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Research Article

Series Solution for the Time-Fractional Coupled mKdV Equation Using the Homotopy Analysis Method

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We present new analytical approximated solutions for the space-time fractional nonlinear partial differential coupled mKdV equation. A homotopy analysis method is considered to obtain an infinite series solution. The effectiveness of this method is demonstrated by finding exact solutions of the fractional equation proposed, for the special case when the limit of the integral order of the time derivative is considered. The comparison shows a precise agreement between these solutions.

1. Introduction

Many important phenomena occurring in different fields of physics, chemistry, biology, signal processing, system identification, control theory, and finance dynamics and many other problems in different areas are frequently modeled through fractional differential equations [1–14]. Several methods have been developed to study the solutions of nonlinear fractional partial differential equations (NFPDEs), for instance, the variational iteration method [15–17], Adomian decomposition method [18–20], fractional subequation method [21–23], the homotopy perturbation technique [24–27], the homotopy analysis method (HAM) [28, 29], and Laplace homotopy analysis method [30, 31]. However, for the nonlinear coupled equations with parameters derivative, especially fractional parameter derivative, not much work has been done [32].

A direct extension to the fractional-order case for the space-time fractional Hirota-Satsuma coupled mKdV equation takes the following form:

$$\begin{aligned} D_t^{\mu} u &= \frac{1}{2} D_x^{3\mu} u - 3u^2 D_x^{\mu} u + \frac{3}{2} D_x^{2\mu} v + 3 D_x^{\mu} (uv) \\ &\quad - 3\lambda D_x^{\mu} u, \\ D_t^{\mu} v &= -D_x^{3\mu} v - 3v D_x^{\mu} v - 3 \left(D_x^{\mu} u\right)\left(D_x^{\mu} v\right) + 3u^2 D_x^{\mu} v \\ &\quad + 3\lambda D_x^{\mu} v, \\ &\qquad t > 0,\ 0 < \mu \leq 1, \end{aligned} \tag{1}$$

where ${}^{C}D_x^{\mu}$ and ${}^{C}D_t^{\mu}$ are the Caputo derivatives [33], λ is a constant, and μ is the order of the fractional derivative.

The aim of this work is to investigate the approximated analytical solutions of (1). The obtained solutions will be compared with the solutions obtained previously in the literature [34–37], for the coupled equation (1). The outline of this work is as follows: in Section 2, we describe the basic tools of fractional calculus. Section 3 contains the application of the method to obtain the approximated solutions for the coupled mKdV fractional system. In Section 4, we compared our results with those reported in the literature [38]. Finally in Section 5 conclusions are given.

2. Fractional Calculus

The Riemann-Liouville operator is defined as [39, 40]

$$ {}^{RL}D^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\,dt, \quad \mu > 0. \tag{2} $$

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$ \begin{aligned} {}^{C}D^{\mu} f(x) &= {}^{RL}D^{n-\mu} \frac{d^n}{dx^n}\left[f(x)\right] \\ &= \frac{1}{\Gamma(n-\mu)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\mu-n+1}} dt, \end{aligned} \tag{3} $$

for $n-1 < \mu \le n, n \in N, x > 0, f \in \mathbb{C}^n$.

The fractional derivative of $f(x)$ in the Caputo sense satisfies the following relations:

$$ \begin{aligned} {}^{RL}D^{\mu C}D^{\mu} f(x) &= f(x) - \sum_{k=0}^{n-1} f^{(n)}(0^+) \frac{x^k}{k!}, \quad x > 0, \\ {}^{C}D^{\mu RL}D^{\mu} f(x) &= f(x). \end{aligned} \tag{4} $$

The Caputo operator is used here because the initial conditions for the fractional differential equations can be handled by using an analogy with the classical case (ordinary derivative).

3. The HAM Applied to Coupled mKdV Equation

Consider the nonlinear space-time fractional coupled mKdV equation

$$ \begin{aligned} {}^{C}D_t^{\mu} u &= \frac{1}{2} {}^{C}D_x^{3\mu} u - 3u^{2C}D_x^{\mu} u + \frac{3}{2} {}^{C}D_x^{2\mu} v + 3 {}^{C}D_x^{\mu}(uv) \\ &\quad - 3\lambda^{C}D_x^{\mu} u, \\ {}^{C}D_t^{\mu} v &= -{}^{C}D_x^{3\mu} v - 3v^{C}D_x^{\mu} v - 3\left({}^{C}D_x^{\mu} u\right)\left({}^{C}D_x^{\mu} v\right) \\ &\quad + 3u^{2C}D_x^{\mu} v + 3\lambda^{C}D_x^{\mu} v, \\ &\qquad t > 0,\ 0 < \mu \le 1, \end{aligned} \tag{5} $$

with the initial conditions

$$ \begin{aligned} u(x,t) &= \frac{b^{\mu}}{2k^{\mu}} + k^{\mu} \tanh_{\mu}(kx), \\ v(x,t) &= \lambda + b^{\mu} \tanh_{\mu}(kx). \end{aligned} \tag{6} $$

The exact solutions obtained by applying the fractional subequation method are given by [41, 42]

$$ \begin{aligned} u(x,t) &= \frac{b^{\mu}}{2k^{\mu}} + k^{\mu} \tanh_{\mu}(kx+ct), \\ v(x,t) &= \lambda + b^{\mu} \tanh_{\mu}(kx+ct), \end{aligned} \tag{7} $$

where

$$ c^{\mu} = -k^{3\mu} + \frac{3}{4} k^{\mu} \left(\frac{b^{\mu}}{k^{\mu}}\right)^2. \tag{8} $$

By means of HAM, we choose the linear operator [43]

$$ \mathscr{L}\left[\Theta(x,t;p)\right] = {}^{C}D_t^{\mu}\Theta(x,t;p), \tag{9} $$

with $\mathscr{L}[d] = 0$ and $\Theta(x,t;p) = u(x,t;p) = u$ or $\Theta(x,t;p) = v(x,t;p) = v$. We define the nonlinear operator $\mathscr{N}_i[{}^{C}D^{\mu}\Theta(x,t;p)]$, $i = 0, 1, 2, 3, \ldots, n$, as

$$ \begin{aligned} \mathscr{N}_1\left[u, {}^{C}D^{\mu}u, v, {}^{C}D^{\mu}v\right] &= {}^{C}D_t^{\mu} u - \left(\frac{1}{2} {}^{C}D_x^{3\mu} u\right. \\ &\quad \left. - 3u^{2C}D_x^{\mu} u + \frac{3}{2} {}^{C}D_x^{2\mu} v + 3 {}^{C}D_x^{\mu} uv - 3\lambda^{C}D_x^{\mu}\right), \\ \mathscr{N}_2\left[u, {}^{C}D^{\mu}u, v, {}^{C}D^{\mu}v\right] &= {}^{C}D_t^{\mu} v - \left(-{}^{C}D_x^{3\mu} v - 3v^{C}D_x^{\mu} v\right. \\ &\quad \left. - 3\left({}^{C}D_x^{\mu} u\right)\left({}^{C}D_x^{\mu} v\right) + 3u^{2C}D_x^{\mu} v + 3\lambda^{C}D_x^{\mu} v\right). \end{aligned} \tag{10} $$

We construct the zero-order deformation equation

$$ \begin{aligned} &(1-p)\mathscr{L}(u-u_0) \\ &\quad = pH(t)h\mathscr{N}_1\left[u, {}^{C}D^{\mu}u, v, {}^{C}D^{\mu}v\right], \\ &(1-p)\mathscr{L}(v-v_0) = pH(t)h\mathscr{N}_2\left[u, {}^{C}D^{\mu}u, v, {}^{C}D^{\mu}v\right], \end{aligned} \tag{11} $$

where h is a nonzero auxiliary parameter; when $p = 0$ and $p = 1$, we have [44]

$$ \begin{aligned} \Theta_i(x,t;0) &= u_{i0}(x,t); \\ \Theta_i(x,t;1) &= u_i(x,t). \end{aligned} \tag{12} $$

Now we obtain the mth order deformation equation

$$ \begin{aligned} \mathscr{L}\left(u_m - \kappa_m u_{m-1}\right) &= hH(t) R_m^{(1)}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right), \\ \mathscr{L}\left(v_m - \kappa_m v_{m-1}\right) &= hH(t) R_m^{(2)}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right), \end{aligned} \tag{13} $$

where

$$ \begin{aligned} R_m^{(1)}\left(\mathbf{u}_{m-1}, \mathbf{v}_{m-1}\right) &= {}^{C}D_t^{\mu} u_{m-1} - \left[\frac{1}{2} {}^{C}D_x^{3\mu} u_{m-1}\right. \\ &\quad - 3\left(\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_{i-j}u_j {}^{C}D_x^{\mu} u_{m-1-i}\right) + \frac{3}{2} {}^{C}D_x^{2\mu} v_{m-1} \end{aligned} $$

$$- 3\sum_{i=0}^{m-1}\left(u_i{}^{C}D_x^{\mu}v_{m-1-i} + v_i{}^{C}D_x^{\mu}u_{m-1-i}\right) - 3\lambda^{C}D_x^{\mu}u_{m-1}\Bigg],$$

$$R_m^{(2)}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1}\right) = {}^{C}D_t^{\mu}v_{m-1} - \Bigg[-{}^{C}D_x^{3\mu}v_{m-1} - 3\sum_{i=0}^{m-1}\left(v_{m-1-i}{}^{C}D_x^{\mu}v_i + \left({}^{C}D_x^{\mu}v_i\right)\left({}^{C}D_x^{\mu}u_{m-1-i}\right)\right) + 3\left(\sum_{i=0}^{m-1}\sum_{j=0}^{i}u_{i-j}u_j{}^{C}D_x^{\mu}v_{m-1-i}\right) + 3\lambda^{C}D_x^{\mu}v_{m-1}\Bigg]. \tag{14}$$

For the auxiliary function $H(t)$, we can take $H(t) = 1$ [44, 45]. Applying the Riemann-Liouville operator ${}^{\mathrm{RL}}D^{\mu}$, the solution of the mth order deformation equation (14) for $m \geq 1$ becomes

$$u_m = \kappa_m u_{m-1} + h^{\mathrm{RL}}D^{\mu}R_m^{(1)}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1}\right),$$
$$v_m = \kappa_m v_{m-1} + h^{\mathrm{RL}}D^{\mu}R_m^{(2)}\left(\mathbf{u}_{m-1},\mathbf{v}_{m-1}\right). \tag{15}$$

Now if we substitute the initial condition (6), system (15) takes the following form:

$$p^0\colon u_0(x,0) = \frac{b^{\mu}}{2k^{\mu}} + k^{\mu}\tanh_{\mu}(kx+ct),$$
$$v_0(x,0) = \lambda + b^{\mu}\tanh_{\mu}(kx+ct),$$
$$p^1\colon u_1(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{-\Big[\frac{1}{2}{}^{C}D_x^{3\mu}u_0 - 3u_0^{2C}D_x^{\mu}u_0 + \frac{3}{2}{}^{C}D_x^{2\mu}v_0 + 3u_0{}^{C}D_x^{\mu}v_0 + 3v_0{}^{C}D_x^{\mu}u_0 - 3\lambda^{C}D_x^{\mu}u_0\Big]\Big\},$$
$$u_1(x,0) = 0,$$
$$v_1(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{-\Big[-{}^{C}D_x^{3\mu}v_0 - 3v_0{}^{C}D_x^{\mu}v_0 - 3\left({}^{C}D_x^{\mu}u_0\right)\left({}^{C}D_x^{\mu}v_0\right) + 3u_0^{2C}D_x^{\mu}v_0 + 3\lambda^{C}D_x^{\mu}v_0\Big]\Big\},$$
$$v_1(x,0) = 0,$$
$$p^2\colon u_2(x,t) - u_1(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{{}^{C}D_t^{\mu}u_1 - \Big[\frac{1}{2}{}^{C}D_x^{3\mu}u_1 - 3u_0^{2C}D_x^{\mu}u_1 - 6u_1u_0{}^{C}D_x^{\mu}u_0 + \frac{3}{2}{}^{C}D_x^{2\mu}v_1 + 3u_0{}^{C}D_x^{\mu}v_1 + 3v_1{}^{C}D_x^{\mu}u_0 + 3v_0{}^{C}D_x^{\mu}u_1 + 3u_1{}^{C}D_x^{\mu}v_0 - 3\lambda^{C}D_x^{\mu}u_1\Big]\Big\},$$
$$u_2(x,0) = 0,$$
$$v_2(x,t) - v_1(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{{}^{C}D_t^{\mu}v_1 - \Big[-{}^{C}D_x^{3\mu}v_1 - 3v_1{}^{C}D_x^{\mu}v_0 - 3v_0{}^{C}D_x^{\mu}v_1 - 3\left({}^{C}D_x^{\mu}u_0\right)\left({}^{C}D_x^{\mu}v_1\right) - 3\left({}^{C}D_x^{\mu}u_1\right)\left({}^{C}D_x^{\mu}v_0\right) + 3u_0^{2C}D_x^{\mu}v_1 + 6u_1u_0{}^{C}D_x^{\mu}v_0 + 3\lambda^{C}D_x^{\mu}v_1\Big]\Big\},$$
$$v_2(x,0) = 0,$$
$$p^3\colon u_3(x,t) - u_2(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{{}^{C}D_t^{\mu}u_2 - \Big[\frac{1}{2}{}^{C}D_x^{3\mu}u_2 - 3u_0^{2C}D_x^{\mu}u_2 - 6u_2u_0{}^{C}D_x^{\mu}u_0 - 6u_1u_0{}^{C}D_x^{\mu}u_1 - 3u_1^{2C}D_x^{\mu}u_0 + \frac{3}{2}{}^{C}D_x^{2\mu}v_2 + 3u_0{}^{C}D_x^{\mu}v_2 + 3v_2{}^{C}D_x^{\mu}u_0 + 3v_1{}^{C}D_x^{\mu}u_1 + 3v_0{}^{C}D_x^{\mu}u_2 + 3u_2{}^{C}D_x^{\mu}v_0 + 3u_1{}^{C}D_x^{\mu}v_1 - 3\lambda^{C}D_x^{\mu}u_2\Big]\Big\},$$
$$u_3(x,0) = 0,$$
$$v_3(x,t) - v_2(x,t) = h^{\mathrm{RL}}D^{\mu}\Big\{{}^{C}D_t^{\mu}v_2 - \Big[-{}^{C}D_x^{3\mu}v_2 - 3v_2{}^{C}D_x^{\mu}v_0 - 3v_1{}^{C}D_x^{\mu}v_1 - 3v_0{}^{C}D_x^{\mu}v_2 - 3\left({}^{C}D_x^{\mu}u_0\right)\left({}^{C}D_x^{\mu}v_2\right) - 3\left({}^{C}D_x^{\mu}u_1\right)\left({}^{C}D_x^{\mu}v_1\right) - 3\left({}^{C}D_x^{\mu}u_2\right)\left({}^{C}D_x^{\mu}v_0\right) + 3u_0^{2C}D_x^{\mu}v_2 + 6u_2u_0{}^{C}D_x^{\mu}v_0 + 6u_1u_0{}^{C}D_x^{\mu}v_1 + 3u_1^{2C}D_x^{\mu}v_0 + 3\lambda^{C}D_x^{\mu}v_2\Big]\Big\},$$
$$v_3(x,0) = 0,$$
$$\vdots \tag{16}$$

Solving with Mathematica software, the approximate solutions of (5), considering $h = -1$ [44], are

$$p^0\colon u_0(x,0) = \frac{b^{\mu}}{2k^{\mu}} + k^{\mu}\tanh_{\mu}(kx+ct),$$
$$v_0(x,0) = \lambda + b^{\mu}\tanh_{\mu}(kx+ct),$$
$$p^1\colon u_1(x,t) = \frac{t^{\mu}\left(4k^{4\mu} - 3b^{2\mu}\right)\left(-1 + \tanh_{\mu}(kx)^2\right)}{4\Gamma(\mu+1)},$$
$$v_1(x,t) = \frac{t^{\mu}(bk)^{\mu}\left(4k^{2\mu} - 3(b/k)^{2\mu}\right)\left(-1 + \tanh_{\mu}(kx)^2\right)}{4\Gamma(\mu+1)},$$

$$p^2: u_2(x,t) = \frac{\left(3b^{2\mu} - 4k^{4\mu}\right)^2 \left(t^2/k\right)^{\mu} \tanh_{\mu}(kx)\left(-1 + \tanh_{\mu}(kx)^2\right)}{8\Gamma(2\mu+1)},$$

$$v_2(x,t) = \frac{b^{\mu}\left(3b^{2\mu} - 4k^{4\mu}\right)^2 (t/k)^{2\mu} \tanh_{\mu}(kx)\left(-1 + \tanh_{\mu}(kx)^2\right)}{8\Gamma(2\mu+1)},$$

$$\vdots \tag{17}$$

4. Discussion

Next, we will compare the approximated analytical solution obtained in the above section with the exact analytical solution previously obtained in [41]. In this comparison, using `Mathematica`, setting $p = 1$ in the solutions obtained by the HAM to the coupled mKdV system eq. (17), we can write for $\mu = 1$ (classical case)

$$\begin{aligned} u(x,t) &= \frac{b}{2k} + k\tanh(kx) + \frac{1}{4}t\left(3b^2 - 4k^4\right)\operatorname{sech}^2(kx) \\ &- \frac{t^2\left(3b^2 - 4k^4\right)^2 \tanh(kx)\operatorname{sech}^2(kx)}{16k} \\ &+ \frac{t^3\left(3b^2 - 4k^4\right)^3 (\cosh(2kx) - 2)\operatorname{sech}^4(kx)}{192k^2} \\ &- \frac{t^4\left(3b^2 - 4k^4\right)^4 (\cosh(2kx) - 5)\tanh(kx)\operatorname{sech}^4(kx)}{1536k^3} \\ &+ \cdots = \frac{b}{2k} + k\tanh\left(kx + \left(-k^3 + \frac{3b^2}{4k}\right)t\right), \\ v(x,t) &= \lambda + b\tanh(kx) + \frac{bt\left(3b^2 - 4k^4\right)\operatorname{sech}^2(kx)}{4k} \\ &- \frac{bt^2\left(3b^2 - 4k^4\right)^2 \tanh(kx)\operatorname{sech}^2(kx)}{16k^2} \\ &+ \frac{bt^3\left(3b^2 - 4k^4\right)^3 (\cosh(2kx) - 2)\operatorname{sech}^4(kx)}{192k^3} \\ &- \frac{bt^4\left(3b^2 - 4k^4\right)^4 (\cosh(2kx) - 5)\tanh(kx)\operatorname{sech}^4(kx)}{1536k^4} \\ &+ \cdots = \lambda + b\tanh\left(kx + \left(-k^3 + \frac{3b^2}{4k}\right)t\right), \end{aligned} \tag{18}$$

which is the exact solution when $\mu = 1$, in the general solutions (7), that has been obtained previously, by applying the fractional subequation method [41]. Similar results have been obtained for the nonlinear fractional-order coupled mKdV equation when only time-fractional dependence has been considered in the HAM [36], but with integer order derivatives for space variables. However, in [36], they have compared their results with the exact analytical results obtained by Fan [38], for the classical case $\mu = 1$, that is,

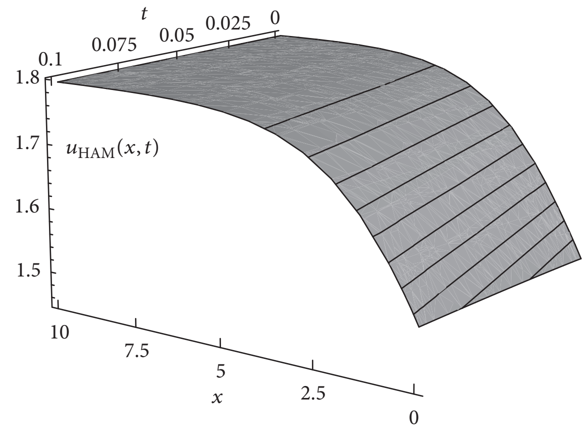

Figure 1: Approximated analytical solutions eq. (17) for $u(x,t)$, with $\lambda = 0.1$, $b = 1$; $k = 1/3$, at $t = 0.001$ and $\mu = 0.97$.

$$\begin{aligned} \tilde{u}(x,t) &= \frac{b}{2k} + k\tanh(kx + \tilde{c}t), \\ \tilde{v}(x,t) &= \frac{\lambda}{2}\left(1 + \frac{k}{b}\right) + b\tanh(kx + \tilde{c}t), \end{aligned} \tag{19}$$

where in the classical solution of Fan the following dispersion relation was obtained [38]:

$$\tilde{c} = -k^3 + \frac{3}{4}k\left(\frac{b}{k}\right)^2 + \frac{6}{4}k\lambda\left(\frac{k}{b} - 1\right). \tag{20}$$

Figures 1 and 2 show the numerical solutions of the nonlinear coupled mKdV equations with $\lambda = 0.1$, $b = 1$; $k = 1/3$, at $t = 0.001$ and $\mu = 0.97$. Figures 3 and 4 depict the exact solutions (7), with $\lambda = 0.1$, $b = 1$, $k = 1/3$, $t = 0.001$, and $\mu = 0.97$.

Figures 5 and 6 show the absolute difference between the exact solutions (7) and the approximated solutions (17), where we have taken into account the first five terms in the infinite series solution, when $\lambda = 0.1$, $b = 1$, $k = 1/3$, $t = 0.001$, and $\mu = 1$. Additionally, we can mention that the results shown in Figures 5 and 6 have the absolute error of the order of 10^{-12}–10^{-13} between the exact and approximated solutions, while the absolute error obtained in [36] turns to be greater, of the order of 10^{-5}–10^{-6}.

Figures 7, 8, and 9 show the absolute error between the approximated solutions (17) and the analytical solutions (7) for the special cases of 5, 7, and 10 perturbative terms in the approximated solutions, when $\mu = 0.97$ and $t = 0.002$. We notice from these figures that the convergency of the solutions obtained by applying the HAM is reached by taking into account only a few number of perturbative terms in the solutions (17).

Similar results have been obtained for different values of the order of the fractional derivative (μ), as we can see in Table 1. In this table we have shown the absolute error for

Table 1: Absolute error between the HAM solution $u_{\text{HAM}}(x,t)$ and the exact solution $u_{\text{exact}}(x,t)$ with $\mu = 0.94$, for the 5th and 10th perturbation terms.

t	x	$\lvert u_{\text{exact}}(x,t) - u_{\text{HAM}}(x,t)\rvert$ (5th terms)	$\lvert u_{\text{exact}}(x,t) - u_{\text{HAM}}(x,t)\rvert$ (10th terms)
1/1000	0	5.59496×10^{-10}	5.59496×10^{-10}
1/1000	1/10	0.000231565	0.000231565
1/1000	1/5	0.000266312	0.000266312
1/1000	3/10	0.000284713	0.000284713
1/1000	2/5	0.000296001	0.000296001
1/1000	1/2	0.000303069	0.000303069
1/1000	3/5	0.00030722	0.00030722
1/1000	7/10	0.000309167	0.000309167
1/1000	4/5	0.000309357	0.000309357
1/1000	9/10	0.000308099	0.000308099
1/1000	1	0.000305621	0.000305621
1/500	0	3.948657×10^{-9}	3.948657×10^{-9}
1/500	1/10	0.000373782	0.000373782
1/500	1/5	0.000441939	0.000441939
1/500	3/10	0.00047875	0.00047875
1/500	2/5	0.000501839	0.000501839
1/500	1/2	0.00051681	0.00051681
1/500	3/5	0.000526196	0.000526196
1/500	7/10	0.000531384	0.000531384
1/500	4/5	0.000533242	0.000533242
1/500	9/10	0.000532363	0.000532363
1/500	1	0.000529188	0.000529188
3/1000	0	1.2381×10^{-8}	1.2381×10^{-8}
3/1000	1/10	0.000486726	0.000486726
3/1000	1/5	0.000586844	0.000586844
3/1000	3/10	0.000641684	0.000641684
3/1000	2/5	0.000676534	0.000676534
3/1000	1/2	0.000699548	0.000699548
3/1000	3/5	0.000714431	0.000714431
3/1000	7/10	0.00072322	0.00072322
3/1000	4/5	0.000727182	0.000727182
3/1000	9/10	0.000727186	0.000727186
3/1000	1	0.000723875	0.000723875

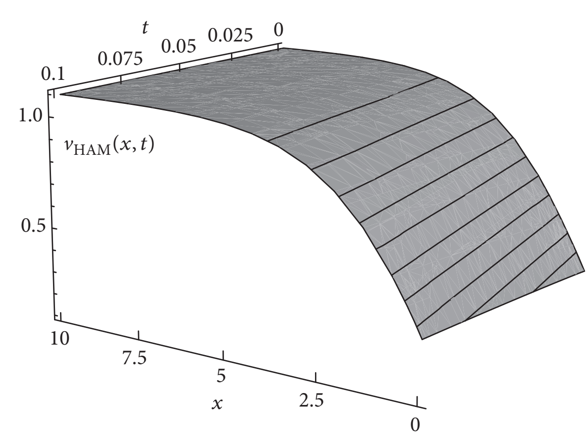

Figure 2: Approximated analytical solutions eq. (17) for $v(x,t)$, with $\lambda = 0.1$, $b = 1$; $k = 1/3$, at $t = 0.001$ and $\mu = 0.97$.

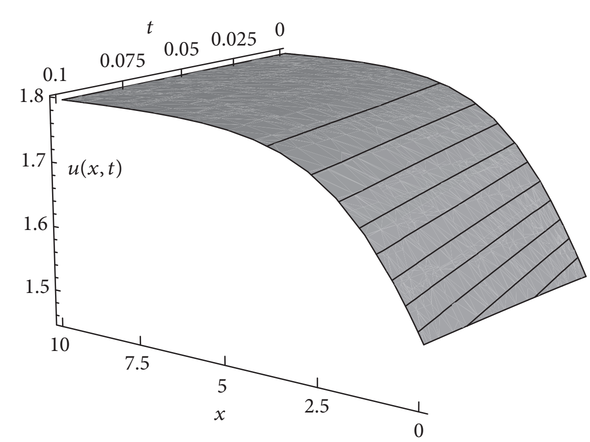

Figure 3: Exact solutions eq. (7) for $u(x,t)$, with $\lambda = 0.1$, $b = 1$; $k = 1/3$, at $t = 0.001$ and $\mu = 0.97$.

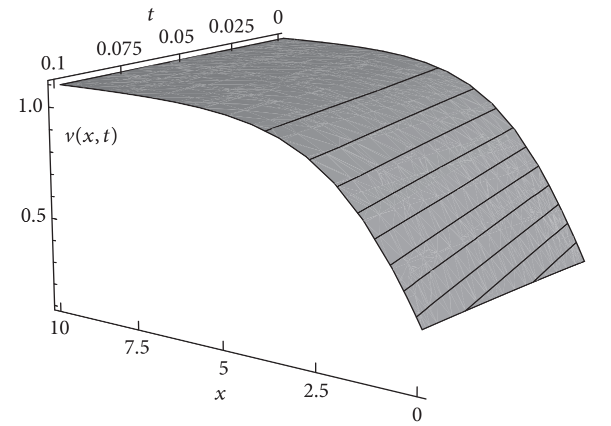

Figure 4: Exact solutions eq. (7) for $v(x,t)$, with $\lambda = 0.1$, $b = 1$; $k = 1/3$, at $t = 0.001$ and $\mu = 0.97$.

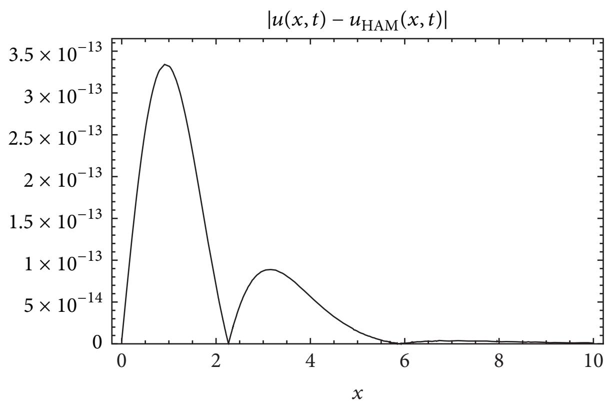

Figure 5: Absolute error between the approximated solution and the exact one for $u(x,t)$ at $t = 0.001$, with $\mu = 1$.

different values of x and t, when $\mu = 0.94$, for the special cases when 5 and 10 perturbative terms in the approximated solutions (17) were considered.

From Figures 7, 8, and 9 and Table 1 we observed the general behavior of the convergency of the solutions obtained when the HAM is applied to the mKdV equation. The accuracy of the solutions requires more terms in the perturbative series (17) as we go from the integer order $\mu = 1$ to lower values. Also the accuracy of the solutions requires more terms in the perturbative series for greater values of the t variable. We have obtained the same general behavior for the solution $v(x,t)$ of the mKdV equation.

5. Conclusion

Based on the HAM, in this paper, we obtain new approximated solutions of nonlinear coupled fractional mKdV equation; the solutions are given in a series form which converges rapidly. The methodologies presented become

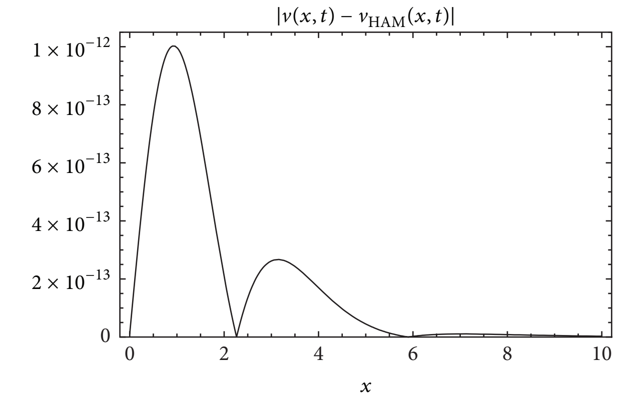

Figure 6: Absolute error between the approximated solution and the exact one for $v(x,t)$ at $t = 0.001$, with $\mu = 1$.

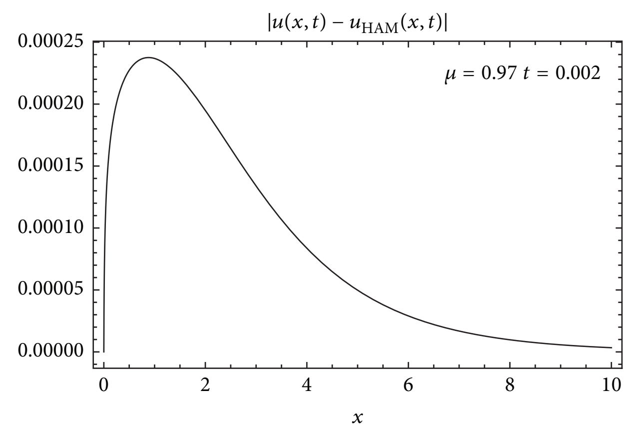

Figure 7: Absolute error between the approximated solution and the exact one for $u(x,t)$ at $t = 0.002$, with $\mu = 0.97$, when 5 perturbative terms in the solutions (17) are considered.

important mathematical tools, motivated by the potential useful for physics and engineers working in various areas of natural sciences. In this work `Mathematica` has been used for algebraic calculations.

To the best of our knowledge, the absolute error reported in Figures 5 and 6, between the exact solutions (7) and the analytical approximated solutions (17), for (1), has not been achieved before in the literature, when the HAM was applied to the mKdV equation.

Competing Interests

The authors declare no conflict of interests.

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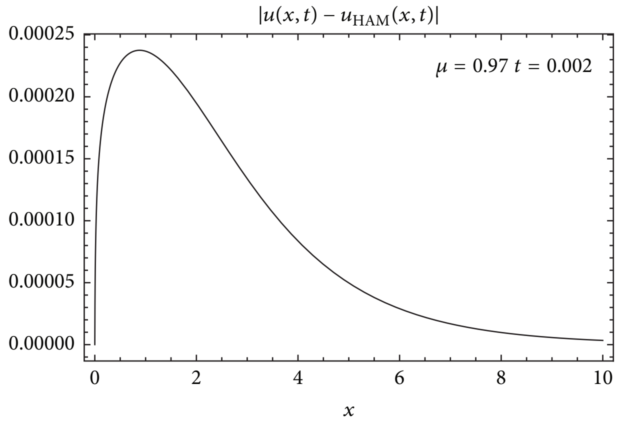

Figure 8: Absolute error between the approximated solution and the exact one for $u(x,t)$ at $t = 0.002$, with $\mu = 0.97$, when 7 perturbative terms in the solutions (17) are considered.

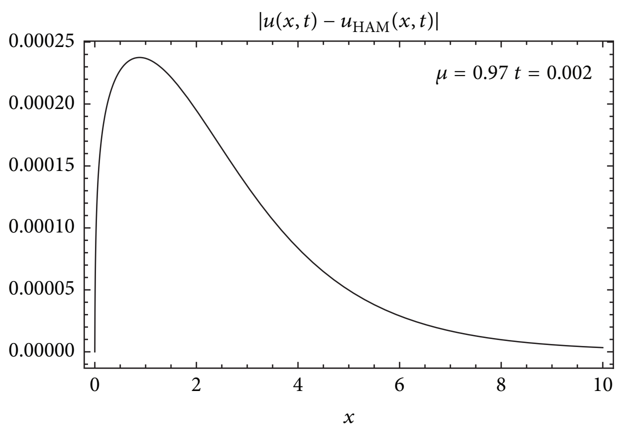

Figure 9: Absolute error between the approximated solution and the exact one for $u(x,t)$ at $t = 0.002$, with $\mu = 0.97$, when 10 perturbative terms in the solutions (17) are considered.

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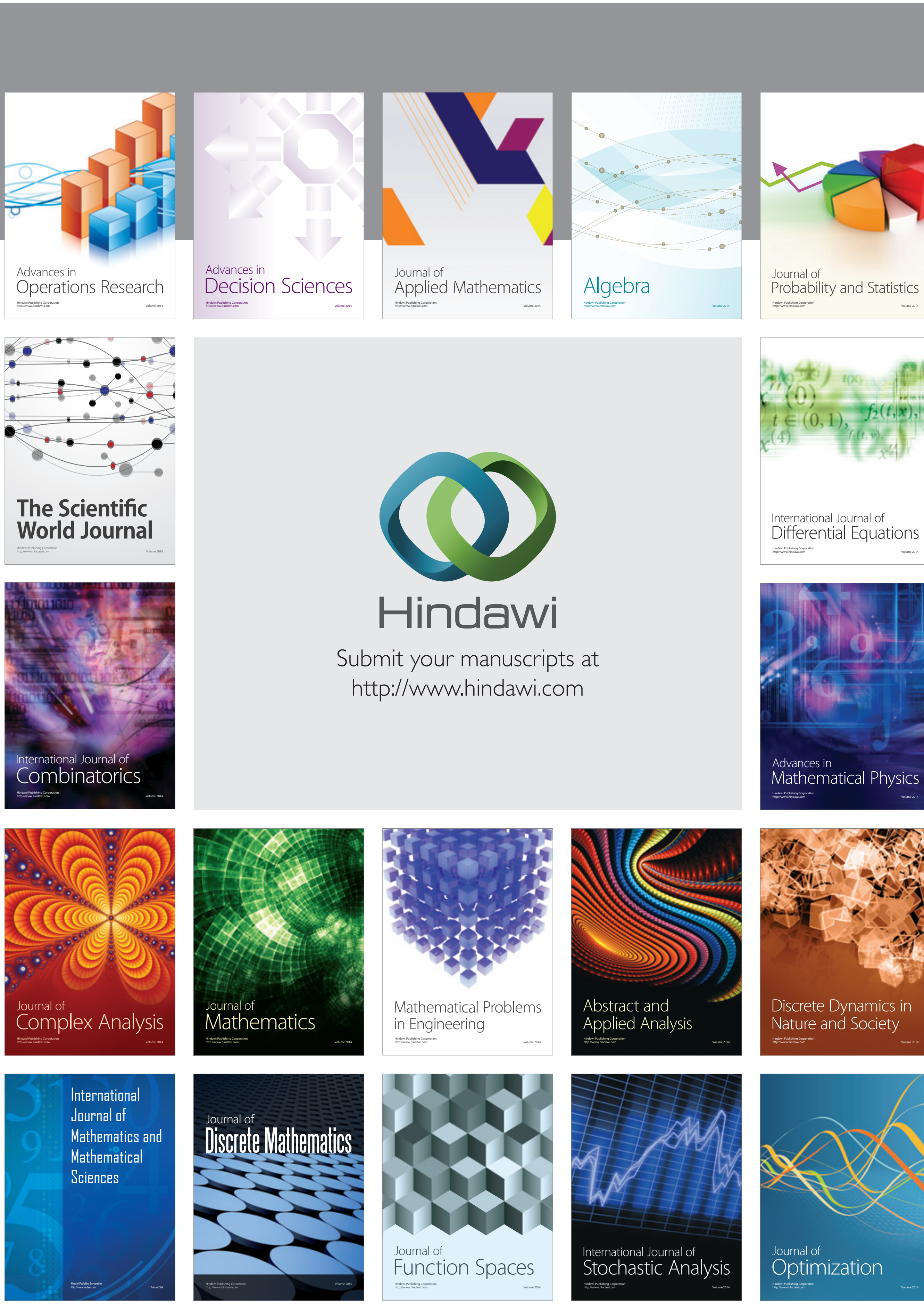
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