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Research Article

Oscillation Criteria for Second-Order Neutral Delay Dynamic Equations with Nonlinearities Given by Riemann-Stieltjes Integrals

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We establish several oscillation criteria for a class of second-order neutral delay dynamic equations with nonlinearities given by Riemann-Stieltjes integrals. Our results extend and unify a number of other existing results and handle the cases which are not covered by known criteria. The new results we obtain are of significance because the equations we study allow an infinite number of nonlinear terms and even a continuum of nonlinearities.

1. Introduction

In this paper, we consider the oscillatory behavior of solutions of the following second-order neutral delay dynamic equations with nonlinearities given by Riemann-Stieltjes integrals:

$$\begin{aligned}\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1}Z^{\Delta}(t)\right)^{\Delta}+f(t,x(\delta(t)))\\ +\int_a^{\sigma(b)}k(t,s)\left|x(g(t,s))\right|^{\theta(s)-1}x(g(t,s))\,\Delta\xi(s)=0,\end{aligned}\tag{1}$$

where $t\in[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$, $t_0\in\mathbb{T}$, $\mathbb{T}$ is a time scale which is unbounded above, $Z(t)=x(t)+p(t)x(\tau(t))$, $\alpha>0$ is a constant, and the following conditions are satisfied:

(H_1) $a,b\in\mathbb{T}_1$, $\mathbb{T}_1$ is another time scale, $C_{rd}(\mathbb{D},\mathbb{S})$ denotes the collection of all functions $f:\mathbb{D}\to\mathbb{S}$ which are right-dense continuous on $\mathbb{D}$;

(H_2) $r(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $R(t,t_0):=\int_{t_0}^t r^{-1/\alpha}(s)\Delta s$, $\lim_{t\to\infty}R(t,t_0)=\infty$, $p(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,1))$, $\theta\in C_{rd}([a,b]_{\mathbb{T}_1},\mathbb{R})$ is a strictly increasing and satisfying $0<\theta(a)<\alpha<\theta(b)$, $k\in C_{rd}([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}_1},[0,\infty))$;

(H_3) $\tau(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},[t_0,\infty)_{\mathbb{T}})$, $\tau(t)\leqslant t$, for $t\in[t_0,\infty)_{\mathbb{T}}$, $\lim_{t\to\infty}\tau(t)=\infty$, $\delta(t)\in C_{rd}([t_0,\infty)_{\mathbb{T}},[t_0,\infty)_{\mathbb{T}})$, $\delta(t)\leqslant t$, for $t\in[t_0,\infty)_{\mathbb{T}}$, $\lim_{t\to\infty}\delta(t)=\infty$, $g(t,s)\in C_{rd}([t_0,\infty)_{\mathbb{T}}\times[a,b]_{\mathbb{T}_1},[t_0,\infty)_{\mathbb{T}})$, $\lim_{t\to\infty}g(t,s)=\infty$ for any $s\in[a,b]_{\mathbb{T}_1}$;

(H_4) $\delta^{\Delta}(t)>0$ is right-dense continuous on $[t_0,\infty)_{\mathbb{T}}$, and $\delta(\sigma(t))=\sigma(\delta(t))$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, where $\sigma(t)$ is the forward jump operator on $[t_0,\infty)_{\mathbb{T}}$;

(H_5) $f(t,u)\in C([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$ is a continuous function such that $uf(t,u)>0$, for all $u\neq0$ and there exists a positive right-dense continuous function $q(t)$ defined on $[t_0,\infty)_{\mathbb{T}}$ such that $|f(t,u)|\geqslant q(t)|u^{\alpha}|$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and for all $u\in\mathbb{R}$;

(H_6) $\xi:[a,b]_{\mathbb{T}_1}\to\mathbb{R}$ is strictly increasing; $\int_a^{\sigma(b)}f(s)\Delta\xi(s)$ denotes the Riemann-Stieltjes integral of the function f on $[a,\sigma(b)]_{\mathbb{T}_1}$ with respect to ξ.

By a solution of (1), we mean a function $x(t)$ such that $x(t)+p(t)x(\tau(t))\in C^1_{rd}(t_x,\infty)_{\mathbb{T}}$ and $r(t)|[x(t)+p(t)x(\tau(t))]^{\Delta}|^{\alpha-1}[x(t)+p(t)x(\tau(t))]^{\Delta}\in C^1_{rd}(t_x,\infty)_{\mathbb{T}}$, $t_x\geqslant t_0$ and satisfying (1) for all $t\geqslant t_x$, where $C^1_{rd}(t_x,\infty)_{\mathbb{T}}$ denotes the set of right-dense continuously Δ-differentiable functions on $(t_x,\infty)_{\mathbb{T}}$. In the sequel, we will restrict our attention to those solutions of (1) which exist on the half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \in [\widetilde{T},\infty)_{\mathbb{T}}\} > 0$ for any $\widetilde{T} \geqslant t_x$. A nontrivial solution of (1) is called oscillatory if it has arbitrary large zeros; otherwise, it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of neutral functional equations on time scales, and we refer the reader to the papers [1–10] and the references cited therein. For an introduction to time scale calculus and dynamic equations, we refer the reader to the landmark paper of Hilger [11] and the seminal book by Bohner and Peterson [12] for a comprehensive treatment of the subject.

Recently, Saker and O'Regan [13] studied the the quasilinear equation of the form

$$\left(p(t)\left(\left[y(t)+r(t)\,y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f\left(t,y(\delta(t))\right)=0,\qquad t\in\mathbb{T},\ t\geqslant t_0, \tag{2}$$

where $|f(t,u)| \geqslant q(t)|u^{\gamma}|$, $\gamma > 0$ is an odd positive integer.

Wu et al. [14] obtained several oscillation criteria for the equation

$$\left(r(t)\left(\left[y(t)+p(t)\,y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f\left(t,y(\delta(t))\right)=0,\qquad t\in\mathbb{T},\ t\geqslant t_0, \tag{3}$$

with $|f(t,u)| \geqslant q(t)|u|^{\gamma}$, $\gamma \geqslant 1$ is a quotient of odd positive integers.

Chen [15] investigated the following second-order Emden-Fowler neutral delay dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta}+f\left(t,y(\delta(t))\right)=0,\qquad t\in\mathbb{T},\ t\geqslant t_0, \tag{4}$$

with $x(t) = y(t) + p(t)y(\tau(t))$, $|f(t,u)| \geqslant q(t)|u|^{\gamma}$, $\gamma > 0$ is a constant.

It is obvious that (2)–(4) are special cases of (1). In the present paper, we will establish several oscillation criteria for the more general (1), which is of significance because (1) allows an infinite number of nonlinear terms and even a continuum of nonlinearities determined by the function ξ. Our results extend and unify a number of other existing results and handle the cases which are not covered by known criteria. Finally, two examples are demonstrated to illustrate the efficiency of our work.

2. Preliminaries

In the sequel, we denote by $L_{\xi}[a,b]$ the set of Riemann-Stieltjes integrable functions on $[a,\sigma(b))_{\mathbb{T}_1}$ with respect to ξ, and we use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$.

Lemma 1 (see [15]). *Suppose that (H_4) holds. Let $x : \mathbb{T} \to \mathbb{R}$. If x^{Δ} exists for all sufficiently large $t \in \mathbb{T}$, then $(x(\delta(t)))^{\Delta} = x^{\Delta}(\delta(t))\delta^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.*

Lemma 2 (see [12]). *Assume that $x(t)$ is Δ-differentiable and eventually positive or eventually negative, then*

$$\left(x^{\alpha}(t)\right)^{\Delta}=\alpha\left\{\int_0^1\left[(1-h)\,x(t)+hx(\sigma(t))\right]^{\alpha-1}dh\right\}x^{\Delta}(t). \tag{5}$$

Lemma 3 (see [16]). *Suppose that X and Y are nonnegative, then*

$$\gamma XY^{\gamma-1}-X^{\gamma}\leqslant(\gamma-1)\,Y^{\gamma},\quad \gamma>1, \tag{6}$$

where equality holds if and only if $X = Y$.

Lemma 4 (see [17]). *Let $u(t) \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}_1},\mathbb{R})$ and $\eta(t) \in L_{\xi}[a,b]$ satisfy $u(t) \geqslant 0\ (\not\equiv 0)$, $\eta(t) > 0$ on $[a,b]_{\mathbb{T}_1}$, and*

$$\int_a^{\sigma(b)}\eta(s)\,\Delta\xi(s)=1. \tag{7}$$

Then,

$$\int_a^{\sigma(b)}\eta(s)\,u(s)\,\Delta\xi(s)\geqslant\exp\left(\int_a^{\sigma(b)}\eta(s)\ln\left[u(s)\right]\Delta\xi(s)\right). \tag{8}$$

Lemma 5 (see [17]). *Let $\tau(t) \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[t_0,\infty)_{\mathbb{T}})$ satisfying $0 \leqslant \tau(t) \leqslant t$, and $\tau_{t_0} = \inf\{\tau(t) : t \in [t_0,\infty)_{\mathbb{T}}\}$. Assume $x(t) \in C_{\mathrm{rd}}([\tau_{t_0},\infty)_{\mathbb{T}},(0,\infty))$ such that $r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)$ is nonincreasing on $[t_0,\infty)_{\mathbb{T}}$, where $r(t) \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\alpha > 0$ is a constant. Then,*

$$\frac{x(\tau(t))}{x(\sigma(t))}\geqslant\frac{R\left(\tau(t),\tau_{t_0}\right)}{R\left(\sigma(t),\tau_{t_0}\right)}. \tag{9}$$

3. Main Results

Theorem 6. *Assume that (H_1)–(H_6) hold. If there exist a function $\phi(t) \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and a function $\eta(s) \in L_{\xi}[a,b]$ such that $\eta(s) > 0$ on $[a,b]_{\mathbb{T}_1}$,*

$$\int_a^{\sigma(b)}\eta(s)\,\Delta\xi(s)=1, \tag{10}$$

$$\int_a^{\sigma(b)}\eta(s)\,\theta(s)\,\Delta\xi(s)=\alpha, \tag{11}$$

$$\overline{\lim_{s\to\infty}}\int_{t_0}^{s}\left(M(t)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\delta(t))\left(\phi^{\Delta}(t)\right)_{+}^{\alpha+1}}{\phi^{\alpha}(t)\left(\delta^{\Delta}(t)\right)^{\alpha}}\right)\Delta t=\infty, \tag{12}$$

where

$$M(t) = \phi(t)\overline{p}(t) + \phi(t) \times \exp\left(\int_a^{\sigma(b)} \eta(s) \times \ln\left(\eta^{-1}(s)k(t,s) \times \left[\frac{R(g(t,s),g_{T_1})(1-p(g(t,s)))}{R(\sigma(t),g_{T_1})}\right]^{\theta(s)}\right) \times \Delta\xi(s)\right),$$

$$\overline{p}(t) = q(t)[1-p(\delta(t))]^{\alpha}, \quad (\phi^{\Delta}(t))_{+} = \max\{\phi^{\Delta}(t), 0\}, \tag{13}$$

then (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$, then there exists $T_0\ (\in \mathbb{T}) \geqslant t_0$ such that $x(t) \neq 0$ for all $t \in [T_0,\infty)_{\mathbb{T}}$. Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$, and $x(g(t,s)) > 0$ for $t \in [T_0,\infty)_{\mathbb{T}}$, $s \in [a,b]_{\mathbb{T}_1}$, because a similar analysis holds for $x(t) < 0$, $x(\tau(t)) < 0$, $x(\delta(t)) < 0$, and $x(g(t,s)) < 0$. Then, from (1), (H_2), and (H_5), we get

$$Z(t) \geqslant x(t) > 0, \qquad \left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1}Z^{\Delta}(t)\right)^{\Delta} \leqslant 0, \quad t \in [T_0,\infty)_{\mathbb{T}}. \tag{14}$$

Therefore, $r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t)$ is a nonincreasing function, and $Z^{\Delta}(t)$ is eventually of one sign.

We claim that

$$Z^{\Delta}(t) > 0 \quad \text{or} \quad Z^{\Delta}(t) = 0, \quad t \in [T_0,\infty)_{\mathbb{T}}. \tag{15}$$

Otherwise, if there exists a $t_1\ (\in \mathbb{T}) \geqslant T_0$ such that $Z^{\Delta}(t) < 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, then, from (14), there exists some positive constant K such that

$$-r(t)\left(-Z^{\Delta}(t)\right)^{\alpha} \leqslant -K, \quad t \in [t_1,\infty)_{\mathbb{T}}, \tag{16}$$

that is,

$$-Z^{\Delta}(t) \geqslant \left(\frac{K}{r(t)}\right)^{1/\alpha}, \quad t \in [t_1,\infty)_{\mathbb{T}}, \tag{17}$$

and integrating the above inequality from t_1 to t, we have

$$Z(t) \leqslant Z(t_1) - K^{1/\alpha}\left(R(t,t_0) - R(t_1,t_0)\right). \tag{18}$$

Letting $t \to \infty$, from (H_2), we get $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts (14). Thus, we have proved (15).

We choose some $T_1\ (\in \mathbb{T}) \geqslant T_0$ such that $\delta(t) \geqslant T_0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Therefore, from (14), (15), and the fact $\delta(t) \leqslant \sigma(t)$, we have that

$$r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \leqslant r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}, \quad t \in [T_1,\infty)_{\mathbb{T}}, \tag{19}$$

which follows that

$$Z^{\Delta}(\delta(t)) \geqslant Z^{\Delta}(\sigma(t))\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}, \quad t \in [T_1,\infty)_{\mathbb{T}}. \tag{20}$$

On the other hand, from (1), (H_5), and (15), we obtain

$$\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)\left(Z(\delta(t)) - p(\delta(t))x(\tau(\delta(t)))\right)^{\alpha} + \int_a^{\sigma(b)} k(t,s)\left(x(g(t,s))\right)^{\theta(s)}\Delta\xi(s) \leqslant 0, \quad t \in [T_1,\infty)_{\mathbb{T}}. \tag{21}$$

Notice (15) and the fact $Z(t) \geqslant x(t)$, we get

$$\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + \overline{p}(t)Z^{\alpha}(\delta(t)) + \int_a^{\sigma(b)} k(t,s)\left(x(g(t,s))\right)^{\theta(s)}\Delta\xi(s) \leqslant 0, \quad t \in [T_1,\infty)_{\mathbb{T}}, \tag{22}$$

where $\overline{p}(t) = q(t)[1-p(\delta(t))]^{\alpha}$.

Define

$$w(t) = \phi(t)\frac{r(t)\left(Z^{\Delta}(t)\right)^{\alpha}}{Z^{\alpha}(\delta(t))}, \quad \text{for } t \in [T_1,\infty)_{\mathbb{T}}. \tag{23}$$

Obviously, $w(t) > 0$. From (22), (23), it follows that

$$\begin{aligned} w^{\Delta}(t) = {} & \frac{\phi(t)}{Z^{\alpha}(\delta(t))}\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + \frac{\phi^{\Delta}(t)Z^{\alpha}(\delta(t)) - \phi(t)\left(Z^{\alpha}(\delta(t))\right)^{\Delta}}{Z^{\alpha}(\delta(t))Z^{\alpha}(\delta(\sigma(t)))} \times r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \\ \leqslant {} & -\phi(t)\overline{p}(t) - \frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)} k(t,s)\left(x(g(t,s))\right)^{\theta(s)}\Delta\xi(s) + \frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t)) \\ & - \frac{\phi(t)\left(Z^{\alpha}(\delta(t))\right)^{\Delta}r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}}{Z^{\alpha}(\delta(t))Z^{\alpha}(\delta(\sigma(t)))}. \end{aligned} \tag{24}$$

Now, we consider the following two cases.

In the first case, $\alpha \geqslant 1$. By (15), (H_4), and Lemmas 1 and 2, we have

$$\begin{aligned}&\left(Z^{\alpha}(\delta(t))\right)^{\Delta}\\&\quad=\alpha\left\{\int_0^1\left[(1-h)Z(\delta(t))+hZ(\delta(\sigma(t)))\right]^{\alpha-1}\mathrm{d}h\right\}\\&\qquad\times\left(Z(\delta(t))\right)^{\Delta}\\&\quad\geqslant\alpha(Z(\delta(t)))^{\alpha-1}Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t).\end{aligned}\tag{25}$$

From (H_4), (20), (23)–(25), and the fact that $Z(t)$ is nondecreasing, we obtain

$$\begin{aligned}&w^{\Delta}(t)\\&\leqslant-\phi(t)\overline{p}(t)\\&\quad-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\phi(t)\,\alpha(Z(\delta(t)))^{\alpha-1}Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t)\,r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}}{Z^{\alpha}(\delta(t))\,Z^{\alpha}(\delta(\sigma(t)))}\\&\leqslant-\phi(t)\overline{p}(t)-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\phi(t)\,\alpha Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t)\,r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}}{Z^{\alpha+1}(\delta(\sigma(t)))}\\&\leqslant-\phi(t)\overline{p}(t)\\&\quad-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\alpha\phi(t)\,\delta^{\Delta}(t)\,r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha+1}}{Z^{\alpha+1}(\delta(\sigma(t)))}\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}\\&=-\phi(t)\overline{p}(t)-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\alpha\phi(t)\,\delta^{\Delta}(t)}{\left(\phi(\sigma(t))\right)^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}}w^{(\alpha+1)/\alpha}(\sigma(t)).\end{aligned}\tag{26}$$

In the second case, $0<\alpha<1$. By (15), (H_4), and Lemmas 1 and 2, we get

$$\begin{aligned}&\left(Z^{\alpha}(\delta(t))\right)^{\Delta}\\&\quad=\alpha\left\{\int_0^1\left[(1-h)Z(\delta(t))+hZ(\delta(\sigma(t)))\right]^{\alpha-1}\mathrm{d}h\right\}\\&\qquad\times\left(Z(\delta(t))\right)^{\Delta}\\&\quad\geqslant\alpha(Z(\delta(\sigma(t))))^{\alpha-1}Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t).\end{aligned}\tag{27}$$

From (H_4), (20), (23)-(24), (27), and the fact that $Z(t)$ is nondecreasing, we have

$$\begin{aligned}&w^{\Delta}(t)\\&\leqslant-\phi(t)\overline{p}(t)\\&\quad-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\phi(t)\,\alpha(Z(\delta(\sigma(t))))^{\alpha-1}Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t)\,r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}}{Z^{\alpha}(\delta(t))\,Z^{\alpha}(\delta(\sigma(t)))}\\&\leqslant-\phi(t)\overline{p}(t)\\&\quad-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\phi(t)\,\alpha Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t)\,r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}}{Z^{\alpha+1}(\delta(\sigma(t)))}\\&\leqslant-\phi(t)\overline{p}(t)\\&\quad-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&\quad+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&\quad-\frac{\alpha\phi(t)\,\delta^{\Delta}(t)}{\left(\phi(\sigma(t))\right)^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}}w^{(\alpha+1)/\alpha}(\sigma(t)).\end{aligned}\tag{28}$$

Therefore, for $\alpha>0$, from (26) and (28), we get

$$\begin{aligned}w^{\Delta}(t)\leqslant&-\phi(t)\overline{p}(t)\\&-\frac{\phi(t)}{Z^{\alpha}(\delta(t))}\int_a^{\sigma(b)}k(t,s)\left(x\left(g(t,s)\right)\right)^{\theta(s)}\Delta\xi(s)\\&+\frac{\phi^{\Delta}(t)}{\phi(\sigma(t))}w(\sigma(t))\\&-\frac{\alpha\phi(t)\,\delta^{\Delta}(t)}{\left(\phi(\sigma(t))\right)^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}}w^{(\alpha+1)/\alpha}(\sigma(t)).\end{aligned}\tag{29}$$

On the other hand, it is obvious that the conditions in Lemma 5 are satisfied with $x(t)$, $\tau(t)$, $p(t)$ replaced by $Z(t)$, $g(t,s)$, and $r(t)$, respectively. So, we have

$$\frac{Z(g(t,s))}{Z(\sigma(t))} \geqslant \frac{R\left(g(t,s), g_{T_1}\right)}{R\left(\sigma(t), g_{T_1}\right)}, \tag{30}$$

in the view of $x(t) \geqslant (1-p(t))Z(t)$, we get

$$\begin{aligned} Z(\delta(t)) &\leqslant Z(\sigma(t)) \\ &\leqslant \frac{R\left(\sigma(t), g_{T_1}\right)}{R\left(g(t,s), g_{T_1}\right)} Z(g(t,s)) \\ &\leqslant \frac{R\left(\sigma(t), g_{T_1}\right)}{R\left(g(t,s), g_{T_1}\right)} \frac{x(g(t,s))}{1-p(g(t,s))}. \end{aligned} \tag{31}$$

From (29) and (31), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant & -\phi(t)\overline{p}(t) \\ & -\phi(t)\int_a^{\sigma(b)} k(t,s) \\ & \times \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)} \\ & \times [Z(\sigma(t))]^{\theta(s)-\alpha} \Delta\xi(s) \\ & + \frac{\phi^{\Delta}(t)}{\phi(\sigma(t))} w(\sigma(t)) \\ & - \frac{\alpha\phi(t)\delta^{\Delta}(t)}{(\phi(\sigma(t)))^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}} w^{(\alpha+1)/\alpha}(\sigma(t)). \end{aligned} \tag{32}$$

By (10) and (11), we have

$$\int_a^{\sigma(b)} \eta(s)[\theta(s)-\alpha]\Delta\xi(s) = 0. \tag{33}$$

Therefore, by Lemma 4 and (33), we have that for $t \in [T_1, \infty)_{\mathbb{T}}$

$$\begin{aligned} & \int_a^{\sigma(b)} k(t,s) \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)} \\ & \quad \times [Z(\sigma(t))]^{\theta(s)-\alpha} \Delta\xi(s) \\ & = \int_a^{\sigma(b)} \eta(s)\eta^{-1}(s)k(t,s) \\ & \qquad \times \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)} \\ & \qquad \times [Z(\sigma(t))]^{\theta(s)-\alpha} \Delta\xi(s) \\ & \geqslant \exp\left(\int_a^{\sigma(b)} \eta(s) \times \ln\left(\eta^{-1}(s)k(t,s) \times \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)} \times [Z(\sigma(t))]^{\theta(s)-\alpha}\right)\Delta\xi(s)\right) \\ & = \exp\left(\int_a^{\sigma(b)} \eta(s) \times \ln\left(\eta^{-1}(s)k(t,s) \times \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)}\right) \times \Delta\xi(s)\right) \\ & \quad \times \exp\left(\ln(Z(\sigma(t)))\int_a^{\sigma(b)} \eta(s)[\theta(s)-\alpha]\Delta\xi(s)\right) \\ & = \exp\left(\int_a^{\sigma(b)} \eta(s) \times \ln\left(\eta^{-1}(s)k(t,s) \times \left[\frac{R\left(g(t,s), g_{T_1}\right)(1-p(g(t,s)))}{R\left(\sigma(t), g_{T_1}\right)}\right]^{\theta(s)}\right) \times \Delta\xi(s)\right). \end{aligned} \tag{34}$$

Substituting (34) into (32), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant & -M(t) + \frac{\phi^{\Delta}(t)}{\phi(\sigma(t))} w(\sigma(t)) \\ & - \frac{\alpha\phi(t)\delta^{\Delta}(t)}{(\phi(\sigma(t)))^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}} w^{(\alpha+1)/\alpha}(\sigma(t)) \\ \leqslant & -M(t) + \frac{\left(\phi^{\Delta}(t)\right)_+}{\phi(\sigma(t))} w(\sigma(t)) \\ & - \frac{\alpha\phi(t)\delta^{\Delta}(t)}{(\phi(\sigma(t)))^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}} w^{(\alpha+1)/\alpha}(\sigma(t)), \end{aligned} \tag{35}$$

where $M(t)$ and $(\phi^{\Delta}(t))_+$ are defined by (13).

Taking

$$X = \left[\frac{\alpha\phi(t)\,\delta^{\Delta}(t)}{(\phi(\sigma(t)))^{1+1/\alpha}(r(\delta(t)))^{1/\alpha}} \right]^{\alpha/(\alpha+1)} w(\sigma(t)),$$

$$\gamma = \frac{\alpha+1}{\alpha}, \quad (36)$$

$$Y = \frac{\alpha^{\alpha/(\alpha+1)}}{(\alpha+1)^{\alpha}} \left[\frac{r(\delta(t))\left(\phi^{\Delta}(t)\right)_+^{\alpha+1}}{\phi^{\alpha}(t)\left(\delta^{\Delta}(t)\right)^{\alpha}} \right]^{\alpha/(\alpha+1)},$$

by Lemma 3 and (35), we obtain

$$w^{\Delta}(t) \leqslant -M(t) + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(t))\left(\phi^{\Delta}(t)\right)_+^{\alpha+1}}{\phi^{\alpha}(t)\left(\delta^{\Delta}(t)\right)^{\alpha}}. \quad (37)$$

Integrating above inequality (37) from T_1 to t, we have

$$\begin{aligned} w(t) &\leqslant w(T_1) \\ &\quad - \int_{T_1}^{t} \left(M(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(s))\left(\phi^{\Delta}(s)\right)_+^{\alpha+1}}{\phi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}} \right) \Delta s. \\ &\leqslant w(T_1) + \int_{t_0}^{T_1} M(s)\,\Delta s \\ &\quad - \int_{t_0}^{t} \left(M(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(s))\left(\phi^{\Delta}(s)\right)_+^{\alpha+1}}{\phi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}} \right) \Delta s. \end{aligned} \quad (38)$$

Since $w(t) > 0$ for $t > T_1$, we have

$$\begin{aligned} &\int_{t_0}^{t} \left(M(s) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(s))\left(\phi^{\Delta}(s)\right)_+^{\alpha+1}}{\phi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}} \right) \Delta s \\ &\quad \leqslant w(T_1) + \int_{t_0}^{T_1} M(s)\,\Delta s - w(t) \leqslant w(T_1) \\ &\qquad + \int_{t_0}^{T_1} M(s)\,\Delta s, \end{aligned} \quad (39)$$

which contradicts (12). This completes the proof of Theorem 6. □

Remark 7. If we take $r(x,s) = 0$ and use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$, then Theorems 6 reduces to [15, Theorems 3.1]. If furthermore $\alpha \geqslant 1$ is a quotient of odd positive integer, then Theorem 6 reduces to [14, Theorem 3.1].

Remark 8. The function $\eta(t)$ satisfying (10) and (11) in Theorem 6 can be constructed explicitly for any nondecreasing function ξ. In fact, if we assume that $\theta(t), 1/\theta(t) \in L_{\xi}[a,b]_{\mathbb{T}_1}$, and let $h = \sup\{s \in (a,b)_{\mathbb{T}_1} : \theta(s) \leqslant \alpha\}$,

$$\begin{aligned} \eta_1(s) &= \begin{cases} \dfrac{\alpha}{\theta(s)} \left(\displaystyle\int_{\sigma(h)}^{\sigma(b)} \Delta\xi(s) \right)^{-1}, & s \in [\sigma(h), b]_{\mathbb{T}}, \\ 0, & s \in [a, \sigma(h))_{\mathbb{T}}, \end{cases} \\ \eta_2(s) &= \begin{cases} 0, & s \in [\sigma(h), b]_{\mathbb{T}}, \\ \dfrac{\alpha}{\theta(s)} \left(\displaystyle\int_{a}^{\sigma(h)} \Delta\xi(s) \right)^{-1}, & s \in [a, \sigma(h))_{\mathbb{T}}. \end{cases} \end{aligned} \quad (40)$$

It is easy to see that $\eta_i(s) \in L_{\xi}[a.b]_{\mathbb{T}_1}$ and

$$\int_{a}^{\sigma(b)} \theta(s)\,\eta_i(s)\,\Delta\xi(s) = \alpha, \quad i = 1, 2. \quad (41)$$

Moreover,

$$\begin{aligned} \int_{a}^{\sigma(b)} \eta_1(s)\,\Delta\xi(s) &= m_1 < 1, \\ \int_{a}^{\sigma(b)} \eta_2(s)\,\Delta\xi(s) &= m_2 > 1. \end{aligned} \quad (42)$$

Let

$$\eta(s,l) = (1-l)\,\eta_1(s) + l\eta_2(s), \quad \text{for } s \in [a,b]_{\mathbb{T}},\ l \in [0,1]. \quad (43)$$

Then, we obtain that

$$\begin{aligned} &\int_{a}^{\sigma(b)} \theta(s)\,\eta(s,l)\,\Delta\xi(s) = \alpha, \\ &\int_{a}^{\sigma(b)} \eta(s,l)\,\Delta\xi(s) \\ &\quad = (1-l) \int_{a}^{\sigma(b)} \eta_1(s)\,\Delta\xi(s) + l \int_{a}^{\sigma(b)} \eta_2(s)\,\Delta\xi(s) \\ &\quad = (1-l)\,m_1 + lm_2. \end{aligned} \quad (44)$$

By the continuous dependence of $\eta(s,l)$ on l, there exists $l^* \in (0,1)$ such that $\eta(s) := \eta(s,l^*)$ satisfies

$$\int_{a}^{\sigma(b)} \eta(s)\,\Delta\xi(s) = \int_{a}^{\sigma(b)} \eta(s,l^*)\,\Delta\xi(s) = 1. \quad (45)$$

Remark 9. Set $\mathbb{T}_1 = \mathbb{N}$, $a = 1$, $b = n$ for $n \in \mathbb{N}$, and

$\xi(s) = s$;

$\theta(s) = \beta_s, (s = 1, 2, \ldots, n)$ satisfying $\beta_1 > \beta_2 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n$;

$k(t,s) = q_s(t), s = 1, 2, \ldots, n$;

$g(t,s) = \tau_s(t), s = 1, 2, \ldots, n$;

Then, (1) reduces to

$$\begin{aligned} &\left(r(t) \left| Z^{\Delta}(t) \right|^{\alpha-1} Z^{\Delta}(t) \right)^{\Delta} + f(t, x(\delta(t))) \\ &\quad + \sum_{j=1}^{n} q_j(t) \left| x\left(\tau_j(t)\right) \right|^{\beta_j - 1} x\left(\tau_j(t)\right) = 0. \end{aligned} \quad (46)$$

So, if we take $\mathbb{T}$ for some peculiar cases in Theorem 6, we can obtain various results. For example, if we take $\mathbb{T}=\mathbb{R}$, $p(t)=0$, $|f(t,u)|=q(t)|u^{\alpha}|$, and $n=2$ in (46), then Theorem 6 generalizes the results by [18, Theorem 2].

Next, we use the general weighted functions from the class $\mathcal{F}$ which will be extensively used in the sequel.

Let $\mathbb{D}\equiv\{(t,s)\in[t_0,\infty)_{\mathbb{T}}\times[t_0,\infty)_{\mathbb{T}}: t\geqslant s\geqslant t_0\}$, we say that a continuous function $H(t,s)\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$ belongs to the class $\mathcal{F}$ if

(i) $H(t,t)=0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ and $H(t,s)>0$ for $t>s\geqslant t_0$ where $t,s\in[t_0,\infty)_{\mathbb{T}}$;

(ii) $H(t,s)$ has a nonpositive right-dense continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable.

Theorem 10. *Assume that* (H_1)–(H_6) *hold. If there exist functions* $H(t,s)\in\mathcal{F}$, $\varphi(t)\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $\eta(t)\in L_{\xi}[a,b]$ *such that* $\eta(s)>0$ *on* $[a,b]_{\mathbb{T}_1}$, (10) *and* (11) *hold, and*

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)M(s)-U(t,s)]\Delta s=\infty, \quad (47)$$

where

$$U(t,s)=\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\Phi_+^{\alpha+1}(t,s)\,r(\delta(s))\left(\phi(\sigma(s))\right)^{\alpha+1}}{(H(t,s))^{\alpha}\phi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}, \quad (48)$$

$$\Phi_+(t,s)=\max\left\{H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\phi^{\Delta}(s)\right)_+}{\phi(\sigma(s))},0\right\}, \quad (49)$$

$M(t)$ *and* $(\phi^{\Delta}(t))_+$ *are defined as in Theorem 6, then* (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 6 to have (35). From (35), we obtain

$$M(t)\leqslant -w^{\Delta}(t)+\frac{\left(\phi^{\Delta}(t)\right)_+}{\phi(\sigma(t))}w(\sigma(t)) - \frac{\alpha\phi(t)\delta^{\Delta}(t)}{\left(\phi(\sigma(t))\right)^{1+1/\alpha}\left(r(\delta(t))\right)^{1/\alpha}}w^{(\alpha+1)/\alpha}(\sigma(t)),$$

$$t\in[T_1,\infty)_{\mathbb{T}}. \quad (50)$$

Multiplying (50) (with t replaced by s) by $H(t,s)$, integrating it with respect to s from T_1 to t for $t\in(T_1,\infty)_{\mathbb{T}}$, and using integration by parts and (i)-(ii), we get

$$\begin{aligned}
\int_{T_1}^{t}H(t,s)M(s)\Delta s
&\leqslant -\int_{T_1}^{t}H(t,s)w^{\Delta}(s)\Delta s + \int_{T_1}^{t}\frac{H(t,s)\left(\phi^{\Delta}(s)\right)_+}{\phi(\sigma(s))}w(\sigma(s))\Delta s \\
&\quad -\int_{T_1}^{t}H(t,s)\frac{\alpha\phi(s)\delta^{\Delta}(s)}{\left(\phi(\sigma(s))\right)^{1+1/\alpha}\left(r(\delta(s))\right)^{1/\alpha}}\times w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s \\
&= H(t,T_1)w(T_1)+\int_{T_1}^{t}H^{\Delta_s}(t,s)w(\sigma(s))\Delta s + \int_{T_1}^{t}\frac{H(t,s)\left(\phi^{\Delta}(s)\right)_+}{\phi(\sigma(s))}w(\sigma(s))\Delta s \\
&\quad -\int_{T_1}^{t}H(t,s)\frac{\alpha\phi(s)\delta^{\Delta}(s)}{\left(\phi(\sigma(s))\right)^{1+1/\alpha}\left(r(\delta(s))\right)^{1/\alpha}}\times w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s \\
&= H(t,T_1)w(T_1) + \int_{T_1}^{t}\left[\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\phi^{\Delta}(s)\right)_+}{\phi(\sigma(s))}\right)w(\sigma(s)) - H(t,s)\frac{\alpha\phi(s)\delta^{\Delta}(s)}{\left(\phi(\sigma(s))\right)^{1+1/\alpha}\left(r(\delta(s))\right)^{1/\alpha}}\times w^{(\alpha+1)/\alpha}(\sigma(s))\right]\Delta s \\
&\leqslant H(t,T_1)w(T_1) + \int_{T_1}^{t}\left[\Phi_+(t,s)w(\sigma(s)) - H(t,s)\frac{\alpha\phi(s)\delta^{\Delta}(s)}{\left(\phi(\sigma(s))\right)^{1+1/\alpha}\left(r(\delta(s))\right)^{1/\alpha}}\times w^{(\alpha+1)/\alpha}(\sigma(s))\right]\Delta s,
\end{aligned} \quad (51)$$

where $\Phi_+(t,s)$ is defined as in (49).

Taking

$$X=\left[\frac{\alpha H(t,s)\phi(s)\delta^{\Delta}(s)}{\left(\phi(\sigma(s))\right)^{1+1/\alpha}\left(r(\delta(s))\right)^{1/\alpha}}\right]^{\alpha/(\alpha+1)}w(\sigma(s)),$$

$$\gamma=\frac{\alpha+1}{\alpha},$$

$$Y=\frac{\alpha^{\alpha/(\alpha+1)}}{(\alpha+1)^{\alpha}}\left[\frac{\Phi_+^{\alpha+1}(t,s)\,r(\delta(s))\left(\phi(\sigma(s))\right)^{\alpha+1}}{(H(t,s))^{\alpha}\phi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}\right]^{\alpha/(\alpha+1)}, \quad (52)$$

by Lemma 3 and (51), we obtain

$$\begin{aligned}
&\int_{T_1}^{t} H(t,s) M(s) \Delta s \\
&\leqslant H(t,T_1) w(T_1) \\
&\quad + \int_{T_1}^{t} \left[\frac{1}{(\alpha+1)^{\alpha+1}} \frac{\Phi_{+}^{\alpha+1}(t,s) r(\delta(s)) (\phi(\sigma(s)))^{\alpha+1}}{(H(t,s))^{\alpha} \phi^{\alpha}(s) (\delta^{\Delta}(s))^{\alpha}} \right] \Delta s \\
&\leqslant H(t,t_0) w(T_1) + \int_{T_1}^{t} U(t,s) \Delta s,
\end{aligned} \tag{53}$$

where $U(t,s)$ is defined as in (48).

Then, it follows that

$$\int_{T_1}^{t} [H(t,s) M(s) - U(t,s)] \Delta s \leqslant H(t,t_0) w(T_1). \tag{54}$$

Thus, from (54), we get

$$\begin{aligned}
&\int_{t_0}^{t} [H(t,s) M(s) - U(t,s)] \Delta s \\
&\quad = \left(\int_{t_0}^{T_1} + \int_{T_1}^{t} \right) [H(t,s) M(s) - U(t,s)] \Delta s \\
&\quad \leqslant \int_{t_0}^{T_1} [H(t,s) M(s)] \Delta s + H(t,t_0) w(T_1) \\
&\quad \leqslant H(t,t_0) \int_{t_0}^{T_1} M(s) \Delta s + H(t,t_0) w(T_1).
\end{aligned} \tag{55}$$

Therefore,

$$\begin{aligned}
&\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s) M(s) - U(t,s)] \Delta s \\
&\quad \leqslant \int_{t_0}^{T_1} M(s) \Delta s + w(T_1) < \infty,
\end{aligned} \tag{56}$$

which contradicts (47). This completes the proof of Theorem 10. □

Remark 11. In the literature, there are so many results for second-order nonlinear neutral functional dynamic equation; however, to the best of our knowledge, there is no work done attempting to study the neutral functional dynamic equation with an infinite number of nonlinear terms. Hence, our paper seems to be the first one dealing with this untouched problem. Our results not only unify the existing results in the literature, but also extend the existing results to a wider class of dynamic equations.

4. Examples

Example 1. Consider the following dynamic equation:

$$\begin{aligned}
&\left[\left| \left(x(t) + \frac{1}{1+t^2} x(\tau(t)) \right)^{\Delta} \right|^{1/2} \left(x(t) + \frac{1}{1+t^2} x(\tau(t)) \right)^{\Delta} \right]^{\Delta} \\
&\quad + \frac{1}{t^2} \left(\frac{1+q_0^2 t^2}{q_0^2 t^2} \right)^{s} \int_{1}^{2} |x(g(t,s))|^{s-1} x(g(t,s)) \Delta s = 0, \\
&\qquad t \in \mathbb{T}.
\end{aligned} \tag{57}$$

In (57), $r(t) = 1$, $\alpha = 3/2$, $p(t) = 1/(1+t^2)$, $f(t,u) = 0$, $a = 1$, $b = 2$, $\theta(s) = s$, $\xi(s) = s$, and $q_0 > 1$ is a constant.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} = \{q_0^n : n \in \mathbb{Z}\} \cup \{0\}$ and $\tau(t) = t/q_0^{k_0}$, $\delta(t) = t/q_0$, $g(t,s) = q_0 t$, and $k(t,s) = (1/t^2)((1+q_0^2 t^2)/q_0^2 t^2)^s$, where k_0 is an arbitrary positive integer, then $\delta^{\Delta}(t) = 1/q_0$. We can choose $\eta(t) = 1$, $\phi(t) = t$. Then, it is easy to get that $M(t) = 1/t$, and therefore,

$$\begin{aligned}
&\overline{\lim_{s\to\infty}} \int_{t_0}^{s} \left(M(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(t)) (\phi^{\Delta}(t))_{+}^{\alpha+1}}{\phi^{\alpha}(t) (\delta^{\Delta}(t))^{\alpha}} \right) \Delta t \\
&\quad = \overline{\lim_{s\to\infty}} \int_{t_0}^{s} \left(\frac{1}{t} - \frac{q_0^{3/2}}{(5/2)^{5/2} t^{3/2}} \right) \Delta t = \infty.
\end{aligned} \tag{58}$$

Hence, by Theorem 6, (57) is oscillatory.

Example 2. Consider on $\mathbb{T} = \mathbb{R}$ the following differential equation:

$$\begin{aligned}
&\left[t^{1/2} \left| \left(x(t) + \left(1 - \frac{1}{1+t^2} \right) x(t-1) \right)' \right|^{1/2} \right. \\
&\quad \left. \times \left(x(t) + \left(1 - \frac{1}{1+t^2} \right) x(t-1) \right)' \right]' \\
&\quad + \frac{\beta}{t^2} \left(\frac{4+t^2}{4} \right)^{3/2} \left| x\left(\frac{t}{2}\right) \right|^{1/2} x\left(\frac{t}{2}\right) \\
&\quad + \frac{\lambda (1+2t^2)^{3s}}{t^2} \int_{0}^{1} |x(t)|^{3s-1} x(t) \, ds = 0.
\end{aligned} \tag{59}$$

In (59), $r(t) = t^{1/2}$, $\alpha = 3/2$, $p(t) = 1 - 1/(1+t^2)$, $\tau(t) = t-1$, $\delta(t) = t/2$, $\beta, \lambda > 0$ are constants, $f(t, x(\delta(t))) = (\beta/t^2)((4+t^2)/4)^{3/2} |x(t/2)|^{1/2} x(t/2)$, $q(t) = (\beta/t^2)((4+t^2)/4)^{3/2}$, $k(t,s) = \lambda(1+2t^2)^{3s}/t^2$, $g(t,s) = t$, $a = 0$, $b = 1$, $\theta(s) = 3s$, and $\xi(s) = s$.

We can choose $\eta(t) = 1, \phi(t) = t$. Then, it is easy to get that $M(t) = (\beta+\lambda)/t, \delta'(t) = 1/2$, and therefore, from Theorem 6,

$$\begin{aligned}
&\overline{\lim_{s\to\infty}} \int_{t_0}^{s} \left(M(t) - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\delta(t)) \left(\phi'(t)\right)_+^{\alpha+1}}{\phi^{\alpha}(t) \left(\delta'(t)\right)^{\alpha}} \right) \mathrm{d}t \\
&\quad = \overline{\lim_{s\to\infty}} \int_{t_0}^{s} \left(\frac{\beta+\lambda}{t} - \frac{2^{7/2}}{5^{5/2} t} \right) \mathrm{d}t \\
&\quad = \left(\beta + \lambda - \frac{2^{7/2}}{5^{5/2}} \right) \overline{\lim_{s\to\infty}} \int_{t_0}^{s} \frac{1}{t} \mathrm{d}t \\
&\quad = \infty, \quad \text{if } \beta + \lambda > \frac{2^{7/2}}{5^{5/2}}.
\end{aligned} \tag{60}$$

Hence, by Theorem 6, (59) is oscillatory if $\beta + \lambda > 2^{7/2}/5^{5/2}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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