

Research Article

Time Scale Inequalities of the Ostrowski Type for Functions Differentiable on the Coordinates

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In 2016, some inequalities of the Ostrowski type for functions (of two variables) differentiable on the coordinates were established. In this paper, we extend these results to an arbitrary time scale by means of a parameter $\lambda \in [0, 1]$. The aforementioned results are regained for the case when the time scale $\mathbb{T} = \mathbb{R}$. Besides extension, our results are employed to the continuous and discrete calculus to get some new inequalities in this direction.

1. Introduction

To find a bound for the difference of a function and its integral mean, the Ukraine-born mathematician Ostrowski [1], in 1938, established the subsequent result which is nowadays celebrated as the Ostrowski inequality.

Theorem 1. Let $G : [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable in (α, β) and its derivative $G' : (\alpha, \beta) \rightarrow \mathbb{R}$ is bounded in (α, β) . If $|G'(s)| \leq \mathcal{M}$ for all $s \in [\alpha, \beta]$, then we have

$$\left| G(x) - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(s) ds \right| \leq \left(\frac{1}{4} + \frac{(x - (\alpha + \beta)/2)^2}{(\beta - \alpha)^2} \right) (\beta - \alpha) \mathcal{M}, \quad (1)$$

for all $x \in [\alpha, \beta]$. The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one.

In 2001, Cheng [2] gave the following improvement of the above inequality.

Theorem 2. Let $G : [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable in (α, β) such that there exist constants $\gamma, \Gamma \in \mathbb{R}$

with $\gamma \leq G'(s) \leq \Gamma$ for all $s \in [\alpha, \beta]$. Then for all $x \in [\alpha, \beta]$, one gets

$$\left| \frac{1}{2} G(x) - \frac{(x - \beta)G(\beta) - (x - \alpha)G(\alpha)}{2(\beta - \alpha)} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(s) ds \right| \leq \frac{(x - \alpha)^2 + (\beta - x)^2}{8(\beta - \alpha)} (\Gamma - \gamma). \quad (2)$$

Recently, Farid [3] extended Theorems 1 and 2 to functions of two variables that are differentiable on their coordinates. Specifically, he proved the following two theorems.

Theorem 3. Let $G : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$, where \mathcal{F}, \mathcal{F} are open intervals in \mathbb{R} , be a mapping such that for $\alpha_1, \beta_1 \in \mathcal{F}$, $\alpha_2, \beta_2 \in \mathcal{F}$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, the partial mappings

$$\begin{aligned} G_y : [\alpha_1, \beta_1] &\longrightarrow \mathbb{R}, \\ G_y(\xi) &:= G(\xi, y), \\ G_x : [\alpha_2, \beta_2] &\longrightarrow \mathbb{R}, \\ G_x(\zeta) &:= G(x, \zeta), \end{aligned} \quad (3)$$

defined for all $y \in [\alpha_2, \beta_2]$ and $x \in [\alpha_1, \beta_1]$, are differentiable, and $|G'_y(s)| \leq \mathcal{M}$, $s \in [\alpha_1, \beta_1]$, $|G'_x(s)| \leq \mathcal{N}$, $s \in [\alpha_2, \beta_2]$. Then we have

$$\left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} dy - \left(\frac{1}{\beta_1 - \alpha_1} + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{\mathcal{M} + \mathcal{N}}{2} (\beta_1 - \alpha_1) (\beta_2 - \alpha_2), \tag{4}$$

$$\left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1 - \alpha_1)} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2 - \alpha_2)} dy - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)}{4}, \tag{5}$$

$$\left| \int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) dx + \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) dy - \left(\frac{1}{\beta_1 - \alpha_1} + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{\mathcal{M} + \mathcal{N}}{4} (\beta_1 - \alpha_1) (\beta_2 - \alpha_2), \tag{6}$$

$$\left| \frac{1}{2(\beta_1 - \alpha_1)} \int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) dx + \frac{1}{2(\beta_2 - \alpha_2)} \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) dy - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)}{8}. \tag{7}$$

Remark 4. Inequality (7) is the correct version of inequality (2.18) as presented in [[3], Theorem 2.6].

Theorem 5. Let $G : \mathcal{I} \times \mathcal{J} \rightarrow \mathbb{R}$, where \mathcal{I}, \mathcal{J} are open intervals in \mathbb{R} , be a mapping such that for $\alpha_1, \beta_1 \in \mathcal{I}$, $\alpha_2, \beta_2 \in \mathcal{J}$, $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$, the partial mappings

$$G_x : [\alpha_1, \beta_1] \rightarrow \mathbb{R}, \\ G_y(\xi) := G(\xi, y),$$

$$G_x : [\alpha_2, \beta_2] \rightarrow \mathbb{R}, \\ G_x(\zeta) := G(x, \zeta), \tag{8}$$

defined for all $y \in [\alpha_2, \beta_2]$ and $x \in [\alpha_1, \beta_1]$, are differentiable with $\gamma_y \leq G'_y(s) \leq \Gamma_y$, $s \in [\alpha_1, \beta_1]$, $\gamma_x \leq G'_x(s) \leq \Gamma_x$, $s \in [\alpha_2, \beta_2]$. Then we have

$$\left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} dy - \left(\frac{1}{\beta_1 - \alpha_1} + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{\Gamma_x + \Gamma_y - (\gamma_x + \gamma_y)}{8} (\beta_1 - \alpha_1) (\beta_2 - \alpha_2), \tag{9}$$

$$\left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1 - \alpha_1)} dx + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2 - \alpha_2)} dy - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \leq \frac{(\beta_1 - \alpha_1)(\Gamma_y - \gamma_y) + (\beta_2 - \alpha_2)(\Gamma_x - \gamma_x)}{16}.$$

In 1988, the idea of time scales [4] was initiated so as to bring together the continuous and discrete analysis into a unified fold. Since the introduction of this subject, many classical integral results have been extended to time scales. “A time scale \mathbb{T} is an arbitrary nonempty closed subset of \mathbb{R} .” We shall presume, all over this work, that the reader is familiar with the theory of time scale (see [5, 6] for more on this subject). We present here a result of Bohner and Matthews [7] which is embedded in Theorem 6 below. This result extends Theorem 1 to time scales. For more improvements and generalizations around this result, we refer the interested reader to see the papers [8–12] and the references therein.

Theorem 6. Let $\alpha, \beta, s, t \in \mathbb{T}$, $\alpha < \beta$ and $G : [\alpha, \beta] \rightarrow \mathbb{R}$ be differentiable. Then for all $t \in [\alpha, \beta]$, we have

$$\left| G(t) - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(\sigma(s)) \Delta s \right| \leq \frac{\mathcal{M}}{\beta - \alpha} (h_2(t, \alpha) + h_2(t, \beta)), \tag{10}$$

where $h_2(t, s) = \int_s^t (\tau - s) \Delta \tau$ for all $s, t \in \mathbb{T}$ and $\mathcal{M} = \sup_{\alpha < t < \beta} |G^\Delta(t)| < \infty$. This inequality is sharp in the sense that the right-hand side of (10) cannot be replaced by a smaller one.

It is our purpose in this paper to extend inequalities (4), (5), (6), (7), and (9) to time scales by means of a parameter $\lambda \in [0, 1]$. In Section 2, we frame and prove the main results followed by applications to the continuous and discrete calculus.

2. Main Results

In this section, we will present our results involving double integrals. For some recent results in this regard, see [13–18]. The proofs of our findings shall be anchored on the subsequent lemmas.

Lemma 7 (see [9]). *Let $\alpha, \beta, t, x \in \mathbb{T}, \alpha < \beta$ and $G : [\alpha, \beta] \rightarrow \mathbb{R}$ be differentiable. Then*

$$\begin{aligned} & \left| (1 - \lambda) G(x) + \lambda \frac{G(\alpha) + G(\beta)}{2} \right. \\ & \quad \left. - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(\sigma(t)) \Delta t \right| \\ & \leq \frac{\mathcal{M}}{\beta - \alpha} \left[h_2 \left(\alpha, \alpha + \lambda \frac{\beta - \alpha}{2} \right) \right. \\ & \quad + h_2 \left(x, \alpha + \lambda \frac{\beta - \alpha}{2} \right) + h_2 \left(x, \beta - \lambda \frac{\beta - \alpha}{2} \right) \\ & \quad \left. + h_2 \left(\beta, \beta - \lambda \frac{\beta - \alpha}{2} \right) \right], \end{aligned} \tag{11}$$

for all $\lambda \in [0, 1]$ such that $\alpha + \lambda((\beta - \alpha)/2)$ and $\beta - \lambda((\beta - \alpha)/2)$ are in \mathbb{T} and $x \in [\alpha + \lambda((\beta - \alpha)/2), \beta - \lambda((\beta - \alpha)/2)] \cap \mathbb{T}$, where $\mathcal{M} := \sup_{\alpha < x < \beta} |G^\Delta(x)| < \infty$. This is sharp provided that

$$\frac{\lambda}{2} \alpha (\beta - \alpha) + \frac{\lambda^2}{4} (\beta - \alpha)^2 \leq \int_{\alpha}^{\alpha + \lambda((\beta - \alpha)/2)} t \Delta t. \tag{12}$$

Lemma 8 (see [8]). *Let $\alpha, \beta, t, x \in \mathbb{T}, \alpha < \beta$ and $G : [\alpha, \beta] \rightarrow \mathbb{R}$ be differentiable. If $G^\Delta \in C_{rd}(\mathbb{T}, \mathbb{R})$ and there exist $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq G^\Delta(t) \leq \Gamma$ for all $t \in [\alpha, \beta]$, then for all $x \in [\alpha, \beta]$, we have*

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2} \right) G(x) + \lambda \frac{(x - \alpha) G(\alpha) + (\beta - x) G(\beta)}{2(\beta - \alpha)} \right. \\ & \quad - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} G(\sigma(t)) \Delta t - \frac{\Gamma + \gamma}{2} \frac{1}{\beta - \alpha} \left[h_2(x, \alpha) \right. \\ & \quad \left. - h_2(x, \beta) - \lambda \left(\frac{(x - \alpha)^2 - (\beta - x)^2}{2} \right) \right] \Big| \\ & \leq \frac{\Gamma - \gamma}{2(\beta - \alpha)} \left[h_2 \left(\alpha, \alpha + \lambda \frac{x - \alpha}{2} \right) + h_2 \left(x, \alpha \right. \right. \\ & \quad \left. \left. + \lambda \frac{x - \alpha}{2} \right) + h_2 \left(x, \beta - \lambda \frac{\beta - x}{2} \right) + h_2 \left(\beta, \beta \right. \right. \\ & \quad \left. \left. - \lambda \frac{\beta - x}{2} \right) \right], \end{aligned} \tag{13}$$

for all $\lambda \in [0, 1]$ such that $\alpha + \lambda((x - \alpha)/2)$ and $\beta - \lambda((\beta - x)/2)$ are in \mathbb{T} .

We now formulate and prove our first result.

Theorem 9. *Let $\alpha_1, \beta_1, x \in \mathbb{T}_1, \alpha_2, \beta_2, y \in \mathbb{T}_2$, with $\alpha_1 < \beta_1, \alpha_2 < \beta_2$ and $G : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \rightarrow \mathbb{R}$ be such that the partial mappings*

$$\begin{aligned} G_y : [\alpha_1, \beta_1] & \longrightarrow \mathbb{R}, \\ G_y(\xi) & := G(\xi, y), \\ G_x : [\alpha_2, \beta_2] & \longrightarrow \mathbb{R}, \\ G_x(\zeta) & := G(x, \zeta), \end{aligned} \tag{14}$$

defined for all $y \in [\alpha_2, \beta_2]$ and $x \in [\alpha_1, \beta_1]$, are differentiable. If $\mathcal{M} = \sup_{\alpha_1 < t < \beta_1} |G_y^\Delta(t)|$ and $\mathcal{N} = \sup_{\alpha_2 < t < \beta_2} |G_x^\Delta(t)|$, then the succeeding inequalities

$$\begin{aligned} & \left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} \Delta x \right. \\ & \quad + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} \Delta y - \frac{1}{\beta_1 - \alpha_1} \\ & \quad \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x - \frac{1}{\beta_2 - \alpha_2} \\ & \quad \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \Big| \\ & \leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{2(\beta_1 - \alpha_1)} \left[3h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right. \\ & \quad + h_2 \left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\ & \quad + 3h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\ & \quad \left. + h_2 \left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right] \\ & \quad + \frac{\mathcal{N}(\beta_1 - \alpha_1)}{2(\beta_2 - \alpha_2)} \left[3h_2 \left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right. \\ & \quad + h_2 \left(\beta_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \\ & \quad + 3h_2 \left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2} \right) \\ & \quad \left. + h_2 \left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right], \\ & \left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1 - \alpha_1)} \Delta x \right. \\ & \quad \left. + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2 - \alpha_2)} \Delta y \right. \end{aligned} \tag{15}$$

$$\begin{aligned}
& - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \\
& \cdot \left| \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [G(x, \sigma(y)) + G(\sigma(x), y)] \Delta y \Delta x \right| \\
& \leq \frac{\mathcal{M}}{4(\beta_1 - \alpha_1)} \left[3h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right. \\
& + h_2 \left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& + 3h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& \left. + h_2 \left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right] \\
& + \frac{\mathcal{N}}{4(\beta_2 - \alpha_2)} \left[3h_2 \left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right. \\
& + h_2 \left(\beta_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \\
& + 3h_2 \left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2} \right) \\
& \left. + h_2 \left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right], \tag{16}
\end{aligned}$$

hold for all $\lambda \in [0, 1]$ such that $(\alpha_1 + \lambda((\beta_1 - \alpha_1)/2), \alpha_2 + \lambda((\beta_2 - \alpha_2)/2)) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(\beta_1 - \lambda((\beta_1 - \alpha_1)/2), \beta_2 - \lambda((\beta_2 - \alpha_2)/2)) \in \mathbb{T}_1 \times \mathbb{T}_2$.

Proof. Applying Lemma 7 to G_y at $x = \beta_1$, we get

$$\begin{aligned}
& \left| \left(1 - \frac{\lambda}{2} \right) G(\beta_1, y) + \frac{\lambda}{2} G(\alpha_1, y) \right. \\
& \left. - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} G(\sigma(x), y) \Delta x \right| \\
& \leq \frac{\mathcal{M}}{\beta_1 - \alpha_1} \left[h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right. \\
& + h_2 \left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& \left. + 2h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right]. \tag{17}
\end{aligned}$$

Integrating (17) over $[\alpha_2, \beta_2]$ gives

$$\begin{aligned}
& \left| \left(1 - \frac{\lambda}{2} \right) \int_{\alpha_2}^{\beta_2} G(\beta_1, y) \Delta y + \frac{\lambda}{2} \int_{\alpha_2}^{\beta_2} G(\alpha_1, y) \Delta y \right. \\
& \left. - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\
& \leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{\beta_1 - \alpha_1} \left[h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right.
\end{aligned}$$

$$\begin{aligned}
& + h_2 \left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& \left. + 2h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right]. \tag{18}
\end{aligned}$$

Applying, again, Lemma 7 to G_y at $x = \alpha_1$, and integrating over $[\alpha_2, \beta_2]$ give

$$\begin{aligned}
& \left| \left(1 - \frac{\lambda}{2} \right) \int_{\alpha_2}^{\beta_2} G(\alpha_1, y) \Delta y + \frac{\lambda}{2} \int_{\alpha_2}^{\beta_2} G(\beta_1, y) \Delta y \right. \\
& \left. - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\
& \leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{\beta_1 - \alpha_1} \left[2h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right. \\
& + h_2 \left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& \left. + h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right]. \tag{19}
\end{aligned}$$

Using (18) and (19), we have

$$\begin{aligned}
& \left| \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} \Delta y \right. \\
& \left. - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\
& \leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{2(\beta_1 - \alpha_1)} \left[3h_2 \left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right. \\
& + h_2 \left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& + 3h_2 \left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \\
& \left. + h_2 \left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2} \right) \right]. \tag{20}
\end{aligned}$$

Similarly, doing the same thing for G_x at $y = \alpha_2$ and $y = \beta_2$, and then integrating the resultant inequality over $[\alpha_1, \beta_1]$, we get

$$\begin{aligned}
& \left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} \Delta x \right. \\
& \left. - \frac{1}{\beta_2 - \alpha_2} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \right| \\
& \leq \frac{\mathcal{N}(\beta_1 - \alpha_1)}{2(\beta_2 - \alpha_2)} \left[3h_2 \left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right. \\
& \left. + h_2 \left(\beta_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2} \right) \right]
\end{aligned}$$

$$\begin{aligned}
 &+ 3h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \\
 &+ h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \Big].
 \end{aligned}
 \tag{21}$$

Using (20) and (21) amounts to (15). Also, from (20) and (21), we get

$$\begin{aligned}
 &\left| \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2(\beta_2 - \alpha_2)} \Delta y \right. \\
 &\quad \left. - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\
 &\leq \frac{\mathcal{M}}{2(\beta_1 - \alpha_1)} \left[3h_2\left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\
 &\quad + h_2\left(\beta_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \\
 &\quad + 3h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \\
 &\quad \left. + h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right],
 \end{aligned}
 \tag{22}$$

$$\begin{aligned}
 &\left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2(\beta_1 - \alpha_1)} \Delta x \right. \\
 &\quad \left. - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \right| \\
 &\leq \frac{\mathcal{N}}{2(\beta_2 - \alpha_2)} \left[3h_2\left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\
 &\quad + h_2\left(\beta_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \\
 &\quad + 3h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \\
 &\quad \left. + h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right].
 \end{aligned}
 \tag{23}$$

Combining (22) and (23), one gets (16). □

Theorem 10. *Under the assumptions of Theorem 9 and suppose also the intervals contain the mid points, then we have*

$$\begin{aligned}
 &\left| (1 - \lambda) \left[\int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) \Delta x \right. \right. \\
 &\quad \left. \left. + \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) \Delta y \right] + \frac{\lambda}{2} \right. \\
 &\quad \cdot \int_{\alpha_1}^{\beta_1} [G(x, \alpha_2) + G(x, \beta_2)] \Delta x + \frac{\lambda}{2} \\
 &\quad \left. \cdot \int_{\alpha_2}^{\beta_2} [G(\alpha_1, y) + G(\beta_1, y)] \Delta y - \frac{1}{\beta_1 - \alpha_1} \right.
 \end{aligned}$$

$$\begin{aligned}
 &\cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x - \frac{1}{\beta_2 - \alpha_2} \\
 &\cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \Big| \\
 &\leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\
 &\quad + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \beta_1 \right. \\
 &\quad \left. - \lambda \frac{\beta_1 - \alpha_1}{2}\right) + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \Big] \\
 &\quad + \frac{\mathcal{N}(\beta_1 - \alpha_1)}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\
 &\quad + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \beta_2 \right. \\
 &\quad \left. - \lambda \frac{\beta_2 - \alpha_2}{2}\right) + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \Big],
 \end{aligned}
 \tag{24}$$

$$\begin{aligned}
 &\left| \frac{1 - \lambda}{2(\beta_1 - \alpha_1)} \int_{\alpha_2}^{\beta_2} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) \Delta x + \frac{1 - \lambda}{2(\beta_2 - \alpha_2)} \right. \\
 &\quad \cdot \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) \Delta y + \frac{\lambda}{4(\beta_1 - \alpha_1)} \\
 &\quad \cdot \int_{\alpha_1}^{\beta_1} [G(x, \alpha_2) + G(x, \beta_2)] \Delta x + \frac{\lambda}{4(\beta_2 - \alpha_2)} \\
 &\quad \cdot \int_{\alpha_2}^{\beta_2} [G(\alpha_1, y) + G(\beta_1, y)] \Delta y \\
 &\quad \left. - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \right. \\
 &\quad \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} [G(\sigma(x), y) + G(x, \sigma(y))] \Delta y \Delta x \Big| \\
 &\leq \frac{\mathcal{M}}{2(\beta_1 - \alpha_1)} \left[h_2\left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\
 &\quad + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \beta_1 \right. \\
 &\quad \left. - \lambda \frac{\beta_1 - \alpha_1}{2}\right) + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \Big] \\
 &\quad + \frac{\mathcal{N}}{2(\beta_2 - \alpha_2)} \left[h_2\left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\
 &\quad + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \beta_2 \right. \\
 &\quad \left. - \lambda \frac{\beta_2 - \alpha_2}{2}\right) + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \Big].
 \end{aligned}
 \tag{25}$$

Proof. Next, we now apply Lemma 7 to G_y at $x = (\alpha_1 + \beta_1)/2$ and thereafter integrate the resulting inequality over $[\alpha_2, \beta_2]$ to get

$$\begin{aligned} & \left| (1 - \lambda) \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) \Delta y \right. \\ & \quad \left. + \lambda \frac{\int_{\alpha_2}^{\beta_2} G(\alpha_1, y) \Delta y + \int_{\alpha_2}^{\beta_2} G(\beta_1, y) \Delta y}{2} \right. \\ & \quad \left. - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\ & \leq \frac{\mathcal{M}(\beta_2 - \alpha_2)}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right]. \end{aligned} \tag{26}$$

Similarly, if one applies Lemma 7 to G_x at $y = (\alpha_2 + \beta_2)/2$ and thereafter integrate the resulting inequality over $[\alpha_1, \beta_1]$, one gets

$$\begin{aligned} & \left| (1 - \lambda) \int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) \Delta x \right. \\ & \quad \left. + \lambda \frac{\int_{\alpha_1}^{\beta_1} G(x, \alpha_2) \Delta x + \int_{\alpha_1}^{\beta_1} G(x, \beta_2) \Delta x}{2} \right. \\ & \quad \left. - \frac{1}{\beta_2 - \alpha_2} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \right| \\ & \leq \frac{\mathcal{N}(\beta_1 - \alpha_1)}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right]. \end{aligned} \tag{27}$$

Combining (26) and (27), we get (24). Finally, from (26) and (27), we get

$$\begin{aligned} & \left| \frac{1 - \lambda}{\beta_2 - \alpha_2} \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) \Delta y \right. \\ & \quad \left. + \frac{\lambda}{2(\beta_2 - \alpha_2)} \int_{\alpha_2}^{\beta_2} [G(\alpha_1, y) + G(\beta_1, y)] \Delta y \right. \end{aligned}$$

$$\begin{aligned} & \left. - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x \right| \\ & \leq \frac{\mathcal{M}}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \alpha_1 + \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_1 + \beta_1}{2}, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right], \end{aligned}$$

$$\begin{aligned} & \left| \frac{1 - \lambda}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) \Delta x \right. \\ & \quad \left. + \frac{\lambda}{2(\beta_1 - \alpha_1)} \int_{\alpha_1}^{\beta_1} [G(x, \alpha_2) + G(x, \beta_2)] \Delta x \right. \\ & \quad \left. - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x \right| \\ & \leq \frac{\mathcal{N}}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \alpha_2 + \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\frac{\alpha_2 + \beta_2}{2}, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right]. \end{aligned} \tag{28}$$

Using (28) amounts to (25). Thus, the proof of Theorem 9 is complete. \square

Corollary 11. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorems 9 and 10, then we obtain the inequality*

$$\begin{aligned} & \left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} dx \right. \\ & \quad \left. + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} dy - \left(\frac{1}{\beta_1 - \alpha_1} \right. \right. \\ & \quad \left. \left. + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \\ & \leq \frac{\lambda^2 - \lambda + 1}{2} (\beta_1 - \alpha_1)(\beta_2 - \alpha_2) (\mathcal{M} + \mathcal{N}), \end{aligned}$$

$$\begin{aligned} & \left| \int_{\alpha_1}^{\beta_1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1 - \alpha_1)} dx \right. \\ & \quad \left. + \int_{\alpha_2}^{\beta_2} \frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2 - \alpha_2)} dy \right. \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \Big| \\
 & \leq \frac{\lambda^2 - \lambda + 1}{4} (\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)), \\
 & \left| (1 - \lambda) \left[\int_{\alpha_1}^{\beta_1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) dx \right. \right. \\
 & \quad \left. \left. + \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) dy \right] + \frac{\lambda}{2} \right. \\
 & \quad \cdot \int_{\alpha_1}^{\beta_1} [G(x, \alpha_2) + G(x, \beta_2)] dx + \frac{\lambda}{2} \\
 & \quad \cdot \int_{\alpha_2}^{\beta_2} [G(\alpha_1, y) + G(\beta_1, y)] dy - \left(\frac{1}{\beta_1 - \alpha_1} \right. \\
 & \quad \left. + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \Big| \\
 & \leq \frac{2\lambda^2 - 2\lambda + 1}{4} (\beta_1 - \alpha_1)(\beta_2 - \alpha_2)(\mathcal{M} + \mathcal{N}), \\
 & \left| \frac{1 - \lambda}{2(\beta_1 - \alpha_1)} \int_{\alpha_2}^{\beta_2} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) dx + \frac{1 - \lambda}{2(\beta_2 - \alpha_2)} \right. \\
 & \quad \cdot \int_{\alpha_2}^{\beta_2} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) dy + \frac{\lambda}{4(\beta_1 - \alpha_1)} \\
 & \quad \cdot \int_{\alpha_1}^{\beta_1} [G(x, \alpha_2) + G(x, \beta_2)] dx + \frac{\lambda}{4(\beta_2 - \alpha_2)} \\
 & \quad \cdot \int_{\alpha_2}^{\beta_2} [G(\alpha_1, y) + G(\beta_1, y)] dy \\
 & \quad \left. - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \right| \\
 & \leq \frac{2\lambda^2 - 2\lambda + 1}{8} (\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)). \tag{29}
 \end{aligned}$$

Remark 12. Corollary 11 becomes Theorem 3 if $\lambda = 0$.

Corollary 13. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorems 9 and 10, then we get the succeeding inequalities*

$$\begin{aligned}
 & \left| \sum_{x=\alpha_1}^{\beta_1-1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{2} \right. \\
 & \quad \left. + \sum_{y=\alpha_2}^{\beta_2-1} \frac{G(\alpha_1, y) + G(\beta_1, y)}{2} - \frac{1}{\beta_1 - \alpha_1} \right. \\
 & \quad \cdot \sum_{x=\alpha_1, y=\alpha_2}^{\beta_1-1, \beta_2-1} G(x+1, y) - \frac{1}{\beta_2 - \alpha_2}
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \sum_{x=\alpha_1, y=\alpha_2}^{\beta_1-1, \beta_2-1} G(x, y+1) \Big| \leq \frac{\lambda^2 - \lambda + 1}{2} (\beta_1 - \alpha_1)(\beta_2 \\
 & - \alpha_2)(\mathcal{M} + \mathcal{N}), \\
 & \left| \sum_{x=\alpha_1}^{\beta_1-1} \frac{G(x, \alpha_2) + G(x, \beta_2)}{4(\beta_1 - \alpha_1)} \right. \\
 & \quad \left. + \sum_{y=\alpha_2}^{\beta_2-1} \frac{G(\alpha_1, y) + G(\beta_1, y)}{4(\beta_2 - \alpha_2)} \right. \\
 & \quad \left. - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \right. \\
 & \quad \cdot \sum_{x=\alpha_1, y=\alpha_2}^{\beta_1-1, \beta_2-1} G[(x, y+1) + G(x+1, y)] \Big| \\
 & \leq \frac{\lambda^2 - \lambda + 1}{4} [\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)], \\
 & \left| (1 - \lambda) \left[\sum_{x=\alpha_1}^{\beta_1-1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) + \sum_{y=\alpha_2}^{\beta_2-1} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) \right] \right. \\
 & \quad \left. + \frac{\lambda}{2} \sum_{x=\alpha_1}^{\beta_1-1} [G(x, \alpha_2) + G(x, \beta_2)] + \frac{\lambda}{2} \right. \\
 & \quad \cdot \sum_{y=\alpha_2}^{\beta_2-1} [G(\alpha_1, y) + G(\beta_1, y)] - \frac{1}{\beta_1 - \alpha_1} \\
 & \quad \cdot \sum_{x=\alpha_1, y=\alpha_2}^{\beta_1-1, \beta_2-1} G(x+1, y) - \frac{1}{\beta_2 - \alpha_2} \\
 & \quad \cdot \sum_{x=\alpha_1, y=\alpha_2}^{\beta_1-1, \beta_2-1} G(x, y+1) \Big| \leq \frac{2\lambda^2 - 2\lambda + 1}{4} (\beta_1 - \alpha_1) \\
 & \quad \cdot (\beta_2 - \alpha_2)(\mathcal{M} + \mathcal{N}), \\
 & \left| \frac{1 - \lambda}{2(\beta_1 - \alpha_1)} \sum_{y=\alpha_2}^{\beta_2-1} G\left(x, \frac{\alpha_2 + \beta_2}{2}\right) + \frac{1 - \lambda}{2(\beta_2 - \alpha_2)} \right. \\
 & \quad \cdot \sum_{y=\alpha_2}^{\beta_2-1} G\left(\frac{\alpha_1 + \beta_1}{2}, y\right) + \frac{\lambda}{4(\beta_1 - \alpha_1)} \\
 & \quad \cdot \sum_{x=\alpha_1}^{\beta_1-1} [G(x, \alpha_2) + G(x, \beta_2)] + \frac{\lambda}{4(\beta_2 - \alpha_2)} \\
 & \quad \cdot \sum_{y=\alpha_2}^{\beta_2-1} [G(\alpha_1, y) + G(\beta_1, y)] \\
 & \quad \left. - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}
 \end{aligned}$$

$$\begin{aligned} & \left| \sum_{x=\alpha_1}^{\beta_1-1} \sum_{y=\alpha_2}^{\beta_2-1} [G(x+1, y) + G(x, y+1)] \right| \\ & \leq \frac{2\lambda^2 - 2\lambda + 1}{8} [\mathcal{M}(\beta_1 - \alpha_1) + \mathcal{N}(\beta_2 - \alpha_2)]. \end{aligned} \tag{30}$$

Theorem 14. Let $\alpha_1, \beta_1, x \in \mathbb{T}_1$, $\alpha_2, \beta_2, y \in \mathbb{T}_2$, with $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ and $G : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \rightarrow \mathbb{R}$ be such that the partial mappings

$$\begin{aligned} G_y : [\alpha_1, \beta_1] &\longrightarrow \mathbb{R}, \\ G_y(\xi) &:= G(\xi, y), \\ G_x : [\alpha_2, \beta_2] &\longrightarrow \mathbb{R}, \\ G_x(\zeta) &:= G(x, \zeta), \end{aligned} \tag{31}$$

defined for all $y \in [\alpha_2, \beta_2]$ and $x \in [\alpha_1, \beta_1]$, are differentiable. If $G_x^\Delta, G_y^\Delta \in C_{rd}(\mathbb{T}, \mathbb{R})$ and there exist $\gamma_x, \gamma_y, \Gamma_x, \Gamma_y \in \mathbb{R}$ such that $\gamma_y \leq G_y^\Delta(t) \leq \Gamma_y$, $t \in [\alpha_1, \beta_1]$, $\gamma_x \leq G_x^\Delta(t) \leq \Gamma_x$, $t \in [\alpha_2, \beta_2]$, then the succeeding inequalities

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) \left[\int_{\alpha_1}^{\beta_1} G(x, \alpha_2) \Delta x + \int_{\alpha_2}^{\beta_2} G(\alpha_1, y) \Delta y \right] \right. \\ & + \frac{\lambda}{2} \left[\int_{\alpha_1}^{\beta_1} G(x, \beta_2) \Delta x + \int_{\alpha_2}^{\beta_2} G(\beta_1, y) \Delta y \right] - \frac{1}{\beta_1 - \alpha_1} \\ & \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(\sigma(x), y) \Delta y \Delta x - \frac{1}{\beta_2 - \alpha_2} \\ & \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, \sigma(y)) \Delta y \Delta x + \frac{\Gamma_x + \gamma_x}{2} \\ & \cdot \frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2} \left[h_2(\alpha_2, \beta_2) - \frac{\lambda}{2} (\beta_2 - \alpha_2)^2 \right] + \frac{\Gamma_y + \gamma_y}{2} \\ & \cdot \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1} \left[h_2(\alpha_1, \beta_1) - \frac{\lambda}{2} (\beta_1 - \alpha_1)^2 \right] \left| \leq \frac{\Gamma_x - \gamma_x}{2} \right. \\ & \cdot \frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \left. \right] + \frac{\Gamma_y - \gamma_y}{2} \\ & \cdot \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \left. \right], \\ & \left| \frac{1}{4(\beta_1 - \alpha_1)} \left[\int_{\alpha_1}^{\beta_1} ((2 - \lambda)G(x, \alpha_2) + \lambda G(x, \beta_2)) \Delta x \right] \right. \\ & + \frac{1}{4(\beta_2 - \alpha_2)} \left[\int_{\alpha_2}^{\beta_2} ((2 - \lambda)G(\alpha_1, y) + \lambda G(\beta_1, y)) \Delta y \right] \\ & \left. - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \right. \end{aligned} \tag{32}$$

$$\begin{aligned} & \cdot \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} (G(x, \sigma(y)) + G(\sigma(x), y)) \Delta y \Delta x + \frac{\Gamma_y + \gamma_y}{4} \\ & \cdot \frac{1}{\beta_1 - \alpha_1} \left[h_2(\alpha_1, \beta_1) - \frac{\lambda}{2} (\beta_1 - \alpha_1)^2 \right] + \frac{\Gamma_x + \gamma_x}{4} \\ & \cdot \frac{1}{\beta_2 - \alpha_2} \left[h_2(\alpha_2, \beta_2) - \frac{\lambda}{2} (\beta_2 - \alpha_2)^2 \right] \left| \leq \frac{\Gamma_y - \gamma_y}{4} \right. \\ & \cdot \frac{1}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \left. \right] + \frac{\Gamma_x - \gamma_x}{4} \\ & \cdot \frac{1}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \left. \right], \end{aligned} \tag{33}$$

hold for all $\lambda \in [0, 1]$ such that $\beta_1 - \lambda((\beta_1 - \alpha_1)/2) \in \mathbb{T}_1$ and $\beta_2 - \lambda((\beta_2 - \alpha_2)/2) \in \mathbb{T}_2$.

Proof. Applying Lemma 8 to the mapping G_y at $x = \alpha_1$ gives

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) G(\alpha_1, y) + \frac{\lambda}{2} G(\beta_1, y) \right. \\ & - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^b f(\sigma(t), y) \Delta y \\ & + \frac{\Gamma_y + \gamma_y}{2} \frac{1}{\beta_1 - \alpha_1} \left[h_2(\alpha_1, \beta_1) - \frac{\lambda}{2} (\beta_1 - \alpha_1)^2 \right] \left| \right. \\ & \leq \frac{\Gamma_y - \gamma_y}{2} \frac{1}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \left. \right]. \end{aligned} \tag{34}$$

Integrating (34) over $[\alpha_2, \beta_2]$ yields

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) \int_{\alpha_2}^d f(\alpha_1, y) \Delta y + \frac{\lambda}{2} \int_{\alpha_2}^d f(\beta_1, y) \Delta y \right. \\ & - \frac{1}{\beta_1 - \alpha_1} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^d f(\sigma(x), y) \Delta y \Delta x \\ & + \frac{\Gamma_y + \gamma_y}{2} \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1} \left[h_2(\alpha_1, \beta_1) - \frac{\lambda}{2} (\beta_1 - \alpha_1)^2 \right] \left| \right. \\ & \leq \frac{\Gamma_y - \gamma_y}{2} \frac{\beta_2 - \alpha_2}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \left. \right]. \end{aligned} \tag{35}$$

Similarly, applying Lemma 8 to the mapping G_x at $y = \alpha_2$ and then integrating the resulting inequality over $[\alpha_1, \beta_1]$ give

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) \int_{\alpha_1}^b f(x, \alpha_2) \Delta x + \frac{\lambda}{2} \int_{\alpha_1}^b f(x, \beta_2) \Delta x \right. \\ & \quad - \frac{1}{\beta_2 - \alpha_2} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^d f(x, \sigma(y)) \Delta y \Delta x \\ & \quad \left. + \frac{\Gamma_x + \gamma_x}{2} \frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2} \left[h_2(\alpha_2, \beta_2) - \frac{\lambda}{2} (\beta_2 - \alpha_2)^2 \right] \right| \quad (36) \\ & \leq \frac{\Gamma_x - \gamma_x}{2} \frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right]. \end{aligned}$$

Using (35) and (36), we get (32). Also, we obtain from (35) and (36) the following inequalities:

$$\begin{aligned} & \left| \frac{(1 - \lambda/2)}{\beta_2 - \alpha_2} \int_{\alpha_2}^d f(\alpha_1, y) \Delta y \right. \\ & \quad + \frac{\lambda}{2(\beta_2 - \alpha_2)} \int_{\alpha_2}^d f(\beta_1, y) \Delta y \\ & \quad - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^d f(\sigma(x), y) \Delta y \Delta x \\ & \quad \left. + \frac{\Gamma_y + \gamma_y}{2} \frac{1}{\beta_1 - \alpha_1} \left[h_2(\alpha_1, \beta_1) - \frac{\lambda}{2} (\beta_1 - \alpha_1)^2 \right] \right| \\ & \leq \frac{\Gamma_y - \gamma_y}{2} \frac{1}{\beta_1 - \alpha_1} \left[h_2\left(\alpha_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_1, \beta_1 - \lambda \frac{\beta_1 - \alpha_1}{2}\right) \right], \quad (37) \end{aligned}$$

$$\begin{aligned} & \left| \frac{(1 - \lambda/2)}{\beta_1 - \alpha_1} \int_{\alpha_1}^b f(x, \alpha_2) \Delta x \right. \\ & \quad + \frac{\lambda}{2(\beta_1 - \alpha_1)} \int_{\alpha_1}^b f(x, \beta_2) \Delta x \\ & \quad - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^d f(x, \sigma(y)) \Delta y \Delta x \\ & \quad \left. + \frac{\Gamma_x + \gamma_x}{2} \frac{1}{\beta_2 - \alpha_2} \left[h_2(\alpha_2, \beta_2) - \frac{\lambda}{2} (\beta_2 - \alpha_2)^2 \right] \right| \\ & \leq \frac{\Gamma_x - \gamma_x}{2} \frac{1}{\beta_2 - \alpha_2} \left[h_2\left(\alpha_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right. \\ & \quad \left. + h_2\left(\beta_2, \beta_2 - \lambda \frac{\beta_2 - \alpha_2}{2}\right) \right]. \end{aligned}$$

Using (37) amounts to (33). That completes the proof of Theorem 14. \square

Corollary 15. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 10, then the succeeding inequalities hold:*

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) \left[\int_{\alpha_1}^{\beta_1} G(x, \alpha_2) dx + \int_{\alpha_2}^{\beta_2} G(\alpha_1, y) dy \right] \right. \\ & \quad + \frac{\lambda}{2} \left[\int_{\alpha_1}^{\beta_1} G(x, \beta_2) dx + \int_{\alpha_2}^{\beta_2} G(\beta_1, y) dy \right] \\ & \quad - \left(\frac{1}{\beta_1 - \alpha_1} + \frac{1}{\beta_2 - \alpha_2} \right) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \\ & \quad \left. + \frac{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}{4} (1 - \lambda) (\Gamma_x + \gamma_x + \Gamma_y + \gamma_y) \right| \\ & \leq \frac{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)}{8} ((\lambda - 1)^2 + 1) (\Gamma_x - \gamma_x + \Gamma_y \\ & \quad - \gamma_y), \quad (38) \\ & \left| \frac{1}{4(\beta_1 - \alpha_1)} \left[\int_{\alpha_1}^{\beta_1} ((2 - \lambda)G(x, \alpha_2) + \lambda G(x, \beta_2)) dx \right] \right. \\ & \quad + \frac{1}{4(\beta_2 - \alpha_2)} \left[\int_{\alpha_2}^{\beta_2} ((2 - \lambda)G(\alpha_1, y) + \lambda G(\beta_1, y)) dy \right] \\ & \quad - \frac{1}{(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} G(x, y) dy dx \\ & \quad \left. + \frac{1 - \lambda}{8} ((\Gamma_y + \gamma_y)(\beta_1 - \alpha_1) + (\Gamma_x + \gamma_x)(\beta_2 - \alpha_2)) \right| \\ & \leq \frac{(\lambda - 1)^2 + 1}{16} ((\Gamma_y - \gamma_y)(\beta_1 - \alpha_1) \\ & \quad + (\Gamma_x - \gamma_x)(\beta_2 - \alpha_2)). \end{aligned}$$

Remark 16. Corollary 15 becomes Theorem 5 if $\lambda = 1$.

Corollary 17. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 10, then the succeeding inequalities hold:*

$$\begin{aligned} & \left| \left(1 - \frac{\lambda}{2}\right) \left[\sum_{x=\alpha_1}^{\beta_1-1} G(x, \alpha_2) + \sum_{y=\alpha_2}^{\beta_2-1} G(\alpha_1, y) \right] \right. \\ & \quad + \frac{\lambda}{2} \left[\sum_{x=\alpha_1}^{\beta_1-1} G(x, \beta_2) + \sum_{y=\alpha_2}^{\beta_2-1} G(\beta_1, y) \right] - \frac{1}{\beta_1 - \alpha_1} \\ & \quad \cdot \sum_{x=\alpha_1}^{\beta_1-1} \sum_{y=\alpha_2}^{\beta_2-1} G(x + 1, y) - \frac{1}{\beta_2 - \alpha_2} \sum_{x=\alpha_1}^{\beta_1-1} \sum_{y=\alpha_2}^{\beta_2-1} G(x, y + 1) \\ & \quad + \frac{\Gamma_x + \gamma_x}{4} (\beta_1 - \alpha_1) [(\beta_2 - \alpha_2 + 1) - \lambda(\beta_2 - \alpha_2)] \\ & \quad \left. + \frac{\Gamma_y + \gamma_y}{4} (\beta_2 - \alpha_2) [(\beta_1 - \alpha_1 + 1) - \lambda(\beta_1 - \alpha_1)] \right| \\ & \leq \frac{\Gamma_x - \gamma_x}{8} (\beta_1 - \alpha_1) ((\beta_2 - \alpha_2)(\lambda^2 - 2\lambda + 2) - 2\lambda \\ & \quad + 2) + \frac{\Gamma_y - \gamma_y}{8} (\beta_2 - \alpha_2) ((\beta_1 - \alpha_1)(\lambda^2 - 2\lambda + 2) \\ & \quad - 2\lambda + 2), \end{aligned}$$

$$\begin{aligned}
& \left| \frac{1}{4(\beta_1 - \alpha_1)} \left[\sum_{x=\alpha_1}^{\beta_1-1} ((2-\lambda)G(x, \alpha_2) + \lambda G(x, \beta_2)) \right] \right. \\
& + \frac{1}{4(\beta_2 - \alpha_2)} \left[\sum_{y=\alpha_2}^{\beta_2-1} ((2-\lambda)G(\alpha_1, y) + \lambda G(\beta_1, y)) \right] \\
& - \frac{1}{2(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)} \\
& \cdot \sum_{x=\alpha_1}^{\beta_1-1} \sum_{y=\alpha_2}^{\beta_2-1} (G(x, y+1) + G(x+1, y)) \\
& + \frac{\Gamma_y + \gamma_y}{8} [(\beta_1 - \alpha_1 + 1) - \lambda(\beta_1 - \alpha_1)] \\
& + \frac{\Gamma_x + \gamma_x}{8} [(\beta_2 - \alpha_2 + 1) - \lambda(\beta_2 - \alpha_2)] \Bigg| \\
& \leq \frac{\Gamma_y - \gamma_y}{16} ((\beta_1 - \alpha_1)(\lambda^2 - 2\lambda + 2) - 2\lambda + 2) \\
& + \frac{\Gamma_x - \gamma_x}{16} ((\beta_2 - \alpha_2)(\lambda^2 - 2\lambda + 2) - 2\lambda + 2).
\end{aligned} \tag{39}$$

3. Conclusion

Three main theorems are hereby established. The results of Farid [3] are obtained as special cases of our results. Loads of interesting new inequalities can be obtained by choosing different values of $\lambda \in [0, 1]$, and considering a different time scale different from \mathbb{R} and \mathbb{Z} .

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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