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Research Article

The Smallest Spectral Radius of Graphs with a Given Clique Number

Jing-Ming Zhang,[1,2] Ting-Zhu Huang,[1] and Ji-Ming Guo[3]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*
[2] *College of Science, China University of Petroleum, Qingdao, Shandong 266580, China*
[3] *College of Science, East China University of Science and Technology, Shanghai 200237, China*

Correspondence should be addressed to Ting-Zhu Huang; tingzhuhuang@126.com

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The first four smallest values of the spectral radius among all connected graphs with maximum clique size $\omega \geq 2$ are obtained.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Its adjacency matrix $A(G) = (a_{ij})$ is defined as $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. Denote by $d(v_i)$ or $d_G(v_i)$ the degree of the vertex v_i. It is well known that $A(G)$ is a real symmetric matrix. Hence, the eigenvalues of $A(G)$ can be ordered as

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G), \quad (1)$$

respectively. The largest eigenvalue of $A(G)$ is called the spectral radius of G, denoted by $\rho(G)$. It is easy to see that if G is connected, then $A(G)$ is nonnegative irreducible matrix. By the Perron-Frobenius theory, $\rho(G)$ has multiplicity one and exists a unique positive unit eigenvector corresponding to $\rho(G)$. We refer to such an eigenvector corresponding to $\rho(G)$ as the Perron vector of G.

Denote by P_n and C_n the path and the cycle on n vertices, respectively. The characteristic polynomial of $A(G)$ is $\det(xI - A(G))$, which is denoted by $\Phi(G)$ or $\Phi(G, x)$. Let X be an eigenvector of G corresponding to $\rho(G)$. It will be convenient to associate with X a labelling of G in which vertex v_i is labelled x_i (or x_{v_i}). Such labellings are sometimes called "valuation" [1].

The investigation on the spectral radius of graphs is an important topic in the theory of graph spectra. The recent developments on this topic also involve the problem concerning graphs with maximal or minimal spectral radius, signless Laplacian spectral radius, and Laplacian spectral radius, of a given class of graphs, respectively. The spectral radius of a graph plays an important role in modeling virus propagation in networks [2]. It has been shown that the smaller the spectral radius, the larger the robustness of a network against the spread of viruses [3]. In [4], the first three smallest values of the Laplacian spectral radii among all connected graphs with maximum clique size ω are given. And, in [5], it is shown that among all connected graphs with maximum clique size ω the minimum value of the spectral radius is attained for a kite graph $PK_{n-\omega,\omega}$, where $PK_{n-\omega,\omega}$ is a graph on n vertices obtained from the path $P_{n-\omega}$ and the complete graph K_ω by adding an edge between an end vertex of $P_{n-\omega}$ and a vertex of K_ω (shown in Figure 1). Furthermore, in this paper, the first four smallest values of the spectral radius are obtained among all connected graphs with maximum clique size ω.

Let $\mathfrak{I}_{n,\omega}$ be the set of all connected graphs of order n with a maximum clique size ω, where $2 \leq \omega \leq n$. It is easy to see that $\mathfrak{I}_{\omega,\omega} = \{K_\omega\}$. By direct calculation, we have $\rho(K_\omega) = \omega - 1$. If $G \in \mathfrak{I}_{\omega+1,\omega}$, then, from the Perron-Frobenius theorem, the first $\omega - 1$ smallest values of the spectral radius of $\mathfrak{I}_{\omega+1,\omega}$ are $PK_{1,\omega;i}$ $(0 \leq i \leq \omega - 2)$, respectively, where $PK_{1,\omega;i}$ is the graph obtained from $PK_{1,\omega}$ by adding i $(0 \leq i \leq \omega - 2)$ edges. So in the following, we consider that $n \geq \omega + 2$.

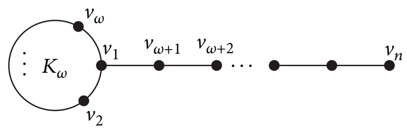


FIGURE 1: Kite graph $PK_{n-\omega,\omega}$.

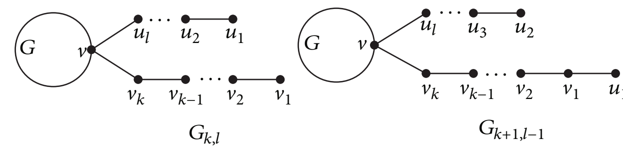


FIGURE 2: Grafting an edge.

2. Preliminaries

In order to complete the proof of our main result, we need the following lemmas.

Lemma 1 (see [6]). *Let v be a vertex of the graph G. Then the inequalities*

$$\begin{aligned}\lambda_1(G) \geq \lambda_1(G-v) \geq \lambda_2(G) \geq \lambda_2(G-v)\\ \geq \cdots \geq \lambda_{n-1}(G-v) \geq \lambda_n(G)\end{aligned} \tag{2}$$

hold. If G is connected, then $\lambda_1(G) > \lambda_1(G-v)$.

For the spectral radius of a graph, by the well-known Perron-Frobenius theory, we have the following.

Lemma 2. *Let G be a connected graph and H a proper subgraph of G. Then $\rho(H) < \rho(G)$.*

Lemma 3 (see [6, 7]). *Let G be a graph on n vertices, then*

$$\rho(G) \leq \max\{d(v) : v \in V(G)\}. \tag{3}$$

The equality holds if and only if G is a regular graph.

Let v be a vertex of a graph G and suppose that two new paths $P = v(v_{k+1})v_k \cdots v_2v_1$ and $Q = v(u_{l+1})u_l \cdots u_2u_1$ of lengths k and l ($k \geq l \geq 1$) are attached to G at $v(= v_{k+1} = u_{l+1})$, respectively, to form a new graph $G_{k,l}$ (shown in Figure 2), where $v_1, v_2, \ldots, v_k$ and $u_1, u_2, \ldots, u_l$ are distinct. Let

$$G_{k+1,l-1} = G_{k,l} - u_1u_2 + v_1u_1. \tag{4}$$

We call that $G_{k+1,l-1}$ is obtained from $G_{k,l}$ by grafting an edge (see Figure 2).

Lemma 4 (see [8, 9]). *Let G be a connected graph on $n \geq 2$ vertices and v is a vertex of G. Let $G_{k,l}$ and $G_{k+1,l-1}$ ($k \geq l \geq 1$) be the graphs as defined above. Then $\rho(G_{k,l}) > \rho(G_{k+1,l-1})$.*

Let v be a vertex of the graph G and $N(v)$ the set of vertices adjacent to v.

Lemma 5 (see [10, 11]). *Let G be a connected graph, and let u, v be two vertices of G. Suppose that $v_1, v_2, \ldots, v_s \in N(v) \setminus (N(u) \bigcup \{u\})$ ($1 \leq s \leq d(v)$) and $x = (x_1, x_2, \ldots, x_n)$ is the Perron vector of G, where x_i corresponds to the vertex v_i ($1 \leq i \leq n$). Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i ($1 \leq i \leq s$). If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 6 (see [12]). *Let v be a vertex of G, let $\varphi(v)$ be the collection of circuits containing v, and let $V(Z)$ denote the set of vertices in the circuit Z. Then the characteristic polynomial $\Phi(G)$ satisfies*

$$\begin{aligned}\Phi(G) = x\Phi(G-v) - \sum_{w}\Phi(G-v-w)\\ - 2\sum_{Z\in\varphi(v)}\Phi(G-V(Z)),\end{aligned} \tag{5}$$

where the first summation extends over those vertices w adjacent to v, and the second summation extends over all $Z \in \varphi(v)$.

An internal path of a graph G is a sequence of vertices $v_1, v_2, \ldots, v_k$ with $k \geq 2$ such that

(1) the vertices in the sequence are distinct (except possibly $v_1 = v_k$);

(2) v_i is adjacent to v_{i+1}, ($i = 1, 2, \ldots, k-1$);

(3) the vertex degrees $d(v_i)$ satisfy $d(v_1) \geq 3, d(v_2) = \cdots = d(v_{k-1}) = 2$ (unless $k = 2$) and $d(v_k) \geq 3$.

Let W_n be the tree on n vertices obtained from P_{n-4} by attaching two new pendant edges to each end vertex of P_{n-4}, respectively.

Lemma 7 (see [13]). *Suppose that $G \neq W_n$ is a connected graph and uv is an edge on an internal path of G. Let G_{uv} be the graph obtained from G by subdivision of the edge uv. Then $\rho(G_{uv}) < \rho(G)$.*

3. Main Results

Let H_1 be the graph obtained from K_ω and a path P_4 : $v_1v_2v_3v_4$ by joining a vertex of K_ω and a nonpendant vertex, say, v_2, of P_4 by a path with length 2 and let H_2 be the graph obtained from K_ω by attaching two pendant edges at two different vertices of K_ω (see Figure 3).

Lemma 8. *Let H_1 and H_2 be the graphs defined as above (see Figure 3). If $\omega \geq 3$, then $\rho(H_2) > \rho(H_1)$.*

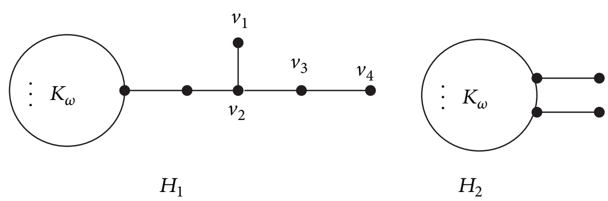


Figure 3: H_1 and H_2.

Proof. For $5 \geq \omega \geq 3$, by direct computations, we have $\rho(H_2) > \rho(H_1)$. In the following, we suppose that $\omega \geq 6$. From Lemma 6, we have

$$\begin{aligned}
\Phi(H_1) &= (x+1)^{\omega-2}\left[x^7-(\omega-2)x^6-(\omega+4)x^5\right.\\
&\quad +(5\omega-10)x^4+(4\omega+1)x^3\\
&\quad \left.-(5\omega-10)x^2-(2\omega-1)x+\omega-2\right]\\
&= (x-\omega+2)^{\omega-2}g_1(x).\\
\Phi(H_2) &= (x+1)^{\omega-3}\left[x^5-(\omega-3)x^4-(2\omega-1)x^3\right.\\
&\quad \left.+(\omega-5)x^2+(2\omega-3)x-\omega+3\right]\\
&= (x+1)^{\omega-3}g_2(x).
\end{aligned} \tag{6}$$

By direct calculation, we have

$$\begin{aligned}
&g_1\left(\omega-1+\frac{1}{\omega^2}\right)\\
&\quad = -\omega^3+2\omega^2+6\omega+\frac{13}{\omega}+\frac{26}{\omega^2}-\frac{54}{\omega^3}\\
&\qquad +\frac{26}{\omega^4}+\frac{34}{\omega^5}-\frac{54}{\omega^6}+\frac{20}{\omega^7}+\frac{20}{\omega^8}-\frac{25}{\omega^9}\\
&\qquad +\frac{5}{\omega^{10}}+\frac{6}{\omega^{11}}-\frac{5}{\omega^{12}}+\frac{1}{\omega^{14}}-20<0;\\
&g_1\left(\omega-1+\frac{2}{\omega^2}\right)\\
&\quad = \omega^4-6\omega^3+7\omega^2+26\omega+\frac{66}{\omega}+\frac{166}{\omega^2}-\frac{416}{\omega^3}\\
&\qquad +\frac{224}{\omega^4}+\frac{432}{\omega^5}-\frac{832}{\omega^6}+\frac{320}{\omega^7}+\frac{560}{\omega^8}-\frac{800}{\omega^9}\\
&\qquad +\frac{160}{\omega^{10}}+\frac{384}{\omega^{11}}-\frac{320}{\omega^{12}}+\frac{128}{\omega^{14}}-91>0;\\
&g_2\left(\omega-1+\frac{2}{\omega^2}\right)\\
&\quad = -2\omega+\frac{12}{\omega}-\frac{18}{\omega^2}-\frac{8}{\omega^3}+\frac{48}{\omega^4}-\frac{48}{\omega^5}\\
&\qquad -\frac{8}{\omega^6}+\frac{64}{\omega^7}-\frac{32}{\omega^8}+\frac{32}{\omega^{10}}<0.
\end{aligned} \tag{7}$$

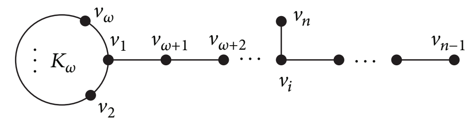


Figure 4: Graph $PK^i_{n-\omega,\omega}$, where $i = \omega+1,\ldots,n-1$.

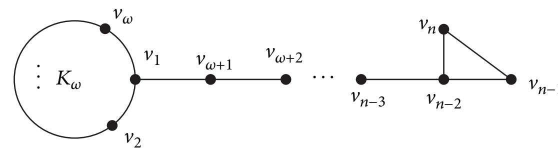


Figure 5: Graph $\overline{PK}^{n-2}_{n-\omega,\omega}$.

From Lemmas 1 and 3, we have $\omega > \rho(H_1) \geq \rho(K_\omega) = \omega-1 \geq \lambda_2(H_1)$ and $\omega > \rho(H_2) \geq \rho(K_\omega) = \omega-1$. Then from (7) we have $\rho(H_2) > \omega - 1 + (2/\omega^2) > \rho(H_1)$. □

Let $PK^i_{n-\omega,\omega}$ be the graph obtained from the kite graph $PK_{n-\omega-1,\omega}$ (see Figure 1) and an isolated vertex v_n by adding an edge v_nv_i $(\omega+1 \leq i \leq n-1)$ (see Figure 4). It is easy to see that $PK^{\omega+2}_{5,\omega} = H_1$ and $PK^{n-1}_{n-\omega,\omega} = PK_{n-\omega,\omega}$.

Let $\overline{PK}^{n-2}_{n-\omega,\omega} = PK^{n-2}_{n-\omega,\omega} + v_{n-1}v_n$ (see Figure 5).

Lemma 9. *Let $PK^i_{n-\omega,\omega}$ be the graphs defined as above (see Figure 4). Then*

$$\rho(P_n) < \rho\left(PK^{n-2}_{n-2,2}\right) < \rho(C_n) = \rho(W_n) < \rho\left(PK^{n-3}_{n-2,2}\right), \quad (n \geq 10). \tag{8}$$

Proof. Clearly, $P_n = P_{2_{n-2,0}}$, $PK^2_{n-2,2} = P_{2_{n-3,1}}$. From Lemma 4, we have

$$\rho(P_n) < \rho\left(PK^{n-2}_{n-2,2}\right) < \rho(W_n) = 2 = \rho(C_n). \tag{9}$$

For $n \geq 10$, from Lemma 2, we have $\rho(PK^{n-3}_{n-2,2}) \geq \rho(PK^7_{8,2}) \approx 2.00659 > \rho(C_n)$. □

Let $G_1 = PK^{n-3}_{n-3,3} - v_{n-1}v_{n-2} + v_{n-3}v_{n-1}$, let $G_2 = PK^{n-3}_{n-3,3} + v_{n-1}v_n$, and let $C_{n-1,1}$ be the graph obtained from C_{n-1} and an isolated vertex by adding an edge between some vertex of C_{n-1} and the isolated vertex (see Figure 6).

Theorem 10. *Among all connected graphs on n vertices with maximum clique size $\omega = 2$ and $n \geq 10$, the first four smallest spectral radii are exactly obtained for P_n, $PK^{n-2}_{n-2,2}$, C_n, W_n, and $PK^{n-3}_{n-2,2}$, respectively.*

Proof. Let G be a connected graph with maximum clique size $\omega = 2$ and $n \geq 10$ vertices. From Lemma 9, we have $\rho(P_n) < \rho(PK^{n-2}_{n-2,2}) < \rho(W_n) = \rho(C_n) < \rho(PK^{n-3}_{n-2,2})$. Thus, we only need to prove that $\rho(G) > \rho(PK^{n-3}_{n-2,2})$ if $G \neq P_n, PK^{n-2}_{n-2,2}, W_n, C_n, PK^{n-3}_{n-2,2}$. If G is a tree, note that $G \neq P_n, PK^{n-2}_{n-2,2}, W_n, PK^{n-3}_{n-2,2}$, then, from Lemma 4, we have $\rho(G) > \rho(PK^{n-3}_{n-2,2})$. If G contains some cycle as a subgraph, then, from Lemmas 2 and 7, we have $\rho(G) \geq \rho(C_{n-1,1}) > \rho(PK^{n-3}_{n-2,2})$. □

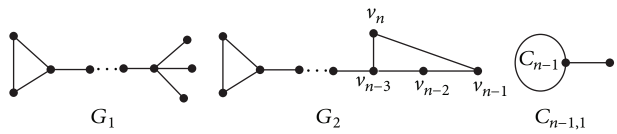


Figure 6: Graphs G_1, G_2, $C_{n-1,1}$.

Lemma 11. *Let $PK^i_{n-\omega,\omega}$, $\overline{PK}^{n-2}_{n-\omega,\omega}$, G_1 and G_2 be the graphs defined as above (see Figures 4, 5, and 6). Then*

$$\rho\left(PK^{n-4}_{n-3,3}\right) < \min\left\{\rho\left(\overline{PK}^{n-2}_{n-3,3}\right), \rho\left(G_1\right), \rho\left(G_2\right)\right\}, \quad (n \geq 8). \tag{10}$$

Proof. For $8 \leq n \leq 11$, by direct calculation, we have $\rho(PK^{n-4}_{n-3,3}) < \rho(G_1)$. If $n \geq 12$, from Lemmas 2 and 7, we have $2.23601 < \rho(PK_{8,3}) < \rho(PK^{n-4}_{n-3,3}) < \rho(PK^8_{9,3}) < 2.23808$. From Lemma 6, we have

$$\begin{aligned}
\Phi\left(PK^{n-4}_{n-3,3}\right) &= \left(x^5 - 4x^3 + 3x\right)\Phi\left(PK_{n-8,3}\right) - \left(x^4 - 2x^2\right)\Phi\left(PK_{n-8,3} - v_{n-5}\right) \\
&= f_1(x)\Phi\left(PK_{n-8,3}\right) - f_2(x)\Phi\left(PK_{n-8,3} - v_{n-5}\right), \\
\Phi\left(G_1\right) &= \left(x^5 - 4x^3\right)\Phi\left(PK_{n-8,3}\right) - \left(x^4 - 3x^2\right)\Phi\left(PK_{n-8,3} - v_{n-5}\right) \\
&= f_3(x)\Phi\left(PK_{n-8,3}\right) - f_4(x)\Phi\left(PK_{n-8,3} - v_{n-5}\right).
\end{aligned} \tag{11}$$

Then we have

$$\begin{aligned}
f_3(x)\Phi\left(PK^{n-4}_{n-3,3}\right) - f_1(x)\Phi\left(G_1\right) &= \left(f_1(x) f_4(x) - f_2(x) f_3(x)\right)\Phi\left(PK_{n-8,3} - v_{n-5}\right) \\
&= \left(-x^7 + 7x^5 - 9x^3\right)\Phi\left(PK_{n-8,3} - v_{n-5}\right) \\
&= R_1(x)\Phi\left(PK_{n-8,3} - v_{n-5}\right).
\end{aligned} \tag{12}$$

For $2.23601 < x < 2.23808$, we have

$$\begin{aligned}
f_1(x) &> 2.23601^5 - 4 \times 2.23808^3 + 3 \times 2.23601 \approx 17 > 0; \\
f_3(x) &> 2.23601^5 - 4 \times 2.23808^3 \approx 11 > 0; \\
R_1(x) &> -2.23808^7 + 7 \times 2.23601^5 - 9 \times 2.23808^3 \approx 9 > 0.
\end{aligned} \tag{13}$$

Note that from Lemma 2, $\rho(PK_{n-8,3} - v_{n-5}) < \rho(PK^{n-4}_{n-3,3})$ and $2.23601 < \rho(PK^{n-4}_{n-3,3}) < 2.23808$. Then, we have

$$f_3(x)\Phi\left(PK^{n-4}_{n-3,3}\right) > f_1(x)\Phi\left(G_1\right), \quad x \in \left[\rho\left(PK^{n-4}_{n-3,3}\right), 2.23808\right). \tag{14}$$

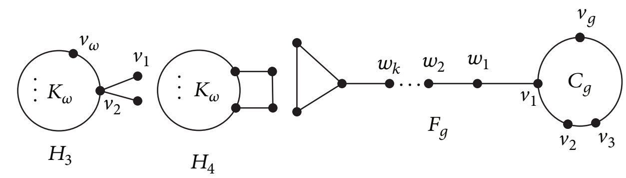


Figure 7: Graphs H_3, H_4, and F_g.

Thus, $\rho(PK^{n-4}_{n-3,3}) < \rho(G_1)$. By similar method, we have for $n \geq 8$

$$\rho\left(PK^{n-4}_{n-3,3}\right) < \rho\left(\overline{PK}^{n-2}_{n-3,3}\right), \qquad \rho\left(PK^{n-4}_{n-3,3}\right) < \rho\left(G_2\right). \tag{15}$$

□

Let H_3 be the graph obtained from K_ω by attaching two pendant edges at some vertex of K_ω; let H_4 be the graph obtained from K_ω and P_2 by adding two edges between two vertices of K_ω and two end vertices of P_2 (see Figure 7).

Theorem 12. *Among all connected graphs on n vertices with maximum clique size $\omega = 3$ and $n \geq 9$, the first four smallest spectral radii are exactly obtained for $PK_{n-3,3}$, $PK^{n-2}_{n-3,3}$, $PK^{n-3}_{n-3,3}$, $PK^{n-4}_{n-3,3}$, respectively.*

Proof. Let G be a connected graph with maximum clique size $\omega = 3$ and $n \geq 9$ vertices. From Lemmas 2 and 7, we have

$$\rho\left(PK^{n-4}_{n-3,3}\right) > \rho\left(PK^{n-3}_{n-3,3}\right) > \rho\left(PK^{n-2}_{n-3,3}\right) > \rho\left(PK_{n-3,3}\right). \tag{16}$$

Thus, we only need to prove that $\rho(G) > \rho(PK^{n-4}_{n-3,3})$ if $G \neq PK_{n-3,3}, PK^{n-2}_{n-3,3}, PK^{n-3}_{n-3,3}, PK^{n-4}_{n-3,3}$.

We distinguish the following three cases.

Case 1. If there exist at least two vertices outside of K_3 that are adjacent to some vertices of K_3, then we have that G contains either H_2 ($\omega = 3$) or H_3 ($\omega = 3$) as a proper subgraph. If G contains H_2 ($\omega = 3$) as a proper subgraph, from Lemmas 2 and 7, we have

$$\begin{aligned}
\rho(G) &> \rho\left(H_2\right) \approx 2.30278 > \rho\left(PK^5_{6,3}\right) \\
&\approx 2.26542 > \rho\left(PK^{n-4}_{n-3,3}\right), \quad (\omega = 3).
\end{aligned} \tag{17}$$

If G contains H_3 ($\omega = 3$) as a proper subgraph, from Lemmas 2 and 7, we have

$$\begin{aligned}
\rho(G) &> \rho\left(H_3\right) \approx 2.34292 \\
&> \rho\left(PK^5_{6,3}\right) > \rho\left(PK^{n-4}_{n-3,3}\right), \quad (\omega = 3).
\end{aligned} \tag{18}$$

Case 2. Suppose that there exists a vertex, say, u, which does not belong to K_3, such that u is adjacent to at least two vertices of K_3. Then G contains C_4^* as a proper subgraph, where C_4^* is obtained from C_4 by adding an edge between two disjoint vertices. From Lemmas 2 and 7, we have

$$\rho(G) > \rho\left(C_4^*\right) \approx 2.56155 > \rho\left(PK^5_{6,3}\right) > \rho\left(PK^{n-4}_{n-3,3}\right). \tag{19}$$

Case 3. Suppose that there uniquely exists a vertex u which does not belong to K_3 such that u is adjacent to a vertex of K_3. We distinguish the following two cases.

Subcase 1. Suppose that $G - V(K_3)$ is a tree. If there exist two vertices $u, r \in V(G - V(K_3))$ such that $d(u) \geq 3$ and $d(r) \geq 3$, then, from Lemmas 2, 4, and 7, we have $\rho(G) > \rho(PK_{n-3,3}^{n-4})$. If there exists only one vertex $u \in V(G - V(K_3))$ such that $d(u) \geq 4$, then, from Lemmas 2, 7, and 11, we have $\rho(G) \geq \rho(G_1) > \rho(PK_{n-3,3}^{n-4})$. If there exists exactly one vertex $u \in V(G - V(K_3))$ such that $d(u) = 3$, note that $G \neq PK_{n-3,3}^{n-2}, PK_{n-3,3}^{n-3}, PK_{n-3,3}^{n-4}$, then from Lemmas 2 and 7 we have $\rho(G) > \rho(PK_{n-3,3}^{n-4})$.

Subcase 2. Suppose that $G - V(K_3)$ contains cycle C_g as a subgraph. If $g = 3, 4$, then, from Lemmas 2, 7 and 11, we have $\rho(G) \geq \rho(\overline{PK}_{n-3,3}^{n-2}) > \rho(PK_{n-3,3}^{n-4})$ or $\rho(G) \geq \rho(G_2) > \rho(PK_{n-3,3}^{n-4})$. If $g \geq 5$, then, from Lemma 2, we can construct a graph F_g from G by deleting vertices such that $\rho(G) \geq \rho(F_g)$, where F_g is the graph obtained from K_3 and a cycle C_g by joining a vertex of K_3 and a vertex of C_g with a path and $|V(F_g)| \leq n$ (see Figure 7). Suppose that C_g is labelled $v_1, v_2, \ldots, v_g$ satisfying $v_i v_{i+1} \in E(C_g)$, $(1 \leq i \leq g - 1)$, $v_1 v_g \in E(C_g)$, and $d(v_1) = 3$. Then, from Lemmas 2 and 7, we have $\rho(F_g - v_2 v_3) > \rho(PK_{n-3,3}^{n-4})$. Thus, we have $\rho(G) > \rho(PK_{n-3,3}^{n-4})$. □

Lemma 13. *Let $PK_{n-\omega,\omega}^{i}$ and $\overline{PK}_{n-\omega,\omega}^{n-2}$ be the graphs defined as above (see Figures 4 and 5). Then $\rho(PK_{n-\omega,\omega}^{n-3}) > \rho(\overline{PK}_{n-\omega,\omega}^{n-2})$ $(\omega \geq 4)$.*

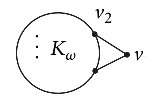
Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $\overline{PK}_{n-\omega,\omega}^{n-2}$, where x_i corresponds to v_i. It is easy to prove that $x_n = x_{n-1}$. From $AX = \rho(\overline{PK}_{n-\omega,\omega}^{n-2})X$, we have

$$\begin{aligned} x_{n-2} &= \left(\rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) - 1\right) x_n, \\ x_{n-3} &= \left[\rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right)\left(\rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) - 1\right) - 2\right] x_n. \end{aligned} \tag{20}$$

From Lemma 2, for $\omega \geq 4$ we have $\rho(\overline{PK}_{n-\omega,\omega}^{n-2}) \geq \rho(K_\omega) = \omega - 1 \geq 3$. Then

$$\begin{aligned} &\rho\left(PK_{n-\omega,\omega}^{n-3}\right) - \rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) \\ &\quad \geq X^T A\left(PK_{n-\omega,\omega}^{n-3}\right) X - X^T A\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) X \\ &\quad = 2x_n\left(x_{n-3} - x_{n-2} - x_n\right) \\ &\quad = 2\left[\rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right)\left(\rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) - 2\right) - 2\right] x_n \\ &\quad \geq 2x_n > 0. \end{aligned} \tag{21}$$

So, $\rho(PK_{n-\omega,\omega}^{n-3}) > \rho(\overline{PK}_{n-\omega,\omega}^{n-2})$. □



M_ω^2

Figure 8: Graph M_ω^2.

Let M_ω^2 $(\omega \geq 4)$ be the graph as shown in Figure 8.

Theorem 14. *Among all connected graphs on n vertices with maximum clique size $\omega \geq 4$ and $n \geq \omega + 5$, the first four smallest spectral radii are exactly obtained for $PK_{n-\omega,\omega}$, $PK_{n-\omega,\omega}^{n-2}$, $\overline{PK}_{n-\omega,\omega}^{n-2}$, $PK_{n-\omega,\omega}^{n-3}$, respectively.*

Proof. Let G be a connected graph with maximum clique size $\omega \geq 4$ and $n \geq \omega + 5$ vertices. Suppose that K_ω is a maximum clique of G. From Lemmas 2, 4, and 13, we have

$$\rho\left(PK_{n-\omega,\omega}^{n-3}\right) > \rho\left(\overline{PK}_{n-\omega,\omega}^{n-2}\right) > \rho\left(PK_{n-\omega,\omega}^{n-2}\right) > \rho\left(PK_{n-\omega,\omega}\right). \tag{22}$$

Thus, we only need to prove that $\rho(G) > \rho(PK_{n-\omega,\omega}^{n-3})$ if $G \neq PK_{n-\omega,\omega}, PK_{n-\omega,\omega}^{n-2}, \overline{PK}_{n-\omega,\omega}^{n-2}, PK_{n-\omega,\omega}^{n-3}$. We distinguish the following three cases.

Case 1. If there exist at least two vertices outside of K_ω that are adjacent to some vertices of K_ω, then G contains either H_2 or H_3 as a proper subgraph. If G contains H_2 as a proper subgraph, from Lemmas 2, 7, and 8, we have

$$\rho(G) > \rho(H_2) > \rho(H_1) \geq \rho\left(PK_{n-\omega,\omega}^{n-3}\right). \tag{23}$$

If G contains H_3 as a proper subgraph, from Lemmas 2, 5, 7, and 8, we have

$$\rho(G) > \rho(H_3) > \rho(H_2) > \rho(H_1) \geq \rho\left(PK_{n-\omega,\omega}^{n-3}\right). \tag{24}$$

Case 2. Suppose that there exists a vertex, say, u, which does not belong to K_ω, such that u is adjacent to at least two vertices of K_ω. From Lemmas 2, 7, and 8, we have

$$\rho(G) > \rho\left(M_\omega^2\right) > \rho(H_4) > \rho(H_2) > \rho(H_1) \geq \rho\left(PK_{n-\omega,\omega}^{n-3}\right). \tag{25}$$

Case 3. Suppose that there uniquely exists a vertex u which does not belong to K_ω such that u is adjacent to a vertex of K_ω. If $G - V(K_\omega)$ is a tree, note that $G \neq PK_{n-\omega,\omega}, PK_{n-\omega,\omega}^{n-2}, PK_{n-\omega,\omega}^{n-3}$, then, from Lemmas 2, 4, and 7, we have $\rho(G) > \rho(PK_{n-\omega,\omega}^{n-3})$. Suppose that $G - V(K_\omega)$ contains cycle C_g as a subgraph. If $g = 3$, note that $G \neq \overline{PK}_{n-\omega,\omega}^{n-2}$, then, from Lemmas 2 and 7, we have $\rho(G) > \rho(G^*) > \rho(PK_{n-\omega,\omega}^{n-3})$, where $G^* = PK_{n-\omega,\omega}^{n-3} + v_{n-1}v_n$. If $g \geq 4$, then by the similar reasoning as that of Subcase 2 of Case 3 of Theorem 12, we have $\rho(G) > \rho(PK_{n-\omega,\omega}^{n-3})$. □

Lemma 15. *Let H_3 and H_4 be the graphs defined as above (see Figure 7). Then*

$$\rho(H_4) > \rho(H_3) \quad (\omega \geq 3). \tag{26}$$

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of H_3, where x_i corresponds to v_i. From $AX = \rho(H_3)X$, we have

$$\rho(H_3) x_1 = x_2,$$
$$\rho(H_3) x_2 = 2x_1 + (\omega - 1) x_\omega, \quad (27)$$
$$\rho(H_3) x_\omega = (\omega - 2) x_\omega + x_2.$$

From above equations, we have

$$\rho^3(H_3) - (\omega - 2)\rho^2(H_3) - (\omega + 1)\rho(H_3) + 2\omega - 4 = 0. \quad (28)$$

Let

$$r_1(x) = x^3 - (\omega - 2)x^2 - (\omega + 1)x + 2\omega - 4. \quad (29)$$

Then

$$r_1(\omega - 1) = -2 < 0. \quad (30)$$

For $x > \omega - 1$ and $\omega \geq 3$, we have

$$r_1'(x) = 3x^2 - 2(\omega - 2)x - (\omega + 1) > 0. \quad (31)$$

Note that $\rho(H_3) > \rho(K_\omega) = \omega - 1$. From (30) and (31), we have $\rho(H_3)$ which is the largest root of equation $r_1(x) = 0$. Similarly, we have $\rho(H_4)$ which is the largest root of equation

$$r_2(x) = x^3 - (\omega - 1)x^2 - 2x + 2\omega - 4 = 0. \quad (32)$$

Then we have, for $x > \omega - 1$,

$$r_1(x) - r_2(x) = x^2 - (\omega - 1)x > 0. \quad (33)$$

Thus, we have $\rho(H_3) < \rho(H_4)$. □

Theorem 16. *Let G be a graph on n vertices with maximum clique size $\omega \geq 3$ and $n = \omega + 2$. Let $PK_{2,\omega}$, H_2, H_3, and H_4 be the graphs defined as above (see Figures 1, 3 and 7). The first four smallest spectral radii are obtained for $PK_{2,\omega}$, H_2, H_3, H_4, respectively.*

Proof. From Lemmas 2, 5, 8, and 15, we have

$$\rho(H_4) > \rho(H_3) > \rho(H_2) > \rho(H_1) > \rho(PK_{2,\omega}). \quad (34)$$

Thus, we only need to prove that, for $G \neq PK_{2,\omega}$, H_2, H_3, and H_4, $\rho(G) > \rho(H_4)$. We distinguish the following two cases.

Case 1. Suppose that there exists exactly one vertex outside of K_ω that is adjacent to at least two vertices of K_ω. Then G contains M_ω^2 (see Figure 8) as a subgraph. From Lemmas 2 and 7, we have $\rho(M_\omega^2) > \rho(H_4)$.

Case 2. Suppose that the two vertices outside of K_ω that are all adjacent to some vertices of K_ω. Note that $G \neq H_2, H_3, H_4$. Then G contains one of graphs $\overline{H}_3$ and M_ω^2 as a subgraph, where $\overline{H}_3$ is obtained from H_3 by adding an edge between two pendant vertices. From Lemma 5, we have $\rho(G) \geq \rho(\overline{H}_3) > \rho(H_4)$. From Lemmas 2 and 7, $\rho(G) > \rho(M_\omega^2) > \rho(H_4)$. □

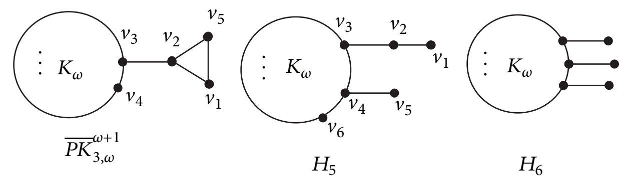


Figure 9: Graphs $\overline{PK}_{3,\omega}^{\omega+1}$, H_5, and H_6.

Let H_5 be the graph obtained from H_2 and an isolated vertex by adding an edge between a pendant vertex of H_2 and the isolated vertex; let $\overline{PK}_{3,\omega}^{\omega+1}$ and H_6 be the graphs as shown in Figure 9.

Lemma 17. *Let $\overline{PK}_{3,\omega}^{\omega+1}$ and H_5 be the graphs defined as above (see Figure 9). Then*

$$\rho(H_5) > \rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right), \quad (\omega \geq 4). \quad (35)$$

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of $\overline{PK}_{3,\omega}^{\omega+1}$, where x_i corresponds to v_i. It is easy to see that $x_1 = x_5$. From $AX = \rho(\overline{PK}_{3,\omega}^{\omega+1})X$, we have

$$\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) x_1 = x_1 + x_2,$$
$$\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) x_2 = 2x_1 + x_3,$$
$$\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) x_3 = x_2 + (\omega - 1) x_4, \quad (36)$$
$$\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) x_4 = x_3 + (\omega - 2) x_4.$$

From above equations, we have

$$x_2 = \left(\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - 1\right) x_1,$$
$$x_4 = \frac{\rho^2\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - \rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - 2}{\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - \omega + 2} x_1. \quad (37)$$

Then for $\omega \geq 4$, we have

$$\begin{aligned} \rho(H_5) - \rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) &\geq X^T A(H_5) X - X^T A\left(\overline{PK}_{3,\omega}^{\omega+1}\right) X \\ &= 2x_1(x_4 - x_2 - x_1) \\ &= 2\frac{(\omega - 3)\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - 2}{\rho\left(\overline{PK}_{3,\omega}^{\omega+1}\right) - \omega + 2} x_1 > 0. \end{aligned} \quad (38)$$

The result follows. □

Lemma 18. *Let H_5 and H_6 be the graphs defined as above (see Figure 9). Then*

$$\rho(H_6) > \rho(H_5), \quad (\omega \geq 4). \quad (39)$$

Proof. For $\omega = 4$, by direct calculation, we have $\rho(H_6) > \rho(H_5)$. In the following, we suppose that $\omega \geq 5$. Then, from Lemmas 2 and 3, we have $\omega > \rho(H_5) > \rho(K_\omega) = \omega - 1 \geq 4$. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the Perron vector of H_5, where x_i corresponds to v_i. From $AX = \rho(H_5)X$, we have

$$\begin{gathered} \rho(H_5)x_1 = x_2, \\ \rho(H_5)x_2 = x_1 + x_3, \\ \rho(H_5)x_3 = x_2 + x_4 + (\omega-2)x_6, \\ \rho(H_5)x_4 = x_3 + x_5 + (\omega-2)x_6, \\ \rho(H_5)x_5 = x_4, \\ \rho(H_5)x_6 = x_3 + x_4 + (\omega-3)x_6. \end{gathered} \tag{40}$$

From above equations, we have for $\omega > \rho(H_5) > \omega - 1 \geq 4$,

$$\begin{aligned} x_6 &= \frac{\rho^2(H_5) - 1}{\rho(H_5) - \omega + 3}x_1 \\ &\quad + \frac{(\rho^2(H_5) + \rho(H_5))(\rho^2(H_5) - 1) - \rho^2(H_5)}{(\rho(H_5) - \omega + 3)(\rho^2(H_5) + \rho(H_5) - 1)}x_1 \\ &> \frac{\rho^2(H_5) - 1}{3}x_1 > \rho(H_5)x_1 = x_2. \end{aligned} \tag{41}$$

Then, from Lemma 5, we have $\rho(H_6) = \rho(H_5 - v_1v_2 + v_1v_6) > \rho(H_5)$. □

Let H_7 be the graph obtained from H_3 and an isolated vertex by adding an edge between v_ω and the isolated vertex; let H_8 be the graph obtained from H_3 and an isolated vertex by adding an edge between v_2 and the isolated vertex; let H_9 be the graph obtained from H_3 and an isolated vertex by adding an edge between one pendant vertex and the isolated vertex; and let H_{10} be the graph obtained from $PK_{3,\omega}^{\omega+1}$ and an isolated vertex by adding an edge between $v_{\omega+1}$ and the isolated vertex (see Figure 10).

Theorem 19. *Let $PK_{3,\omega}$, $PK_{3,\omega}^{\omega+1}$, $\overline{PK}_{3,\omega}^{\omega+1}$, and H_5 be the graphs defined as above (see Figures 1, 4, 5, and 9). Among all connected graphs on n vertices with maximum clique size ω and $n = \omega + 3$ ($\omega \geq 4$), the first four smallest spectral radii are obtained for $PK_{3,\omega}$, $PK_{3,\omega}^{\omega+1}$, $\overline{PK}_{3,\omega}^{\omega+1}$, and H_5, respectively.*

Proof. From Lemmas 2, 4, and 17, we have

$$\rho(H_5) > \rho(\overline{PK}_{3,\omega}^{\omega+1}) > \rho(PK_{3,\omega}^{\omega+1}) > \rho(PK_{3,\omega}). \tag{42}$$

Thus, we only need to prove that $\rho(G) > \rho(H_5)$ if $G \neq PK_{3,\omega}$, $PK_{3,\omega}^{\omega+1}$, $\overline{PK}_{3,\omega}^{\omega+1}$, and H_5. We distinguish the following four cases.

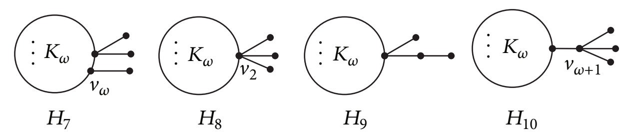


Figure 10: Graphs H_7, H_8, H_9, H_{10}.

Case 1. There exists exactly one vertex outside of K_ω that is adjacent to only one vertex of K_ω. Then G must be one of graphs $PK_{3,\omega}$, $PK_{3,\omega}^{\omega+1}$, and $\overline{PK}_{3,\omega}^{\omega+1}$.

Case 2. There exists one vertex outside of K_ω that is adjacent to at least two vertices of K_ω. Then G contains M_ω^2 (see Figure 8) as a proper subgraph. From Lemmas 2 and 7, we have $\rho(G) > \rho(M_\omega^2) > \rho(H_5)$.

Case 3. If there exactly exist two vertices outside of K_ω that are adjacent to some vertices of K_ω, then G contains H_5 or H_9 (see Figures 9 and 10) as a subgraph. If G contains H_9 as a subgraph, then, from Lemmas 2 and 5, we have $\rho(G) \geq \rho(H_9) > \rho(H_5)$. If G contains H_5 as a subgraph, note that $G \neq H_5$, then, from Lemma 2, we have $\rho(G) > \rho(H_5)$.

Case 4. If there exist three vertices outside of K_ω that are adjacent to some vertices of K_ω, then G contains one of graphs H_6, H_7, and H_8 (see Figures 9 and 10) as a subgraph. From Lemmas 5 and 18, we have $\rho(H_8) > \rho(H_7) > \rho(H_6) > \rho(H_5)$. Then, from Lemma 2, we have $\rho(G) > \rho(H_5)$. □

Lemma 20. *Let $PK_{n-\omega,\omega}^{i}$ and $\overline{PK}_{4,\omega}^{\omega+2}$ be the graphs defined as above (see Figures 4 and 5). Then*

$$\rho(PK_{4,\omega}^{\omega+1}) > \rho(\overline{PK}_{4,\omega}^{\omega+2}), \quad (\omega \geq 4). \tag{43}$$

Proof. From Lemma 6, we have

$$\begin{aligned} \Phi(\overline{PK}_{4,\omega}^{\omega+2}) &= (x^4 - 4x^2 - 2x + 1)\Phi(K_\omega) \\ &\quad - (x^3 - 3x - 2)\Phi(K_{\omega-1}) \\ &= f_5(x)\Phi(K_\omega) - f_6(x)\Phi(K_{\omega-1}); \\ \Phi(PK_{4,\omega}^{\omega+1}) &= (x^4 - 3x^2 + 1)\Phi(K_\omega) \\ &\quad - (x^3 - x)\Phi(K_{\omega-1}) \\ &= f_7(x)\Phi(K_\omega) - f_8(x)\Phi(K_{\omega-1}). \end{aligned} \tag{44}$$

Then, we have

$$\begin{aligned} &f_7(x)\Phi(\overline{PK}_{4,\omega}^{\omega+2}) - f_5(x)\Phi(PK_{4,\omega}^{\omega+1}) \\ &\quad = (f_5(x)f_8(x) - f_6(x)f_7(x))\Phi(K_{\omega-1}) \\ &\quad = (x^5 - 5x^3 - 4x^2 + 2x + 2)\Phi(K_{\omega-1}) \\ &\quad = R_2(x)\Phi(K_{\omega-1}). \end{aligned} \tag{45}$$

For $x > \omega - 1$ ($\omega \geq 4$), we have

$$f_5(x) > 0, \qquad f_7(x) > 0, \qquad R_2(x) > 0, \qquad \Phi(K_{\omega-1}) > 0. \tag{46}$$

From Lemma 2, we have $\rho(\overline{PK}_{4,\omega}^{\omega+2}) > \rho(K_\omega) = \omega - 1$ and $\rho(PK_{4,\omega}^{\omega+1}) > \rho(K_\omega) = \omega - 1$. Thus, for $x > \omega - 1$ ($\omega \geq 4$), we have $f_7(x)\Phi(\overline{PK}_{4,\omega}^{\omega+2}) - f_5(x)\Phi(PK_{4,\omega}^{\omega+1}) > 0$. Then $\rho(PK_{4,\omega}^{\omega+1}) > \rho(\overline{PK}_{4,\omega}^{\omega+2})$, ($\omega \geq 4$). □

Lemma 21. *Let $PK_{n-\omega,\omega}^{i}$ and H_2 be the graphs defined as above (see Figures 3 and 4). Then*

$$\rho(H_2) > \rho(PK_{4,\omega}^{\omega+1}), \quad (\omega \geq 3). \tag{47}$$

Proof. For $\omega = 3, 4, 5$, by direct calculation, we have $\rho(H_2) > \rho(PK_{4,\omega}^{\omega+1})$. In the following, we suppose that $\omega \geq 6$. From Lemma 6, we have

$$\begin{aligned}\Phi(PK_{4,\omega}^{\omega+1}) &= (x+1)^{\omega-2}\left[x^6 - (\omega-2)x^5 - (\omega+3)x^4 \right.\\ &\quad + (4\omega-8)x^3 + (3\omega-1)x^2 \\ &\quad \left. - (2\omega-4)x - \omega + 1\right] \\ &= (x+1)^{\omega-2} g_3(x).\end{aligned} \tag{48}$$

For $\omega \geq 6$, we have

$$\begin{aligned} g_3\left(\omega - 1 + \frac{1}{\omega^2}\right) &= -\omega^2 - \frac{13}{\omega} + \frac{4}{\omega^2} + \frac{17}{\omega^3} - \frac{24}{\omega^4} + \frac{6}{\omega^5} + \frac{14}{\omega^6} \\ &\quad - \frac{16}{\omega^7} + \frac{2}{\omega^8} + \frac{5}{\omega^9} - \frac{4}{\omega^{10}} + \frac{1}{\omega^{12}} + 7 < 0; \\ g_3\left(\omega - 1 + \frac{2}{\omega^2}\right) &= \omega^3 - 5\omega^2 + 2\omega - \frac{58}{\omega} + \frac{20}{\omega^2} + \frac{108}{\omega^3} - \frac{192}{\omega^4} + \frac{48}{\omega^5} \\ &\quad + \frac{192}{\omega^6} - \frac{256}{\omega^7} + \frac{32}{\omega^8} + \frac{160}{\omega^9} - \frac{128}{\omega^{10}} + \frac{64}{\omega^{12}} + 24 > 0. \end{aligned} \tag{49}$$

From Lemmas 1 and 3, we have $\omega > \rho(PK_{4,\omega}^{\omega+1}) \geq \rho(K_\omega) = \omega - 1 \geq \lambda_2(PK_{4,\omega}^{\omega+1})$. Then from (49) we have $\omega - 1 + 2/\omega^2 > \rho(PK_{4,\omega}^{\omega+1}) > \omega - 1 + 1/\omega^2$. From the proof of Lemma 8, we have $\rho(H_2) > \omega - 1 + 2/\omega^2$ ($\omega \geq 6$). The result follows. □

Theorem 22. *Among all connected graphs on n vertices with maximum clique size ω and $n = \omega + 4$ ($\omega \geq 4$), the first four smallest spectral radii are obtained for $PK_{4,\omega}$, $PK_{4,\omega}^{\omega+2}$, $\overline{PK}_{4,\omega}^{\omega+2}$, and $PK_{4,\omega}^{\omega+1}$ (see Figures 1, 4, and 5), respectively.*

Proof. Let G be a connected graph with maximum clique size $\omega \geq 4$ and $n = \omega + 4$ vertices. Suppose that K_ω is a maximum clique of G. From Lemmas 2, 4, and 20, we have

$$\rho(PK_{4,\omega}^{\omega+1}) > \rho(\overline{PK}_{4,\omega}^{\omega+2}) > \rho(PK_{4,\omega}^{\omega+2}) > \rho(PK_{4,\omega}). \tag{50}$$

Thus, we only need to prove that $\rho(G) > \rho(PK_{4,\omega}^{\omega+1})$ if $G \neq PK_{4,\omega}$, $PK_{4,\omega}^{\omega+2}$, $\overline{PK}_{4,\omega}^{\omega+2}$, $PK_{4,\omega}^{\omega+1}$. We distinguish the following three cases.

Case 1. There exists exactly one vertex outside of K_ω that is adjacent to one vertex of K_ω.

Subcase 1. Suppose that $G - V(K_\omega)$ is a tree. If G contains exactly one pendant vertex, then $G = PK_{4,\omega}$. If G contains exactly two pendant vertices, then $G = PK_{4,\omega}^{\omega+1}$ or $G = PK_{4,\omega}^{\omega+2}$. If G contains three pendant vertices, then $G = H_{10}$ (see Figure 10). From Lemma 4, we have $\rho(H_{10}) > \rho(PK_{4,\omega}^{\omega+1})$.

Subcase 2. Suppose that $G - V(K_\omega)$ contains a cycle. If $G - V(K_\omega)$ contains C_4, then G contains H_{11} as a subgraph, where H_{11} is obtained from $PK_{4,\omega}^{\omega+1}$ by adding an edge between two pendant vertices. From Lemma 2, we have $\rho(H_{11}) > \rho(PK_{4,\omega}^{\omega+1})$. If $G - V(K_\omega)$ does not contain C_4, then $G = \overline{PK}_{4,\omega}^{\omega+2}$ or G contains $\overline{PK}_{3,\omega}^{\omega+1}$ as a proper subgraph. From Lemmas 2 and 7, we have $\rho(\overline{PK}_{3,\omega}^{\omega+1}) > \rho(H_{11}) > \rho(PK_{4,\omega}^{\omega+1})$. Note that $G \neq \overline{PK}_{4,\omega}^{\omega+2}$. Thus, we have $\rho(G) > \rho(PK_{4,\omega}^{\omega+1})$.

Case 2. There exists at least one vertex outside of K_ω that is adjacent to at least two vertices of K_ω. Then G contains M_ω^2 (see Figure 8) as a subgraph. From Lemmas 2, 7, and 21, we have $\rho(G) > \rho(M_\omega^2) > \rho(H_2) > \rho(PK_{4,\omega}^{\omega+1})$.

Case 3. There exist at least two vertices outside of K_ω that are adjacent to some vertices of K_ω. Then G contains H_2 or H_3 as a subgraph (see Figures 3 and 7). From Lemmas 2, 5, and 21, we have $\rho(H_3) > \rho(H_2) > \rho(PK_{4,\omega}^{\omega+1})$. Thus, from Lemma 2, we have $\rho(G) > \rho(PK_{4,\omega}^{\omega+1})$. □

4. Conclusion

In this paper, the first four graphs, which have the smallest values of the spectral radius among all connected graphs of order n with maximum clique size $\omega \geq 2$, are determined.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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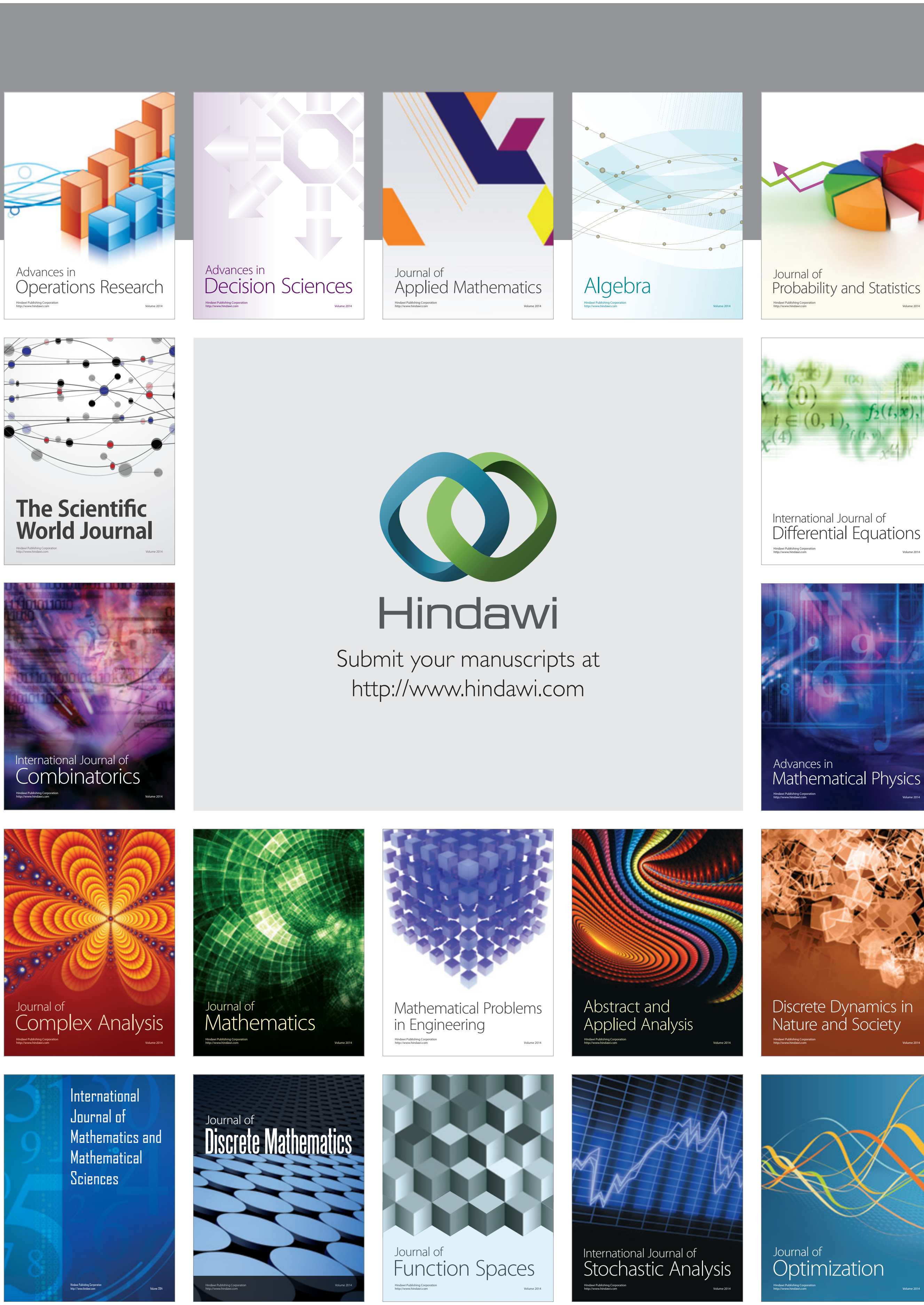
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