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Research Article

Conservation Laws, Symmetry Reductions, and New Exact Solutions of the (2 + 1)-Dimensional Kadomtsev-Petviashvili Equation with Time-Dependent Coefficients

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The (2 + 1)-dimensional Kadomtsev-Petviashvili equation with time-dependent coefficients is investigated. By means of the Lie group method, we first obtain several geometric symmetries for the equation in terms of coefficient functions and arbitrary functions of t. Based on the obtained symmetries, many nontrivial and time-dependent conservation laws for the equation are obtained with the help of Ibragimov's new conservation theorem. Applying the characteristic equations of the obtained symmetries, the (2 + 1)-dimensional KP equation is reduced to (1 + 1)-dimensional nonlinear partial differential equations, including a special case of (2 + 1)-dimensional Boussinesq equation and different types of the KdV equation. At the same time, many new exact solutions are derived such as soliton and soliton-like solutions and algebraically explicit analytical solutions.

1. Introduction

The Lie group method is a powerful tool to perform Lie symmetry analysis, study conservation laws, and look for exact solutions of nonlinear partial differential equations (NLPDEs) [1–4]. The notion of conservation laws, which plays an important role in the study of nonlinear science, is used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [5, 6]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. On the other hand, seeking exact solutions of NLPDEs has become one central theme of perpetual interest in mathematical physics as explicit solutions will be helpful to better understand the phenomena described by the equations. To get exact solutions of NLPDEs, many effective methods have been presented such as inverse scattering method [7], Hirota's bilinear method [8], and Painlevé expansion method [9]. Among them the Lie group method offers a systematic algorithmic procedure to find the symmetry reductions and exact solutions of a partial differential equation. In this paper, we use the Lie group method to consider a time-dependent Kadomtsev-Petviashvili equation:

$$E_1 \equiv u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + e(t)\, u_x + n(t)\, u_{yy} = 0, \tag{1}$$

with time-dependent coefficient functions $e(t)$, $n(t)$, and $n(t) \neq 0$.

The above equation was also called "a 2D KdV equation with time-dependent coefficients" by Hereman and Zhuang [10]; they performed Painlevé analysis for (1) and found that (1) was Painlevé integrable when $e_t + 2e^2 = 0$, $n_t + 4ne = 0$. Equation (1) can be reduced to the KdV equation ($e(t) = 0, n(t) = 0$) or the KP equation ($e(t) = 0, n(t) = \pm 1$). Equation (1) can also be reduced to the cylindrical KdV equation

$$u_t + 6uu_x + u_{xxx} + \frac{1}{2t}u = 0, \tag{2a}$$

when $e(t) = 1/2t$, $n(t) = 0$ or the cylindrical KP equation

$$u_{xt} + 6{u_x}^2 + 6uu_{xx} + u_{xxxx} + \frac{1}{2t}u_x \pm 3\frac{1}{t^2}u_{yy} = 0, \tag{2b}$$

when $e(t)=1/2t$, $n(t)=\pm 3/t^2$. The KdV and KP equations and their cylindrical generalizations (2a) and (2b) are all known to be completely integrable [10]. Zhang et al. [11] performed Painlevé analysis for (1) and constructed bilinear auto-Bäcklund, analytic solutions in the Wronskian form. Soliton-like solutions, Jacobi elliptic function-like solutions, and other exact solutions have been obtained by the method of auxiliary equations [12–15]. Elwakil et al. [16] used the homogeneous balance method to study the exact solutions of (1). Based on the homogeneous balance method and Clarkson-Kruskal method, direct reduction and exact solutions have been obtained in [17] by Moussa and El-Shiekh. The bilinear formalism, bilinear Bäcklund transformation, and Lax pair of (1) have been obtained by the binary Bell polynomial approach in [18]. As far as we know, conservation laws and symmetry reductions for (1) have not been studied.

The rest of the paper is organized as follows. In Section 2, the Lie group method is applied to the time-dependent Kadomtsev-Petviashvili equation (1) and thus Lie symmetries of (1) are obtained. In Section 3, using the obtained symmetries and the general theorem on conservation laws by Ibragimov, nontrivial and time-dependent conservation laws are derived. In Section 4, we use the symmetry to get symmetry reductions and new exact solutions of (1). The last section is a short summary and discussion.

2. Lie Symmetry Analysis of (1)

Generally speaking, Lie symmetry denotes a transformation that leaves the solution manifold of a system invariant; that is, it maps any solution of the system into a solution of the same system, so it is also called geometric symmetry. In this section, we will perform Lie symmetry analysis for (1) by the classical Lie group method. Suppose that Lie symmetry of (1) is expressed as follows:

$$V=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}, \tag{3}$$

where ξ, η, τ, and ϕ are undetermined functions with respect to x, y, t, and u. According to the procedures of Lie group method, the vector field (3) can be determined by applying the fourth prolongation of V to (1) and thus the undetermined functions ξ, η, τ, and ϕ must satisfy the following invariant condition:

$$\begin{aligned}&\phi^{xt}+12u_x\phi^{x}+6u_{xx}\phi+6u\phi^{xx}+\phi^{xxxx}\\&\quad+e'(t)\tau u_x+e(t)\phi^{x}+n'(t)\tau u_{yy}+n(t)\phi^{yy}=0,\end{aligned} \tag{4}$$

where

$$\begin{aligned}\phi^{x}&=D_x\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{xx}+\eta u_{xy}+\tau u_{xt},\\ \phi^{xt}&=D_{xt}\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)+\xi u_{xxt}+\eta u_{xty}+\tau u_{xtt},\\ \phi^{xx}&=D_{xx}\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)\\&\quad+\xi u_{xxx}+\eta u_{xxy}+\tau u_{xxt},\\ \phi^{yy}&=D_{yy}\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)\\&\quad+\xi u_{xyy}+\eta u_{yyy}+\tau u_{yyt},\\ \phi^{xxxx}&=D_{xxxx}\left(\phi-\xi u_x-\eta u_y-\tau u_t\right)\\&\quad+\xi u_{xxxxx}+\eta u_{xxxxy}+\tau u_{xxxxt}.\end{aligned} \tag{5}$$

Substituting (5) into (4) with u being a solution of (1), that is,

$$u_{xxxx}=-u_{xt}-6u_x^2-6uu_{xx}-e(t)u_x-n(t)u_{yy}, \tag{6}$$

we obtain the determining equations of symmetry (3). Solving the determining equations with the aid of Maple, we can get the following cases.

Case 1. When $e(t)$ and $n(t)$ are arbitrary functions,

$$\begin{aligned}\xi&=-\frac{g_t y}{2n(t)}+f(t),\qquad \eta=g(t),\qquad \tau=0,\\ \phi&=\frac{f_t}{6}-\frac{g_{tt}}{12n(t)}y+\frac{g_t n_t}{12n^2(t)}y,\end{aligned} \tag{7}$$

where $f(t)$ and $g(t)$ are arbitrary functions. It shows that (1) admits an infinite-dimensional Lie algebra of symmetries

$$V=V_f+V_g, \tag{8}$$

where

$$\begin{aligned}V_f&=f(t)\frac{\partial}{\partial x}+\frac{f_t}{6}\frac{\partial}{\partial u},\\ V_g&=-\frac{g_t y}{2n(t)}\frac{\partial}{\partial x}+g(t)\frac{\partial}{\partial y}+\left(\frac{g_t n_t}{12n^2(t)}y-\frac{g_{tt}}{12n(t)}y\right)\frac{\partial}{\partial u}.\end{aligned} \tag{9}$$

Case 2. When $e(t)=0$, $n(t)=(t-m)^pC_1$, $p\neq 0$, $C_1\neq 0$, and $C_2\neq 0$,

$$\begin{aligned}\xi&=\frac{C_2x}{3p}-\frac{g_t y}{2C_1(t-m)^p}+f(t),\\ \eta&=\left(\frac{2C_2}{3p}+\frac{C_2}{2}\right)y+g(t),\qquad \tau=\frac{C_2(t-m)}{p},\\ \phi&=-\frac{2C_2}{3p}u+\frac{g_t}{12C_1(t-m)^{p+1}}yp-\frac{g_{tt}}{12C_1(t-m)^p}y+\frac{f_t}{6},\end{aligned} \tag{10}$$

where m, p, C_1, and C_2 are constants and $f(t)$ and $g(t)$ are arbitrary functions. This shows that the symmetries of equation

$$u_{xt}+6u_x^2+6uu_{xx}+u_{xxxx}+C_1(t-m)^pu_{yy}=0 \tag{11}$$

have the form of

$$V = V_1 + V_f + V_g, \tag{12}$$

where

$$V_1 = \frac{x}{3p}\frac{\partial}{\partial x} + \left(\frac{2}{3p} + \frac{1}{2}\right) y \frac{\partial}{\partial y} + \frac{(t-m)}{p}\frac{\partial}{\partial t} - \frac{2}{3p} u \frac{\partial}{\partial u} \tag{13}$$

is a one-dimensional Lie algebra of symmetries and V_f and V_g are two infinite-dimensional Lie algebra of symmetries as expressed by (9) with $n(t) = (t-m)^p C_1$.

Case 3. When $e(t) = 0$, $n(t) = \text{Const.}$, and $\tau(t) \neq 0$,

$$\begin{aligned} \xi &= \frac{\tau_t}{3} x - \frac{\tau_{tt}}{6n} y^2 - \frac{g_t}{2n} y + f(t), \\ \eta &= \frac{2}{3}\tau_t y + g(t), \qquad \tau = \tau(t), \\ \phi &= -\frac{2\tau_t}{3} u + \frac{\tau_{tt}}{18} x - \frac{\tau_{ttt}}{36n} y^2 - \frac{g_{tt}}{12n} y + \frac{f_t}{6}, \end{aligned} \tag{14}$$

where $f(t)$ and $g(t)$ are arbitrary functions. It shows that the KP equation

$$u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + Cu_{yy} = 0 \tag{15}$$

admits an infinite-dimensional Lie algebra of symmetries

$$V = V_f + V_g + V_\tau, \tag{16}$$

where C is a constant and $C \neq 0$; V_f and V_g are expressed by (9) with $n(t) = \text{Const.}$,

$$\begin{aligned} V_\tau = &\left(\frac{\tau_t}{3} x - \frac{\tau_{tt}}{6n} y^2\right)\frac{\partial}{\partial x} + \frac{2}{3}\tau_t y \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} \\ &+ \left(-\frac{2\tau_t}{3} u + \frac{\tau_{tt}}{18} x - \frac{\tau_{ttt}}{36n} y^2\right)\frac{\partial}{\partial u}. \end{aligned} \tag{17}$$

Case 4. When $e(t) = -n_t/4n + C_3/\tau(t)$, $\tau(t) \neq 0$, and $n_t \neq 0$,

$$\begin{aligned} \xi = &\frac{\tau_t}{3} x - \frac{\tau_{tt}}{6n(t)} y^2 - \frac{g_t}{2n(t)} y - \frac{\tau_t n_t}{8n^2(t)} y^2 \\ &- \frac{\tau(t) n_{tt}}{8n^2(t)} y^2 + \frac{\tau(t) n_t^2}{8n^3(t)} y^2 + f(t), \\ \eta = &\left(\frac{\tau(t) n_t}{2n(t)} + \frac{2}{3}\tau_t\right) y + g(t), \\ \phi = &-\frac{2\tau_t}{3} u + \frac{\tau_{tt}}{18} x + \frac{\tau(t) n_{tt} n_t}{12n^3(t)} y^2 \\ &- \frac{\tau(t) n_{ttt}}{48n^2(t)} y^2 - \frac{\tau(t) n_t^3}{16n^4(t)} y^2 + \frac{\tau_{tt} n_t}{144n^2(t)} y^2 \\ &- \frac{\tau_{ttt}}{36n(t)} y^2 + \frac{\tau_t n_t^2}{16n^3(t)} y^2 - \frac{\tau_t n_{tt}}{24n^2(t)} y^2 \\ &+ \frac{f_t}{6} - \frac{g_{tt}}{12n(t)} y + \frac{g_t n_t}{12n^2(t)} y, \end{aligned} \tag{18}$$

where $f(t)$ and $g(t)$ are arbitrary functions, C_3 is an integral constant, and $n(t)$ and $\tau(t)$ satisfy the following ordinary differential equation:

$$\begin{aligned} &n_{ttt} + \frac{2n_{tt}\tau_t}{\tau(t)} - \frac{3\tau_t n_t^2}{n(t)\tau(t)} + \frac{3n_t^3}{n^2(t)} \\ &- \frac{4n_{tt} n_t}{n(t)} - \frac{4C_3 n(t) \tau_{tt}}{3\tau^2(t)} = 0. \end{aligned} \tag{19}$$

This shows that, under the condition (19), the equation

$$\begin{aligned} &u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} \\ &+ \left(-\frac{n_t}{4n} + \frac{C_3}{\tau(t)}\right) u_x + n(t) u_{yy} = 0 \end{aligned} \tag{20}$$

admits an infinite-dimensional Lie algebra of symmetries

$$V = V_f + V_g + V_{0\tau}, \tag{21}$$

where V_f and V_g are expressed by (9):

$$\begin{aligned} V_{0\tau} = &\left(\frac{\tau_t}{3} x - \frac{\tau_{tt}}{6n} y^2 - \frac{\tau_t n_t}{8n^2(t)} y^2 - \frac{\tau(t) n_{tt}}{8n^2(t)} y^2 \right. \\ &\left. + \frac{\tau(t) n_t^2}{8n^3(t)} y^2\right)\frac{\partial}{\partial x} + \left(\frac{\tau(t) n_t}{2n(t)} + \frac{2}{3}\tau_t\right) y \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} \\ &+ \left(-\frac{2\tau_t}{3} u + \frac{\tau_{tt}}{18} x + \frac{\tau(t) n_{tt} n_t}{12n^3(t)} y^2 - \frac{\tau(t) n_{ttt}}{48n^2(t)} y^2 \right. \\ &- \frac{\tau(t) n_t^3}{16n^4(t)} y^2 + \frac{\tau_{tt} n_t}{144n^2(t)} y^2 - \frac{\tau_{ttt}}{36n(t)} y^2 \\ &\left. + \frac{\tau_t n_t^2}{16n^3(t)} y^2 - \frac{\tau_t n_{tt}}{24n^2(t)} y^2\right)\frac{\partial}{\partial u}. \end{aligned} \tag{22}$$

3. Conservation Laws for (1)

3.1. A General Theorem on Conservation Laws. As expressed through the famous Noether theorem, for a given differential equation, there is a close connection between Lie symmetries and conservation laws. To derive conservation laws of (1), we use the following conclusion proved by Ibragimov in [19].

Theorem 1. *Every Lie point, Lie-Bäcklund, and nonlocal symmetry*

$$V = \xi^i (x, u, u_{(1)}, \dots) \frac{\partial}{\partial x^i} + \eta^s (x, u, u_{(1)}, \dots) \frac{\partial}{\partial u^s} \quad (23)$$

of a system of m equations

$$F_s (x, u, u_{(1)}, \dots, u_{(N)}) = 0, \quad s = 1, \dots, m, \quad (24)$$

with n independent variables $x = (x^1, \dots, x^n)$ *and m dependent variables;* $u = (u^1, \dots, u^m)$ *provides a conservation law for system* (24) *and the corresponding adjoint system*

$$F_s^* (x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(N)}, v_{(N)}) \equiv \frac{\delta (v^i F_i)}{\delta u^s} = 0, \quad s = 1, \dots, m. \quad (25)$$

Then the elements of the conservation vector $T = (T^1, \dots, T^n)$ *are defined by the following expression:*

$$\begin{aligned} T^i = {} & \xi^i L + W^s \\ & \times \left[\frac{\partial L}{\partial u_i^s} - D_{x^j} \left(\frac{\partial L}{\partial u_{ij}^s} \right) + D_{x^j} D_{x^k} \left(\frac{\partial L}{\partial u_{ijk}^s} \right) - \cdots \right] \\ & + D_{x^j} (W^s) \\ & \times \left[\frac{\partial L}{\partial u_{ij}^s} - D_{x^k} \left(\frac{\partial L}{\partial u_{ijk}^s} \right) + D_{x^k} D_{x^r} \left(\frac{\partial L}{\partial u_{ijkr}^s} \right) - \cdots \right] \\ & + D_{x^j} D_{x^k} (W^s) \left[\frac{\partial L}{\partial u_{ijk}^s} - D_{x^r} \left(\frac{\partial L}{\partial u_{ijkr}^s} \right) + \cdots \right] + \cdots , \end{aligned} \quad (26)$$

with

$$W^s = \eta^s - \xi^i u_i^s, \quad s = 1, \dots, m. \quad (27)$$

3.2. Conservation Laws for (1). To search for conservation laws of (1) by Theorem 1, adjoint equation and formal Lagrangian of (1) must be known. We first construct its adjoint equation. Following the idea in [19], the adjoint equation of (1) is

$$E_1^* \equiv v_{xt} + 6uv_{xx} + v_{xxxx} - e(t) v_x + n(t) v_{yy} = 0, \quad (28)$$

where v is a new dependent variable with respect to x, y, and t.

According to the method of constructing Lagrangian in [19], the formal Lagrangian for the system consisting of (1) and (28) is

$$L = v \left(u_{xt} + 6u_x^2 + 6uu_{xx} + u_{xxxx} + e(t) u_x + n(t) u_{yy} \right). \quad (29)$$

By means of the symmetries of (1), conservation laws of the system consisting of (1) and (28) can be derived by Theorem 1. However, we are only interested in the conservation laws of (1). Therefore one has to eliminate the nonlocal variable v which is introduced in the adjoint equation. To solve this problem, the concepts of self-adjointness, quasi-self-adjointness, and nonlinear self-adjointness are developed [20–24]. In the following, we will discuss the adjointness and nonlinear adjointness using these definitions.

Equation (1) is said to be self-adjoint if the equation obtained from the adjoint equation (28) by the substitution $v = u$ is identical with the original equation (1). It is easy to see that (28) is not identical with (1) when $v = u$, so (1) is not a self-adjoint equation. According to the definition of nonlinear self-adjointness [24], (1) is said to be nonlinearly self-adjoint if its adjoint equation (28) is satisfied for all solutions u of (1) upon a substitution

$$v = H(x, y, t, u), \quad H(x, y, t, u) \neq 0. \quad (30)$$

In other words, (1) is nonlinearly self-adjoint if and only if

$$E_1^* \big|_{v=H(x,y,t,u)} = \lambda \left(x, y, t, u, u_x, u_y, u_t, u_{xx}, \dots \right) E_1, \quad (31)$$

where λ is an undetermined and smooth function.

From (31), we can get the following equation:

$$\begin{aligned} & (H_u - \lambda) u_{xxxx} + n(t) (H_u - \lambda) u_{yy} + (H_u - \lambda) u_{xt} \\ & + 4H_{uu} u_x u_{xxx} + 4H_{xu} u_{xxx} + 2n(t) H_{yu} u_y \\ & + u_x^2 (6uH_{uu} + 6H_{uuu} u_{xx} - 6\lambda + 6H_{xxuu}) \\ & + u (12u_x H_{xu} + 6H_u u_{xx} - 6\lambda u_{xx} + 6H_{xx}) + H_{tu} u_x \\ & - \lambda e(t) u_x - e(t) u_x H_u + n(t) u_y^2 H_{uu} \\ & + 12H_{xuu} u_x u_{xx} + u_x u_t H_{uu} + 6H_{xxu} u_{xx} + 4H_{xxxu} u_x \\ & + H_{xu} u_t + 4H_{xuuu} u_x^3 + 3H_{uu} u_{xx}^2 + H_{uuuu} u_x^4 \\ & + \left(-e(t) H_x + n(t) H_{yy} + H_{xt} + H_{xxxx} \right) = 0. \end{aligned} \quad (32)$$

Solving the above system with the aid of Maple, the final results read as

$$\lambda = 0, \quad (33)$$

$$\begin{aligned} H = {} & (a(t) y + b(t)) x - \frac{a_t y^3}{6n(t)} - \frac{b_t y^2}{2n(t)} \\ & + \frac{e(t) a(t) y^3}{6n(t)} + \frac{e(t) b(t) y^2}{2n(t)} + k(t) y + l(t), \end{aligned} \quad (34)$$

where $a(t)$, $b(t)$, $k(t)$, and $l(t)$ are arbitrary functions. In summary, we have the following statements.

Theorem 2. *The time-dependent KP equation* (1) *is nonlinearly self-adjoint.*

In the following, we first construct the conservation laws for the system consisting of the initial equation (1) and its adjoint (28).

For the symmetry in Case 1, the corresponding components of the conservation laws are

$$\begin{aligned}
X_1 = {} & f(t)u_xv_t + f_tu_xv - g(t)u_{xxxy}v - g(t)u_{xy}v_{xx} \\
& - f_tv_xu + f(t)u_{xt}v + g(t)u_yv_{xxx} + \frac{f_te(t)v}{6} \\
& + g(t)u_{xxy}v_x + g(t)u_yv_t \\
& + f(t)u_xv_{xxx} + f(t)v_xu_{xxx} \\
& - f(t)u_{xx}v_{xx} + \frac{g_tyu_{xx}v_{xx}}{2n(t)} - \frac{1}{6}f_tv_{xxx} - \frac{g_tyu_{yy}v}{2} \\
& + f(t)n(t)u_{yy}v + \frac{g_{tt}yv_t}{12n(t)} + \frac{g_{tt}yv_{xxx}}{12n(t)} + 6f(t)u_xv_xu \\
& - g(t)e(t)u_yv - 6g(t)u_yu_xv + 6g(t)u_yv_xu \\
& - 6g(t)u_{xy}uv - \frac{1}{6}f_tv_t - \frac{g_{tt}ye(t)v}{12n(t)} - \frac{g_tyu_xv_t}{2n(t)} \\
& - \frac{g_tyu_{xt}v}{2n(t)} - \frac{g_{tt}yu_xv}{2n(t)} + \frac{g_{tt}yuv_x}{2n(t)} + \frac{g_tyn_te(t)v}{12n^2(t)} \\
& + \frac{g_tyn_tu_xv}{2n^2(t)} - \frac{g_tyn_tv_xu}{2n^2(t)} - \frac{g_tyn_tv_t}{12n^2(t)} - \frac{g_tyn_tv_{xxx}}{12n^2(t)} \\
& - \frac{3g_tyu_xv_xu}{n(t)} - \frac{g_tyu_xv_{xxx}}{2n(t)} - \frac{g_tyu_{xxx}v_x}{2n(t)}, \\
Y_1 = {} & -\frac{n(t)f_tv_y}{6} + \frac{yg_{tt}v_y}{12} - \frac{g_tyn_tv_y}{12n(t)} - \frac{1}{2}g_tyu_xv_y \\
& + f(t)n(t)u_xv_y + g(t)n(t)u_yv_y - \frac{g_{tt}v}{12} + \frac{g_tn_tv}{12n(t)} \\
& + \frac{1}{2}g_tu_xv + \frac{1}{2}g_tyu_{xy}v - f(t)n(t)u_{xy}v \\
& - g(t)vn(t)u_{yy}, \\
T_1 = {} & \frac{g_tyu_{xx}v}{2n(t)} - f(t)u_{xx}v - g(t)u_{xy}v.
\end{aligned} \tag{35}$$

For the symmetry in Case 2, the corresponding components of the conservation laws are

$$\begin{aligned}
X_2 = {} & -6\frac{C_2mu_tv_xu}{p} - \frac{1}{6}f_tv_{xxx} - \frac{1}{6}f_tv_t + \frac{g_ty}{2n(t)}u_{xx}v_{xx} \\
& + \frac{C_2mu_{txxx}v}{p} - \frac{g_{tt}y}{2n(t)}u_xv + \frac{g_{tt}y}{12n(t)}v_t + \frac{g_{tt}y}{12n(t)}v_{xxx} \\
& + \frac{C_2x}{3p}u_xv_t + \frac{C_2x}{3p}u_xv_{xxx} - \frac{3g_ty}{n(t)}u_xv_xu - \frac{g_ty}{2n(t)}u_xv_t \\
& + 4\frac{C_2y}{p}u_yv_xu + \frac{2C_2y}{3p}u_yv_t + \frac{2C_2y}{3p}u_yv_{xxx} \\
& - 3C_2yu_yu_xv + 3C_2yu_yv_xu - 6\frac{C_2t}{p}u_tu_xv \\
& + 6\frac{C_2t}{p}u_tu_xu + \frac{C_2t}{p}u_tv_t + \frac{C_2t}{p}u_tv_{xxx} + 6\frac{C_2mu_tu_xv}{p} \\
& - \frac{C_2mu_tv_{xxx}}{p} - \frac{C_2x}{3p}u_{xx}v_{xx} - \frac{2C_2y}{3p}u_{xy}v_{xx} \\
& - 3C_2yuvu_{xy} - 6\frac{C_2t}{p}u_{xt}uv - \frac{C_2t}{p}u_{xt}v_{xx} \\
& - \frac{g_typ}{12n(t)(t-m)}v_t - \frac{10C_2uu_xv}{p} - 4\frac{C_2yu_yu_xv}{p} \\
& + \frac{C_2xu_{xt}v}{3p} + \frac{C_2xn(t)u_{yy}v}{3p} - \frac{g_ty}{2n(t)}u_{xt}v \\
& + f(t)vn(t)u_{yy} + \frac{g_typ}{2n(t)(t-m)}u_xv \\
& - \frac{g_typ}{2n(t)(t-m)}v_xu + \frac{g_{tt}y}{2n(t)}v_xu \\
& - \frac{g_typ}{12n(t)(t-m)}v_{xxx} + 6\frac{C_2m}{p}u_{xt}uv \\
& + \frac{C_2m}{p}u_{xt}v_{xx} + \frac{C_2x}{3p}v_xu_{xxx} - \frac{g_ty}{2n(t)}u_{xxx}v_x \\
& + \frac{2C_2y}{3p}u_{xxy}v_x + \frac{C_2t}{p}u_{xxt}v_x - \frac{2C_2yu_{xxxy}v}{3p} \\
& - \frac{C_2tu_{xxxt}v}{p} - \frac{C_2mu_tv_t}{p} + g(t)u_yv_{xxx} \\
& - f_tv_xu - g(t)u_{xxxy}v - f(t)u_{xx}v_{xx} + f(t)u_{xxx}v_x \\
& + g(t)u_{xxy}v_x + f_tu_xv + f(t)u_xv_{xxx} - g(t)u_{xy}v_{xx} \\
& + vf(t)u_{xt} + g(t)u_yv_t + f(t)u_xv_t + \frac{2C_2}{3p}uv_t \\
& + \frac{4C_2}{p}u^2v_x - \frac{g_ty}{2}u_{yy}v - \frac{C_2u_xv_{xx}}{p} + \frac{2C_2x}{p}u_xv_xu \\
& + 6g(t)u_yv_xu - 6g(t)u_yu_xv + \frac{C_2y}{2}u_yv_{xxx} \\
& + \frac{C_2y}{2}u_yv_t + 6f(t)u_xv_xu + \frac{2C_2}{3p}uv_{xxx} \\
& - \frac{C_2y}{2}vu_{xxxy} - \frac{5C_2u_{xxx}v}{3p} \\
& + \frac{C_2y}{2}v_xu_{xxy} + \frac{4C_2}{3p}u_{xx}v_x - 6g(t)u_{xy}vu \\
& - \frac{C_2y}{2}v_{xx}u_{xy} - \frac{4C_2yu_{xy}uv}{p} - \frac{C_2mu_{xxt}v_x}{p},
\end{aligned}$$

$$
\begin{aligned}
Y_2 = & -\frac{g_{tt}v}{12}-\frac{C_2tn(t)\,u_{yt}v}{p}+\frac{C_2mn(t)\,u_{yt}v}{p}\\
& -f(t)\,n(t)\,vu_{xy}+\frac{g_tvp}{12(t-m)}+\frac{2C_2n(t)\,uv_y}{3p}\\
& -\frac{g_typv_y}{12(t-m)}+\frac{C_2xn(t)\,u_xv_y}{3p}+f(t)\,n(t)\,u_xv_y\\
& +\frac{2C_2yn(t)\,u_yv_y}{3p}+\frac{C_2yn(t)}{2}u_yv_y+g(t)\,n(t)\,u_yv_y\\
& +\frac{C_2tn(t)\,u_tv_y}{p}-\frac{C_2mn(t)\,u_tv_y}{p}+\frac{g_t}{2}u_xv\\
& -\frac{4C_2n(t)\,u_yv}{3p}-\frac{C_2xn(t)\,u_{xy}v}{3p}+\frac{g_{tt}}{12}yv_y+\frac{g_tyu_{xy}v}{2}\\
& -\frac{g_tyu_xv_y}{2}-\frac{1}{6}f_tn(t)\,v_y-\frac{C_2n(t)\,u_yv}{2}\\
& -g(t)\,vn(t)\,u_{yy}-\frac{2C_2yvn(t)}{3p}u_{yy}-\frac{C_2yvn(t)}{2}u_{yy},\\
T_2 = & -\frac{C_2u_xv}{p}-\frac{C_2xu_{xx}v}{3p}+\frac{g_tyu_{xx}v}{2n(t)}-f(t)\,u_{xx}v\\
& -\frac{2C_2yu_{xy}v}{3p}-\frac{C_2yu_{xy}v}{2}-g(t)\,u_{xy}v-\frac{C_2vtu_{xt}}{p}\\
& +\frac{C_2vmu_{xt}}{p}.
\end{aligned}\tag{36}
$$

Here we should note that the coefficient function $n(t)$ in the expression of X_2, Y_2, and T_2 satisfies $n(t)=(t-m)^pC_1$, m, p, and C_1 are constants, and $p\neq 0$, $C_1\neq 0$.

For the symmetry in Case 3, the corresponding components of the conservation laws are

$$
\begin{aligned}
X_3 = & -\frac{x}{18}\tau_{tt}v_{xxx}+f(t)\,v_xu_{xxx}+g(t)\,v_xu_{xxy}\\
& +f(t)\,u_xv_{xxx}+\tau(t)\,u_tv_t-\tau_tu_xv_{xx}+g(t)\,u_yv_{xxx}\\
& +g(t)\,u_yv_t+\frac{2}{3}\tau_tuv_t-\tau(t)\,u_{xt}v_{xx}+\frac{2}{3}\tau_tuv_{xxx}\\
& +\tau(t)\,v_xu_{txx}-\frac{5}{3}\tau_tvu_{xxx}-\tau(t)\,vu_{txxx}+4\tau_tu^2v_x\\
& -f_tv_xu-g(t)\,u_{xy}v_{xx}+\frac{1}{3}\tau_{tt}uv+f_tu_xv\\
& -f(t)\,u_{xx}v_{xx}-\frac{1}{6}f_tv_{xxx}-g(t)\,vu_{xxxy}+f(t)\,vu_{tx}\\
& +f(t)\,v_tu_x+\tau(t)\,u_tv_{xxx}-\frac{x}{18}\tau_{tt}v_t+\frac{y^2}{6n}\tau_{ttt}uv_x\\
& +\frac{y}{2n}g_{tt}uv_x-\frac{y^2}{6n}\tau_{tt}vu_{xt}-\frac{1}{6}f_tv_t+2x\tau_tu_xv_xu\\
& +\frac{1}{18}\tau_{tt}v_{xx}-\frac{y^2}{6n}\tau_{ttt}u_xv+\frac{x}{3}n\tau_tvu_{yy}-\frac{y}{2n}g_{tt}u_xv\\
& +\frac{x}{3}\tau_tvu_{xt}-\frac{y^2}{6}\tau_{tt}vu_{yy}-\frac{y}{2}g_tvu_{yy}+f(t)\,vnu_{yy}\\
& -10\tau_tuu_xv+\frac{x}{3}\tau_{tt}u_xv-\frac{x}{3}\tau_{tt}v_xu+\frac{y^2}{36n}\tau_{ttt}v_t\\
& +\frac{y^2}{36n}\tau_{ttt}v_{xxx}+\frac{y}{12n}g_{tt}v_t+\frac{y}{12n}g_{tt}v_{xxx}+\frac{x}{3}\tau_tu_xv_t\\
& +\frac{x}{3}\tau_tu_xv_{xxx}+6f(t)\,u_xv_xu+\frac{2y}{3}\tau_tu_yv_t\\
& +\frac{2y}{3}\tau_tu_yv_{xxx}-6g(t)\,u_xu_yv+6g(t)\,v_xu_yu\\
& -6\tau(t)\,u_tu_xv+6\tau(t)\,u_tv_xu-\frac{x}{3}\tau_tu_{xx}v_{xx}\\
& -\frac{2y}{3}\tau_tu_{xy}v_{xx}-6g(t)\,u_{xy}uv-6\tau(t)\,u_{xt}uv\\
& +\frac{x}{3}\tau_tu_{xxx}v_x+\frac{2y}{3}\tau_tu_{xxy}v_x-\frac{2y}{3}\tau_tu_{xxxy}v\\
& -\frac{y}{2n}g_tu_{tx}v-\frac{y^2}{6n}\tau_{tt}v_xu_{xxx}-\frac{y}{2n}g_tu_{xxx}v_x\\
& -\frac{y^2}{n}\tau_{tt}u_xv_xu-\frac{y^2}{6n}\tau_{tt}u_xv_t-\frac{y^2}{6n}\tau_{tt}u_xv_{xxx}\\
& -\frac{3y}{n}g_tu_xv_xu-\frac{y}{2n}g_tu_xv_t-\frac{y}{2n}g_tu_xv_{xxx}\\
& -4y\tau_tu_yu_xv+4y\tau_tu_yv_xu+\frac{y^2}{6n}\tau_{tt}u_{xx}v_{xx}\\
& +\frac{y}{2n}g_tu_{xx}v_{xx}-4y\tau_tu_{xy}uv+\frac{4}{3}\tau_tu_{xx}v_x,\\
Y_3 = & \frac{1}{2}g_tvu_x+\frac{y}{12}g_{tt}v_y-\frac{y}{18}\tau_{ttt}v+\frac{y^2}{36}\tau_{ttt}v_y\\
& -\frac{1}{12}g_{tt}v+\frac{x}{3}n\tau_tu_xv_y+\frac{y}{3}\tau_{tt}u_xv+\frac{y^2}{6}\tau_{tt}u_{xy}v\\
& +\frac{2}{3}n\tau_tuv_y-\frac{x}{18}n\tau_{tt}v_y+nf(t)\,u_xv_y+g(t)\,nv_yu_y\\
& +n\tau(t)\,u_tv_y-\frac{4}{3}n\tau_tvu_y-nf(t)\,vu_{xy}-\tau(t)\,nvu_{ty}\\
& -\frac{y^2}{6}\tau_{tt}u_xv_y-\frac{y}{2}g_tu_xv_y+\frac{y}{2}g_tvu_{xy}-\frac{1}{6}nf_tv_y\\
& +\frac{2y}{3}n\tau_tu_yv_y-\frac{x}{3}n\tau_tvu_{xy}-\frac{2y}{3}\tau_tvnu_{yy}\\
& -g(t)\,vnu_{yy},
\end{aligned}
$$

$$\begin{aligned} T_3 = & -\tau(t)\, v u_{xt} - \tau_t v u_x + \frac{1}{18}\tau_{tt} v - \frac{x}{3}\tau_t v u_{xx} \\ & + \frac{y^2}{6n}\tau_{tt} u_{xx} v + \frac{y}{2n} g_t u_{xx} v - f(t)\, v u_{xx} \\ & - \frac{2y}{3}\tau_t u_{xy} v - \tau(t)\, v u_{xy}. \end{aligned} \tag{37}$$

For the fourth symmetry, the two functions $\tau(t)$ and $n(t)$ are determined by the differential equation (19) and they have many explicit solutions. For simplicity, we take $\tau(t) = 1$; then $n(t) = 1 + \tan^2 t$ and $e(t) = (-\tan t/2) + C_3$. When $f(t) = g(t) = 0$, the corresponding Lie symmetry is

$$V = -\frac{y^2}{4}\frac{\partial}{\partial x} + y\tan t\frac{\partial}{\partial y} + \frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}, \tag{38}$$

and the components of the conservation laws are

$$\begin{aligned} X_4 = & -v u_{txxx} - \frac{y^2}{4} u_{xxx} v_x - 6uvu_{xt} + \frac{y^2}{4} u_{xx} v_{xx} \\ & + 6uu_t v_x - 6vu_t u_x - C_3 u_t v + v_x u_{txx} + u_t v_t \\ & + \frac{\tan t}{2} u_t v - \frac{y^2}{4} u_x v_{xxx} - \frac{y^2}{4} u_x v_t - \frac{y^2}{4} u_{yy} v \\ & - \frac{y^2}{4} v u_{xt} + u_t v_{xxx} - \frac{3y^2}{2} u_x v_x u + \frac{\tan^2 t}{2} y v u_y \\ & + y u_y v_{xxx}\tan t + y u_y v_t \tan t + y v_x u_{xxy}\tan t \\ & - 6yvu_y u_x \tan t - C_3 y v u_y \tan t - \frac{\tan^2 t}{4} y^2 v u_{yy} \\ & - u_{xt} v_{xx} - y v_{xx} u_{xy}\tan t - y v u_{xxxy}\tan t \\ & - 6yuvu_{xy}\tan t + 6yuu_y v_x \tan t, \\ Y_4 = & -\frac{y^2}{4} u_x v_y - \frac{y^2}{4} v_y u_x \tan^2 t + y v_y u_y \tan t \\ & + y v_y u_y \tan^3 t + v_y u_t + v_y u_t \tan^2 t + \frac{1}{2} v y u_x \\ & + \frac{y^2}{4} v u_{xy} + \frac{y^2}{4} v u_{xy}\tan^2 t - v u_y \tan t - v u_y \tan^3 t \\ & - v u_{ty} - v u_{ty}\tan^2 t + \frac{1}{2} v y u_x \tan^2 t \\ & - y v \tan t u_{yy} - y v \tan^3 t u_{yy}, \\ T_4 = & \frac{y^2}{4} v u_{xx} - y v u_{xy}\tan t - v u_{xt}. \end{aligned} \tag{39}$$

We should mention that in the above components of the conservation laws for (1) and (28), u is a solution of (1) and v is a solution of the adjoint equation (28). Making use of the explicit solutions of (28), local conservation laws for (1) can be obtained. For example, when $a(t) = 0$ and $b(t) = 0$ in (34),

$$v = k(t)\, y + l(t), \tag{40}$$

where $k(t)$ and $l(t)$ are arbitrary functions, is an exact solution of (28). Substituting (40) into the above four conservation laws, we can obtain time-dependent and local conservation laws for (1). Here we take (X_4, Y_4, T_4) as an illustrative example; when $v = k(t)y + l(t)$, the components of the conservation laws (X_4, Y_4, T_4) become

$$\begin{aligned} \bar{X}_4 = & -C_3 y^2 u_y k(t)\tan t - C_3 l(t)\, y u_y \tan t \\ & - 6k(t)\, y^2 u_y u_x \tan t - 6l(t)\, y u_y u_x \tan t + l'(t)\, u_t \\ & - l(t)\, u_{xxxt} - 6k(t)\, y^2 u_{xy} u \tan t - \frac{y^3}{4} k(t)\, u_{yy} \\ & - \frac{y^2}{4} l'(t)\, u_x - C_3 l(t)\, u_t + \frac{1}{2} l(t)\, u_t \tan t - 6l(t)\, u_t u_x \\ & - 6l(t)\, u_{xt} u - \frac{y^2}{4} l(t)\, u_{yy} - k(t)\, y u_{xxxt} - \frac{y^3}{4} k'(t)\, u_x \\ & + k'(t)\, y u_t - \frac{y^2}{4} l(t)\, u_{xt} - \frac{y^3}{4} k(t)\, u_{xt} \\ & - y l(t)\, u_{xxxy}\tan t - k(t)\, y^2 u_{xxxy}\tan t \\ & - \frac{y^3}{4} k(t)\, u_{yy}\tan^2 t + \frac{y}{2} k(t)\, u_t \tan t + l'(t)\, y u_y \tan t \\ & - 6k(t)\, y u u_{xt} + \frac{y}{2} l(t)\, u_y \tan^2 t + \frac{y^2}{2} k(t)\, u_y \tan^2 t \\ & - 6k(t)\, y u_t u_x - C_3 k(t)\, y u_t - \frac{y^2\tan^2 t}{4} l(t)\, u_{yy} \\ & + k'(t)\, y^2 u_y \tan t - 6l(t)\, y u_{xy} u \tan t, \\ \bar{Y}_4 = & -l(t)\, y u_{yy}\tan^3 t - l(t)\, y u_{yy}\tan t - l(t)\, u_{yt} t + k(t)\, u_t \\ & - k(t)\, y^2 \tan^3 t u_{yy} - k(t)\, y \tan^2 t u_{yt} + \frac{y}{2} l(t)\, u_x \tan^2 t \\ & + \frac{y^2}{4} l(t)\, u_{xy}\tan^2 t + \frac{y^3}{4} k(t)\, u_{xy} + k(t)\, u_t \tan^2 t \\ & + \frac{y^2}{4} k(t)\, u_x - l(t)\, u_{yt}\tan^2 t - y k(t)\, u_{yt} - l(t)\, u_y \tan^3 t \\ & - l(t)\, u_y \tan t + \frac{y^2}{4} l(t)\, u_{xy} + \frac{y}{2} l(t)\, u_x \\ & + \frac{y^2}{4} k(t)\, u_x \tan^2 t + \frac{y^3}{4} k(t)\, u_{xy}\tan^2 t \\ & - k(t)\, y^2 u_{yy}\tan t, \\ \bar{T}_4 = & \frac{1}{4}\left(k(t)\, y + l(t)\right)\left(y^2 u_{xx} - 4 y u_{xy}\tan t - 4u_{xt}\right). \end{aligned} \tag{41}$$

These are local and explicit conservation laws of (1). Next we show that the above conservation laws $(\bar{X}_4, \bar{Y}_4, \bar{T}_4)$ are nontrivial:

$$\begin{aligned}
&D_x\left(\bar{X}_4\right)+D_y\left(\bar{Y}_4\right)+D_t\left(\bar{T}_4\right)\\
&\quad=-C_3 y^2 k(t) u_{xy}\tan t-l(t)u_{xxxxt}-l(t)u_{xtt}\\
&\quad-k(t)yu_{xtt}-12l(t)u_xu_{xt}-2l(t)u_{yy}\tan^3 t\\
&\quad+\frac{1}{2}l(t)u_x\tan^2 t-2l(t)u_{yy}\tan t+\frac{1}{2}yk(t)u_x\\
&\quad-k(t)yu_{xxxxt}-6l(t)uu_{xxt}-6l(t)u_tu_{xx}\\
&\quad+\frac{1}{2}l(t)u_{xt}\tan t-C_3 l(t)u_{xt}-l(t)u_{yyt}\tan^2 t\\
&\quad-k(t)yu_{yyt}-6y^2k(t)uu_{xxy}\tan t\\
&\quad-12y^2k(t)u_xu_{xy}\tan t-12yl(t)u_xu_{xy}\tan t\\
&\quad-6l(t)yuu_{xxy}\tan t-6k(t)y^2u_yu_{xx}\tan t\\
&\quad-6l(t)yu_yu_{xx}\tan t-l(t)u_{yyt}-C_3l(t)yu_{xy}\tan t\\
&\quad+\frac{1}{2}l(t)u_x-k(t)yu_{yyt}\tan^2 t-l(t)yu_{yyy}\tan^3 t\\
&\quad+\frac{1}{2}yl(t)u_{xy}\tan^2 t-C_3k(t)yu_{xt}-6yk(t)u_tu_{xx}\\
&\quad+\frac{1}{2}y^2k(t)u_{xy}\tan^2 t-l(t)yu_{xxxxy}\tan t\\
&\quad-12k(t)yu_xu_{xt}-6k(t)yuu_{xxt}+\frac{1}{2}yk(t)u_{xt}\tan t\\
&\quad-k(t)y^2u_{xxxxy}\tan t-k(t)y^2u_{xyt}\tan t\\
&\quad-l(t)yu_{xyt}\tan t-2k(t)yu_{yy}\tan^3 t\\
&\quad+\frac{1}{2}yk(t)u_x\tan^2 t-2k(t)yu_{yy}\tan t\\
&\quad-k(t)y^2u_{yyy}\tan t-l(t)yu_{yyy}\tan t\\
&\quad-k(t)y^2u_{yyy}\tan^3 t.
\end{aligned}\tag{42}$$

Obviously, if $k(t), l(t)$ are not zero at the same time, $D_x(\bar{X}_4)+D_y(\bar{Y}_4)+D_t(\bar{T}_4)\neq 0$. And we can easily check that

$$\begin{aligned}
&\left(D_x\left(X_4\right)+D_y\left(Y_4\right)\right.\\
&\quad\left.+D_t\left(T_4\right)\right)\Big|_{u_{xxxx}=-u_{xt}-6u_x{}^2-6uu_{xx}-e(t)u_x-n(t)u_{yy}}\equiv 0.
\end{aligned}\tag{43}$$

4. Symmetry Reductions and New Exact Solutions of (1)

In Section 2, we obtain the Lie symmetries of (1). In this section, we will investigate the symmetry reductions and exact solutions for the equation. Using the obtained symmetries (3), similarity variables and symmetry reductions can be found by solving the corresponding characteristic equation:

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dt}{\tau}=\frac{du}{\phi}.\tag{44}$$

For the four different cases, we determine the following symmetry reductions and exact solutions of (1).

4.1. For the Symmetry in Case 1, Where $e(t)$ and $n(t)$ ($n(t)\neq 0$) Are Arbitrary Functions.

(i) When $g(t)=0$, $f(t)\neq 0$, we can obtain

$$u=\frac{f_t x}{6f}+\Omega\left(y,t\right),\tag{45}$$

and $\Omega(y,t)$ is a solution of the following reduction equation:

$$\frac{f_{tt}}{6f}+\frac{ef_t}{6f}+n\Omega_{yy}=0.\tag{46}$$

From the above equation, we can obtain an algebraically explicit analytical solution for (1):

$$u=\frac{f_t x}{6f}-\frac{f_{tt}+ef_t}{12nf}y^2+F_1(t)\,y+F_2(t),\tag{47}$$

where $F_1(t)$ and $F_2(t)$ are arbitrary functions of t.

(ii) When $f(t)=0$, $g(t)=t$, the corresponding symmetry is

$$V=-\frac{y}{2n}\frac{\partial}{\partial x}+t\frac{\partial}{\partial y}+0\frac{\partial}{\partial t}+\frac{n_t}{12n^2}y\frac{\partial}{\partial u}.\tag{48}$$

By the characteristic equations of the symmetry, we have $u=\Omega(\theta,t)$, $\theta=y^2/2+2nxt$. Substituting it into (1), we get a symmetry reduction of (1):

$$\begin{aligned}
&\Omega_{\theta t}+\frac{\theta}{t}\Omega_{\theta\theta}+12nt\left(\Omega_\theta\Omega\right)_\theta+8n^3t^3\Omega_{\theta\theta\theta\theta}\\
&\quad+\left(\frac{3}{2t}-\frac{n_t}{n}+e(t)\right)\Omega_\theta+\frac{n_t^2}{6n^3t}-\frac{n_{tt}}{12n^2t}\\
&\quad-\frac{e(t)\,n_t}{12n^2t}=0.
\end{aligned}\tag{49}$$

If the coefficient functions $e(t)=0$, $n(t)=$ Const., the obtained symmetry reduction can be simplified to

$$\Omega_{\theta t}+\frac{\theta}{t}\Omega_{\theta\theta}+\frac{3}{2t}\Omega_\theta+12nt\left(\Omega_\theta\Omega\right)_\theta+8n^3t^3\Omega_{\theta\theta\theta\theta}=0.\tag{50}$$

Integrating (50) with respect to θ and taking the constant of integration to zero, we get the following equation:

$$\Omega_t+12nt\Omega_\theta\Omega+8n^3t^3\Omega_{\theta\theta\theta}+\frac{\theta}{t}\Omega_\theta+\frac{1}{2t}\Omega=0.\tag{51}$$

Equation (51) is the (1 + 1)-dimensional generalized KdV equation with variable coefficients. To the best of our knowledge, exact solutions of (51) have not been studied up to now. Solving (51) by the method in [25], we can get the following solutions for (1):

$$u = \Omega(\theta, t) = \frac{\theta}{24nt^2} + \frac{M_3}{24ntM_1} - \frac{8n^2M_1^2c_2}{3t} - \frac{8n^2M_1^2c_4}{t}P^2(\varphi), \qquad \varphi = M_1\theta t^{-3/2} + M_3t^{-1/2} + M_2, \tag{52}$$

where M_1, M_2, and M_3 are arbitrary constants and the function $P(\varphi)$ satisfies

$$P'^2 = c_0 + c_2P^2 + c_4P^4, \tag{53}$$

where c_0, c_2, and c_4 are constants; solutions of (53) have been given in [26]. By means of the solutions of (53), plenty of solutions for (1) can be obtained; for example,

$$\begin{aligned}
u_1 &= \frac{y^2/2 + 2nxt}{24nt^2} + \frac{M_3}{24ntM_1} - \frac{8n^2M_1^2(-k^2-1)}{3t} - \frac{8n^2M_1^2k^2\operatorname{sn}^2(\varphi)}{t}, \quad (c_0 = 1, c_2 = -1-k^2, c_4 = k^2),\\
u_2 &= \frac{y^2/2 + 2nxt}{24nt^2} + \frac{M_3}{24ntM_1} - \frac{8n^2M_1^2(-k^2-1)}{3t} - \frac{8n^2M_1^2\operatorname{ns}^2(\varphi)}{t}, \quad (c_0 = k^2, c_2 = -1-k^2, c_4 = 1),\\
u_3 &= \frac{y^2/2 + 2nxt}{24nt^2} + \frac{M_3}{24ntM_1} - \frac{8n^2M_1^2c_2}{3t} + \frac{8n^2M_1^2c_2\operatorname{sech}^2(\varphi)}{t}, \quad (c_0 = 0, c_2 > 0, c_4 < 0),\\
u_4 &= \frac{y^2/2 + 2nxt}{24nt^2} + \frac{M_3}{24ntM_1} - \frac{8n^2M_1^2c_2}{3t} + \frac{4n^2M_1^2c_2\tanh^2(\varphi)}{t}, \quad \left(c_0 = \frac{c_2^2}{4c_4}, c_2 < 0, c_4 > 0\right),
\end{aligned} \tag{54}$$

where k $(0 < k < 1)$ denotes the modulus of the Jacobi elliptic function.

(iii) When $e(t) = 0, n(t) = (t-m)^pC_1, p \neq 0, C_1 \neq 0, f(t) = M_0$, and $g(t) = 1$, we can get

$$u = \Omega(\theta, t), \quad \theta = x - M_0y. \tag{55}$$

And $\Omega(\theta, t)$ satisfies the following reduction equation:

$$\Omega_{\theta t} + 6\left(\Omega_\theta^2 + \Omega\Omega_{\theta\theta}\right) + \Omega_{\theta\theta\theta\theta} + M_0^2C_1(t-m)^p\Omega_{\theta\theta} = 0. \tag{56}$$

The above equation can be integrated by θ and, when we take the constant of integration to zero, we get a reduced reduction equation:

$$\Omega_t + 6\Omega\Omega_\theta + \Omega_{\theta\theta\theta} + M_0^2C_1(t-m)^p\Omega_\theta = 0. \tag{57}$$

Equation (57) is variable coefficient KdV equation and soliton-like solutions have been obtained in [27]. By means of the known solutions, many explicit solutions of (1) can be obtained. For example,

$$\begin{aligned}
u_1 &= k_1 + 2ck_4^2\operatorname{sech}^2(\sqrt{c}\varphi),\\
\varphi &= k_4(x - M_0y) - 6k_1k_4t - 4ck_4^3t - \frac{M_0^2C_1k_4}{p+1}(t-m)^{p+1},\\
u_2 &= k_1 - 2ck_4^2\tanh^2(\varphi),\\
\varphi &= k_4(x - M_0y) - 6k_1k_4t + 8k_4^3t - \frac{M_0^2C_1k_4}{p+1}(t-m)^{p+1},
\end{aligned} \tag{58}$$

where k_1, k_4, and c are constants.

(iv) When $e(t) \neq 0$ and $n(t) = N_0\exp((\int(e_t - 2e^2)/e)dt)$, $f(t) = N_1$, $g(t) = 1$. By the corresponding characteristic equation of the symmetry, we have

$$u = \Omega(\theta, t), \quad \theta = x - N_1y. \tag{59}$$

Substituting it into (1), we get the following symmetry reduction of (1):

$$\Omega_{\theta t} + 6\left(\Omega_\theta^2 + \Omega\Omega_{\theta\theta}\right) + \Omega_{\theta\theta\theta\theta} + e(t)\Omega_\theta + N_1^2N_0\exp\left(\int\frac{e_t - 2e^2}{e}dt\right)\Omega_{\theta\theta}. \tag{60}$$

Integrating the above equation with respect to θ and taking the constant of integration to zero, the obtained reduction equation becomes

$$\Omega_t + 6\Omega\Omega_\theta + \Omega_{\theta\theta\theta} + e(t)\Omega + N_1^2N_0\exp\left(\int\frac{e_t - 2e^2}{e}dt\right)\Omega_\theta. \tag{61}$$

Equation (61) is a variable coefficient KdV equation [28, 29].

4.2. For the Symmetry in Case 2, $e(t)=0$, $n(t)=(t-m)^pC_1$, $p \neq 0$, $C_1 \neq 0$. When $f(t) = g(t) = 0$, $m = 0$, $p = C_2 = 2/3$, then

$n(t) = C_1 t^{2/3}$, and $C_1 \neq 0$; the corresponding symmetry of (1) is

$$V = \frac{x}{3}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u}. \quad (62)$$

By the characteristic equations of the symmetry, we can get the explicit solutions for (1)

$$u = \Omega(\theta,\delta)\, t^{-2/3}, \quad \theta = \frac{x^3}{t}, \ \delta = \frac{y}{t}, \quad (63)$$

where the function $\Omega(\theta,\delta)$ satisfies the following reduction equation:

$$\begin{aligned} &-3\theta^{5/3}\Omega_{\theta\theta} - 3\theta^{2/3}\delta\Omega_{\theta\delta} - 5\theta^{2/3}\Omega_\theta \\ &\quad + 54\theta^{4/3}\left(\Omega_\theta^2 + \Omega_{\theta\theta}\Omega\right) + 36\theta^{1/3}\Omega_\theta\Omega \\ &\quad + 81\theta^{8/3}\Omega_{\theta\theta\theta\theta} + 324\theta^{5/3}\Omega_{\theta\theta\theta} + 180\theta^{2/3}\Omega_{\theta\theta} \\ &\quad + C_1\Omega_{\delta\delta} = 0. \end{aligned} \quad (64)$$

Equation (64) is difficult to solve and we will study its exact solutions in a future paper.

4.3. For the Symmetry in Case 3, $e(t)=0$, $n(t)=$Const., and $\tau(t)\neq 0$. When $f(t) = 0$, $g(t) = 0$, the corresponding symmetry is

$$\begin{aligned} V = &\left(\frac{\tau_t}{3}x - \frac{\tau_{tt}}{6n}y^2\right)\frac{\partial}{\partial x} + \frac{2}{3}\tau_t y\frac{\partial}{\partial y} + \tau(t)\frac{\partial}{\partial t} \\ &+ \left(-\frac{2\tau_t}{3}u + \frac{\tau_{tt}}{18}x - \frac{\tau_{ttt}}{36n}y^2\right)\frac{\partial}{\partial u}. \end{aligned} \quad (65)$$

By the characteristic equation of the symmetry, we have

$$\begin{aligned} u = \frac{1}{18\tau}x\tau_t - \frac{1}{36n\tau}y^2\tau_{tt} + \frac{1}{54n\tau^2}y^2\tau_t^2 + \Omega(\theta,\delta)\,\tau^{-2/3}, \\ \theta = x\tau^{-1/3} + \frac{1}{6n}y^2\tau_t\tau^{-4/3}, \ \delta = y\tau^{-2/3}. \end{aligned} \quad (66)$$

Substituting it into (1), we get a symmetry reduction of (1):

$$6\Omega_\theta^2 + 6\Omega_{\theta\theta}\Omega + \Omega_{\theta\theta\theta\theta} + n\Omega_{\delta\delta} = 0. \quad (67)$$

Equation (67) is the special case of (2 + 1)-dimensional Boussinesq equation and exact solutions of (67) have been studied by Chen and Zhang in [30] (with $a = 0$, $b = 0$, $r = -3/n$, and $s = -1/n$). With the help of the known solutions in [30], many explicit solutions of (1) can be obtained. We list the following soliton solutions (u_1–u_4) and Jacobi elliptic function solutions (u_5–u_{17}):

$$\begin{aligned} u_1 = &\left(\frac{-n\omega^2}{6\alpha^2} + \frac{4}{3}\alpha^2\right)\tau^{-2/3} - 2\alpha^2\tau^{-2/3}\tanh^2(\varphi) \\ &+ \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_2 = &\left(\frac{-n\omega^2}{6\alpha^2} - \frac{2}{3}\alpha^2\right)\tau^{-2/3} + 2\alpha^2\tau^{-2/3}\operatorname{sech}^2(\varphi) \\ &+ \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_3 = &\left(\frac{-n\omega^2}{6\alpha^2} + \frac{1}{3}\alpha^2\right)\tau^{-2/3} \\ &- \frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon\tanh^4(\varphi) + \beta(1+\operatorname{sech}(\varphi))^4}{\tanh^2(\varphi)(1+\operatorname{sech}(\varphi))^2} \\ &+ \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_4 = &\left(\frac{-n\omega^2}{6\alpha^2} - \frac{2}{3}\alpha^2\right)\tau^{-2/3} - 2\alpha^2\tau^{-2/3}\frac{\operatorname{sech}^2(\varphi)}{\tanh^2(\varphi)} \\ &+ \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_5 = &\left(\frac{-n\omega^2}{6\alpha^2} + \frac{2}{3}\alpha^2 + \frac{2}{3}\alpha^2 m^2\right)\tau^{-2/3} - 2\alpha^2\tau^{-2/3} \\ &\times \frac{\varepsilon + \beta m^2\operatorname{sn}^4(\varphi)}{\operatorname{sn}^2(\varphi)} + \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_6 = &\left(\frac{-n\omega^2}{6\alpha^2} + \frac{2}{3}\alpha^2 + \frac{2}{3}\alpha^2 m^2\right)\tau^{-2/3} - 2\alpha^2\tau^{-2/3} \\ &\times \frac{\varepsilon\operatorname{dn}^4(\varphi) + \beta m^2\operatorname{cn}^4(\varphi)}{\operatorname{cn}^2(\varphi)\operatorname{dn}^2(\varphi)} + \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_7 = &\left(\frac{-n\omega^2}{6\alpha^2} - \frac{4}{3}\alpha^2 m^2 + \frac{2}{3}\alpha^2\right)\tau^{-2/3} - 2\alpha^2\tau^{-2/3} \\ &\times \frac{\varepsilon\left(1-m^2\right) - \beta m^2\operatorname{cn}^4(\varphi)}{\operatorname{cn}^2(\varphi)} + \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$\begin{aligned} u_8 = &\left(\frac{-n\omega^2}{6\alpha^2} - \frac{4}{3}\alpha^2 + \frac{2}{3}\alpha^2 m^2\right)\tau^{-2/3} + 2\alpha^2\tau^{-2/3} \\ &\times \frac{\varepsilon\left(1-m^2\right) + \beta\operatorname{dn}^4(\varphi)}{\operatorname{dn}^2(\varphi)} + \frac{\tau_t x}{18\tau} - \frac{\tau_{tt}y^2}{36n\tau} + \frac{\tau_t^2 y^2}{54n\tau^2}, \end{aligned}$$

$$u_9=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{4}{3}\alpha^2+\frac{2}{3}\alpha^2m^2\right)\tau^{-2/3}-2\alpha^2\tau^{-2/3}\frac{\varepsilon\left(1-m^2\right)\operatorname{sn}^4(\varphi)+\beta\operatorname{cn}^4(\varphi)}{\operatorname{sn}^2(\varphi)\operatorname{cn}^2(\varphi)}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{10}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{4}{3}\alpha^2m^2+\frac{2}{3}\alpha^2\right)\tau^{-2/3}-2\alpha^2\tau^{-2/3}\frac{\varepsilon\operatorname{dn}^4(\varphi)-\beta m^2\left(1-m^2\right)\operatorname{sn}^4(\varphi)}{\operatorname{sn}^2(\varphi)\operatorname{dn}^2(\varphi)}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{11}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2+\frac{2}{3}\alpha^2m^2\right)\tau^{-2/3}-\frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon\operatorname{sn}^4(\varphi)+\beta(1\pm\operatorname{cn}(\varphi))^4}{\operatorname{sn}^2(\varphi)(1\pm\operatorname{cn}(\varphi))^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{12}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2+\frac{2}{3}\alpha^2m^2\right)\tau^{-2/3}-\frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon\operatorname{cn}^4(\varphi)+\beta\left(\sqrt{1-m^2}\operatorname{sn}(\varphi)\pm\operatorname{dn}(\varphi)\right)^4}{\operatorname{cn}^2(\varphi)\left(\sqrt{1-m^2}\operatorname{sn}(\varphi)\pm\operatorname{dn}(\varphi)\right)^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{13}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2-\frac{1}{3}\alpha^2m^2\right)\tau^{-2/3}+\frac{\alpha^2}{2}\left(1-m^2\right)\tau^{-2/3}\frac{\varepsilon\operatorname{dn}^4(\varphi)+\beta(1\pm m\operatorname{sn}(\varphi))^4}{\operatorname{dn}^2(\varphi)(1\pm m\operatorname{sn}(\varphi))^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{14}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2-\frac{1}{3}\alpha^2m^2\right)\tau^{-2/3}-\frac{\alpha^2}{2}\left(1-m^2\right)\tau^{-2/3}\frac{\varepsilon\operatorname{cn}^4(\varphi)+\beta(1\pm\operatorname{sn}(\varphi))^4}{\operatorname{cn}^2(\varphi)(1\pm\operatorname{sn}(\varphi))^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{15}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2-\frac{1}{3}\alpha^2m^2\right)\tau^{-2/3}+\frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon\left(1-m^2\right)^2+\beta(m\operatorname{cn}(\varphi)\pm\operatorname{dn}(\varphi))^4}{(m\operatorname{cn}(\varphi)\pm\operatorname{dn}(\varphi))^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{16}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2-\frac{2}{3}\alpha^2m^2\right)\tau^{-2/3}-\frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon\left(1-m^2\right)^2\operatorname{sn}^4(\varphi)+\beta(\operatorname{dn}(\varphi)\pm\operatorname{cn}(\varphi))^4}{\operatorname{sn}^2(\varphi)(\operatorname{dn}(\varphi)\pm\operatorname{cn}(\varphi))^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},$$

$$u_{17}=\left(\frac{-n\omega^2}{6\alpha^2}-\frac{1}{3}\alpha^2m^2+\frac{2}{3}\alpha^2\right)\tau^{-2/3}-\frac{\alpha^2}{2}\tau^{-2/3}\frac{\varepsilon m^4\operatorname{cn}^4(\varphi)+\beta\left(\sqrt{1-m^2}\pm\operatorname{dn}(\varphi)\right)^4}{\operatorname{cn}^2(\varphi)\left(\sqrt{1-m^2}\pm\operatorname{dn}(\varphi)\right)^2}+\frac{\tau_tx}{18\tau}-\frac{\tau_{tt}y^2}{36n\tau}+\frac{\tau_t^2y^2}{54n\tau^2},\tag{68}$$

where $\varphi=\alpha(x\tau^{-1/3}+(1/6n)y^2\tau_t\tau^{-4/3})+\omega(y\tau^{-2/3})$, α and ω are constants, $k(0<k<1)$ denotes the modulus of the Jacobi elliptic function, and ε and β are arbitrary elements of $\{0,1\}$. We should mention that the soliton solution u_1 is the limit of u_5 when $m\to 1$, $\varepsilon=0$, $\beta=1$. The solutions u_2, u_3, and u_4 are the limit of u_7, u_{11}, and u_9, respectively, when $m\to 1$, $\beta=1$.

4.4. For the Symmetry in Case 4, $e(t)=-n_t/4n+C_3/\tau(t)$, $n(t)$, and $\tau(t)$ Satisfy (19). For simplicity, we take $f(t)=g(t)=0$, $\tau(t)=1$; then $n(t)=1+\tan^2 t$ and $e(t)=-\tan t/2+C_3$. Solving the corresponding characteristic equation, we get

$$u=\Omega(\theta,\delta),\quad \theta=x+\frac{y^2}{4}\sin t\cos t,\ \delta=y\cos t.\tag{69}$$

Substituting it into (1), we get a symmetry reduction of (1):

$$\frac{\delta^2}{4}\Omega_{\theta\theta}+6\Omega_{\theta\theta}\Omega+6\Omega_\theta^2+\Omega_{\theta\theta\theta\theta}+C_3\Omega_\theta+\Omega_{\delta\delta}=0.\tag{70}$$

Obviously, $\Omega=-(C_3/6)\theta+N_1\delta+N_2$ is a solution of (70). From that, we can get an algebraically explicit analytical solution for (1) as follows:

$$u=-\frac{C_3}{6}\left(x+\frac{y^2}{4}\sin t\cos t\right)+N_1y\cos t+N_2,\tag{71}$$

where N_1 and N_2 are integral constants. And, if $C_3=0$, (70) becomes the following (2 + 1)-dimensional variable coefficient Boussinesq equation:

$$\frac{\delta^2}{4}\Omega_{\theta\theta}+6\Omega_{\theta\theta}\Omega+6\Omega_\theta^2+\Omega_{\theta\theta\theta\theta}+\Omega_{\delta\delta}=0.\tag{72}$$

Remark 3. To the best of our knowledge, the symmetry reductions obtained in this paper have not been reported in the existent literature, so they are completely new. The exact solutions of (1) obtained here are all different from the known solutions and they are also new. All the solutions and conservation laws obtained in this paper for (1) have been checked by Maple software.

5. Conclusions

In summary, by performing Lie symmetry analysis to (1), four cases of geometric symmetries are obtained when the coefficient functions satisfy four different constraint conditions. According to the relationship between symmetry and conservation laws given by Ibragimov, many explicit and nontrivial conservation laws, which includes arbitrary functions of t, are derived. These conservation laws may be useful for the explanation of some practical physical problems. Using the associated vector fields of the obtained symmetry, (1) is reduced to $(1 + 1)$-dimensional nonlinear partial differential equations including different types of variable coefficient KdV equation (see (51), (57), and (61)), special case of $(2 + 1)$-dimensional Boussinesq equation (see (67) and (72)), and other reduction equations (see (64) and (70)). Many new explicit solutions of (1) have been derived by solving the reduction equations. These solutions, including soliton solutions, Jacobi doubly periodic solutions, and algebraically explicit analytical solutions, can make one discuss the behavior of solutions and also provide mathematical foundation for the explanation of some interesting physical phenomena.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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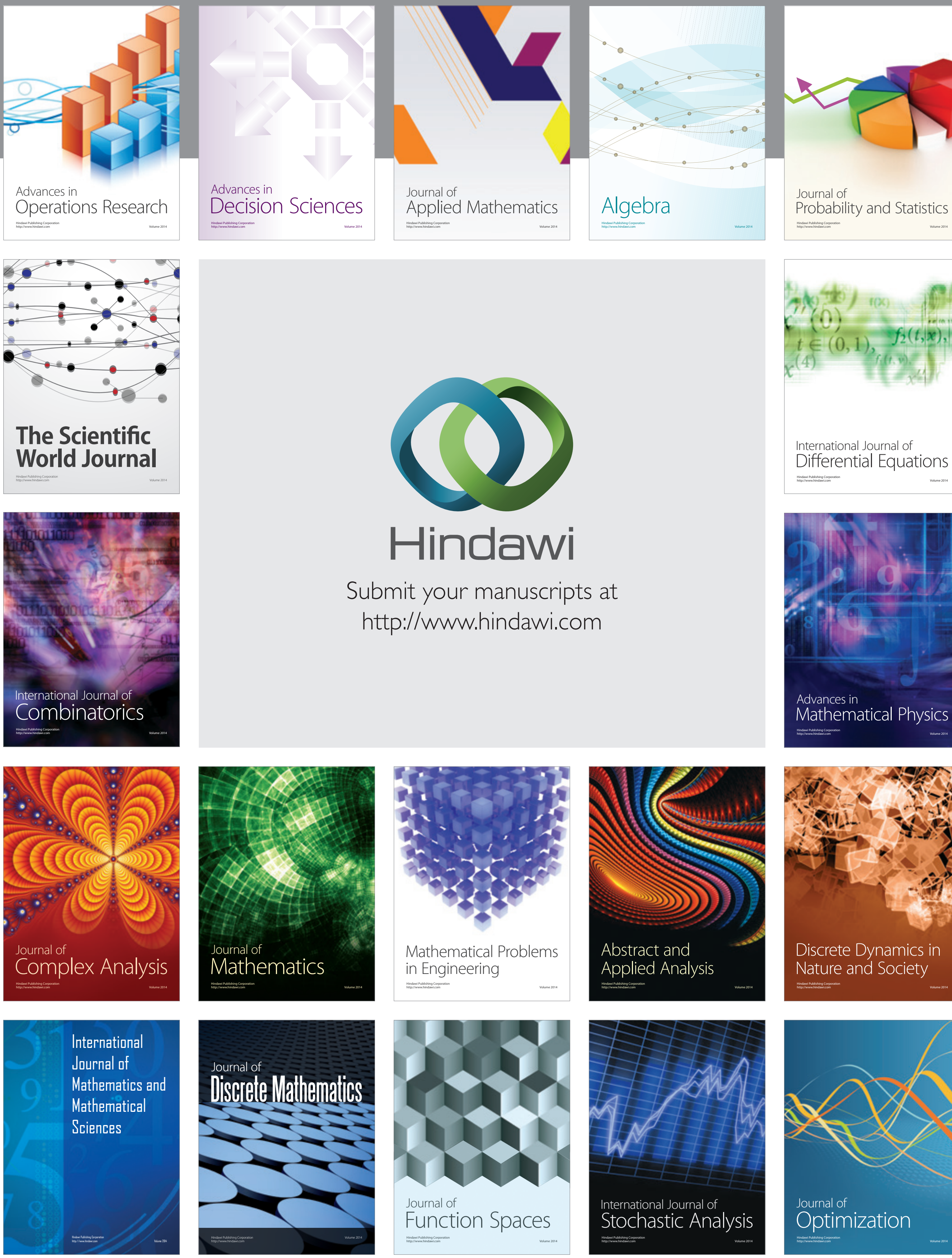
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