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Research Article

On Improving Ratio/Product Estimator by Ratio/Product-cum-Mean-per-Unit Estimator Targeting More Efficient Use of Auxiliary Information

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To achieve a more efficient use of auxiliary information we propose single-parameter ratio/product-cum-mean-per-unit estimators for a finite population mean in a simple random sample without replacement when the magnitude of the correlation coefficient is not very high (less than or equal to 0.7). The first order large sample approximation to the bias and the mean square error of our proposed estimators are obtained. We use simulation to compare our estimators with the well-known sample mean, ratio, and product estimators, as well as the classical linear regression estimator for efficient use of auxiliary information. The results are conforming to our motivating aim behind our proposition.

1. Introduction and Notation

This paper addresses the problem of efficiently estimating the population mean, using auxiliary information. A fairly large simple random sample of size n is selected without replacement from, say, a large bivariate population of size N (which could, reasonably, be thought to have come from a normal superpopulation), with the sampling fraction $f = n/N$, $N \gg n$, so that f is negligible. Quite often, we have surveys where some auxiliary variable X may be relatively less expensive to observe than the main variable Y. In order to have a survey estimate of the population mean $\overline{Y}$ of the main variable, assuming knowledge of the population mean $\overline{X}$ of the auxiliary variable, the following estimators are well known.

The ratio estimator:

$$\hat{\overline{Y}}_R = \hat{R}\overline{X}. \quad (1)$$

The product estimator:

$$\hat{\overline{Y}}_P = \frac{\hat{P}}{\overline{X}}. \quad (2)$$

Here $\hat{R} = \overline{y}/\overline{x}$ is the estimate of the ratio of the population means and $\hat{P} = \overline{y} \cdot \overline{x}$ is the estimate of the product of the population means, $\overline{y}$ and $\overline{x}$ being unweighted sample means of the two variables, respectively. Usually, the variability of $\overline{x}$ is less than that of $\overline{y}$.

It is straightforward to derive first order approximations to the bias and mean square error of these estimators. Let $C_Y = S_Y/\overline{Y}$ and $C_X = S_X/\overline{X}$ be the population coefficients of variation of Y and X, respectively, where

$$S_Y^2 = (N-1)^{-1} \sum \left(Y_i - \overline{Y}\right)^2, \quad S_X^2 = (N-1)^{-1} \sum \left(X_i - \overline{X}\right)^2 \quad (3)$$

are the population variances of Y and X, respectively. Let $\overline{y} = \overline{Y}(1+e)$ and $\overline{x} = \overline{X}(1+e')$, where the errors e and e' can be positive or negative, so that $E(e) = E(e') = 0$. It is known that, for simple random sample without replacement, $\text{Var}(e) = (1-f)C_Y^2/n$, $\text{Var}(e') = (1-f)C_X^2/n$, and $\text{Cov}(e,e') = (1-f)\rho C_X C_Y/n$ where ρ is the correlation coefficient between the variables (P. V. Sukhatme and B. V. Sukhatme [1]). Further, to validate our first order large sample approximations, we assume that the sample is large enough to make $|e|$ and $|e'|$ so small that the terms involving e and/or e' to a degree higher than two are negligible, an assumption which is not unrealistic.

Substituting the expressions for $\overline{y}$ and $\overline{x}$ in terms of e and e' in (1) we have

$$\hat{\overline{Y}}_R = (1+e)\left(1+e'\right)^{-1}\overline{Y}. \tag{4}$$

Assuming that $|e'| < 1$, we expand $(1+e')^{-1}$ to obtain

$$\text{Bias}\left(\hat{\overline{Y}}_R\right) = \frac{(1-f)}{n}\overline{Y}C_X\left(C_X - \rho C_Y\right), \quad \text{up to the first order of approximation, } O\left(\frac{1}{n}\right). \tag{5}$$

Since $(1-f)/n \rightarrow 0$ as $n \rightarrow \infty$, we have that the ratio estimator is asymptotically unbiased up to $O(1/n)$. Similarly we have that the product estimator is asymptotically unbiased (Murthy [2]). Also,

$$\begin{aligned} \text{MSE}\left(\hat{\overline{Y}}_R\right) &= E\left(\hat{\overline{Y}}_R - \overline{Y}\right)^2 \approx E\left[\hat{\overline{Y}}^2\left(e^2 + e'^2 - 2ee'\right)\right], \\ &\quad \text{up to the first order of approximation} \\ &= \left\{\overline{Y}^2\frac{(1-f)}{n}\right\}\left\{C_Y^2 + C_X^2 - 2\rho C_X C_Y\right\} \\ &= \text{Var}\left(\overline{y}\right) + \frac{(1-f)}{n}\overline{Y}^2\left\{C_X^2 - 2\rho C_X C_Y\right\}. \end{aligned} \tag{6}$$

Thus up to order $O(1/n)$ of approximation, $\text{MSE}(\hat{\overline{Y}}_R) < \text{Var}(\overline{y})$ if and only if $C_X^2 - 2\rho C_X C_Y < 0$, or if and only if $C > 1/2$, where $C = \rho C_Y/C_X$. (It is worth noting here that because of long association with the experimental data, C is guessable.) Similarly we have $\text{MSE}(\hat{\overline{Y}}_P) < \text{Var}(\overline{y})$ if and only if $C < -1/2$ (Murthy [2]). Thus the ratio and product estimators are relatively more efficient than the usual unbiased estimator (u.u.e) sample mean when $C > 1/2$ and $C < -1/2$, respectively. Consequently, $\hat{\overline{Y}}_R/\hat{\overline{Y}}_P$ fail to improve $\overline{y}$ (by using auxiliary information) when $-1/2 \leq C \leq 1/2$.

Also we cannot ignore the classically well-known linear regression estimator, say $\hat{\overline{Y}}_{\text{LR}}$:

$$\hat{\overline{Y}}_{\text{LR}} = \overline{y} + \hat{\beta}\left(\overline{X} - \overline{x}\right), \quad \text{where } \hat{\beta} = \frac{\sum_{i=1}^{n}\left(y_i - \overline{y}\right)\left(x_i - \overline{x}\right)}{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}. \tag{7}$$

If we recall the ANOVA of linear regression analysis, we must remember that the residual sum of squares for $\hat{\overline{Y}}_{\text{LR}}$ is $S_Y^2(1-\rho^2)$ (Cochran [3]). Thus when $|\rho|$ is high (say $\rho > 0.7$ or $\rho < -0.7$), linear regression estimator is most likely to be more efficient than ratio/product estimators in using the auxiliary information (via auxiliary variable X). We aim at improving use of auxiliary information on $\overline{y}$ when $-1/2 \leq C \leq 1/2$; $\hat{\overline{Y}}_R$ when $C > 1/2$; $\hat{\overline{Y}}_P$ when $C < -1/2$; and $\hat{\overline{Y}}_{\text{LR}}$ when $|\rho| \leq 0.7$.

2. Our Proposed Estimators

Because $\hat{\overline{Y}}_R$ and $\hat{\overline{Y}}_P$ are relatively more efficient than $\overline{y}$ when $C > 1/2$ and $C < -1/2$, respectively, we try the following single parameter linear combinations of $\hat{\overline{Y}}_R$ and $-\overline{y}$, as well as $\hat{\overline{Y}}_P$ and $-\overline{y}$ to propose the estimators:

(i) Shirley-Sahai-Dialsingh-ratio-cum-mean, say $\hat{\overline{Y}}_{\text{SSDR}}$:

$$\hat{\overline{Y}}_{\text{SSDR}} = (1+\theta)\hat{\overline{Y}}_R - \theta\overline{y}. \tag{8}$$

(ii) Shirley-Sahai-Dialsingh-product-cum-mean, say $\hat{\overline{Y}}_{\text{SSDP}}$:

$$\hat{\overline{Y}}_{\text{SSDP}} = (1+\theta)\hat{\overline{Y}}_P - \theta\overline{y}. \tag{9}$$

In (8) and (9), θ is the design parameter for our proposed estimators to be assigned an optimal value, for example, so as to minimize the first order of MSE, $M_1(\circ)$, as in our case. Note that when $\theta = 0$, $\hat{\overline{Y}}_{\text{SSDR}} = \hat{\overline{Y}}_R$ and $\hat{\overline{Y}}_{\text{SSDP}} = \hat{\overline{Y}}_P$. As remarked earlier, quite often a good guess of C is available from which we can give a suitable value to θ.

3. Sampling Bias and Mean Square Error of the Proposed Estimators

We derive the first order approximation, $B_1(\hat{\overline{Y}}_{\text{SSDR}})$, to the bias of $\hat{\overline{Y}}_{\text{SSDR}}$. Using the notation introduced in Section 1 and substituting the expressions for $\overline{y}$ and $\overline{x}$ in terms of e and e' in (8) we have

$$\hat{\overline{Y}}_{\text{SSDR}} = \overline{Y}(1+e)\left(1-\theta e'\right)\left(1+e'\right)^{-1}. \tag{10}$$

It is realistic practically to suppose that $|e'| < 1$ so that $(1+e')^{-1}$ is expandable. Then to the first order of approximation $\{O(1/n)\}$, the bias of $\hat{\overline{Y}}_{\text{SSDR}}$ is given by

$$\begin{aligned} B_1\left(\hat{\overline{Y}}_{\text{SSDR}}\right) &= E\left(\hat{\overline{Y}}_{\text{SSDR}} - \overline{Y}\right) \\ &\approx \overline{Y}\left\{(1-\theta)E\left(e'^2\right) - (1+\theta)E\left(ee'\right)\right\} \\ &= B_0\left\{a(2-F) - \rho F\right\}, \end{aligned} \tag{11}$$

where $a = C_X/C_Y$, $F = 1+\theta$, and $B_0 = a(1-f)\overline{Y}C_Y^2/n$. $B_0 \rightarrow 0$, as $n \rightarrow \infty$; therefore, $\hat{\overline{Y}}_{\text{SSDR}}$ is asymptotically unbiased up to $O(1/n)$.

To compute the MSE of $\hat{\overline{Y}}_{\text{SSDR}}$ we have

$$\begin{aligned} M_1\left(\hat{\overline{Y}}_{\text{SSDR}}\right) &= E\left(\hat{\overline{Y}}_{\text{SSDR}} - \overline{Y}\right)^2 \approx \overline{Y}^2 E\left\{\left(e - Fe'\right)^2\right\}, \\ &\quad \text{up to the first order of approximation } \left\{O\left(\frac{1}{n}\right)\right\} \\ &= \left\{\overline{Y}^2\frac{(1-f)}{n}\right\}\left\{C_Y^2 + F^2C_X^2 - 2F\rho C_X C_Y\right\} \\ &= V_0\left\{1 + a^2F^2 - 2aF\rho\right\}, \end{aligned} \tag{12}$$

where $V_0 = \text{Var}(\overline{y}) = ((1-f)/n)S_Y^2$ and $S_Y = \overline{Y}C_Y$.

Table 1: Relative efficiencies (in %) of the estimators when $n = 30$ and $R = 0$.

ρ	C	$\hat{\hat{Y}}_{LR}$	$\hat{\hat{Y}}_{R}$	$\hat{\hat{Y}}_{SSDR}$
0.1	0.225	97.00	89.71	100.88
0.2	0.45	100.35	98.09	104.22
0.3	0.675	105.26	106.32	109.10
0.4	0.90	114.29	117.98	118.34
0.5	1.125	128.73	133.07	133.58
0.6	1.35	152.63	152.08	159.40
0.7	1.575	186.79	172.17	193.33

Table 2: Relative efficiencies (in %) of the estimators when $n = 30$ and $R = 0$.

ρ	C	$\hat{\hat{Y}}_{LR}$	$\hat{\hat{Y}}_{P}$	$\hat{\hat{Y}}_{SSDP}$
−0.1	−0.225	97.66	89.14	100.73
−0.2	−0..45	100.42	98.05	104.14
−0.3	−0.675	106.48	108.23	110.39
−0.4	−0.90	114.87	119.11	119.43
−0.5	−1.125	127.32	131.54	132.05
−0.6	−1.35	150.37	150.36	155.88
−0.7	−1.575	186.53	172.80	194.68

For large sample size, $M_1(\hat{\hat{Y}}_{SSDR})$ is minimum for $aF = \rho$. The optimal value of F is thus $\rho/a = C$. If a good guess of C, say C^*, is available, we use $\theta = C^* - 1$ in our proposed estimator (8), so that

$$\hat{\hat{Y}}_{SSDR} \approx C^* \frac{\overline{y}\overline{X}}{\overline{x}} - (C^* - 1)\,\overline{y}. \quad (13)$$

We deduce the large sample approximation for bias of $\hat{\hat{Y}}_{SSDP}$ in a similar manner:

$$\begin{aligned} \hat{\hat{Y}}_{SSDP} &= \overline{Y}(1+e)\left(1+(1+\theta)e'\right), \\ B_1\left(\hat{\hat{Y}}_{SSDP}\right) &= E\left(\hat{\hat{Y}}_{SSDP} - \overline{Y}\right) = \overline{Y}(1+\theta)E\left(ee'\right) \\ &= B_0\rho F, \end{aligned} \quad (14)$$

where F and B_0 are as before. $B_0 \to 0$, as $n \to \infty$; therefore $\hat{\hat{Y}}_{SSDP}$ is asymptotically unbiased.

To compute the MSE of $\hat{\hat{Y}}_{SSDP}$ we have

$$\begin{aligned} M_1\left(\hat{\hat{Y}}_{SSDP}\right) &= E\left(\hat{\hat{Y}}_{SSDP} - \overline{Y}\right)^2 \approx \overline{Y}^2 E\left\{\left(e + Fe'\right)^2\right\}, \\ &\text{up to the first order of approximation } \left\{O\left(\frac{1}{n}\right)\right\} \\ &= \left\{\overline{Y}^2 \frac{(1-f)}{n}\right\}\left\{C_Y^2 + F^2 C_X^2 + 2F\rho C_X C_Y\right\} \\ &= V_0\left\{1 + a^2F^2 + 2aF\rho\right\}, \\ &\quad \text{where } V_0 = \text{Var}\left(\overline{y}\right). \end{aligned} \quad (15)$$

Up to $O(1/n)$, $M_1(\hat{\hat{Y}}_{SSDP})$ is a minimum for $aF = -\rho$. The optimal value of F is thus $-\rho/a = -C$.

We use $\theta = -C^* - 1$ in our proposed estimator (9), where C^* is the guess of C. Thus,

$$\hat{\hat{Y}}_{SSDP} \approx -C^* \frac{\overline{x}\overline{y}}{\overline{X}} + (C^* + 1)\,\overline{y}. \quad (16)$$

4. Comparison of the Estimators

Apparently, no algebraic comparison of mean square errors is feasible. We, therefore, have a numerical setup under simulation to do so. Knowing C exactly is seldom tenable in practice. Consequently, we have to assume the availability of a guess value of C, which we have called C^*, defined by $C^* = C(1+R)$, where R designates the quantum of relative under guess/overguess. We have taken the following R values: 0, ±0.02, ±0.04, ±0.06, ±0.08, and ±0.10. We have also assumed that the parent population is very large, envisaged to have come from a superpopulation which is bivariate normal with the following parameters, therefore having the same parametric values:

$$\begin{aligned} \overline{Y} &= 20.00, \\ \overline{X} &= 15.00, \\ S_Y &= 3.00, \\ S_X &= 1.00. \end{aligned} \quad (17)$$

Consequently we have used the software **R** [4] to calculate the MSEs of each of the following estimators:

$$\begin{aligned} \hat{\hat{Y}}_R &= \hat{R}\overline{X}; \\ \hat{\hat{Y}}_P &= \frac{\hat{P}}{\overline{X}}; \\ \hat{\hat{Y}}_{SSDR} &= (1+\theta)\hat{\hat{Y}}_R - \theta\overline{y}; \\ \hat{\hat{Y}}_{SSDP} &= (1+\theta)\hat{\hat{Y}}_P - \theta\overline{y}, \\ \hat{\hat{Y}}_{LR} &= \overline{y} + \hat{\beta}\left(\overline{X} - \overline{x}\right). \end{aligned} \quad (18)$$

We use 10,000 replications of simulated sample sizes $n = 30$, 40, 60, 80, and 100. Hence we have compared the efficiencies of these estimators relative to $\overline{y}$ by using

$$\text{REff}(\circ) = \frac{\text{MSE}\left(\overline{y}\right)}{\text{MSE}(\circ)} \times 100\%. \quad (19)$$

Motivated by our desire to beat ratio/product estimators (implicitly, therefore $\hat{\hat{Y}}_{LR}$ also), we have, therefore, taken up

Table 3: Relative efficiencies (in %) of $\hat{\hat{Y}}_{\text{LR}}$, $\hat{\hat{Y}}_{R}$, and $\hat{\hat{Y}}_{\text{SSDR}}$ when $n = 30$.

$n = 30$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
$\hat{\hat{Y}}_{\text{LR}}$	97.00734	100.3539	105.2556	114.2872	128.7345	152.627	186.7908
$\hat{\hat{Y}}_{R}$	89.71481	98.09601	106.3240	117.9799	133.0705	152.0778	172.174
R value	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$
−0.1	100.8797	104.1726	109.0713	118.1939	133.1713	157.8555	191.6527
−0.08	100.8806	104.1888	109.0944	118.2593	133.3251	158.3010	192.2725
−0.06	100.8808	104.2015	109.109	118.3068	133.4430	158.6793	192.7517
−0.04	100.8801	104.2107	109.1151	118.3364	133.5250	158.9895	193.0882
−0.02	100.8787	104.2164	109.1125	118.3480	133.5709	159.2308	193.2804
0	100.8764	104.2186	109.1014	118.3416	133.5805	159.4025	193.3276
0.02	100.8733	104.2173	109.0818	118.3172	133.5539	159.5041	193.2295
0.04	100.8694	104.2125	109.0536	118.2749	133.4912	159.5355	192.9865
0.06	100.8646	104.2043	109.0169	118.2147	133.3924	159.4964	192.5998
0.08	100.8591	104.1925	108.9717	118.1366	133.2576	159.3871	192.0710
0.1	100.8527	104.1772	108.9179	118.0408	133.0872	159.2078	191.4025

Table 4: Relative efficiencies (in %) of $\hat{\hat{Y}}_{\text{LR}}$, $\hat{\hat{Y}}_{R}$, and $\hat{\hat{Y}}_{\text{SSDR}}$ when $n = 40$.

$n = 40$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
$\hat{\hat{Y}}_{\text{LR}}$	98.56872	101.6871	106.8427	116.5161	131.2267	153.2434	195.3283
$\hat{\hat{Y}}_{R}$	89.4024	97.99826	107.7805	119.1094	133.8761	150.5256	175.4088
R value	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$
−0.1	100.763	104.0057	110.1035	119.1169	133.9862	155.4611	197.6259
−0.08	100.7616	104.02	110.1465	119.1571	134.1563	155.7931	198.4408
−0.06	100.7593	104.0309	110.1807	119.2293	134.29	156.0572	199.1088
−0.04	100.7563	104.0385	110.206	119.2836	134.3872	156.2525	199.6271
−0.02	100.7525	104.0426	110.2224	119.3198	134.4476	156.3785	199.9932
0	100.7479	104.0433	110.2299	119.3379	134.4712	156.435	200.2054
0.02	100.7425	104.0406	110.2285	119.3379	134.4579	156.4217	200.2627
0.04	100.7362	104.0345	110.2182	119.3198	134.4078	156.3387	200.1649
0.06	100.7292	104.025	110.199	119.2837	134.3209	156.1862	199.9125
0.08	100.7214	104.0121	110.171	119.2295	134.1974	155.9647	199.5065
0.1	100.7127	103.9959	110.134	119.1573	134.0375	155.6746	198.9489

Table 5: Relative efficiencies (in %) of $\hat{\hat{Y}}_{\text{LR}}$, $\hat{\hat{Y}}_{R}$, and $\hat{\hat{Y}}_{\text{SSDR}}$ when $n = 60$.

$n = 60$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
$\hat{\hat{Y}}_{\text{LR}}$	99.78738	102.7511	106.9806	115.7236	133.4915	154.5581	191.2787
$\hat{\hat{Y}}_{R}$	90.7815	98.23825	106.5209	117.4447	135.1337	150.7982	172.6729
R value	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$	$\hat{\hat{Y}}_{\text{SSDR}}$
−0.1	101.0542	104.2512	109.2088	117.8291	135.2587	155.6597	192.4683
−0.08	101.0596	104.2689	109.2346	117.8792	135.4549	155.9771	193.1069
−0.06	101.0642	104.2832	109.2519	117.9111	135.6138	156.2254	193.6036
−0.04	101.068	104.2939	109.2606	117.925	135.7354	156.4038	193.9563
−0.02	101.0711	104.3011	109.2606	117.9206	135.8192	156.5118	194.1633
0	101.0733	104.3048	109.2521	117.8982	135.8653	156.5493	194.2238
0.02	101.0747	104.3051	109.2349	117.8576	135.8735	156.516	194.1373
0.04	101.0754	104.3017	109.2092	117.7989	135.8438	156.4121	193.9044
0.06	101.0753	104.2949	109.1749	117.7222	135.7762	156.2379	193.5261
0.08	101.0743	104.2846	109.132	117.6275	135.6709	155.9937	193.004
0.1	101.0726	104.2707	109.0806	117.515	135.5281	155.6804	192.3406

Table 6: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{\text{LR}}$, $\hat{\hat{\bar{Y}}}_{R}$, and $\hat{\hat{\bar{Y}}}_{\text{SSDR}}$ when $n = 80$.

$n = 80$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
$\hat{\hat{\bar{Y}}}_{\text{LR}}$	99.83416	102.7266	106.6169	115.6179	131.7743	154.1441	194.0998
$\hat{\hat{\bar{Y}}}_{R}$	91.45779	97.66782	105.0443	116.7154	133.2987	150.0163	174.2752
R value	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$
−0.1	101.3714	103.9561	108.1498	117.1127	133.4017	154.9814	194.8863
−0.08	101.3831	103.9678	108.1557	117.1581	133.5593	155.3240	195.5633
−0.06	101.3939	103.9759	108.1534	117.186	133.6809	155.6000	196.0938
−0.04	101.4039	103.9806	108.1429	117.1966	133.7663	155.8088	196.4753
−0.02	101.4131	103.9819	108.1240	117.1897	133.8154	155.9498	196.7060
0	101.4214	103.9796	108.0970	117.1654	133.828	156.0227	196.7849
0.02	101.4289	103.974	108.0617	117.1238	133.8042	156.0272	196.7115
0.04	101.4355	103.9648	108.0181	117.0648	133.744	155.9634	196.4864
0.06	101.4414	103.9522	107.9664	116.9885	133.6476	155.8314	196.1104
0.08	101.4463	103.9361	107.9065	116.8949	133.5149	155.6315	195.5853
0.1	101.4505	103.9165	107.8385	116.7842	133.3464	155.3643	194.9135

Table 7: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{\text{LR}}$, $\hat{\hat{\bar{Y}}}_{R}$, and $\hat{\hat{\bar{Y}}}_{\text{SSDR}}$ when $n = 100$.

$n = 100$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
$\hat{\hat{\bar{Y}}}_{\text{LR}}$	100.3458	102.4952	108.1571	117.9797	130.9376	154.1079	194.4502
$\hat{\hat{\bar{Y}}}_{R}$	90.15273	96.99257	106.6954	119.0344	131.7312	150.3096	174.1364
R value	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDR}}$
−0.1	101.0155	103.6111	109.4732	119.0467	131.8139	155.0562	195.1446
−0.08	101.0193	103.6156	109.4997	119.0954	131.9356	155.3614	195.8685
−0.06	101.0224	103.6166	109.5173	119.1661	132.0222	155.5981	196.4477
−0.04	101.0246	103.6142	109.526	119.2189	132.0735	155.7656	196.8795
−0.02	101.026	103.6084	109.5259	119.2537	132.0895	155.8634	197.1620
0	101.0265	103.5992	109.5169	119.2704	132.0701	155.8914	197.2938
0.02	101.0263	103.5865	109.499	119.2690	132.0154	155.8494	197.2745
0.04	101.0252	103.5704	109.4722	119.2496	131.9254	155.7375	197.104
0.06	101.0233	103.5510	109.4366	119.212	131.8303	155.5561	196.7832
0.08	101.0206	103.5281	109.3922	119.1565	131.7603	155.3057	196.3135
0.1	101.0170	103.5018	109.3389	119.083	131.7357	154.9868	195.6971

Table 8: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{\text{LR}}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{\text{SSDP}}$ when $n = 30$.

$n = 30$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -0.3$	$\rho = -0.4$	$\rho = -0.5$	$\rho = -0.6$	$\rho = -0.7$
$\hat{\hat{\bar{Y}}}_{\text{LR}}$	97.65509	100.4233	106.4819	114.8732	127.3161	150.3700	186.5302
$\hat{\hat{\bar{Y}}}_{P}$	89.13544	98.04504	108.2301	119.1056	131.536	150.3559	172.7989
R value	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\text{SSDP}}$
−0.1	100.7439	104.0964	110.2180	119.2223	131.6342	155.0711	192.8206
−0.08	100.7418	104.1117	110.2696	119.3008	131.7849	155.371	193.4796
−0.06	100.7389	104.1235	110.3126	119.3606	131.9022	155.6019	193.9968
−0.04	100.7351	104.1319	110.347	119.4016	131.9858	155.7635	194.3700
−0.02	100.7305	104.1368	110.3728	119.4239	132.0357	155.8551	194.5975
0	100.7252	104.1383	110.3899	119.4275	132.0516	155.8766	194.6782
0.02	100.719	104.1363	110.3984	119.4122	132.0337	155.8278	194.6118
0.04	100.7119	104.1309	110.3982	119.3781	131.9819	155.7090	194.3986
0.06	100.7041	104.122	110.3893	119.3253	131.8963	155.5205	194.0396
0.08	100.6955	104.1096	110.3718	119.2538	131.777	155.2627	193.5363
0.1	100.6860	104.0938	110.3456	119.1637	131.6243	154.9363	192.8910

Table 9: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{LR}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{SSDP}$ when $n = 40$.

$n = 40$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -0.3$	$\rho = -0.4$	$\rho = -0.5$	$\rho = -0.6$	$\rho = -0.7$
$\hat{\hat{\bar{Y}}}_{LR}$	98.81814	100.6730	107.4055	116.5276	128.5121	148.6026	187.6882
$\hat{\hat{\bar{Y}}}_{P}$	91.74324	97.31070	107.6696	119.5246	131.7449	148.3706	171.5955
R value	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$
−0.1	101.2921	103.7255	109.9459	119.5251	131.8443	152.1479	190.7650
−0.08	101.3028	103.733	109.9884	119.5365	131.9973	152.3256	191.3691
−0.06	101.3128	103.7371	110.0222	119.6137	132.1165	152.4345	191.8342
−0.04	101.322	103.7377	110.0472	119.6726	132.2019	152.4742	192.1584
−0.02	101.3303	103.735	110.0636	119.7132	132.2532	152.4448	192.3400
0	101.3379	103.7289	110.0711	119.7354	132.2704	152.3461	192.3784
0.02	101.3447	103.7194	110.0700	119.7392	132.2535	152.1787	192.2734
0.04	101.3506	103.7064	110.0601	119.7247	132.2025	151.9428	192.0254
0.06	101.3558	103.6901	110.0415	119.6918	132.1174	151.6391	191.6355
0.08	101.3602	103.6703	110.0141	119.6406	131.9984	151.2684	191.1055
0.1	101.3638	103.6472	109.9781	119.5711	131.8457	150.8318	190.4377

Table 10: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{LR}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{SSDP}$ when $n = 60$.

$n = 60$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -0.3$	$\rho = -0.4$	$\rho = -0.5$	$\rho = -0.6$	$\rho = -0.7$
$\hat{\hat{\bar{Y}}}_{LR}$	99.45873	102.7277	107.7386	113.9839	131.7095	153.2816	192.1807
$\hat{\hat{\bar{Y}}}_{P}$	90.94703	99.18717	107.1685	115.5287	133.5669	149.5516	173.2163
R value	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$
−0.1	101.1575	104.661	109.4991	116.2884	133.6678	154.36	193.0719
−0.08	101.1648	104.6883	109.5359	116.3009	133.821	154.6835	193.7006
−0.06	101.1714	104.7122	109.5642	116.2955	133.9376	154.9408	194.1851
−0.04	101.1771	104.7325	109.5841	116.2724	134.0173	155.1312	194.5234
−0.02	101.1821	104.7493	109.5955	116.2316	134.0601	155.2542	194.7139
0	101.1862	104.7626	109.5984	116.173	134.0659	155.3096	194.7557
0.02	101.1895	104.7724	109.5929	116.0968	134.0346	155.2971	194.6486
0.04	101.192	104.7786	109.5789	116.0031	133.9663	155.2168	194.3932
0.06	101.1937	104.7813	109.5564	115.8918	133.8612	155.0689	193.9905
0.08	101.1946	104.7805	109.5255	115.7631	133.7193	154.8538	193.4425
0.1	101.1947	104.7761	109.4862	115.6172	133.571	154.572	192.7515

Table 11: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{LR}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{SSDP}$ when $n = 80$.

$n = 80$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -0.3$	$\rho = -0.4$	$\rho = -0.5$	$\rho = -0.6$	$\rho = -0.7$
$\hat{\hat{\bar{Y}}}_{LR}$	100.2239	102.4000	108.7921	116.2027	131.1301	154.2985	190.0055
$\hat{\hat{\bar{Y}}}_{P}$	91.07041	96.95651	107.8218	117.6336	132.9936	150.1114	172.3139
R value	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$	$\hat{\hat{\bar{Y}}}_{SSDP}$
−0.1	101.2429	103.7526	110.1224	117.7630	133.1038	155.1954	191.5790
−0.08	101.2519	103.7577	110.1661	117.8328	133.2751	155.5554	192.1719
−0.06	101.2600	103.7593	110.2009	117.8855	133.4115	155.8493	192.6230
−0.04	101.2673	103.7574	110.2269	117.921	133.5129	156.0764	192.9303
−0.02	101.2738	103.752	110.2441	117.9394	133.579	156.236	193.0923
0	101.2795	103.743	110.2523	117.9404	133.6098	156.3278	193.1084
0.02	101.2843	103.7305	110.2516	117.9243	133.6052	156.3516	192.9785
0.04	101.2883	103.7144	110.2421	117.891	133.5652	156.3072	192.7031
0.06	101.2915	103.6948	110.2237	117.8404	133.4899	156.1948	192.2835
0.08	101.2939	103.6717	110.1964	117.7727	133.3793	156.0147	191.7216
0.1	101.2954	103.6451	110.1602	117.6880	133.2338	155.7673	191.0199

Table 12: Relative efficiencies (in %) of $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$ when $n = 100$.

$n = 100$	$\rho = -0.1$	$\rho = -0.2$	$\rho = -0.3$	$\rho = -0.4$	$\rho = -0.5$	$\rho = -0.6$	$\rho = -0.7$
$\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$	100.3811	102.6419	107.7698	118.1677	130.9040	152.8819	196.5500
$\hat{\hat{\bar{Y}}}_{P}$	90.55378	97.02887	106.4837	119.4323	131.4591	150.0207	173.0612
R value	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$	$\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$
−0.1	101.0559	103.5352	109.2741	119.4377	131.5400	154.2308	196.5671
−0.08	101.0611	103.5394	109.2983	119.4612	131.6587	154.4565	197.2139
−0.06	101.0654	103.5402	109.3137	119.5365	131.7426	154.6114	197.7604
−0.04	101.0689	103.5377	109.3204	119.5936	131.7916	154.6951	198.1543
−0.02	101.0716	103.5318	109.3184	119.6323	131.8055	154.7074	198.3935
0	101.0734	103.5226	109.3076	119.6528	131.7843	154.6483	198.4771
0.02	101.0745	103.5100	109.2881	119.6548	131.7281	154.5179	198.4046
0.04	101.0748	103.4941	109.2599	119.6385	131.6170	154.3166	198.1764
0.06	101.0742	103.4748	109.2229	119.6039	131.5610	154.0450	197.7935
0.08	101.0729	103.4522	109.1772	119.5509	131.5105	153.7037	197.2577
0.1	101.0707	103.4262	109.1229	119.4797	131.4656	153.2938	196.5715

the numerical simulation comparisons, for example, values of $|\rho|$: 0.1 (0.1) 0.7. For ρ positive, we compare $\hat{\hat{\bar{Y}}}_{R}$, $\hat{\hat{\bar{Y}}}_{\mathrm{SSDR}}$, and $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$, while for ρ negative we compare $\hat{\hat{\bar{Y}}}_{P}$, $\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$, and $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$.

5. Results and Discussion

The results of our simulations are tabulated in the Appendix. For a given value of n the relative efficiencies of $\hat{\hat{\bar{Y}}}_{R}$, $\hat{\hat{\bar{Y}}}_{P}$, and $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$ do not depend on R; they are, therefore, not included in the main body of the tables but are stated at the top for each value of ρ. For $n = 30$, 40, 60, 80, and 100, for each value of $R = 0$, ± 0.02, ± 0.04, ± 0.06, ± 0.08, and ± 0.10, we have given the values of $\mathrm{REff}(\hat{\hat{\bar{Y}}}_{\mathrm{SSDR}})$ for $\rho = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$, and 0.7 and of $\mathrm{REff}(\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}})$ for $\rho = -0.1, -0.2, -0.3, -0.4, -0.5, -0.6$, and -0.7.

As apparent, our proposed estimators $\hat{\hat{\bar{Y}}}_{\mathrm{SSDR}}$ and $\hat{\hat{\bar{Y}}}_{\mathrm{SSDP}}$ are consistently significantly better than $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$ and $\hat{\hat{\bar{Y}}}_{R}$ (or $\hat{\hat{\bar{Y}}}_{P}$, as the case may be).

For illustrative purposes, we highlight below the relative efficiency values for the various values of ρ, for the cases when $n = 30$ and $R = 0$. To lessen the obscurity in the results, we have rounded these values to two decimal places. We also include a column for the value of $C = \rho C_Y / C_X$.

Tables 1 and 2 illustrate very well the relative betterment achieved by our proposed estimators vis-à-vis $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$ and $\hat{\hat{\bar{Y}}}_{R}$ (or $\hat{\hat{\bar{Y}}}_{P}$, as the case may be). Notably, when $|C|$ is *not* greater than 1/2, our estimators are more efficient than $\bar{y}$ even though $\hat{\hat{\bar{Y}}}_{R}$ or $\hat{\hat{\bar{Y}}}_{P}$ (as the case may be) is worse than $\bar{y}$ which does not even use auxiliary information. Also when $|C|$ is significantly less than 1/2, our estimators are more efficient than $\bar{y}$ even though $\hat{\hat{\bar{Y}}}_{\mathrm{LR}}$ is worse than $\bar{y}$ (i.e., it fails to use auxiliary information rightly)!

6. Conclusion

Our results conform to our motivating aim of achieving more efficient use of auxiliary information. Many other authors, such as Sahai [5] and Chami et al. [6], have suggested efficient variants of ratio and product estimators. In future work we are engaged in comparing these estimators and in trying even better estimators, like the proposed ones, which will be not only more efficient relatively, but also, possibly, more robust against the possible over/underguess of the key-population parameter C.

Appendix

See Tables 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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