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PRINCIPAL TOROIDAL BUNDLES OVER CAUCHY-RIEMANN PRODUCTS

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ABSTRACT. The main result we obtain is that given $\pi : N \to M$ a T^s-subbundle of the generalized Hopf fibration $\bar{\pi} : H^{2n+s} \to \mathbb{C}P^n$ over a Cauchy-Riemann product $i : M \subseteq \mathbb{C}P^n$, i.e. $j : N \subseteq H^{2n+s}$ is a diffeomorphism on fibres and $\bar{\pi} \circ j = i \circ \pi$, if s is even and N is a closed submanifold tangent to the structure vectors of the canonical $\mathcal{S}$-structure on H^{2n+s} then N is a Cauchy-Riemann submanifold whose Chen class is non-vanishing.

KEY WORDS AND PHRASES. Principal toroidal bundle, $\mathcal{S}$-manifold, generalized Hopf fibration, framed C.R. submanifold, characteristic form.
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1.- INTRODUCTION AND STATEMENT OF RESULTS.

As a tentative of unifying the concepts of complex and anti-invariant submanifolds of an almost Hermitian manifold, A. BEJANCU, [1], has introduced the notion of Cauchy-Riemann (C.R.) submanifold. This has soon proved to possess a largely rich number of geometrical properties; e.g. by a result of D.E.BLAIR & B. Y.CHEN, [2], any C.R. submanifold of a Hermitian manifold is a Cauchy-Riemann manifold, in the sense of A.ANDREOTTI & C.D.HILL, [3].

Let M^{2n+s} be a real (2n+s)-dimensional manifold carrying a metrical f-structure $(\underline{f}, \xi_a, \eta_a, \mathcal{G})$, $1 \le a \le s$, with complemented frames, cf. [4]. A submanifold $j : N \to M^{2n+s}$ is said to be a *framed C.R. submanifold* if it is tangent to each structure vector ξ_a of M^{2n+s} and it carries a pair of complementary (with respect to $G = j^* \mathcal{G}$) smooth distributions $\mathcal{D}$, $\mathcal{D}^\perp$ such that $\underline{f}_x(\mathcal{D}_x) \subseteq \mathcal{D}_x$, $\underline{f}_x(\mathcal{D}^\perp_x) \subseteq T_x(N)^\perp$, for all $x \in N$, where $T(N)^\perp \to N$ stands for the normal bundle of j. Cf. I.MIHAI, [5], L.ORNEA, [6]. Since f-structures are known to generalize both almost complex (s=0) structures and almost contact (s=1) structures, the notion of framed C.R. submanifold containes those of a C.R. submanifold (see e.g. [7], p.83) of an almost Hermitian manifold and of a contact C.R. submanifold (see e.g. [7], p.48) of an almost contact metrical manifold.

Let $\bar{\pi} : H^{2n+s} \to \mathbb{C}P^n$ be the generalized Hopf fibration, as given by D.E.BLAIR, [8]. Leaving definitions momentarily aside we may formulate the following:

THEOREM A

i) *Let* N *be a framed C.R. submanifold of an* $\mathcal{S}$*-manifold* M^{2n+s}. *Then the* f*-anti-invariant distribution* $\mathcal{D}^{\perp}$ *of* N *is completely integrable.*

ii) *Any framed C.R. submanifold of* H^{2n+s}, *(carrying the standard* $\mathcal{S}$*-structure) is either a C.R. submanifold* (s *even) or a contact C.R. submanifold* (s *odd). The converse holds.*

iii) *Let* N *be an* f*-invariant (i.e.* $\mathcal{D}^{\perp} = (0)$*) submanifold of* H^{2n+s}. *Then* N *is totally-geodesic if and only if it is an* $\mathcal{S}$*-manifold of constant* f*-sectional curvature* $1 - \frac{3}{4}s$.

iv) *Any* f*-invariant submanifold of* H^{2n+s} *having a parallel second fundamental form is totally-geodesic.*

It is known that compact regular contact manifolds are S^1- principal fibre bundles over symplectic manifolds, cf. W.M. BOOTHBY & H.C.WANG, [9]. Eversince this (today classical) paper has been published, several "Boothby-Wang type" theorems have been established, cf. e.g. A.MORIMOTO, [10], for the case of normal almost contact manifolds, S.TANNO, [11], for contact manifolds in the non-compact case; more recently, we may cite a result of I.VAISMAN, [12], asserting that compact generalized Hopf manifolds with a regular Lee field may be fibred over Sasakian manifolds, etc.

There exists today a large literature, cf. K.YANO & M.KON, [7], concerned with the study of the geometry (of the second fundamental form) of a C.R. submanifold of a Kaehlerian ambient space. In particular, following the method of Riemannian fibre bundles (such as introduced by H.B.LAWSON, [13], towards studying submanifolds of complex space-forms, and developed successively by Y.MAEDA, [14], M.OKUMURA, [15]), K.YANO & M.KON, [16], have taken under study contact C.R. submanifolds of a Sasakian manifold M^{2n+1} (where M^{2n+1} is previously fibred over a Kaehlerian manifold M^{2n}) which are themselves S^1-fibrations over C.R. submanifolds of M^{2n}.

The last piece of the mosaic we are going to mend is the concept of canonical cohomology class (here after refered to as the *Chen class*) of a C.R. submanifold . Cf. B.Y.CHEN, [17], with any C.R. submanifold M of a Kaehlerian manifold there may be associated a cohomology class $c(M) \in H^{2p}(M; \mathbb{R})$, where p stands for the complex dimension of the holomorphic distribution of M. Although the canonical Hermitian structure (cf. [18]) of H^{2n+s} is never Kaehlerian (cf. [8], p.174) we show that the Chen class of a C.R. submanifold may be constructed as well and obtain the following :

THEOREM B

Let $j : N \to H^{2n+s}$ *be a closed (i.e. compact, orientable) submanifold tangent to the vector fields* ξ_a , $1 \le a \le s$, *of the canonical* $\mathcal{S}$*-structure on* H^{2n+s} *and assume there exists a* T^s*- principal bundle* $\pi : N \to M$ *over a Cauchy-*

Riemann product $(M, \mathscr{D}, \mathscr{D}^{\perp})$, $i : M \to \mathbb{C}P^n$, *($\mathscr{D}$ is the holomorphic distribution), such that* $\bar{\pi} \circ j = i \circ \pi$ *and* j *is a diffeomorphism on fibres. If* s *is even then* N *is a C.R. submanifold whose totally-real foliation is normal to the characteristic field of* H^{2n+s} *and whose Chen class* $c(N) \in H^{2p+s}(N; \mathbb{R})$, $p = \dim_{\mathbb{C}} \mathscr{D}$, *is non-vanishing.*

2.- NOTATIONS, CONVENTIONS AND BASIC FORMULAE.

Let M^{2n+s} be a real (2n+s)-dimensional C^{∞}-differentiable connected manifold. Let $\underline{f}$ be an *f-structure* on M^{2n+s}; i.e. a (1,1)-tensor field such that $\underline{f}^3 + \underline{f} = 0$ and $\mathrm{rank}(\underline{f}) = 2n$ everywhere on M^{2n+s}, cf. K.YANO, [19]. Assume that $\underline{f}$ has *complemented frames,* i.e. there exist the differential 1-forms η'_a and the dual vector fields ξ'_a on M^{2n+s}, i.e. $\eta'_a(\xi'_b) = \delta_{ab}$, $1 \leq a,b \leq s$, such that the following formulae hold:

$$\eta'_a \circ \underline{f} = 0, \qquad \underline{f}(\xi'_a) = 0, \qquad \underline{f}^2 = - I + \eta'_a \otimes \xi'^a. \tag{2.1}$$

Throughout, one adopts the convention $\eta'_a = \eta'^a$, $\xi'_a = \xi'^a$.The f- structure is *normal* if $[\underline{f}, \underline{f}] + (d\eta'_a) \otimes \xi'^a = 0$, where $[\underline{f}, \underline{f}]$ denotes the Nijenhuis torsion of $\underline{f}$, see e.g. H.NAKAGAWA, [20]. Let $\mathscr{G}$ be a *compatible* Riemaniann metric on M^{2n+s}, i.e. one satisfying:

$$\mathscr{G}(\underline{f}X, \underline{f}Y) = \mathscr{G}(X, Y) - \eta'_a(X)\, \eta'^a(Y). \tag{2.2}$$

Compatible metrics always exist, cf. D.E.BLAIR, [4]. Such $(\underline{f}\ , \xi'_a\ , \eta'_a\ , \mathscr{G})$ has often been called a *metrical f-structure with complemented frames.* Let $\underline{F}(X, Y) = \mathscr{G}(X, \underline{f}Y)$ be its *fundamental 2-form.* Throughout we assume M^{2n+s} to be an $\mathscr{S}$*-manifold,* cf. the terminology in [4], i.e. the given f-structure is normal, its fundamental 2-form is closed and there exist s smooth real-valued functions $\alpha_a \in C^{\infty}(M^{2n+s})$, $1 \leq a \leq s$, such that:

$$d\, \eta'_a = \alpha_a\, \underline{F}\ . \tag{2.3}$$

We shall need, cf. [4], [21], the following result. Let M^{2n+s}, $n > 1$, be a connected manifold carrying the $\mathscr{S}$-structure $(\underline{f}, \xi'_a, \eta'_a, \mathscr{G})$, $1 \leq a \leq s$. Then α_a are real constants, ξ'_a are Killing vector fields (with respect to $\mathscr{G}$) and the following relations hold:

$$\underline{D}_X\, \xi'_a = - \tfrac{1}{2}\, \alpha_a\, \underline{f}\, X \tag{2.4}$$

$$(\underline{D}_X\, \underline{f})\, Y = \tfrac{1}{2}\, \alpha^a\{[\mathscr{G}(X, Y) - \eta'_b(X)\, \eta'^b(Y)]\, \xi'_a - [X - \eta'_b(X)\, \xi'^b]\, \eta'_a(Y)\} \tag{2.5}$$

for any tangent vector fields X, Y on M^{2n+s}. Here $\underline{D}$ denotes the Riemannian connection of $(M^{2n+s}, \mathscr{G})$, and $\alpha^a = \alpha_a$, $1 \leq a \leq s$.

Let M^{2n+s} be an $\mathscr{S}$-manifold with the structure tensors $(\underline{f}, \xi'_a, \eta'_a, \mathscr{G})$. Let $\mathfrak{M}$ be the smooth s-distribution on M^{2n+s} spanned by ξ'_a , $1 \leq a \leq s$. By normality one has $[\xi'_a, \xi'_b] = 0$, i.e. $\mathfrak{M}$ is involutive. If both $\mathfrak{M}$ and the structure vector fields ξ'_a are regular (in the sense of R.PALAIS, [22]) then the $\mathscr{S}$-structure itself is termed *regular.* We shall need the main result of D.E. BLAIR & G.D.LUDDEN & K.YANO, ([21], p.175). That is, let M^{2n+s} be a compact connected (2n+s)-dimensional, $n > 1$, $\mathscr{S}$-manifold; then there is a T^s-principal fibre bundle $\bar{\pi} : M^{2n+s} \to M^{2n} = M^{2n+s}/\mathfrak{M}$ and M^{2n} is a Kaehlerian manifold. Here M^{2n} denotes the leaf space of the s-dimensional foliation $\mathcal{M}$ and T^s is the s-torus. Also, cf. ([21], p.178), $\gamma = (\eta'_1, \dots, \eta'_s)$ is a connection 1-form in $M^{2n+s}(M^{2n}, \bar{\pi}, T^s)$. If X is a tangent vector field on M^{2n}, let X^H denote its horizontal lift with respect to γ. The Kaehlerian structure (J, g) of M^{2n} is expressed by:

$$J X = \bar{\pi}_* \underline{f} X^H \tag{2.6}$$

$$\bar{g}(X, Y) = \mathcal{G}(X^H, Y^H). \tag{2.7}$$

Let $\mathcal{L}$ be the smooth 2n-distribution on M^{2n+s} defined by the Pfaffian equations $\eta'_a = 0$, $1 \le a \le s$. Then $\mathcal{L}$ is precisely the horizontal distribution of γ. Since $\eta'_a \circ \underline{f} = 0$, the f-structure preserves the horizontal distribution. Therefore (2.6) may be also written $(J X)^H = \underline{f} X^H$. Let $\bar{\nabla}$ be the Riemannian connection of $(M^{2n}, \bar{g})$. By ([21], p.179) one has:

$$\underline{D}_{X^H} Y^H = (\nabla_X Y)^H + \tfrac{1}{2}\alpha^a \mathcal{G}(\underline{f} X^H, Y^H)\, \xi'_a . \tag{2.8}$$

REMARK

Let $\pi : N \to M$ be a Riemannian submersion, cf. B.O'NEILL, [23]. Then $\mathrm{Ker}(\pi_*)$ is the *vertical distribution*, while its complement (with respect to the Riemannian metric of N) is the *horizontal distribution* of the Riemannian submersion. As to our $\bar{\pi}: M^{2n+s} \to M^{2n}$ a number of important coincidences occur. Firstly, if M^{2n} is assigned the Riemannian metric (2.7), then $M^{2n+s} \to M^{2n}$ is a Riemannian submersion. Moreover $\mathcal{M} = \mathrm{Ker}(\bar{\pi}_*)$ and therefore the horizontal distribution of the Riemannian submersion is precisely $\mathcal{L}$.

Let N be an (m+s)-dimensional submanifold of M^{2n+s}, and M an m-dimensional submanifold of M^{2n}, such that there exists a fibering $\pi : N \to M$ such that $\bar{\pi} \circ j = i \circ \pi$ and j is a diffeomorphism on fibres. Both $i : M \to M^{2n}$, $j : N \to M^{2n+s}$ stand for canonical inclusions. Let $g = i^* \bar{g}$, $G = j^* \mathcal{G}$ be the induced metrics on M and N, respectively. Also we denote by ∇, D the corresponding Riemannian connections of (M, g) and (N, G), respectively. Let B (resp. h) be the second fundamental form of i (resp. j) and denote by A (resp. W) the Weingarten forms. Let $T(M)^{\perp} \to M$ (resp. $T(N)^{\perp} \to N$) be the normal bundle of i (resp. j). We put $\xi_a = \tan(\xi'_a)$, $\xi^{\perp}_a = \mathrm{nor}(\xi'_a)$, where $\tan_x$, nor_x stand for the projections associated with the direct sum decomposition $T_x(M^{2n+s}) = T_x(N) \oplus T_x(N)^{\perp}$, $x \in N$. Then the Gauss and Weingarten formulae, (cf. e.g. [24],p.39-40), of i, j and our (2.8) lead to:

$$D_{X^H} Y^H = (\nabla_X Y)^H + \tfrac{1}{2}\alpha^a \mathcal{G}(\underline{f} X^H, Y^H)\, \xi_a \tag{2.9}$$

$$h(X^H, Y^H) = B(X, Y)^H + \tfrac{1}{2}\alpha^a \mathcal{G}(\underline{f} X^H, Y^H)\, \xi_a \tag{2.10}$$

$$W_{V^H} Y^H = (A_V X)^H - \tfrac{1}{2}\alpha^a \mathcal{G}(\underline{f} X^H, V^H)\, \xi_a \tag{2.11}$$

$$D^{\perp}_{X^H} V^H = (\nabla^{\perp}_X V)^H + \tfrac{1}{2}\alpha^a \mathcal{G}(\underline{f} X^H, V^H)\, \xi^{\perp}_a \tag{2.12}$$

for any tangent vector fields X, Y on M, respectively any cross-section V in $T(M)^{\perp} \to M$. Here $\nabla^{\perp}$, $D^{\perp}$ stand for the normal connections of i, j. Of course, towards obtaining our (2.9) - (2.12) one exploits the fact that $(i_* X)^H$ is tangent to N, while V^H is a cross-section in $T(N)^{\perp} \to N$.

REMARKS

1) Let $H(i) = \frac{1}{m}$ Trace (B) (resp. $H(j) = \frac{1}{m+s}$ Trace(h)) be the mean curvature vector of i (resp. j). As an application of our (2.9) - (2.12) one may derive:

$$(m+s)\, H(j) = m\, H(i)^H + \sum_{a=1}^{s} \left[\tfrac{1}{2}\, \alpha^a\, \mathrm{nor}(\underline{f}\, \xi_a^{\perp}) - D^{\perp}_{\xi_a}\, \xi_a^{\perp} \right] \tag{2.13}$$

provided that $\{\xi_a : 1 \le a \le s\}$ consists of mutually orthogonal unit vector fields. In particular, *if* N *is tangent to each structure vector* ξ'_a , $1 \le a \le s$, *then* N *is minimal if and only if* M *is minimal.* Indeed, if X is tangent to N, then (2.4) and the Gauss - Weingarten formulae lead to:

$$D_X \xi_a = W_{\xi_a^{\perp}} X - \tfrac{1}{2}\, \alpha_a\, \tan(\underline{f}\, X) \tag{2.14}$$

$$h(X, \xi_a) + D^{\perp}_X\, \xi_a^{\perp} = - \tfrac{1}{2}\, \alpha_a\, \mathrm{nor}\,(\underline{f}\, X). \tag{2.15}$$

Now, if $\{\xi_a : 1 \le a \le s\}$ are orthonormal, one uses a frame $\{X_i, \xi_a^H\}$ (where $\{X_i : 1 \le i \le m\}$ is an orthonormal tangential frame of M) such as to compute H(j).

2) Generally, if N is a submanifold of the $\mathcal{S}$-manifold M^{2n+s} and N is normal to some ξ'_a with $\alpha_a = 0$ then tangent spaces at points of N are *f-anti-invariant*, i.e. $\underline{f}_x(T_x(N)) \subseteq T_x(N)^{\perp}$, $x \in N$. Indeed, by (2.4) and the Weingarten formula of N in M^{2n+s}, one has $\mathcal{G}(\alpha_a\, \underline{f}\, X, Y) = -2\, \mathcal{G}\, (\underline{D}_X\, \xi'_a\, , Y) = 2\, \mathcal{G}(W_{\xi_a^{\perp}} X, Y)$ wherefrom $W_{\xi_a^{\perp}} = 0$ and $\underline{f}\, X$ is normal to N.

3. $\mathcal{S}$-MANIFOLDS AS HERMITIAN OR NORMAL ALMOST CONTACT METRICAL MANIFOLDS.

We denote by $\mathbb{C}P^n$ the complex projective space with constant holomorphic sectional curvature 1 (with Fubini - Study metric) and complex dimension n, and by S^{2n+1} the (2n+1)-dimensional unit sphere carrying the standard Sasakian structure. Let $\pi^1 : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration and set

$H^{2n+s} = \{(p_1,\ldots,p_s) \in S^{2n+1} \times \ldots \times S^{2n+1} \mid \pi^1(p_1) = \ldots = \pi^1(p_s)\}$.

We define a principal toroidal bundle by the commutative diagram:

$$\begin{array}{ccc} H_H^{2n+s} & \xrightarrow{\ \hat{\Delta}\ } & S^{2n+1} \times \ldots \times S^{2n+1} \\ \bar{\pi} \downarrow & & \downarrow\ \pi^1 \times \ldots \times \pi^1 \\ \mathbb{C}P^n & \xrightarrow[\ \Delta\ ]{} & \mathbb{C}P^n \times \ldots \times \mathbb{C}P^n \end{array}$$

where Δ denotes the diagonal map, while $\hat{\Delta}$ stands for the canonical inclusion. Let η' be the standard contact 1-form on S^{2n+1}. We put $\eta'_a = \hat{\Delta}^* \Delta_a^* \eta'$, $1 \le a \le s$ where $\Delta_a : S^{2n+1} \times \ldots \times S^{2n+1} \to S^{2n+1}$ are natural projections. Let Ω be the Kaehler 2-form of $\mathbb{C}P^n$. Then on one hand $\gamma = (\eta'_1,\ldots,\eta'_s)$ is a connection 1-form in $H^{2n+s}(\mathbb{C}P^n, \bar{\pi}, T^s)$, and on the other $d\eta'_a = \bar{\pi}^* \Omega$, such that one may apply theorem 3.1 of [8], (p.163) such as to yield a natural $\mathcal{S}$-structure on H^{2n+s}. (Cf also [4], p.173). Let $(\underline{f}, \xi_a, \eta_a, \mathcal{G})$ be the canonical $\mathcal{S}$-structure of H^{2n+s}. If s is even one sets:

$$\mathcal{J} = \underline{f} + \sum_{i=1}^{s} \{\eta_i \otimes \xi_{i*} - \eta_{i*} \otimes \xi_i\} \tag{3.1}$$

where $i* = i + \frac{s}{2}$, $1 \le i \le \frac{s}{2}$. If s is odd, one labels the 1-forms η_a as follows: η_0, η_i, η_{i*}, $i* = i+r$, $1 \le i \le r$, $s = 2r+1$, and similarly for the tangent vector fields ξ_a. We consider:

$$\varphi = \underline{f} + \sum_{i=1}^{r} \eta_i \otimes \xi_{i*} - \eta_{i*} \otimes \xi_i\}. \tag{3.2}$$

The *characteristic 1-form* of H^{2n+s}, s even, is defined by:

$$\omega = 2 \sum_{i=1}^{s/2} \{\eta_i - \eta_{i*}\}. \tag{3.3}$$

Let $B = \omega^{\dagger}$ be the *characteristic field*, where $\dagger$ means raising of indices by $\mathcal{G}$.

REMARKS

1) *If s is even then* $(H^{2n+s}, \mathcal{J}, \mathcal{G})$ *is a Hermitian non-Kaehlerian manifold and its characteristic form is parallel.* Indeed, if s is even, then $\mathcal{J}$ given by (3.1) is a complex structure and $(H^{2n+s}, \mathcal{J}, \mathcal{G})$ turns to be a Hermitian manifold, (cf. prop.4.1 in [8], p.174). Let $\tilde{F}(X, Y) = \mathcal{G}(X, \mathcal{J} Y)$ be its Kaehler 2-form. By (3.1) it follows that $\tilde{F} = F - 2 \sum_{i=1}^{s/2} \eta_i \wedge \eta_{i*}$; consequently (3.3) leads to

$$dF = \omega \wedge \underline{F} \tag{3.4}$$

i.e. $\mathcal{G}$ is not a Kaehler metric. Now our (2.4) yields $\underline{D}\,\omega = \frac{1}{2} \sum_{i=1}^{s/2} (\alpha_i - \bar{\alpha}_{i*})\, \underline{F}$ on an arbitrary $\mathcal{S}$-manifold, provided s is even. Yet for H^{2n+s} one has $\alpha_1 = \dots = \alpha_s$, (cf.[8],p.173), i.e. ω is parallel.

2) Since $d\,\eta' = \bar{\pi}^* \Omega$, $1 \le a \le s$, it follows that ω is closed. Therefore H^{2n+s}, s even, admits the canonical foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$. *Each leaf of* $\mathcal{F}$ *is a totally-geodesic real hypersurface normal to the characteristic field of* H^{2n+s}.

3) Consider the submanifolds $i : M \to \mathbb{C}P^n$ and $j : N \to H^{2n+s}$ and assume that a T^s-subbundle $\pi : N \to M$ of the generalized Hopf fibration, i.e. $\bar{\pi} \circ j = i \circ \pi$ and j is a diffeomorphism on fibres. Suppose N is tangent to the structure vectors ξ_a of the $\mathcal{S}$-manifold H^{2n+s}. *Then* M *is a C.R. submanifold of* $\mathbb{C}P^n$ *if and only if* N *is either a C.R. submanifold of* $(H^{2n+s}, \mathcal{J}, \mathcal{G})$ *or a contact C.R. submanifold of* $(H^{2n+s}, \varphi, \xi_0, \eta_0, \mathcal{G})$. Note firstly that, if s is odd, then $(\varphi, \xi_0, \eta_0, \mathcal{G})$ is a normal almost contact metrical (a. ct. m.) structure on H^{2n+s}, (cf. [8], p.175). If $\xi_a^{\perp} = 0$, $1 \le a \le s$, and s is even then:

$$\mathcal{J} \xi_i = \xi_{i*}, \quad \mathcal{J} \xi_{i*} = -\xi_i, \quad \mathcal{J} X^H = (J X)^H \tag{3.5}$$

for any tangent vector field X on M, cf.(2.6). Let us define $\mathcal{P} Y = \tan(\mathcal{J} Y)$, $\mathcal{P}^{\perp} Y = \mathrm{nor}(\mathcal{J} Y)$, for any tangent vector field Y on N. Then:

$$\mathcal{P}^{\perp} \mathcal{P} \xi_i = 0, \quad \mathcal{P}^{\perp} \mathcal{P} \xi_{i*} = 0, \quad \mathcal{P}^{\perp} \mathcal{P} X^H = (F P X)^H \tag{3.6}$$

where F, P are defined by (1.1) in [7] (p.76). Suppose for instance that $(M, \mathcal{D}, \mathcal{D}^{\perp})$ is a C.R. submanifold of $\mathbb{C}P^n$. Then P is $\mathcal{D}$-valued, while F vanishes on

$\mathcal{D}$, i.e. FP = 0. By (3.6) one has $\mathcal{P}^{\perp}\ \mathcal{P} = 0$, and thus one may apply theor. 3.1 in [7] (p.87), such as to conclude that N is a C.R. submanifold of $(H^{2n+s}, \mathcal{J}, \mathcal{G})$. Note that, although stated for submanifolds in Kaehlerian manifolds, theor.3.1 of [7] (p.87) actually holds for the general case of an arbitrary almost Hermitian ambient space. The case s odd follows similarly from theor. 2.1 of [7] (p.55) which may be easily refined from the Sasakian case to the general case of a. ct. m. structures.

4) Let $(M, \mathcal{D}, \mathcal{D}^{\perp})$ be a C.R. submanifold of CP^n, where $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) denotes the holomorphic (resp. totally-real) distribution. Let $\pi : N \to M$ be a T^s-bundle as in Remark 3). Let $\mathcal{D}_N$, $\mathcal{D}^{\perp}_N$ be the holomorphic and totally-real (resp. the φ-invariant and φ-anti- invariant) distributions of N, provided that s is even (resp. s is odd). Let $\ell_{N,x}$ the natural projection on the first term of the direct sum decomposition $T_x(N) = \mathcal{D}_{N,x} \oplus \mathcal{D}^{\perp}_{N,x}$, $x \in N$. Cf. (3.7) in [7] (p.86), (resp. cf. (2.10) in [7] (p.53)) if s is even (resp. if s is odd) then ℓ_N is expessed by $\ell_N = -\mathcal{P}^2$ (resp. by $\ell_N = -\mathcal{P}^2 + \eta_0 \otimes \xi_0$) where $\mathcal{P}Y = \tan(\mathcal{J}Y)$, (resp. $\mathcal{P}Y = \tan(\varphi Y)$). In both cases one has:

$$\ell_N \xi_a = \xi_a, \quad 1 \le a \le s, \qquad \ell_N X^H = (\ell X)^H \tag{3.7}$$

where $\ell = -P^2$. As the sum $\mathcal{D}^H_x + \mathcal{M}_x$, $x \in N$, is direct one obtaines $\mathcal{D}_{N,x} = \mathcal{D}^H_x \oplus \mathcal{M}_x$, $x \in N$. Indeed, one inclusion follows from (3.7). Conversely, let $X' \in \mathcal{D}_N$, then $X' = (\ell X)^H + (\ell^{\perp} X)^H + \lambda^a \xi_a$, $\lambda^a \in C^{\infty}(N)$, $\ell^{\perp} = I - \ell$. By applying ℓ_N to both members one proves $X' \in \mathcal{D}^H \oplus \mathcal{M}$. It is also straightforward that $(\mathcal{D}^{\perp})^H = \mathcal{D}^{\perp}_N$.

4.- FRAMED CAUCHY-RIEMANN SUBMANIFOLDS

S. GOLDBERG, [25], has inaugurated a program of unifying the treatment of the cases s even, and s odd, and studied f-invariant submanifolds of codimension 2 of an $\mathcal{S}$-manifold. To make the terminology precise, let $(N, \mathcal{D}, \mathcal{D}^{\perp})$ be a framed C.R. submanifold of M^{2n+s}; we call N an f-invariant (resp. f-anti-invariant) submanifold if $\mathcal{D}^{\perp}_x = (0)$, (resp. if $\mathcal{D}_x = (0)$), for any $x \in N$.

Let M^{2n+s} be an $\mathcal{S}$-manifold; let $x \in M^{2n+s}$ and $p \subseteq T_x(M^{n+s})$ a 2-plane. (Cf.[8], p.159), p is an *f-section* if it is spanned by $\{X, f X_x\}$ for some unit tangent vector $X \in \mathcal{L}_x$. The Riemannian sectional curvature of $(M^{2n+s}, \mathcal{G})$ restricted to f-sections is refered to as the *f-sectional curvature* of the $\mathcal{S}$-manifold. (Cf. also [21], p.183).

At this point we may establish i) of theor. A. Let X, V be respectively a tangent vector field on N and a cross-section in $T(N)^{\perp} \to N$. We set $P X = \tan(\underline{f} X)$, $F X = \mathrm{nor}(\underline{f} V)$ and $f V = \mathrm{nor}(\underline{f} V)$. The following identities hold as direct consequences of definitions:

$$P^2 + t F = -I + \eta_a \otimes \xi^a, \qquad F P + f F = 0, \qquad P t + t f = 0,$$
$$F t + f^2 = -I, \qquad \underline{f}\, \ell = P \ell, \qquad F \ell = 0, \tag{4.1}$$
$$\underline{f}\, \ell^{\perp} = F \ell^{\perp}, \qquad P \ell^{\perp} = 0.$$

Using (2.5) and the Gauss - Weingarten formulae of N in M^{2n+s} one obtaines:

$$(D_X P) Y = W_{FY} X + t\, h(X, Y) +$$
$$+ \tfrac{1}{2}\alpha^a \{[G(X, Y) - \eta_b(X)\, \eta^b(Y)]\, \xi_a - [X - \eta_b(X)\, \xi^b]\, \eta_a(Y)\} \tag{4.2}$$

for any tangent vector fields X, Y on N. Let $X, Y \in \mathcal{D}^{\perp}$. As D is torsion-free and by (4.2) one obtains:

$$P[X, Y] = W_{FX} Y - W_{FY} X + \alpha^a \{\tfrac{1}{2} (X \wedge Y) \xi_a + (\eta_a \wedge \eta_b) (X, Y) \xi^b \} \quad (4.3)$$

At this point we may establish the following:

LEMMA

Let $(N, \mathcal{D}, \mathcal{D}^{\perp})$ *be a framed C.R. submanifold of the* $\mathcal{S}$*-manifold* M^{2n+s}. *Then:*

$$W_{FX} Y = W_{FY} X + \tfrac{1}{2}\alpha^a\{\eta_a(X) Y - \eta_a(Y) X - [\eta_a(X) \eta_a(Y) - \eta_a(Y) \eta_a(X)] \xi^b\} \quad (4.4)$$

for any $X, Y \in \mathcal{D}^{\perp}$.

Proof. By (4.1), P vanishes on $\mathcal{D}^{\perp}$. Using (4.2), for any $X, Y \in \mathcal{D}^{\perp}$, $Z \in T(N)$, one has:

$$0 = G((D_Z P)X, Y) = G(W_{FX} Z, Y) + G(t\, h(Z, X), Y) +$$
$$+ \tfrac{1}{2} \alpha^a \{G(Z, X) \eta_a(Y) - G(Z, Y) \eta_a(X) + [\eta_a(X) \eta^b(Y) - \eta_a(Y) \eta^b(X)] \eta_b(Z)\}$$

and finally $G(t\, h(Z, X), Y) = - G(W_{FY} X, Z)$ leads to (4.4).

By (4.3) and the above lemma we conclude $P[X, Y] = 0$, i.e. $D^{\perp}$ is involutive.

Let us prove now ii) in theor. A. We analyse for instance the case s even. Let N a framed C.R. submanifold of H^{2n+s}. Let

$$\mathcal{P} = P + \sum_{i=1}^{s/2} \eta_i \otimes \xi_{i*} - \eta_{i*} \otimes \xi_i \}, \qquad \mathcal{P}^{\perp} = F \quad (4.5)$$

Next $\mathcal{P}^{\perp} \mathcal{P} = F P = 0$, and one applies theor.3.1 of [7], p.87. The case s odd being similar is left as an exercise to the reader. To prove the converse of ii) in theor.A we need to characterize framed C.R. submanifolds as follows. Let N be a framed C.R. submanifold of an $\mathcal{S}$-manifold M^{2n+s}. Then (4.1) leads to $P \ell = P$, $F P = 0$, $f F = 0$, etc. One obtaines the following statement. *Let* N *be a submanifold of the* $\mathcal{S}$*-manifold* M^{2n+s} *such that* N *is tangent to the structure vectors* ξ_a. *Then* N *is a framed C.R. submanifold of* M^{2n+s} *if and only if* $F P = 0$. We have proved the necessity already. Viceversa, let us put by definition $\ell = - P^2 + \eta_a \otimes \xi^a$, $\ell^{\perp} = I - \ell$. Since $F P = 0$, the projections ℓ, $\ell^{\perp}$ make N into a framed C.R. submanifold, Q.E.D. Now the converse of ii) in theor. A is easily seen to hold, i.e. both C.R. submanifolds of $(H^{2n+s}, \mathcal{J}, \mathcal{G})$, s even, and contact C.R. submanifolds of $(H^{2n+s}, \varphi, \xi_0, \eta_0, \mathcal{G})$, s odd, are framed C.R. submanifolds.

REMARKS

1) Let $(N, \mathcal{D}, \mathcal{D}^{\perp})$ be a framed C.R. submanifold of H^{2n+s}. By (4.5) one obtains:

$$\mathcal{P}^2 = P^2 - \eta^a \otimes \xi^a. \quad (4.6)$$

Now the notion of framed C.R. submanifold appears to be essentially on old concept. For not only N becomes a C.R. submanifold of the Hermitian manifold H^{2n+s}, if for instance s is even, but its holomorphic and totally-real distributions are precisely $\mathcal{D}$, $\mathcal{D}^{\perp}$. Indeed, by (4.6) one has $\ell_N = \ell$, Q.E.D.

2) Due to (3.4) there is a certain similarity between $\mathcal{S}$-manifolds and locally conformal Kaehler manifolds, cf. P.LIBERMANN, [26]. See also [12]. For instance, we may use the ideas in [2] (cf. also theor. 3.4 of [7], p.89) to

give an other proof of the integrability of the f-anti-invariant distribution of a framed C.R. submanifold. Indeed, let N be a framed C.R. submanifold of H^{2n+s}, s even. Let $X \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$. By (3.4) one has $0 = = 3(d\,\tilde{F})(X, Y, W) = - G([Z, W], J X)$. Hence $[Z, W] \in \mathcal{D}^{\perp}$. Note that, although N is C.R. in the usual sense one could not apply theor.3.4 or theor.4.1 of [7] (p.89-90) since H^{2n+s} is neither locally conformal Kaehler nor Kaehler.

To establish iii) let N be an f-invariant submanifold of H^{2n+s}. As a consequence of (2.5), for any tangent vector fields X, Y on N one has:

$$(D_X f) Y = \frac{1}{2} \alpha^a \{[G(X, Y) - \eta_b(X) \eta^b(Y)] \xi_a - [X - \eta_b(X) \xi^b] \eta_a(Y)\} \tag{4.7}$$

$$h(X, \underline{f} Y) = \underline{f}\, h(X, Y). \tag{4.8}$$

Let k(X, Y) be the Riemannian sectional curvature of the 2-plane spanned by the orthonormal pair {X, Y} on N; using the Gauss equation, i.e. equation (2.6) in [24], (p.45), and the notations in [4], (p.161), i.e. $H(X) = k(X, fX)$, $X \in \mathcal{L}$, one obtains:

$$1 - \frac{3}{4} s = H(X) + 2 \| h(X, X) \|^2 \tag{4.9}$$

as H^{2n+s} has constant f-sectional curvature, (cf.[8], p.173). By (2.15) and f-invariance one has $h(X, \xi_a) = - \frac{1}{2} \alpha_a \operatorname{nor}(\underline{f} X) = 0$; a standard argument based on (4.8) leads to the proof.

To prove iv) one uses $D h = 0$, (2.15) and f-invariance, i.e. one has $h((D_X \xi_a, Y) = 0$. Thus $\alpha_a h(\underline{f} X, Y) = 0$, by (2.14). For some $\alpha_a = 0$ one uses (4.7). Finally, apply once more $\underline{f}$ and notice that η'_a vanish on normal vectors. Thus $h = 0$.

REMARK

Let $\mathcal{F}$ be the canonical foliation of H^{2n+s}. Let N be a framed C.R. submanifold of H^{2n+s}, as above. *Then* $\mathcal{D}^{\perp} \subseteq \mathcal{F}$, *i.e. the totally-real foliation of* N *(regarded as a C.R. submanifold,* s *even) is normal to the characteristic field* $2 \sum_{i=1}^{s/2} (\xi_i - \xi_{i*})$ *of* H^{2n+s}. Indeed, since $\xi_a \in \mathcal{D}^{\perp}$, the η_a vanish on $\mathcal{D}^{\perp}$. Thus $\omega \circ \iota^{\perp} = 0$.

5.- THE CHEN CLASS OF A CAUCHY-RIEMANN SUBMANIFOLD.

Let M be a C.R. submanifold of $\mathbb{C}P^n$. Let $\pi : N \to M$ be a T^s- fibration, as in theor. B. Assume s is even. Then N is a C.R.submanifold of H^{2n+s} and its totally-real distribution is integrable. We shall need the following:

LEMMA

The holomorphic distribution of N is minimal.

Proof.

Note that we may not use lemma 4. in [17] (p.169) since its proof makes essential use of the Kaehler property. Neither could one use corollary 2.3 of [27] (p.291), (although $\mathcal{D}^{\perp}_N \subseteq \mathcal{F}$) since $(\mathcal{J}, \mathcal{G})$ fails to be locally conformal Kaehler. Now (2.4) - (2.5), (3.1) lead to:

$$(D_X \mathcal{J}) Y = \frac{1}{2} \{[\mathcal{G}(X, Y) - \eta_b(X) \eta^b(Y)] \xi - [X - \eta_b(X) \xi^b] \eta(Y)\} - \frac{1}{4} \{\underline{F}(X, Y) B + \omega(Y) \underline{f} X \} \tag{5.1}$$

where $\eta = \sum_{a=1}^{s} \eta_a$, $\xi = \eta^{\dagger}$. Let $X \in \mathscr{D}_N$, $Z \in \mathscr{D}_N^{\perp}$. Using (5.1) we have:

$$(Z, \underline{D}_X X) = \mathscr{G}(\mathscr{J} Z, \mathscr{J} \underline{D}_X X) = \mathscr{G}(\mathscr{J} Z, \underline{D}_X \mathscr{J} X) = \mathscr{G}(W_{\mathscr{J}X} X, \mathscr{J} X).$$

Thus: $\mathscr{G}(Z, \underline{D}_X X + \underline{D}_{\mathscr{J}X} \mathscr{J} X) = 0$ and $\mathscr{D}_N^{\perp}$ follows to be minimal. Let $p = \dim_{\mathbb{C}} \mathscr{D}$. Let $\{X_A : 1 \le A \le 2p\}$ be a real orthonormal frame of $\mathscr{D}$, where $X_{i+p} = \mathscr{J} X_i$, $1 \le i \le p$. Then $\{X_A^H , \xi_a\}$ is an orthonormal frame of $\mathscr{D}_N$. Let λ^A, $1 \le A \le 2p$, be differential 1-forms on N defined by $\lambda^A(X_B) = \delta_B^A$, $\lambda^A(Y) = 0$, for any $Y \in \mathscr{D}_N^{\perp}$. Let $\lambda = = \lambda^1 \wedge \ldots \wedge \lambda^{2p} \wedge \eta^1 \wedge \ldots \wedge \eta^s$. Then λ is a globally defined $(2p+s)$-form on N, as $\mathscr{D}_N$ is orientable. We leave it as an exercise to the reader to follow the ideas in [17] (p.170) and show that since $\mathscr{D}_N$ is minimal and $\mathscr{D}_N^{\perp}$ integrable the $(2p+s)$-form λ is closed. Thus λ determines a cohomology class $c(N) = [\lambda] \in H^{2p+s}(N; \mathbb{R})$ refered to as the *Chen class* of N.

To prove theor. B suppose M is a C.R. product, i.e. M is locally a product of a complex submanifold and a totally-real submanifold of $\mathbb{C}P^n$, see e.g. [28], (p.63). Now C.R. products have an integrable holomorphic distribution and a minimal totally-real distribution. By (2.8), for any tangent vector fields X, Y on $\mathbb{C}P^n$ one has:

$$[X^H, Y^H] = [X, Y]^H - \alpha^a \underline{F}(X^H, Y^H)\, \xi_a . \qquad (5.2)$$

Then (5.2) used for $X = X_A$, $Y = X_B$ leads to $[X_A^H, X_B^H] \in \mathscr{D}_N$. Next, as $\mathscr{P}^{\perp} X_A^H = 0$ one has

$$\mathscr{P}^{\perp} [X_A^H, \xi_a] = (\underline{D}_{\xi_a} \mathscr{P}^{\perp})\, X_A^H - \mathscr{P}_A^{\perp}\, \underline{D}_{X^H}\, \xi_a . \qquad (5.3)$$

We need the following :

LEMMA

The covariant derivative $(D_X \mathscr{P}^{\perp})\, Y = D_X^{\perp} \mathscr{P}^{\perp} Y - \mathscr{P}^{\perp} D_X Y$ *of* $\mathscr{P}^{\perp}$ *is expressed by:*

$$(D_X \mathscr{P}^{\perp})\, Y = - h(X, \mathscr{P} Y) + \tilde{f}\, h(X, Y) - \tfrac{1}{4}\, \omega(Y)\, F X \qquad (5.4)$$

for any tangent vector fields X, Y *on* N. *Here* $\tilde{f}\, V = \mathrm{nor}(\mathscr{J} V)$ *for any cross-section* V *in* $T(N)^{\perp} \to N$.

Proof.

Let also $\tilde{t}\, V = \tan(\mathscr{J} V)$. Using the Gauss and Weingarten formulae of N in H^{2n+s} one has:

$$(\underline{D}_X \mathscr{J})Y = (D_X \mathscr{P})Y - W_{\mathscr{P}^{\perp}Y} X - \tilde{t}h(X, Y) + (D_X \mathscr{P}^{\perp})Y + h(X, \mathscr{P} Y) - \tilde{f}\, h(X, Y) \qquad (5.5)$$

Let us use (5.1) to substitute in (5.5); a comparisson between the normal components in (5.5) leads to (5.4), Q.E.D.

Now we may use the above lemma to end the proof of the involutivity of $\mathscr{D}_N$. Indeed, by (5.4) and (2.4) our (5.3) turns into:

$$\mathscr{P}^{\perp}[X_A^H , \xi_a] = - h(\xi_a, \mathscr{P} X_A^H) + \tilde{f}\, h(\xi_a, X_A^H) - \tfrac{1}{4}\, \omega(X_A^H) F\, \xi_a + \tfrac{1}{2}\, \alpha^a\, \mathscr{P}^H \underline{f}\, X_A \qquad (5.6)$$

and by (2.15) one obtaines $\mathscr{P}^{\perp} [X_A^H , \xi_a] = 0$.

The last step is to establish minimality of $\mathscr{D}_N^{\perp}$. Let $q = \dim_{\mathbb{R}} \mathscr{D}_x^{\perp}$, $x \in M$.

If $\{E_i : 1 \le i \le q\}$ is an orthonormal frame of $\mathcal{D}^{\perp}$ then (2.8) yields:

$$\ell_N \sum_{i=1}^{q} D_{E_i^H} E_i^H = \{\ell \sum_{i=i}^{q} \bar{\nabla}_{E_i} E_i\}^H . \tag{5.7}$$

But $\mathcal{D}^{\perp}$ is minimal, so the right hand member of (5.7) is zero. Finally, one may follow the ideas in [17], (p.170) to show that since $\mathcal{D}_N$ is integrable and $\mathcal{D}_N^{\perp}$ minimal the (2p+s)-form λ is coclosed. As N is compact, λ is harmonic. Thus $c(N) = [\lambda] \neq 0$, and our theor. B is completely proved.

REFERENCES

1 A.BEJANCU, C.R. submanifolds of a Kaehler manifold, I-II, Proc. A.M.S., 69 (1978), 134-142, Trans. A.M.S., 250 (1979),335-345.

2 D.E.BLAIR & B.Y.CHEN, On C.R. submanifolds of Hermitian manifolds, Israel J. Math., 34 (1979), 353-363

3 A.ANDREOTTI & C.D.HILL, Complex characteristic coordinates and tangential Cauchy-Riemann equations, Ann. Scuola Norm. Sup. Pisa, 26 (1972), 299-324.

4 D.E.BLAIR, Geometry of manifolds with structural group U(n) x O(s), J. Diff. Geometry, (2) 4 (1970), 155-167.

5 I.MIHAI, C.R. Submanifolds of a framed f-manifold, Stud. Cerc. Mat., 35 (1982), 127-136.

6 L.ORNEA, Generic C.R. submanifolds of $\mathcal{S}$-manifolds, Stud. Cerc. Mat., 36 (1984), 135-443.

7 K.YANO & M.KON, C.R. submanifolds of Kaehlerian and Sasakian manifolds, Progress in Math., Vol. 30, Birkhäuser, Boston-Bascel-Stuttgart, 1983.

8 D.E.BLAIR, On a generalization of the Hopf fibration, An. St. Univ. "Al. I. Cuza", 17 (1971), 171-177.

9 W.M.BOOTHBY & H.C.WANG, On contact manifolds, Ann. Math., (3) 68 (1958), 721-734.

10 A.MORIMOTO, On normal almost contact stuctures, J. Math. Soc. Japan, 15 (1963), 420-436.

11 S.TANNO, On fiberings of some noncompact contact manifolds, Tôhoku Math. J., 15 (1963), 289-297.

12 I.VAISMAN, Locally conformal Kaehler manifolds with parallel Lee form, Rendiconti di Matem., 12 (1979), 263-284.

13 H.B. LAWSON, Rigidity theorems in rank-1 symmetric spaces, J. Diff. Geometry, 4 (1970), 349-357.

14 Y.MAEDA, On real hypersurfces of a complex projective space, J. Math. Soc. Japan, 28 (1976), 529-540.

15 M.OKUMURA, On some real hypersurfaces of a complex projective space, Trans. A.M.S., 212 (1975), 355-364.

16 K.YANO & M.KON, Differential geometry of C.R. submanifolds, Geom. Dedicata, 10 (1981), 369-391.

17 B.Y.CHEN, Cohomology of C.R. submanifolds, Ann. Fac. Sci. Toulose, 3 (1981), 167-172.

18 S.GOLDBERG & K.YANO, On normal globally framed f-manifolds, Tôhoku Math. J., 22 (1970), 362-370.

19 K.YANO, On a structure defined by a tensor field f of type (1,1) satisfying $f^3 + f = 0$, Tensor, N.S.,14 (1963), 99-109.

20 H.NAKAGAWA, f-structures induced on submanifolds in spaces, almost Hermitian or almost Kaehlerian, Kōdai Math. Sem. Rep., 18 (1966), 161-183.

21 D.E.BLAIR & G.D.LUDDEN & K.YANO, Differential geometric stuctures on principal toroidal bundles, Trans.A.M.S., 181 (1973), 175-184.

22 R.S.PALAIS, A global formulation of the Lie theory of transformation groups, Memoirs A.M.S., vol.22, 1957.

23 B.O'NEILL, The fundamental equations of a submersion, Michigan Math. J., 13 (1966), 459-469.

24 B.Y.CHEN, Geometry of submanifolds, M.Dekker, Inc., New York, 1973.

25 S.GOLDBERG, Invariant submanifolds of codimension 2 of framed manifolds, An. St. Univ. "Al. I. Cuza",'17 (1971), 157-165.

26 P.LIBERMANN, Sur les structures presque complexes et autres structures infinitesimales reguliers, Bull. Soc. Math. France, 83 (1985), 195-224.

27 B.Y.CHEN & P.PICCINNI, The canonical foliations of a locally conformal Kaehler manifold, Ann. Matem. pura appl., (4) CXLI (1985), 289-305.

28 A.BEJANCU, Geometry of C.R. submanifolds, D. Reidel Publ. Co., Dordrecht- Boston-Lancaster-Tokyo, 1986.