see, e.g., [50]).

The derivation of the normal form coefficients  $K_1$  and  $K_2$  are given in Appendix G. According to Appendix G, we have

$$\begin{aligned}
 K_1 &= \frac{\omega_0^2 \tau_{k,j}}{(1 + \tau_{k,j})^2 + (\omega_0 \tau_{k,j})} > 0, \\
 K_2 &= 3\gamma \tau_{k,j} \operatorname{Re} \{(\bar{a})^{-1} (1 + i\omega_0)\} \\
 &= \frac{1}{2} f'''(0) \tau_{k,j} \frac{1 + \tau_{k,j} + \omega_0^2 \tau_{k,j}}{(1 + \tau^*)^2 + (\omega_0 \tau_{k,j})^2}.
 \end{aligned}
 \tag{20}$$

Consequently, we have the following theorem.

**Theorem 12.** *If  $f'''(0) < 0$  (resp.,  $f'''(0) > 0$ ), then the Hopf bifurcations occurring on the center manifold of system (2) at  $x = 0, \tau = \tau_{k,j}$  are supercritical (resp., subcritical), with nontrivial periodic stable (resp., unstable) orbits on the center manifold.*

*Remark 13.* From Lemma 3, we know that the characteristic equation (8) has at least one root with positive real part if one of the following conditions is satisfied:

- (i)  $n$  is even and  $|\alpha| > 1$ ;
- (ii)  $n$  is odd and either  $\alpha < \sec((n - 1)\pi/n)$  or  $\alpha > 1$ ;
- (iii)  $n$  is odd,  $\tau > \tau_{(n-1)/2,0}$ , and  $\sec((n - 1)\pi/n) < \alpha < -1$ .

Therefore, if one of the above three conditions is satisfied, then the periodic solutions bifurcating from the rest state at the critical value of  $\tau$  must be unstable in the whole phase space although they are stable on the center manifold.

However, when  $n$  is odd and  $\sec((n - 1)\pi/n) < \alpha < -1$ , all roots of the characteristic equation (8) except  $\pm i\omega_0$  have negative real parts at  $\tau = \tau_{(n-1)/2,0}$ . Thus in this case the stability of periodic solutions bifurcating at  $\tau = \tau_{(n-1)/2,0}$  on the center manifold is equivalent to that of periodic solutions in the whole phase space.

**5.2. The Steady State, Pitchfork Bifurcation:  $|\alpha| = 1$ .** We first consider the case when  $n$  is even and  $\alpha = -1$ . From Theorem 5, we know that the characteristic equation (8) has a simple root  $\lambda = 0$  for all  $\tau \geq 0$ . To study the bifurcation in

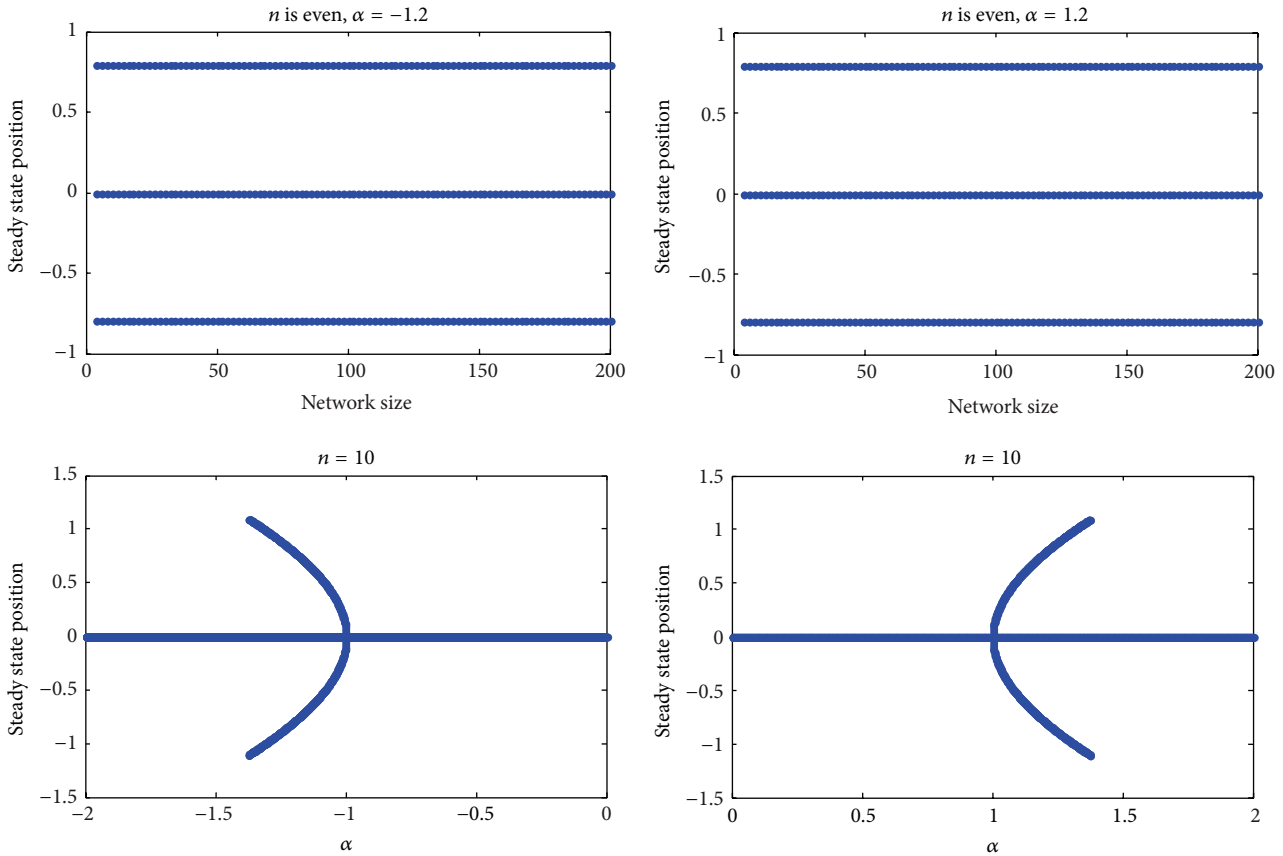


FIGURE 3: Steady state pitchfork bifurcation of system (2) for  $n$  even.

detail we introduce a new bifurcation parameter  $\alpha = -1 + \nu$ . Then the normal form (H.4) becomes (see Appendix H)

$$\dot{x} = -\frac{\tau}{1+\tau}\nu x + \frac{\gamma\tau}{1+\tau}x^3 + \dots \quad (21)$$

This normal form, together with the results in [51], permits the following theorem to hold.

**Theorem 14.** *Suppose that  $n$  is even and  $\alpha = -1$ . Then the rest state  $x_0$  of system (2) is stable if  $f'''(0) < 0$  and unstable if  $f'''(0) > 0$  for any  $\tau \geq 0$ .*

Next if  $\alpha = 1$ , letting  $\alpha = 1 + \nu$ , one can obtain

$$\begin{aligned} \Phi(\theta) &= (1, 1, \dots, 1)^T, \quad -1 \leq \theta \leq 0, \\ \Psi(s) &= \frac{1}{n(1+\tau)}(1, 1, \dots, 1), \quad 0 \leq s \leq 1, \end{aligned} \quad (22)$$

and then get the same normal form (21). So, we can obtain the following theorem.

**Theorem 15.** *The rest state  $x_0$  of system (2) is stable if  $f'''(0) < 0$  and unstable if  $f'''(0) > 0$  for any  $\tau \geq 0$  and  $\alpha = 1$ .*

Now we can explicitly determine the stability of the zero solution of the system (2) on the  $(\alpha, \tau)$  plane. According to the results stated in Theorems 4, 5, 14, and 15, the stable regions of

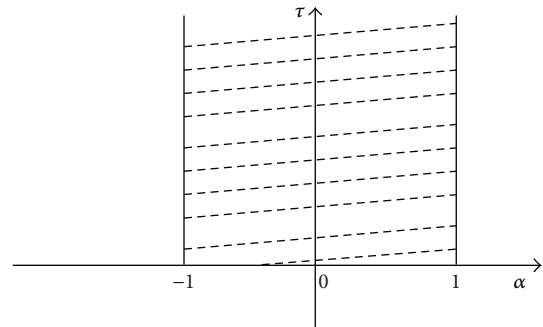


FIGURE 4: Stable region (shaded area) of the rest state  $x_0$  of system (2) for  $n$  even.

the rest state of system (2) with  $f'''(0) < 0$  can be illustrated graphically in Figures 4 and 5.

## 6. Spatiotemporal Patterns of Bifurcating Periodic Solutions

Earlier we have described the Hopf bifurcation at the critical value  $\tau_{k,j}$  leading to a family of periodic solutions. In this section, we will investigate the spatiotemporal patterns of

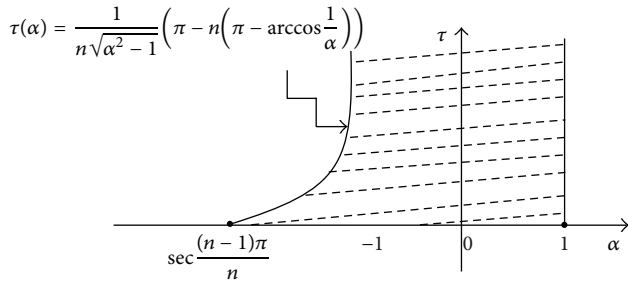


FIGURE 5: Stable region (shaded area) of the rest state  $x_0$  of system (2) for  $n$  odd.

these bifurcating periodic solutions and refer to [22] for explanations of notations involved.

Let us first introduce notations from the theory of compact groups.

- (i)  $Z_n$  is the cyclic group of order  $n$ , which corresponds to rotations of  $2\pi/n$ . Denoting the generator of this group by  $\rho$ , its action on  $\mathbb{R}^n$  is given by

$$(\rho x_i) = x_{i+1}, \quad \forall i \pmod n, \quad x \in \mathbb{R}^n. \quad (23)$$

- (ii) Let  $G : C \rightarrow \mathbb{R}^n$  and  $\Gamma$  be a compact group. The system  $\dot{x}(t) = G(x_t)$  is said to be  $\Gamma$ -equivariant if  $G(\gamma x_t) = \gamma G(x_t)$  for all  $\gamma \in \Gamma$ .

It is clear that system (2) is  $Z_n$  equivariant. Let  $\omega = 2\pi/\omega_0$ . Denote by  $P_\omega$  the Banach space of all continuous  $\omega$ -periodic mappings from  $\mathbb{R}$  into  $\mathbb{R}^n$ , equipped with the supremum norm. Then, for the circle group  $S^1$ ,  $Z_n \times S^1$  acts on  $P_\omega$  by

$$\begin{aligned} (\rho, e^{i\theta}) x(t) &= \rho x\left(t + \frac{\omega}{2\pi}\theta\right), \\ (\rho, e^{i\theta}) &\in Z_n \times S^1, \quad x \in P_\omega. \end{aligned} \quad (24)$$

Denote by  $SP_\omega$  the subspace of  $P_\omega$  of all  $\omega$ -periodic solutions of (4) with  $\tau = \tau_{k,j}$ . Then

$$SP_\omega = \{y_1 e^1(t) + y_2 e^2(t), y_1, y_2 \in \mathbb{R}\}, \quad (25)$$

where  $e^1(t)$  and  $e^2(t)$  are  $n$ -dimensional vector functions with the  $j$ th components defined by  $e_j^1(t) = \cos(\omega_0 t + 2(j-1)k\pi/n)$ ,  $e_j^2(t) = \sin(\omega_0 t + 2(j-1)k\pi/n)$ , respectively. For each subgroup  $\Sigma \leq Z_n \times S^1$ , it is clear that the fixed point set

$$\text{Fix}(\Sigma, SP_\omega) = \{x \in SP_\omega; (\rho, e^{i\theta}) x = x \quad \forall (\rho, e^{i\theta}) \in \Sigma\} \quad (26)$$

is a subspace.

**Lemma 16.** Consider

$$\rho e^1(t) = e^1\left(t + \frac{k\omega}{n}\right), \quad \rho e^2(t) = e^2\left(t + \frac{k\omega}{n}\right). \quad (27)$$

The proof is given in Appendix I.

It is known from [19] that the subgroups of  $Z_n \times S^1$ , up to conjugacy, describe the symmetry of periodic solutions of system (2) which exhibit certain spatiotemporal patterns given by

$$\Sigma = \{(\rho, e^{(2k\pi/n)i}); 0 \leq k \leq n-1\}. \quad (28)$$

From Lemma 16, it follows that  $\text{Fix}(\Sigma, SP_\omega) = SP_\omega$  which means that

$$\dim \text{Fix}(\Sigma, SP_\omega) = 2. \quad (29)$$

This, together with Lemma 16, allows us to apply the symmetric Hopf bifurcation theorem for delay differential equations due to Wu [22] to obtain the following results.

**Theorem 17.** Suppose that  $|\alpha| > 1$  and  $\tau_{k,j}$  is defined by (13) with  $k = 0, 1, \dots, n-1$  and  $j \in \{0, 1, \dots\}$  such that  $\tau_{k,j} > 0$ . Then near  $\tau_{k,j}$  there exists a branch of small-amplitude periodic solutions of system (2) with period  $p$  near  $2\pi/\omega_0$ , satisfying

$$x_{i+1}(t) = x_i\left(t - \frac{k}{n}p\right), \quad i \pmod n. \quad (30)$$

## 7. Examples and Numerical Simulations

In this section, we consider two examples with four neurons and seven neurons, respectively, to justify our theoretical results.

**7.1. Four-Neural Network Model ( $n$ , Even).** Consider a four-neuron network model

$$\frac{dx_i}{dt} = -x_i(t) + \alpha \tanh(x_{i+1}(t - \tau)), \quad i \pmod 4. \quad (31)$$

From (13), we get the Hopf bifurcation curves

$$\tau_{k,j} = \begin{cases} \frac{1}{\sqrt{\alpha^2 - 1}} \\ \times \left\{ -\frac{(k-2)\pi}{2} - \arccos\frac{1}{|\alpha|} + 2j\pi \right\}, & \text{for } \alpha < -1, \\ \frac{1}{\sqrt{\alpha^2 - 1}} \left\{ \frac{k\pi}{2} - \arccos\frac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha > 1, \end{cases} \quad (32)$$

with  $k = 0, 1, 2$  and  $j \in \{0, 1, \dots\}$  such that  $\tau_{k,j} > 0$ . It follows that these Hopf bifurcation curves are symmetric about  $\tau$ -axis (see Figure 6). From Sections 2 and 3, we can obtain the following results.

**Corollary 18.** (i) If  $-1 \leq \alpha \leq 1$ , then the rest state  $x_0$  of system (31) is stable for all  $\tau \geq 0$ .

(ii) If  $|\alpha| > 1$ , then the rest state  $x_0$  of system (31) is unstable but the other two nonzero steady states are both stable for all  $\tau \geq 0$ .

(iii) System (31) undergoes a Hopf bifurcation at  $\tau_{k,j}$  (as shown in Figure 6) and bifurcating periodic solutions are all unstable in the phase space (although stable on the center manifold).



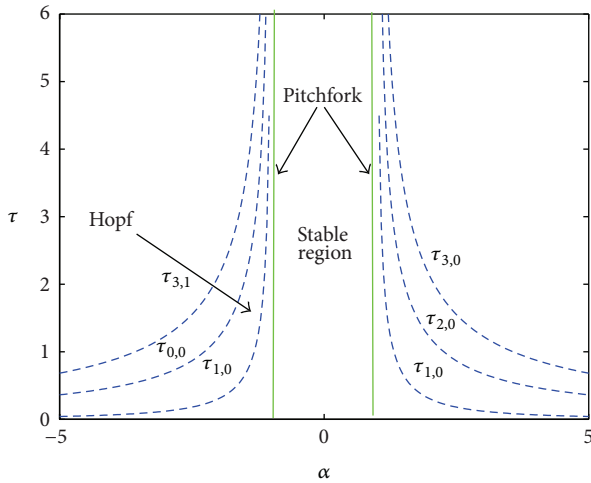


FIGURE 6: The curves of Hopf and pitchfork bifurcations of system (31).

For  $\alpha = -1.5$ , the system (31) has three equilibria  $x_0, x_1^* = (1.2878, -1.2878, 1.2878, -1.2878)$  and  $x_2^* = (-1.2878, 1.2878, -1.2878, 1.2878)$ . Figure 7 shows the evolution of system (31) starting with the initial value  $(0.3, 0.5, -0.5, 1)$  and  $\tau = 2$ .

Taking  $\alpha = 1.5$ , system (31) has three equilibria  $x_0, x_1^* = (1.2878, 1.2878, 1.2878, 1.2878)$  and  $x_2^* = (-1.2878, -1.2878, -1.2878, -1.2878)$ . The zero equilibrium is unstable and the other two nonzero equilibria are both stable (see Figure 8).

7.2. Seven-Neuron Network Model ( $n, \text{Odd}$ ). Consider a seven-neuron network model

$$\frac{dx_i}{dt} = -x_i(t) + \alpha \tanh(x_{i+1}(t - \tau)), \quad i \pmod{7}. \quad (33)$$

Let

$$\tau_{k,j} = \begin{cases} \frac{1}{\sqrt{\alpha^2 - 1}} \left\{ -\frac{2k\pi}{7} + \arccos \frac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha < -1 \\ \frac{1}{\sqrt{\alpha^2 - 1}} \left\{ \frac{2k\pi}{7} - \arccos \frac{1}{\alpha} + 2j\pi \right\}, & \text{for } \alpha > 1, \end{cases} \quad (34)$$

with  $k = 0, 1, \dots, 6$  and  $j \in \{0, 1, \dots\}$  such that  $\tau_{k,j} > 0$ . Then from Sections 2 and 3, we have the following results.

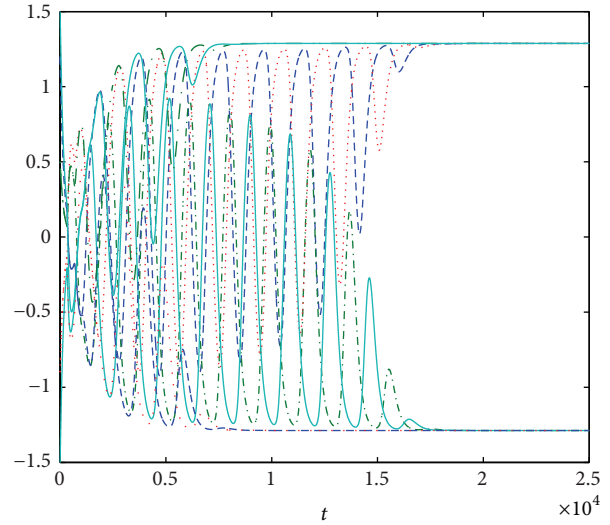
**Corollary 19.** (i) If  $-1 \leq \alpha \leq 1$ , then the rest state  $x_0$  of system (33) is stable for all  $\tau \geq 0$ .

(ii) If  $\alpha < \sec(6\pi/7)$  or  $\alpha > 1$ , then the rest state  $x_0$  of system (33) is unstable for all  $\tau \geq 0$ .

(iii) If  $\sec(6\pi/7) < \alpha < -1$ , then the rest state  $x_0$  of system (33) is asymptotically stable for  $\tau \in [0, \tau_{3,0})$  and unstable for  $\tau > \tau_{3,0}$ .

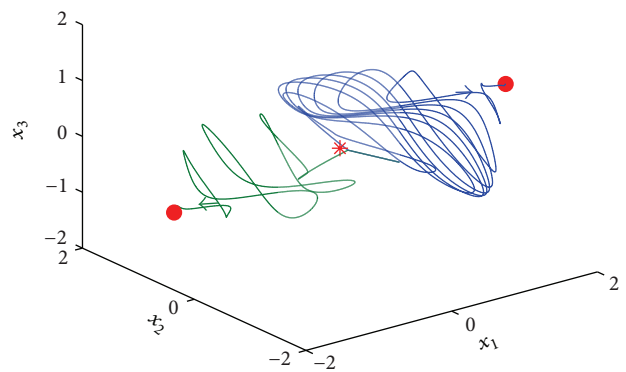
(iv) System (33) undergoes a Hopf bifurcation at  $\tau_{k,j}$  (as shown in Figure 9) and the bifurcating periodic solutions satisfy

$$x_{i+1} = x_i \left( t - \frac{k}{7}p \right), \quad i \pmod{7}. \quad (35)$$



---  $x_1$       ····  $x_3$   
 ····  $x_2$       —  $x_4$

(a)



(b)

FIGURE 7: Evolution of system (31) with  $\alpha = -1.5, \tau = 2$ . The rest state  $x_0$  of system (2) is unstable and the other two nonzero equilibria are stable. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2,$  and  $x_3$  of system (2).

Moreover, the bifurcating periodic solutions at the first bifurcating point  $\tau_{3,0}$  are stable and others unstable in the phase space.

To perform the numerical calculation, we consider the parameter value  $\sec(6\pi/7) < \alpha = -1.08 < -1$ . It follows from (34) that

$$0 < \tau_{3,0} \approx 0.1507 < \tau_{2,0} \approx 2.3511 < \tau_{1,0} \approx 4.5516 < \dots \quad (36)$$

Taking  $\tau = 0.1 < \tau_{3,0}$ , Figure 10 shows that the rest state  $x_0$  of system (33) is asymptotically stable.

For  $\tau_{3,0} < \tau = 0.2 < \tau_{2,0}$ , it follows from Corollary 19 that the rest state  $x_0$  of system (33) becomes unstable and there exists a small-amplitude phase-locked oscillation with

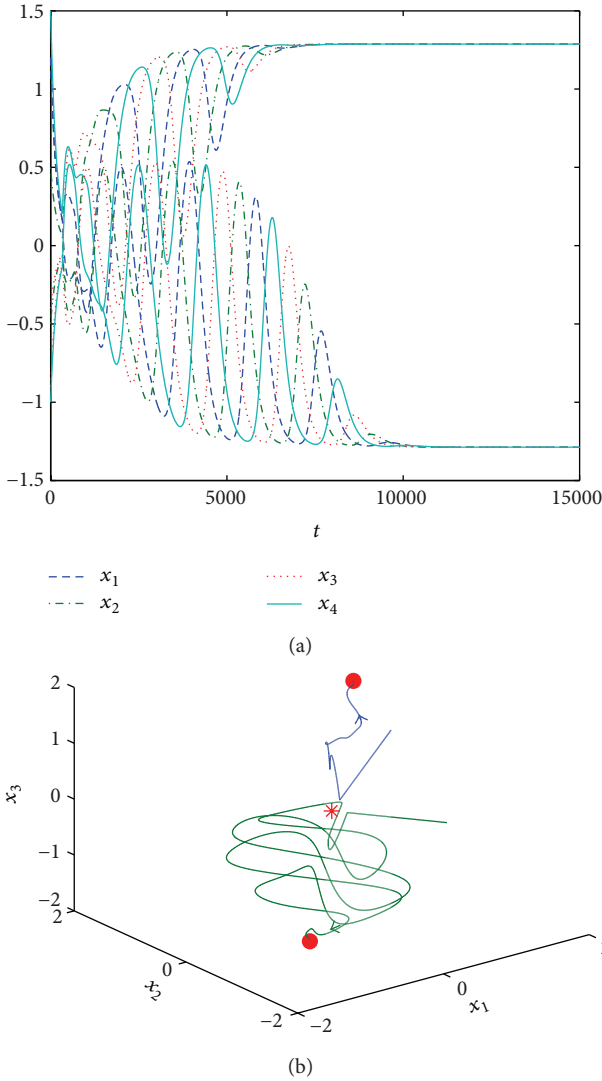


FIGURE 8: Evolution of (31) with  $\alpha = 1.5, \tau = 2$ . The rest state  $x_0$  of system (2) is unstable and the other two nonzero equilibria are stable. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2$ , and  $x_3$  of system (2).

period  $p$  satisfying  $x_{i+1} = x_i(t - (3/7)p), i(\text{mod } 7)$ . This result is illustrated in Figure 11.

Taking  $\tau = 4.5$  far away from the first critical value  $\tau_{3,0}$ , the numerical simulation shows the existence of phase-locked oscillations with period  $p$  satisfying  $x_{i+1} = x_i(t - (3/7)p), i(\text{mod } 7)$  (see Figure 12). The amplitude of this periodic oscillation is larger than that of periodic oscillation shown in Figure 11. This large-amplitude phase-locked oscillation is very similar to the square waves reported in [52, 53] for singularly perturbed delay equations. This phase-locked oscillation with period  $p$  satisfying  $x_{i+1} = x_i(t - (3/7)p), i(\text{mod } 7)$  also shows that this stable bifurcating periodic solution comes from the first critical value  $\tau_{3,0}$  not from other critical values.

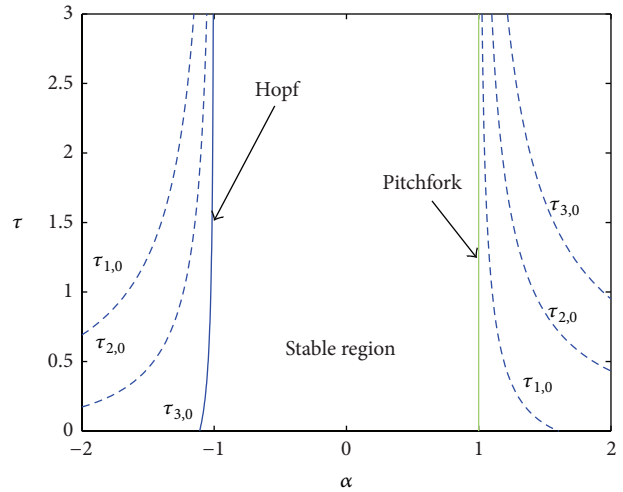


FIGURE 9: Stable region and curves of Hopf and pitchfork bifurcations of system (33) for the rest state  $x_0$ .

Taking  $\alpha = 1.5$ , system (33) has three synchronous equilibria  $x_0, x_1^* = (1.2878, 1.2878, \dots, 1.2878)$  and  $x_2^* = (-1.2878, -1.2878, \dots, -1.2878)$ . The zero equilibrium is unstable and the other two nonzero equilibria are both stable (see Figure 13).

### 8. Conclusions

In this paper, we have studied a lattice ring of  $n$  identical neurons coupled with the same delay. We have analyzed the global stability and delay dependent local stability regions and conditions and explored the different types of bifurcation after the stability is lost. We also have shown that the stability of the zero equilibrium state depends not only on the synaptic strength  $\alpha$  and time delay  $\tau$  but also on the parity of the network size. However, if the nonzero equilibrium exists, it is always stable.

We have also studied the stability of the zero equilibrium on the pitchfork values  $|\alpha| = 1$  and have given easy-to-check conditions on the stability and direction of Hopf bifurcations. For example, if  $f'''(0) < 0$ , then when  $n$  is *odd* the periodic orbits bifurcating from the first critical value  $\tau_{(n-1)/2,0}$  are stable in the phase space, but when  $n$  is *even* all bifurcating periodic orbits are unstable in the phase space.

On the other hand, we have analyzed the spatiotemporal patterns of nonlinear oscillations by using the symmetric bifurcation theory of delay differential equations coupled with representation theory of cyclic groups. Finally, using MATLAB software, we have done some numerical simulations showing the existence of the stable equilibrium and phase-locked periodic waves arising from Hopf bifurcations.

We have generalized recently results for lattice rings with a few units to an arbitrary number. The results obtained are expected to be of interest to neurodynamics and to scientists interested in robots. Indeed we have shown that there are significant features different for  $n$  even at qualitative level and odd which do not disappear when  $n$  goes to large. A finding of

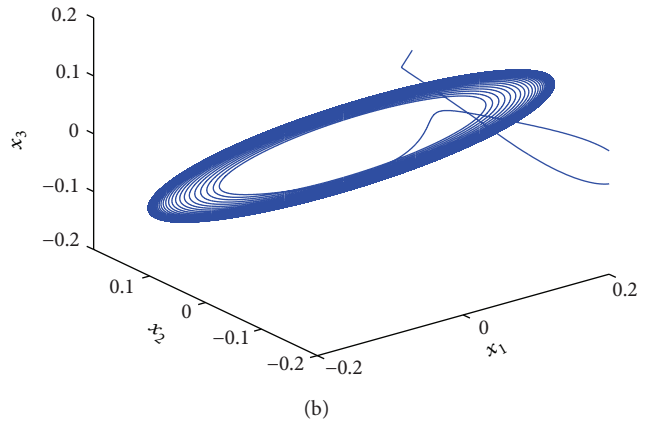
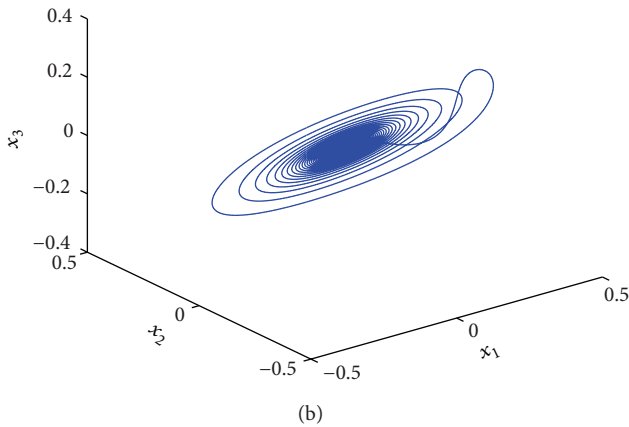
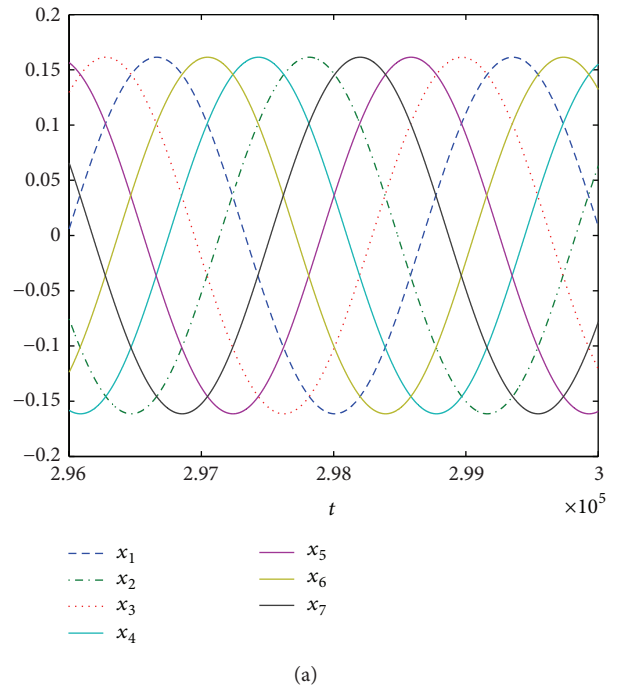
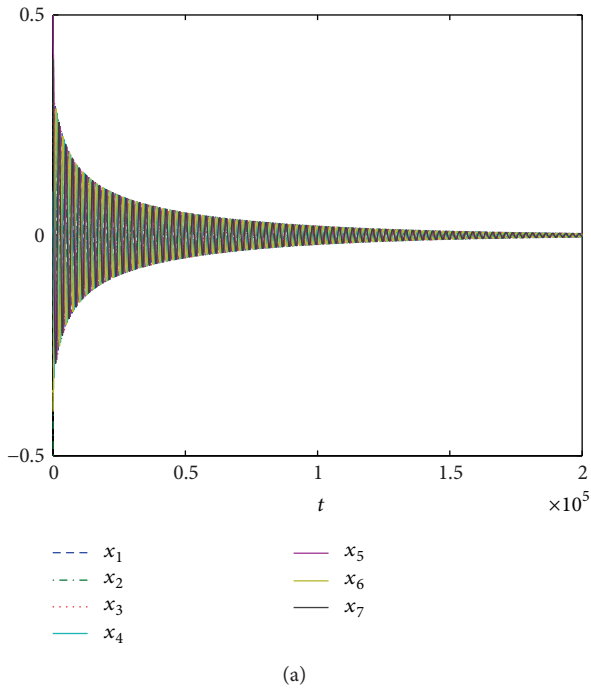


FIGURE 10: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2,$  and  $x_3$  of system (33) at  $\tau = 0.1 < \tau_{3,0}$ .

FIGURE 11: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2,$  and  $x_3$  of system (33) at  $\tau = 0.2 \in (\tau_{3,0}, \tau_{2,0})$ .

practical interest is that for what enough time intervals these differences are not so relevant.

### Appendices

#### A. Proof of Lemma 2

Using the assumption (H1), we can write  $f(x_i(t - \tau))$  as

$$f(x_i(t - \tau)) = p_i(t) x_i(t - \tau), \tag{A.1}$$

where

$$p_i(t) = \int_0^1 f'(sx_i(t - \tau)) ds, \tag{A.2}$$

and find  $p^* \in (0, 1]$  such that  $p_i(t) \leq p^*$  for all  $t \geq 0$  and  $i$ . Thus, system (2) becomes

$$\frac{dx_i}{dt} = -x_i(t) + \alpha p_{i+1}(t) x_{i+1}(t - \tau), \quad i \pmod{n}. \tag{A.3}$$

Using the Lyapunov functional

$$V(t) = \sum_{i=1}^n x_i^2(t) + |\alpha| \sum_{i=1}^n \int_{t-\tau}^t x_{i+1}^2(s) ds, \tag{A.4}$$

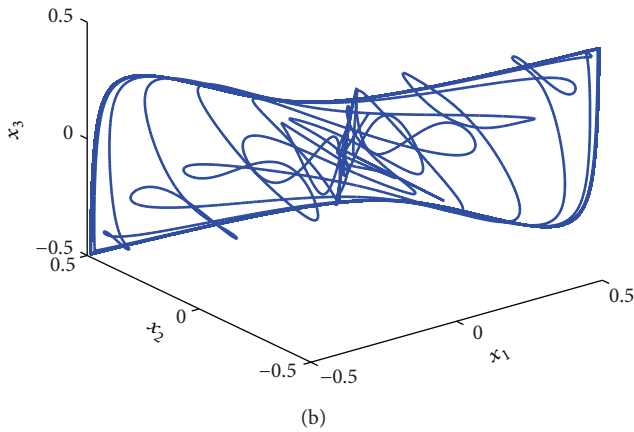
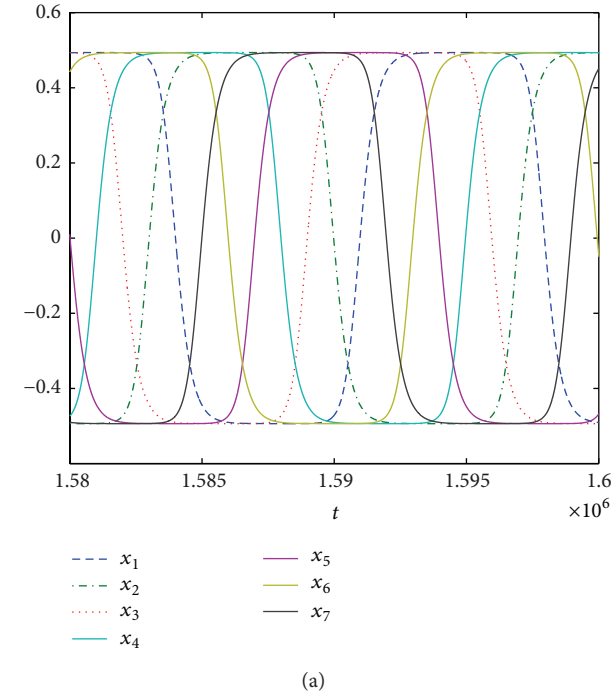


FIGURE 12: (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2$ , and  $x_3$  of system (33) at  $\tau = 4.5$  far away from the first critical value  $\tau_{3,0}$ .

we obtain

$$\begin{aligned} & \left. \frac{dV}{dt} \right|_{(A.3)} \\ &= 2 \sum_{i=1}^n x_i(t) \dot{x}_i(t) + |\alpha| \sum_{i=1}^n (x_{i+1}^2(t) - x_{i+1}^2(t-\tau)) \\ &= 2 \sum_{i=1}^n x_i(t) (-x_i(t) + \alpha p_{i+1}(t) x_{i+1}(t-\tau)) \\ & \quad + |\alpha| \sum_{i=1}^n (x_{i+1}^2(t) - x_{i+1}^2(t-\tau)) \\ &\leq -2 \sum_{i=1}^n x_i^2(t) + \sum_{i=1}^n 2 |\alpha| |x_i(t)| |x_{i+1}(t-\tau)| \end{aligned}$$

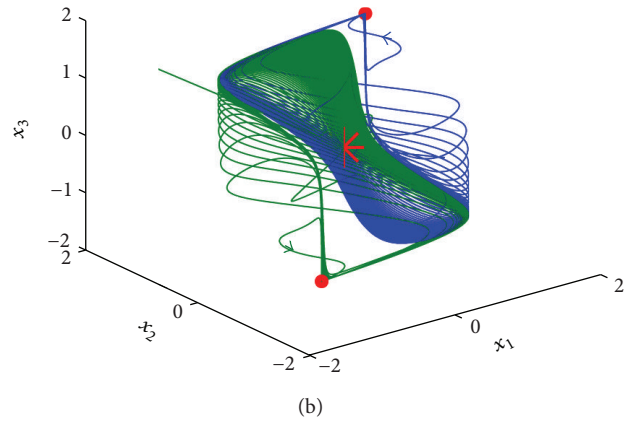
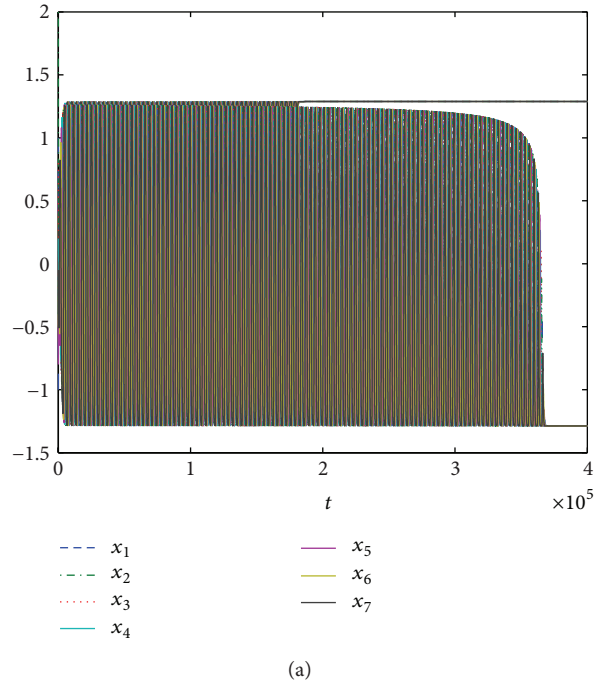


FIGURE 13: Evolution of system (33) with  $\alpha = 1.5, \tau = 2$  starting from different initial values. (a) shows the time evolution for system (33) and (b) is the phase portrait for the first three components  $x_1, x_2$ , and  $x_3$ . The rest state  $x_0$  of system (2) is unstable and two nonzero equilibria  $x_1^*$  and  $x_2^*$  are stable.

$$\begin{aligned} & + |\alpha| \sum_{i=1}^n (x_{i+1}^2(t) - x_{i+1}^2(t-\tau)) \\ &\leq -2 \sum_{i=1}^n x_i^2(t) + |\alpha| \sum_{i=1}^n (x_i^2(t) + x_{i+1}^2(t-\tau)) \\ & \quad + |\alpha| \sum_{i=1}^n (x_{i+1}^2(t) - x_{i+1}^2(t-\tau)) \\ &= -2(1 - |\alpha|) \sum_{i=1}^n x_i^2(t). \end{aligned} \tag{A.5}$$

For  $x(t) \neq 0$  and  $|\alpha| < 1$ ,  $dV/dt|_{(A.3)} < 0$ , which completes the proof.

### B. Proof of Lemma 3

Conclusion (III) follows from the fact that the characteristic equation (8) has a pair of simple purely imaginary roots  $\pm i\omega_0$  at  $\tau = \tau_{k,j}$  and the transversality condition (15) holds.

Denote by  $\lambda_k$  a zero of  $\Delta_k(\tau, \lambda)$  with  $\tau = 0$ . From (9), we have

$$\operatorname{Re} \lambda_k = \alpha \cos \frac{2k\pi}{n} - 1. \tag{B.1}$$

Clearly,  $\operatorname{Re} \lambda_0 = \alpha - 1 > 0$  for  $\alpha > 1$ . If  $n$  is even, then  $\operatorname{Re} \lambda_{n/2} = -\alpha - 1 > 0$  for  $\alpha < -1$ . These, together with (15) and the Rouché-Frobenius theorem, imply that conclusion (I) is true.

If  $n$  is odd with  $\alpha < \sec((n-1)\pi/n)$ , then  $\operatorname{Re} \lambda_{(n-1)/2} > 0$ . If  $n$  is odd with  $\sec((n-1)\pi/n) < \alpha < -1$ , then

$$\operatorname{Re} \lambda_{(n-1)/2} < 0. \tag{B.2}$$

Just then from (B.1), we have

$$\begin{aligned} \operatorname{Re} \lambda_0 &< \operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_{n-1} < \operatorname{Re} \lambda_2 \\ &= \operatorname{Re} \lambda_{n-2} < \dots < \operatorname{Re} \lambda_{(n-1)/2} \\ &= \operatorname{Re} \lambda_{(n+1)/2}. \end{aligned} \tag{B.3}$$

It follows that the root of  $\Delta_k(0, \lambda) = 0$  has negative real parts for any  $k \in \{0, 1, 2, \dots, n-1\}$ . In addition, from (13), we have

$$\tau_{k,j} < \tau_{k-1,j}, \quad \tau_{0,j} < \tau_{n-1,j+1}, \tag{B.4}$$

which means

$$\begin{aligned} \tau_{n-1,0} &< \dots < \tau_{(n+1)/2,0} < \tau_{(n-1)/2,0} < \dots < \tau_{0,0} \\ &< \tau_{n-1,1} < \dots < \tau_{(n+1)/2,1} < \tau_{(n-1)/2,1} < \dots < \tau_{0,1} < \dots \end{aligned} \tag{B.5}$$

From (13), we also have, for  $n$  being odd with  $\sec((n-1)\pi/n) < \alpha < -1$ ,

$$\begin{aligned} \tau_{(n-1)/2,0} &= \frac{1}{\omega_0} \left[ -\pi + \frac{\pi}{n} + \arccos \frac{1}{\alpha} \right] \\ &= \frac{1}{n\omega_0} \left[ \pi - n \left( \pi - \arccos \frac{1}{\alpha} \right) \right] > 0, \end{aligned} \tag{B.6}$$

but

$$\tau_{(n+1)/2,0} = \frac{1}{\omega_0} \left[ -\pi - \frac{\pi}{n} + \arccos \frac{1}{\alpha} \right] < 0. \tag{B.7}$$

Consequently,  $\tau_{(n-1)/2,0}$  is the first positive critical value for the occurrence of purely imaginary roots  $\pm i\omega_0$ . Thus, by (15) and the Rouché-Frobenius theorem, conclusion (II) is proved.

### C. Proof of Theorem 5

It can be verified that  $\lambda = 0$  is a simple zero of  $\Delta_k(\tau, \lambda)$  only for  $k = 0$  when  $\alpha = 1$ . When  $\alpha = -1$  and  $n$  is even,  $\lambda = 0$  is a simple zero of  $\Delta_k(\tau, \lambda)$  with  $k = n/2$ .

On the other hand, suppose that  $\lambda = u + iv \neq 0$  is a root of  $\Delta_k(\tau, \lambda) = 0$ . Then from (9)

$$\begin{aligned} u + 1 &= \alpha e^{-\tau u} \cos \left( \frac{2k\pi}{n} - \tau v \right), \\ v &= \alpha e^{-\tau u} \sin \left( \frac{2k\pi}{n} - \tau v \right). \end{aligned} \tag{C.1}$$

For  $\alpha = \pm 1$ , the latter implies

$$(u + 1)^2 + v^2 = e^{-2\tau u}, \tag{C.2}$$

which can be only satisfied for  $u < 0$ . This completes the proof.

### D. Proof of Theorem 7

For  $1 \leq i \leq n$ , every equilibrium  $x$  of system (2) must satisfy

$$x_i - x_{i+1} = \alpha (f(x_{i+1}) - f(x_{i+2})). \tag{D.1}$$

Using the monotonicity of  $f$  and the assumption  $\alpha > 1$ , we obtain that if  $x_i \neq x_{i+1}$ , say  $x_i > x_{i+1}$ , then  $x_{i+1} > x_{i+2}$ . Repeating the above procedure, we have

$$x_i > x_{i+1} > x_{i+2} > x_{i+3} > \dots > x_n > x_1 \dots > x_{i-1} > x_i, \tag{D.2}$$

which is a contradiction, implying  $x_i = x_{i+1}$ . Thus,  $x$  is an equilibrium of system (2) if and only if  $x_i, i = 1, 2, \dots, n$ , satisfy the equation  $u = \alpha f(u)$ . According to the assumption (H2) when  $\alpha = 1$  it has just one zero root, and when  $\alpha > 1$  it has exactly three roots  $u_-, 0, u_+$  and  $\alpha f'(u_{\pm}) < 1$ . Consequently, when  $\alpha > 1$  system (2) has exactly three equilibria:

$$\begin{aligned} x_- &= (u_-, u_-, \dots, u_-), \quad x_0 = (0, 0, \dots, 0), \\ x_+ &= (u_+, u_+, \dots, u_+). \end{aligned} \tag{D.3}$$

In addition, note that the characteristic equation of system (2) at  $x_{\pm}$  is

$$\det M_n(\tau, \lambda) = \prod_{k=0}^{n-1} \Delta_k(\tau, \lambda) = 0, \tag{D.4}$$

where

$$\Delta_k(\tau, \lambda) = \lambda + 1 - \alpha f'(u_{\pm}) e^{(2k\pi/n)i} e^{-\lambda\tau}. \tag{D.5}$$

Thus, from Section 3 and the fact that  $0 < \alpha f'(u_{\pm}) < 1$ , we have these two equilibria  $x_-$  and  $x_+$  both asymptotically stable for all  $\tau \geq 0$ . This completes the proof.



**E. Proof of Theorem 8**

Lemma 2 says that system (2) has just the zero equilibrium  $x_0$  for  $-1 < \alpha < 1$ . In fact, from the proof of Theorem 7 we also obtain that system (2) has just the zero equilibrium when  $\alpha = 1$ . We now consider the case  $\alpha \leq -1$ . From system (2),  $(x_1, x_2, \dots, x_n)$  is an equilibrium of system (2) if and only if  $(x_1, x_2, \dots, x_n)$  satisfies

$$x_1 = \alpha f(x_2), \quad x_2 = \alpha f(x_3), \dots, x_n = \alpha f(x_1). \quad (E.1)$$

This means that  $x_i, i = 1, 2, \dots, n$ , satisfy

$$x - \alpha f(\alpha f(\alpha(\dots \alpha f(\alpha f(x)) \dots))) = 0. \quad (E.2)$$

Let

$$g(x) = x - \alpha f(\alpha f(\alpha(\dots \alpha f(\alpha f(x)) \dots))). \quad (E.3)$$

Then we have  $g(0) = 0$  and

$$g'(x) = 1 - \alpha^n f'(\alpha f(\alpha(\dots \alpha f(\alpha f(x)) \dots))) \dots f'(\alpha f(x)) f'(x). \quad (E.4)$$

Note that  $f'(x) > 0$  for all  $x \in \mathbb{R}$ ,  $n$  is odd, and  $\alpha \leq -1 < 0$ . Thus, we have  $g'(x) > 0$  for all  $x \in \mathbb{R}$  and hence  $x = 0$  is the unique root of  $g(x)$ . This completes the proof.

**F. Proof of Theorem 9**

Clearly,  $x_0$  is an equilibrium. Let  $(x_1, x_2, \dots, x_n)$  be a nonzero equilibrium of system (2). At first, suppose that  $x_1 = x_2$ , which, together with (D.1), implies  $x_2 = x_3$ , then  $x_3 = x_4$ , and so on. Consequently, we obtain that  $x_1 = x_2 = \dots = x_n$ . Thus,  $x_i, i = 1, 2, \dots, n$ , satisfy  $u = \alpha f(u)$ . Using the monotonicity of  $f$  and  $\alpha < -1 < 0$ , we get  $x_i = 0$  for any  $i = 1, 2, \dots, n$ . This contradiction leads to  $x_1 \neq x_2$ . In fact, we can also obtain  $x_i \neq x_{i+1}, i = 1, 2, \dots, n, i(\text{mod}n)$ . By (E.4), we know that  $g'(0) < 0$  when  $n$  is even and  $\alpha < -1$ . Therefore, there exists a sufficiently small positive number  $\delta$  such that  $g'(x) < 0$  for  $x \in (-\delta, +\delta)$ . From  $g(0) = 0$  we have  $g(x) < 0$  for  $x \in (0, +\delta)$  and  $g(x) > 0$  for  $x \in (-\delta, 0)$ . On the other hand, since  $f(x)$  is bounded, we have  $g(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$  and  $g(x) \rightarrow -\infty$  as  $x \rightarrow -\infty$ . This means that there exist  $u_- < 0$  and  $u_+ > 0$  such that  $g(u_-) = g(u_+) = 0, g(x) > 0$  for  $x \in (u_-, 0)$  and  $g(x) < 0$  for  $x \in (0, u_+)$ , and  $g'(u_-), g'(u_+) \geq 0$ . In addition, by  $f''(x) < 0$  for  $x > 0$  and  $f''(x) > 0$  for  $x < 0$ , we obtain, for  $x > u_+$ ,

$$\begin{aligned} f'(x) &< f'(u_+), \\ f'(\alpha f(x)) &< f'(\alpha f(u_+)), \dots, \\ f'(\alpha f(\alpha(\dots \alpha f(\alpha f(x)) \dots))) & \\ &< f'(\alpha f(\alpha(\dots \alpha f(\alpha f(u_+)) \dots))). \end{aligned} \quad (F.1)$$

So,

$$g'(x) > g'(u_+) \geq 0, \quad \text{for } x > u_+. \quad (F.2)$$

Similarly, we can obtain

$$g'(x) > g'(u_-) \geq 0, \quad \text{for } x < u_-. \quad (F.3)$$

These imply that when  $n$  is even and  $\alpha < -1, g(x)$  has exactly three roots:  $u_-, 0, u_+$ . Thus, from Lemma 6 and  $x_i \neq x_{i+1}$  it follows that system (2) has exactly three roots: the zero equilibrium  $x_0 = (0, 0, \dots, 0)$  and

$$\begin{aligned} x_1^* &= (u_+, u_-, u_+, u_-, \dots, u_+, u_-), \\ x_2^* &= (u_-, u_+, u_-, u_+, \dots, u_-, u_+). \end{aligned} \quad (F.4)$$

From (E.1) and (F.4) we can get  $u_+ = \alpha f(u_-), u_- = \alpha f(u_+)$ , and  $u_-, u_+$  are roots of the following equation:

$$u = \alpha f(\alpha f(u)). \quad (F.5)$$

So,

$$\alpha^2 f'(u_+) f'(u_-) = \alpha^2 f'(\alpha f(u_-)) f'(u_-) = 1. \quad (F.6)$$

Again using the assumption (H1), we have

$$0 < f'(u_+), \quad f'(u_-) < 1. \quad (F.7)$$

Therefore,

$$\alpha^2 f'^2(u_+) f'^2(u_-) < \alpha^2 f'(u_+) f'(u_-) = 1, \quad (F.8)$$

and then

$$-1 < \alpha f'(u_+) f'(u_-) < 0. \quad (F.9)$$

Letting  $v_k = (f'(u_-), f'(u_+) \chi^k, f'(u_-) \chi^{2k}, \dots, f'(u_+) \chi^{(n-1)k})$ , we can obtain that the characteristic equation of the system at  $x_1^*$  is

$$\det M_n(\tau, \lambda) = \prod_{k=0}^{n-1} \Delta_k(\tau, \lambda) = 0, \quad (F.10)$$

where

$$\Delta_k(\tau, \lambda) = \lambda + 1 - \alpha f'(u_+) f'(u_-) e^{(2k\pi/n)i} e^{-\lambda\tau}. \quad (F.11)$$

Thus, from Section 3 and (F.9) we get that the equilibrium  $x_1^*$  is asymptotically stable for all  $\tau \geq 0$ . Similarly, we can also obtain that the equilibrium  $x_2^*$  is asymptotically stable for all  $\tau \geq 0$ . This completes the proof.

**G. Derivation of Coefficients  $K_1, K_2$  of the Normal Form (19)**

The linearized equation at the zero equilibrium of (16) is

$$\dot{z}(t) = L(\tau) z_t, \quad (G.1)$$

where

$$L(\tau) \varphi = -\tau \varphi(0) + \tau \alpha M \varphi(-1). \quad (G.2)$$

The linear operator  $L(\tau)$  can be expressed in the integral form

$$L(\tau)\varphi = \int_{-1}^0 [d\eta_\tau(\theta)]\varphi(\theta), \quad (G.3)$$

where  $\eta_\tau : [-1, 0] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$  is a function of bounded variation.

Denoting  $\tau^* = \tau_{k,j}$  and introducing the new parameter  $\mu = \tau - \tau^*$  so that  $\mu = 0$  corresponds to the Hopf bifurcation, we rewrite (16) as

$$\dot{z}(t) = L(\tau^*)z_t + \tilde{F}(z_t, \mu), \quad (G.4)$$

where

$$\begin{aligned} \tilde{F}(z_t, \mu) &= L(\mu)z_t \\ &+ \gamma\alpha\tau^*M(z_1^3(t-1), z_2^3(t-1), \dots, z_n^3(t-1))^T \\ &+ \text{h.o.t.} \end{aligned} \quad (G.5)$$

Let  $\omega_* = \omega_0\tau^*$  and  $\Lambda_0 = \{i\omega_*, -i\omega_*\}$ . It is clear from Lemma 3 that the characteristic equation of  $\dot{z}(t) = L(\tau^*)z_t$  has a pair of simple imaginary roots  $\pm i\omega_*$  and no other roots in the imaginary axis which are multiple of  $\pm i\omega_*$ . Thus, the nonresonance conditions relative to  $\Lambda_0$  are satisfied.

Let  $\Phi = (\phi_1, \phi_2)$  be a matrix whose columns form a basis of the center space  $P$  of  $\dot{z}(t) = L(\tau^*)z_t$  with  $\phi_1(\theta) = e^{i\omega_*\theta}\nu$ ,  $\phi_2(\theta) = e^{-i\omega_*\theta}\bar{\nu}$ , where the bar means complex conjugation and  $\nu$  is a vector in  $\mathbb{C}^n$  such that

$$L(\tau^*)\phi_1 = i\omega_*\nu. \quad (G.6)$$

Suppose  $\Psi = \text{col}(\psi_1, \psi_2)$  is a matrix whose rows form a basis for the adjoint space  $P^*$  with  $\langle \Psi(s), \Phi(\theta) \rangle = I$  (the  $n \times n$  identity matrix) for the adjoint bilinear form on  $C^* \times C$  defined by

$$\begin{aligned} \langle \psi(s), \phi(\theta) \rangle &= \psi(0)\phi(0) \\ &- \int_{-1}^0 \int_0^\theta \psi(\xi - \theta) d\eta_{\tau^*}(\theta)\phi(\xi) d\xi, \end{aligned} \quad (G.7)$$

$$\text{for } \psi \in C^*, \phi \in C$$

with  $C^* = C([0, 1], \mathbb{R}^{n*})$ , where  $\mathbb{R}^{n*}$  is the  $n$ -dimensional space of row vectors.

Then we have  $\Psi(s) = \text{col}(\psi_1(s), \psi_2(s)) = \text{col}(u^T e^{-i\omega_*s}, \bar{u}^T e^{i\omega_*s})$ ,  $s \in [0, 1]$ , for  $u \in \mathbb{C}^n$  such that

$$\langle \psi_1, \phi_1 \rangle = 1, \quad \langle \psi_1, \phi_2 \rangle = 0. \quad (G.8)$$

From (7), (G.6), and (G.7), we can choose

$$\nu = \nu_k, \quad u = \frac{1}{n}(\bar{a})^{-1}\bar{\nu}_k, \quad (G.9)$$

where  $a = 1 + \tau^* - \omega_*i$ ,  $\nu_k = (1, \chi^k, \chi^{2k}, \dots, \chi^{(n-1)k})^T$ .

Following the procedure [48, 49] very closely, we can obtain the normal form associated with the Hopf singularity

$$\dot{x} = Bx + \frac{1}{2!}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, \mu) + \text{h.o.t.}, \quad (G.10)$$

where

$$\begin{aligned} \frac{1}{2}g_2^1(x, 0, \mu) &= \begin{pmatrix} i\omega_0(\bar{a})^{-1}x_1\mu \\ -i\omega_0a^{-1}x_2\mu \end{pmatrix}, \\ \frac{1}{3!}g_3^1(x, 0, 0) &= 3\gamma\tau^* \begin{pmatrix} (\bar{a})^{-1}(1+i\omega_0)x_1^2x_2 \\ a^{-1}(1-i\omega_0)x_1x_2^2 \end{pmatrix}. \end{aligned} \quad (G.11)$$

Moreover, the normal form (G.10) can be written in real coordinates  $w$  through the change of variables  $x_1 = w_1 - iw_2$ ,  $x_2 = w_1 + iw_2$ . Transformed to polar coordinates  $w_1 = \rho \cos \xi$ ,  $w_2 = \rho \sin \xi$ , this normal form becomes (19), where  $K_1 = \text{Re}A_1$  and  $K_2 = \text{Re}A_2$ .

## H. Derivation of the Normal Form (21)

It is convenient to write (16) as the following delay system:

$$\dot{z}(t) = L_0z_t + \tilde{F}(z_t, \nu), \quad (H.1)$$

with the phase space  $C = C([-1, 0], \mathbb{R}^n)$ , where, for  $\varphi = (\varphi_1, \varphi_2)^T \in C$ ,

$$L_0(\varphi) = -\tau\varphi(0) - \tau M\varphi(-1),$$

$$\begin{aligned} \tilde{F}(z_t, \nu) &= L_1(\nu)\varphi(-1) + \tau(-1 + \nu)\gamma M(\varphi(-1))^3 \\ &+ \text{h.o.t.}, \quad i \pmod{n} \end{aligned} \quad (H.2)$$

with  $L_1(\nu)\varphi(-1) = \tau\nu M\varphi(-1)$ .

Let  $\Lambda_0 = \{0\}$  and consider the center space  $P$  for  $\dot{x}(t) = L_0x_t$  and its dual space  $P^*$ , as in the previous appendix. We can choose normalized dual bases  $\Phi$  of  $P$  and  $\Psi$  of  $P^*$  as follows:

$$\begin{aligned} \Phi(\theta) &= (1, -1, \dots, 1, -1)^T, \quad -1 \leq \theta \leq 0, \\ \Psi(s) &= \frac{1}{n(1+\tau)}(1, -1, \dots, 1, -1), \quad 0 \leq s \leq 1, \end{aligned} \quad (H.3)$$

with  $\dot{\Phi} = \Phi B$ ,  $-\dot{\Psi} = B\Psi$ , and  $B = 0$ .

As the procedure introduced in Section 5.1, decomposing  $z_t$  in (H.1) according to the decomposition of  $BC$  as the form  $z_t = \Phi x(t) + y_t$ , with  $x \in \mathbb{R}$  and  $y_t \in \text{Ker } \pi \cap D(A) = Q \cap C^1 \stackrel{\text{def}}{=} Q^1$ , we can obtain the normal form of (H.1) on the center manifold of the origin

$$\dot{x} = \frac{1}{2}g_2^1(x, 0, \nu) + \frac{1}{3!}g_3^1(x, 0, \nu) + \text{h.o.t.}, \quad (H.4)$$

where  $x, \nu \in \mathbb{R}$ , h.o.t. stands for higher order terms, and  $g_2^1(x, 0, \nu), g_3^1(x, 0, \nu)$  are the second and third order terms in  $(x, \nu)$ , respectively. Then it follows from [48, 49] that

$$\begin{aligned} g_2^1(x, 0, \nu) &= \text{Proj}_{(\text{Im}(M_2^1))^c} f_2^1(x, 0, \nu), \\ g_3^1(x, 0, \nu) &= \text{Proj}_{(\text{Im}(M_3^1))^c} \tilde{f}_3^1(x, 0, \nu), \end{aligned} \quad (H.5)$$

where  $(\text{Im}(M_j^1))^c$  is a complementary space of  $\text{Im}(M_j^1)$  in  $V_j^2(\mathbb{R})$  with  $j = 2, 3$  and  $(1/3!)\tilde{f}_3^1(x, 0, \nu)$  denotes the third

order terms after the calculation of the normal form up to the second order terms.

Since  $B$  is the  $n \times n$  zero matrix, it is easy to check that

$$\begin{aligned} (\text{Im}(M_2^1))^c &= \text{span} \{x^2, x\nu, \nu^2\} = V_2^2(\mathbb{R}), \\ (\text{Im}(M_3^1))^c &= \text{span} \{x^3, x^2\nu, x\nu^2, \nu^3\} = V_3^2(\mathbb{R}). \end{aligned} \tag{H.6}$$

From (H.2), we get

$$\begin{aligned} \frac{1}{2!} f_2^1(x, 0, \nu) &= \Psi(0) L_1(\nu)(\Phi x) \\ &= \frac{\tau\nu x}{n(1+\tau)} (1, -1, \dots, 1, -1) \\ &\quad \times \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} \\ &= -\frac{\tau}{1+\tau} \nu x, \end{aligned} \tag{H.7}$$

which, together with (H.4) and (H.5), leads to

$$\frac{1}{2!} g_2^1(x, 0, \nu) = -\frac{\tau}{1+\tau} \nu x. \tag{H.8}$$

For the bifurcation analysis, it is sufficient to compute the coefficient of  $x^3$  in the normal form (see, e.g., [51]). Hence, we write

$$\begin{aligned} \frac{1}{3!} g_3^1(x, 0, \nu) &= \frac{1}{3!} \text{Proj}_{S_1} \tilde{f}_3^1(x, 0, 0) + \frac{1}{3!} \text{Proj}_{S_2} \tilde{f}_3^1(x, 0, \nu), \end{aligned} \tag{H.9}$$

where  $S_1 = \text{span}\{x^3\}$ ,  $S_2 = \text{span}\{x^2\nu, x\nu^2, \nu^3\}$ . It is clear that  $f_2^1(x, 0, 0) = g_2^1(x, 0, 0) = 0$ . Thus, we have

$$\begin{aligned} \frac{1}{3!} \tilde{f}_3^1(x, 0, 0) &= \frac{1}{3!} f_3^1(x, 0, 0) = -\Psi(0) \tau \gamma M(\Phi(-1)x)^3 \\ &= \frac{-\tau \gamma x^3}{n(1+\tau)} (1, -1, \dots, 1, -1) \\ &\quad \times \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix} \\ &= \frac{\gamma \tau}{1+\tau} x^3. \end{aligned} \tag{H.10}$$

Consequently, the normal form (H.4) becomes (21).

### I. Proof of Lemma 16

It is easy to verify that, for the generator  $\rho$  of  $z_n$  and  $j \pmod n$ ,

$$\begin{aligned} (\rho e^1(t))_j &= \epsilon_{j+1}^1(t) \\ &= \cos\left(\omega_0 t + \frac{2jk\pi}{n}\right) \\ &= \cos\left(\omega_0 \left(t + \frac{2k\pi}{n\omega_0}\right) + \frac{2(j-1)k\pi}{n}\right) \\ &= \epsilon_j^1\left(t + \frac{2k\pi}{n\omega_0}\right), \\ &= \epsilon_j^1\left(t + \frac{k\omega}{n}\right), \\ (\rho e^2(t))_j &= \epsilon_{j+1}^2(t) \\ &= \sin\left(\omega_0 t + \frac{2jk\pi}{n}\right) \\ &= \sin\left(\omega_0 \left(t + \frac{2k\pi}{n\omega_0}\right) + \frac{2(j-1)k\pi}{n}\right) \\ &= \epsilon_j^2\left(t + \frac{2k\pi}{n\omega_0}\right) \\ &= \epsilon_j^2\left(t + \frac{k\omega}{n}\right). \end{aligned} \tag{I.1}$$

This completes the proof.

### Conflict of Interests

The authors certify that there is no conflict of interests with any financial organization regarding the material discussed in the paper.

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### References

- [1] J. Lisman, "The theta/gamma discrete phase code occurring during the hippocampal phase precession may be a more general brain coding scheme," *Hippocampus*, vol. 15, no. 7, pp. 913–922, 2005.
- [2] J. Csicsvari, B. Jamieson, K. D. Wise, and G. Buzsáki, "Mechanisms of gamma oscillations in the hippocampus of the behaving rat," *Neuron*, vol. 37, no. 2, pp. 311–322, 2003.

- [3] E. R. Kandel, J. H. Schwartz, and T. M. Jessell, *Principles of Neural Science*, McGraw-Hill, New York, NY, USA, 4th edition, 2000.
- [4] P. L. Venetianer and T. Roska, "Image compression by cellular neural networks," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 45, no. 3, pp. 205–215, 1998.
- [5] S. A. Campbell, J. Bélair, T. Ohira, and J. Milton, "Limit cycles, tori, and complex dynamics in a second-order differential equation with delayed negative feedback," *Journal of Dynamics and Differential Equations*, vol. 7, no. 1, pp. 213–236, 1995.
- [6] M. G. Velarde, V. I. Nekorkin, V. A. Makarov, V. I. Makarenko, and R. R. Llinás, "Clustering behavior in a three-layer system mimicking olivo-cerebellar dynamics," *Neural Networks*, vol. 17, no. 2, pp. 191–203, 2004.
- [7] B. Jakimovski, B. Meyer, and E. Maehle, "Firefly flashing synchronization as inspiration for selfsynchronization of walking robot gait patterns using a decentralized robot control architecture," in *Architecture of Computing Systems—ARCS 2010*, vol. 5974 of *Lecture Notes in Computer Science*, pp. 61–72, 2010.
- [8] A. J. Ijspeert, "Central pattern generators for locomotion control in animals and robots: a review," *Neural Networks*, vol. 21, no. 4, pp. 642–653, 2008.
- [9] V. A. Makarov, E. Del Rio, M. G. Bedia, M. G. Velarde, and W. Ebeling, "Central pattern generator incorporating the actuator dynamics for a hexapod robot," *International Journal of Applied Mathematics and Computer Science*, vol. 3, no. 2, pp. 97–102, 2007.
- [10] C. M. Marcus and R. M. Westervelt, "Stability of analog neural networks with delay," *Physical Review A*, vol. 39, no. 1, pp. 347–359, 1989.
- [11] P. Baldi and A. F. Atiya, "How delays affect neural dynamics and learning," *IEEE Transactions on Neural Networks*, vol. 5, no. 4, pp. 612–621, 1994.
- [12] J. Wu, *Introduction to Neural Dynamics and Signal Transmission Delay*, vol. 6 of *de Gruyter Series in Nonlinear Analysis and Applications*, Walter de Gruyter, Berlin, Germany, 2001.
- [13] P. P. Civalleri, M. Gilli, and L. Pandolfi, "On stability of cellular neural networks with delay," *IEEE Transactions on Circuits and Systems. I. Fundamental Theory and Applications*, vol. 40, no. 3, pp. 157–165, 1993.
- [14] J. Cao and M. Xiao, "Stability and Hopf bifurcation in a simplified BAM neural network with two time delays," *IEEE Transactions on Neural Networks*, vol. 18, no. 2, pp. 416–430, 2007.
- [15] Y. Song, M. Han, and J. Wei, "Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays," *Physica D: Nonlinear Phenomena*, vol. 200, no. 3–4, pp. 185–204, 2005.
- [16] M. Xiao and J. Cao, "Genetic oscillation deduced from Hopf bifurcation in a genetic regulatory network with delays," *Mathematical Biosciences*, vol. 215, no. 1, pp. 55–63, 2008.
- [17] M. Xiao, W. Zheng, and J. Cao, "Bifurcation and control in a neural network with small and large delays," *Neural Networks*, vol. 44, pp. 132–142, 2013.
- [18] M. Xiao, W. Zheng, and J. Cao, "Hopf bifurcation of an -neuron bidirectional associative memory neural network model with delays," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 24, pp. 118–132, 2013.
- [19] M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and groups in bifurcation theory. Vol. II*, vol. 69 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1988.
- [20] W. Krawcewicz, P. Vivi, and J. Wu, "Computation formulae of an equivariant degree with applications to symmetric bifurcations," *Nonlinear Studies*, vol. 4, no. 1, pp. 89–119, 1997.
- [21] W. Krawcewicz and J. Wu, "Theory and applications of Hopf bifurcations in symmetric functional-differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 7, pp. 845–870, 1999.
- [22] J. Wu, "Symmetric functional-differential equations and neural networks with memory," *Transactions of the American Mathematical Society*, vol. 350, no. 12, pp. 4799–4838, 1998.
- [23] S. A. Campbell, I. Ncube, and J. Wu, "Multistability and stable asynchronous periodic oscillations in a multiple-delayed neural system," *Physica D: Nonlinear Phenomena*, vol. 214, no. 2, pp. 101–119, 2006.
- [24] S. A. Campbell, Y. Yuan, and S. D. Bungay, "Equivariant Hopf bifurcation in a ring of identical cells with delayed coupling," *Nonlinearity*, vol. 18, no. 6, pp. 2827–2846, 2005.
- [25] D. Fan and J. Wei, "Equivariant Hopf bifurcation in a ring of identical cells with delay," *Mathematical Problems in Engineering*, vol. 2009, Article ID 950251, 34 pages, 2009.
- [26] S. Guo and L. Huang, "Hopf bifurcating periodic orbits in a ring of neurons with delays," *Physica D: Nonlinear Phenomena*, vol. 183, no. 1–2, pp. 19–44, 2003.
- [27] S. Guo and L. Huang, "Stability of nonlinear waves in a ring of neurons with delays," *Journal of Differential Equations*, vol. 236, no. 2, pp. 343–374, 2007.
- [28] L. Huang and J. Wu, "Nonlinear waves in networks of neurons with delayed feedback: pattern formation and continuation," *SIAM Journal on Mathematical Analysis*, vol. 34, no. 4, pp. 836–860, 2003.
- [29] M. Peng, "Bifurcation and stability analysis of nonlinear waves in  $\mathbb{D}_n$  symmetric delay differential systems," *Journal of Differential Equations*, vol. 232, no. 2, pp. 521–543, 2007.
- [30] Y. Song, M. O. Tadé, and T. Zhang, "Bifurcation analysis and spatio-temporal patterns of nonlinear oscillations in a delayed neural network with unidirectional coupling," *Nonlinearity*, vol. 22, no. 5, pp. 975–1001, 2009.
- [31] Y. Song, T. Zhang, and M. O. Tadé, "Stability switches, Hopf bifurcations, and spatio-temporal patterns in a delayed neural model with bidirectional coupling," *Journal of Nonlinear Science*, vol. 19, no. 6, pp. 597–632, 2009.
- [32] Y. Song and J. Xu, "Inphase and antiphase synchronization in a delay-coupled system with applications to a delay-coupled FitzHugh-Nagumo system," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 23, pp. 1659–1670, 2012.
- [33] L. P. Shayer and S. A. Campbell, "Stability, bifurcation, and multistability in a system of two coupled neurons with multiple time delays," *SIAM Journal on Applied Mathematics*, vol. 61, no. 2, pp. 673–700, 2000.
- [34] J. Wu, T. Faria, and Y. S. Huang, "Synchronization and stable phase-locking in a network of neurons with memory," *Mathematical and Computer Modelling*, vol. 30, no. 1–2, pp. 117–138, 1999.
- [35] Y. Yuan and S. A. Campbell, "Stability and synchronization of a ring of identical cells with delayed coupling," *Journal of Dynamics and Differential Equations*, vol. 16, pp. 709–744, 2004.
- [36] J. Wei, M. G. Velarde, and V. A. Makarov, "Oscillatory phenomena and stability of periodic solutions in a simple neural network with delay," *Nonlinear Phenomena in Complex Systems*, vol. 5, no. 4, pp. 407–417, 2002.

- [37] J. Wei and M. G. Velarde, "Bifurcation analysis and existence of periodic solutions in a simple neural network with delays," *Chaos*, vol. 14, no. 3, pp. 940–953, 2004.
- [38] J. Wei and M. Y. Li, "Global existence of periodic solutions in a tri-neuron network model with delays," *Physica D: Nonlinear Phenomena*, vol. 198, no. 1-2, pp. 106–119, 2004.
- [39] L. Xiuling and W. Junjie, "Stability and bifurcation analysis on a delayed neural network model," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 15, no. 9, pp. 2883–2893, 2005.
- [40] S. Guo, "Spatio-temporal patterns of nonlinear oscillations in an excitatory ring network with delay," *Nonlinearity*, vol. 18, no. 5, pp. 2391–2407, 2005.
- [41] K. Gopalsamy, I. K. C. Leung, and P. Liu, "Global Hopf-bifurcation in a neural netlet," *Applied Mathematics and Computation*, vol. 94, no. 2-3, pp. 171–192, 1998.
- [42] Y. Han and Y. Song, "Stability and Hopf bifurcation in a three-neuron unidirectional ring with distributed delays," *Nonlinear Dynamics*, vol. 69, no. 1-2, pp. 357–370, 2012.
- [43] Y. Song, Y. Han, and Y. Peng, "Stability and Hopf bifurcation in an unidirectional ring of  $n$  neurons with distributed delays," *Neuroncomputing*, vol. 122, pp. 442–452, 2013.
- [44] J. Mallet-Paret and R. D. Nussbaum, "Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation," *Annali di Matematica Pura ed Applicata*, vol. 145, pp. 33–128, 1986.
- [45] T. Krisztin, H.-O. Walther, and J. Wu, *Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback*, vol. 11 of *Fields Institute Monographs*, American Mathematical Society, Providence, RI, USA, 1999.
- [46] T. Krisztin and H.-O. Walther, "Unique periodic orbits for delayed positive feedback and the global attractor," *Journal of Dynamics and Differential Equations*, vol. 13, no. 1, pp. 1–57, 2001.
- [47] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.
- [48] T. Faria and L. T. Magalhães, "Normal forms for retarded functional-differential equations with parameters and applications to Hopf bifurcation," *Journal of Differential Equations*, vol. 122, no. 2, pp. 181–200, 1995.
- [49] T. Faria and L. T. Magalhães, "Normal forms for retarded functional-differential equations and applications to Bogdanov-Takens singularity," *Journal of Differential Equations*, vol. 122, no. 2, pp. 201–224, 1995.
- [50] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, vol. 251 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*, Springer, New York, NY, USA, 1982.
- [51] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, vol. 2 of *Texts in Applied Mathematics*, Springer, New York, NY, USA, 2nd edition, 2003.
- [52] J. K. Hale and S. M. Tanaka, "Square and pulse waves with two delays," *Journal of Dynamics and Differential Equations*, vol. 12, no. 1, pp. 1–30, 2000.
- [53] W. Huang, "Stability of square wave periodic solution for singularly perturbed delay differential equations," *Journal of Differential Equations*, vol. 168, no. 2, pp. 239–269, 2000.





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