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Research Article

Permanence and Almost Periodic Solutions of a Discrete Ratio-Dependent Leslie System with Time Delays and Feedback Controls

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We consider a discrete almost periodic ratio-dependent Leslie system with time delays and feedback controls. Sufficient conditions are obtained for the permanence and global attractivity of the system. Furthermore, by using an almost periodic functional Hull theory, we show that the almost periodic system has a unique globally attractive positive almost periodic solution.

1. Introduction

Among the relationships between the species living in the same outer environment, the predator-prey theory plays an important and fundamental role. The predator-prey models have been extensively studied by many scholars [1–4]. Under the assumption that reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food, Leslie [5] proposed the famous Leslie-Gower predator-prey model where the carry capacity of the predator's environment is proportional to the number of prey:

$$\begin{aligned} \dot{x}_1 &= x_1[b - ax_1] - cx_1x_2, \\ \dot{x}_2 &= x_2\left[g - f\frac{x_2}{x_1}\right], \end{aligned} \tag{1.1}$$

where x_1 and x_2 represent prey and predator densities at time t, respectively. cx_1 is the predator's rate of feeding upon prey, that is, the so-called predator's functional response.

If we assume that the predator consumes the prey according to the functional response $p(x_1)$, then system (1.1) formulates as the following:

$$\dot{x}_1 = x_1[b - ax_1] - p(x_1)x_2,$$
$$\dot{x}_2 = x_2\left[g - f\frac{x_2}{x_1}\right], \tag{1.2}$$

where $p(x_1)$ is prey-dependent functional responses. Owing to its theoretical and practical significance, system (1.2) and its various generalized forms have been studied extensively and seen great progress (see, for example, [6–9]).

However, in the study of the dynamic behaviors of predator-prey system, many scholars argued that the ratio-dependent predator-prey systems are more realistic [10, 11]. A ratio-dependent predator-prey system with Leslie-Gower term takes the form of

$$\dot{x}_1 = x_1[b - ax_1] - p\left(\frac{x_1}{x_2}\right)x_2,$$
$$\dot{x}_2 = x_2\left[g - f\frac{x_2}{x_1}\right]. \tag{1.3}$$

Though much progress has been seen in the asymptotic convergence of solutions of population systems, such systems are not well studied in the sense that most results are continuous-time cases related. Already, many scholars have paid attention to the nonautonomous discrete population models, since the discrete time models governed by difference equation are more appropriate than the continuous ones when the populations have a short life expectancy, nonoverlapping generations in the real world (see [12–23]). Furthermore, the asymptotic convergence of solutions of difference equations with delay is one of the most important topics in the study of population dynamics, and many excellent results have already been obtained and seen great progress (see [24–26] and the references cited therein).

Since time delays occur so often in nature, a number of ecological systems can be described as systems with time delays (see [14, 15, 19, 21–27]). One of the most important problems for this type of system is to analyze the effect of time delays on the stability of the system. Furthermore, as we know, ecological systems in the real world are often distributed by unpredictable forces which can result in changes in biological parameters such as survival rates, so it is reasonable to study models with control variables which are so-called disturbance functions [22, 23].

So it is very interesting to study dynamics of the following discrete ratio-dependent Leslie system with time delays and feedback controls:

$$x_1(k+1) = x_1(k)\exp\left[b(k) - a(k)x_1(k-\tau_1) - \frac{c(k)x_1(k)x_2(k)}{h^2x_2^2(k) + x_1^2(k)} - d(k)u_1(k-\sigma_1)\right],$$
$$x_2(k+1) = x_2(k)\exp\left[g(k) - f(k)\frac{x_2(k-\tau_2)}{x_1(k-\tau_2)} - p(k)u_2(k-\sigma_2)\right], \tag{1.4}$$
$$\Delta u_i(k) = -\alpha_i(k)u_i(k) + \beta_i(k)x_i(k-\rho_i), \quad i = 1, 2,$$

where $x_i(k),\ i = 1,2$ stand for the density of the prey and the predator at time k, respectively. $u_i(k),\ i = 1,2$ are the control variables at time k. h^2 is a positive constant, denoting the constant of capturing half-saturation.

In this paper, we are concerned with the effects of the almost periodicity of ecological and environmental parameters and time delays on the global dynamics of the discrete ratio-dependent Leslie systems with feedback controls. To do so, for system (1.4) we always assume that for $i = 1,2, k \in Z$

(H_1) $a(k),\ b(k),\ c(k),\ d(k),\ g(k),\ f(k),\ p(k),\ \alpha_i(k),\ \beta_i(k)$ are all bounded nonnegative almost periodic sequences such that

$$
\begin{gathered}
0 < a^l \le a^u, \quad 0 < b^l \le b^u, \quad 0 < c^l \le c^u, \quad 0 < d^l \le d^u, \quad 0 < g^l \le g^u, \\
0 < f^l \le f^u, \quad 0 < p^l \le p^u, \quad 0 < \alpha_i^l \le \alpha_i^u < 1, \quad 0 < \beta_i^l \le \beta_i^u.
\end{gathered} \tag{1.5}
$$

Here, we let Z, Z^+ denote the sets of all integers, nonnegative integers, respectively, and use the notations: $f^u = \sup_{k\in Z}\{f(k)\},\ f^l = \inf_{k\in Z}\{f(k)\}$, for any bounded sequence $\{f(k)\}$ defined on Z.

Let $\tau = \max\{\tau_i, \sigma_i, \rho_i, i = 1,2\}$, we consider system (1.4) with the following initial conditions:

$$
\begin{gathered}
x_i(\theta) = \phi_i(\theta), \quad \theta \in [-\tau, 0] \cap Z,\ \phi_i(0) > 0, \\
u_i(\theta) = \varphi_i(\theta), \quad \theta \in [-\tau, 0] \cap Z,\ \varphi_i(0) > 0,\ i = 1,2.
\end{gathered} \tag{1.6}
$$

One can easily show that the solutions of system (1.4) with initial condition (1.5) are defined and remain positive for $k \in Z^+$.

The principle aim of this paper is to study the dynamic behaviors of system (1.4), such as permanence, global attractivity, and existence of a unique globally attractive positive almost periodic solution of the system. To the best of our knowledge, no work has been done for the nonautonomous difference system (1.4).

The organization of this paper is as follows. In the next section, we introduce some definitions and several useful lemmas. In Section 3, we explore the permanent property of system (1.4). We study globally attractive property of system (1.4) in Section 4 and the almost periodic property of system (1.4) in Section 5. Finally, the conclusion ends with brief remarks.

2. Preliminaries

In this section, we will introduce some basic definitions and several useful lemmas.

Definition 2.1. System (1.4) is said to be permanent, if there are positive constants m_i and M_i, such that for each positive solution $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ of system (1.4) satisfies

$$
\begin{gathered}
m_i \le \liminf_{k\to+\infty} x_i(\mathrm{k}) \le \limsup_{k\to+\infty} x_i(k) \le M_i, \\
n_i \le \liminf_{k\to+\infty} u_i(k) \le \limsup_{k\to+\infty} u_i(k) \le N_i, \quad i = 1,2.
\end{gathered} \tag{2.1}
$$

Definition 2.2. Suppose that $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))^T$ is any solution of system (1.4). $X(k)$ is said to be a strictly positive solution in Z if for $k \in Z$ and $i = 1, 2$ such that

$$0 < \inf_{k \in Z} x_i(k) \le \sup_{k \in Z} x_i(k) < \infty, \qquad 0 < \inf_{k \in Z} u_i(k) \le \sup_{k \in Z} u_i(k) < \infty. \tag{2.2}$$

Definition 2.3 (see [16]). A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : |x(k+\tau) - x(k)| < \varepsilon,\ \forall k \in Z\} \tag{2.3}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$|x(k+\tau) - x(k)| < \varepsilon, \quad \forall k \in Z, \tag{2.4}$$

τ is called an ε-translation number of $x(k)$.

Definition 2.4 (see [17]). The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \left\{ g(k,x) : \lim_{n \to \infty} f(k+\tau_n, x) = g(k,x) \text{ uniformly on } Z \times S \right\}, \tag{2.5}$$

for some sequence $\{\tau_n\}$, where S is any compact set in D.

Lemma 2.5 (see [20]). *Assume that* $\{y(k)\}$ *satisfies* $y(k_1) > 0$ *and*

$$y(k+1) \le y(k) \exp\{r(k)(1 - ay(k))\}, \tag{2.6}$$

for $k \in [k_1, +\infty)$, *where* a *is a positive constant and* $k_1 \in Z^+$. *Then*

$$\limsup_{k \to +\infty} y(k) \le \frac{1}{ar^u} \exp(r^u - 1). \tag{2.7}$$

Lemma 2.6 (see [20]). *Assume that* $\{y(k)\}$ *satisfies* $y(k_2) > 0$ *and*

$$y(k+1) \ge y(k) \exp\{r(k)(1 - ay(k))\}, \tag{2.8}$$

for $k \in [k_2, +\infty)$, $\limsup_{k \to +\infty} y(k) \le M$, *where* a *is a constant such that* $aM > 1$ *and* $k_2 \in Z^+$. *Then*

$$\liminf_{k \to +\infty} y(k) \ge \frac{1}{a} \exp(r^u(1 - aM)). \tag{2.9}$$

Lemma 2.7 (see [22]). *Assume that $A > 0$ and $y(0) > 0$. Suppose that*

$$y(k+1) \leq Ay(k) + B(k), \quad n = 1, 2, \ldots. \tag{2.10}$$

Then for any integer $m \leq k$,

$$y(k) \leq A^m y(k-m) + \sum_{j=0}^{m-1} A^j B(k-j-1). \tag{2.11}$$

Especially, if $A < 1$ and B is bounded above with respect to M, then

$$\limsup_{k \to +\infty} y(k) \leq \frac{M}{1-A}. \tag{2.12}$$

Lemma 2.8 (see [22]). *Assume that $A > 0$ and $y(0) > 0$. Suppose that*

$$y(k+1) \geq Ay(k) + B(k), \quad n = 1, 2, \ldots. \tag{2.13}$$

Then for any integer $m \leq k$,

$$y(k) \geq A^m y(k-m) + \sum_{j=0}^{m-1} A^j B(k-j-1). \tag{2.14}$$

Especially, if $A < 1$ and B is bounded below with respect to N, then

$$\liminf_{k \to +\infty} y(k) \geq \frac{N}{1-A}. \tag{2.15}$$

3. Permanence

In this section, we establish a permanent result for system (1.4).

Theorem 3.1. *Assume that (H_1) holds; assume further that*

(H_2) $b^l > c^u/2|h|$

holds. Then system (1.4) is permanent.

Proof. Let $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))^T$ be any positive solution of system (1.4), from the first equation of system (1.4), it follows that

$$x_1(k+1) \leq x_1(k) \exp[b(k)] \leq x_1(k) \exp[b^u]. \tag{3.1}$$

It follows from (3.1) that

$$\prod_{j=k-\tau_1}^{k-1} \frac{x_1(j+1)}{x_1(j)} \le \prod_{j=k-\tau_1}^{k-1} \exp[b^u] \le \exp[b^u \tau_1], \tag{3.2}$$

which implies that

$$x_1(k-\tau_1) \ge x_1(k)\exp[-b^u \tau_1]. \tag{3.3}$$

Substituting (3.3) into the first equation of (1.4), it immediately follows that

$$\begin{aligned} x_1(k+1) &\le x_1(k)\exp[b(k) - a(k)x_1(k-\tau_1)] \\ &\le x_1(k)\exp\big[b(k) - a(k)\exp[-b^u\tau_1]x_1(k)\big]. \end{aligned} \tag{3.4}$$

By applying Lemma 2.5 to (3.4), we have

$$\limsup_{k\to+\infty} x_1(k) \le \frac{1}{a^l}\exp[b^u(\tau_1+1)-1] =: M_1. \tag{3.5}$$

For any $\varepsilon > 0$ small enough, it follows from (3.5) that there exists enough large K_1 such that for $k \ge K_1$,

$$x_1(k) \le M_1 + \varepsilon. \tag{3.6}$$

From the second equation of system (1.4) it follows that

$$\begin{gathered} x_2(k+1) \le x_2(k)\exp\big[g(k)\big] \le x_2(k)\exp\big[g^u\big], \\ x_2(k+1) \le x_2(k)\exp\left[g(k) - \frac{f(k)}{M_1+\varepsilon}x_2(k-\tau_2)\right], \quad \text{for } k \ge K_1 + \tau. \end{gathered} \tag{3.7}$$

From (3.7), similar to the argument of (3.1), one has

$$x_2(k+1) \le x_2(k)\exp\left[g(k) - \frac{f(k)}{M_1+\varepsilon}\exp[-g^u\tau_2]x_2(k)\right]. \tag{3.8}$$

By applying Lemma 2.5 to (3.8) again, we have

$$\limsup_{k\to+\infty} x_2(k) \le \frac{M_1+\varepsilon}{f^l}\exp\big[g^u(\tau_2+1)-1\big]. \tag{3.9}$$

Setting $\varepsilon \to 0$ in the above inequality, we have

$$\limsup_{k\to+\infty} x_2(k) \le \frac{M_1}{f^l}\exp\big[g^u(\tau_2+1)-1\big] =: M_2. \tag{3.10}$$

For any $\varepsilon > 0$ small enough, it follows from (3.5) and (3.10) that there exists enough large $K_2 > K_1 + \tau$ such that for $i = 1,2$ and $k \geq K_2$

$$x_i(k) \leq M_i + \varepsilon. \tag{3.11}$$

For $k > K_2 + \tau$, (3.11) combining with the third and fourth equations of system (1.4) leads to

$$\Delta u_i(k) \leq -\alpha_i(k)u_i(k) + \beta_i(k)(M_i + \varepsilon), \quad i = 1,2, \tag{3.12}$$

that is,

$$u_i(k+1) \leq \left(1 - \alpha_i^l\right)u_i(k) + \beta_i^u(M_i + \varepsilon), \quad i = 1,2. \tag{3.13}$$

By applying Lemma 2.7, it follows from (3.13) that

$$\limsup_{k \to +\infty} u_i(k) \leq \frac{\beta_i^u(M_i + \varepsilon)}{\alpha_i^l}, \quad i = 1,2. \tag{3.14}$$

Letting $\varepsilon \to 0$ in the above inequality, we have

$$\limsup_{k \to +\infty} u_i(k) \leq \frac{\beta_i^u M_i}{\alpha_i^l} =: N_i, \quad i = 1,2. \tag{3.15}$$

For any $\varepsilon > 0$ small enough, it follows from (3.11) and (3.15) that there exists enough large $K_3 > K_2 + \tau$ such that for $i = 1,2$ and $k \geq K_3$

$$x_i(k) \leq M_i + \varepsilon, \qquad u_i(k) \leq N_i + \varepsilon. \tag{3.16}$$

Thus, from (3.16) and the first equation of system (1.4), it follows that

$$\begin{aligned} x_1(k+1) &\geq x_1(k)\exp\left[b^l - a^u(M_1 + \varepsilon) - \frac{c^u}{2|h|} - d^u(N_1 + \varepsilon)\right] \\ &=: x_1(k)\exp[D_{1\varepsilon}], \quad \text{for } k \geq K_3 + \tau, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} D_{1\varepsilon} &= b^l - a^u(M_1 + \varepsilon) - \frac{c^u}{2|h|} - d^u(N_1 + \varepsilon) \\ &\leq b^l - a^u M_1 \end{aligned}$$

$$\begin{aligned}&\leq b^l - \frac{a^u \exp[b^u - 1]}{a^l \exp[-b^u \tau_1]}\\&\leq b^l - b^u\\&\leq 0.\end{aligned} \tag{3.18}$$

For any integer $\eta \leq k$, it follows from (3.17) that

$$\prod_{j=k-\eta}^{k-1} \frac{x_1(j+1)}{x_1(j)} \geq \prod_{j=k-\eta}^{k-1} \exp[D_{1\varepsilon}] \geq \exp[D_{1\varepsilon}\eta], \tag{3.19}$$

and hence

$$x_1(\eta) \leq x_1(k) \exp[-(k-\eta)D_{1\varepsilon}]. \tag{3.20}$$

From the third equation of system (1.4), we have

$$\begin{aligned}u_1(k+1) &\leq \left(1-\alpha_1^l\right)u_1(k) + \beta_1^u x_1(k-\rho_1)\\&\leq \left(1-\alpha_1^l\right)u_1(k) + \beta_1^u \exp[-\rho_1 D_{1\varepsilon}]x_1(k)\\&= A_1 u_1(k) + B_1(k),\end{aligned} \tag{3.21}$$

where $A_1 = 1 - \alpha_1^l$ and $B_1(k) = \beta_1^u \exp[-\rho_1 D_{1\varepsilon}]x_1(k)$. Therefore, for any $m \leq k - \tau$, by Lemma 2.7, (3.20), and (3.21), one has

$$\begin{aligned}u_1(k) &\leq A_1^m u_1(k-m) + \sum_{j=0}^{m-1} A_1^j B_1(k-j-1)\\&\leq A_1^m u_1(k-m) + \sum_{j=0}^{m-1} A_1^j \beta_1^u \exp[-\rho_1 D_{1\varepsilon}]x_1(k-j-1)\\&\leq A_1^m u_1(k-m) + x_1(k)\sum_{j=0}^{m-1} A_1^j \beta_1^u \exp[-(j+1+\rho_1)D_{1\varepsilon}].\end{aligned} \tag{3.22}$$

Since $\alpha_1(k) \in (0,1)$, we have $0 < A_1 < 1$. So

$$0 \leq A_1^m u_1(k-m) \leq A_1^m(N_1+\varepsilon) \longrightarrow 0, \quad \text{for } m \longrightarrow \infty. \tag{3.23}$$

By the conditions (H_1), for any solution $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ of system (1.4), there exists a positive integer M such that $d^u A_1^m u_1(k-m) < (1/2)(b^l - c^u/2|h|)$ for $m > M$. In fact, we can choose $M = \max\{1, [\log_{A_1}((b^l - c^u/2|h|)/2d^u N_1)] + 1\}$. Then we get

$$\begin{aligned} u_1(k) &\le A_1^M(N_1+\varepsilon) + x_1(k)\sum_{j=0}^{M-1} A_1^j \beta_1^u \exp[-(j+1+\rho_1)D_{1\varepsilon}] \\ &= A_1^M(N_1+\varepsilon) + F_{1\varepsilon}x_1(k), \end{aligned} \tag{3.24}$$

where $F_{1\varepsilon} = \sum_{j=0}^{M-1} A_1^j \beta_1^u \exp[-(j+1+\rho_1)D_{1\varepsilon}]$.

Substituting (3.20) and (3.24) into the first equation of system (1.4), one has

$$\begin{aligned} x_1(k+1) &\ge x_1(k)\exp\left[b(k) - a(k)x_1(k-\tau_1) - \frac{c(k)}{2|h|} - d(k)u_1(k-\sigma_1)\right] \\ &\ge x_1(k)\exp\left[b(k) - \frac{c(k)}{2|h|} - d(k)A_1^M(N_1+\varepsilon)\right. \\ &\qquad \left. -\left(a(k)\exp[-\tau_1 D_{1\varepsilon}] + d(k)F_{1\varepsilon}\exp[-\sigma_1 D_{1\varepsilon}]\right)x_1(k)\right] \\ &=: x_1(k)\exp\left[S_{1\varepsilon}(k)\left(1 - \left(\frac{S_{2\varepsilon}(k)}{S_{1\varepsilon}(k)}\right)x_1(k)\right)\right], \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} S_{1\varepsilon}(k) &= b(k) - \frac{c(k)}{2|h|} - d(k)A_1^M(N_1+\varepsilon), \\ S_{2\varepsilon}(k) &= a(k)\exp[-\tau_1 D_{1\varepsilon}] + d(k)F_{1\varepsilon}\exp[-\sigma_1 D_{1\varepsilon}]. \end{aligned} \tag{3.26}$$

In particular, we have

$$\begin{aligned} \frac{S_{2\varepsilon}^u}{S_{1\varepsilon}^l}M_1 &= \frac{a^u\exp[-\tau_1 D_{1\varepsilon}] + d^u F_{1\varepsilon}\exp[-\sigma_1 D_{1\varepsilon}]}{b^l - c^u/2|h| - d^u A_1^M(N_1+\varepsilon)}M_1 \\ &\ge \frac{a^u\exp[-\tau_1 D_{1\varepsilon}]}{b^l}\frac{\exp[b^u(\tau_1+1)-1]}{a^l} \\ &\ge \frac{a^u\exp\left[-\tau_1 b^l\right]}{b^l}\frac{\exp\left[b^l(\tau_1+1)-1\right]}{a^u} \\ &\ge \frac{\exp\left[b^l-1\right]}{b^l} \\ &> 1, \end{aligned} \tag{3.27}$$

where we use the inequality $\exp(x-1) > x$ for $x > 0$. By applying Lemma 2.6 to (3.25), it follows that

$$\liminf_{k\to+\infty} x_1(k) \geq \frac{S_{1\varepsilon}^l}{S_{2\varepsilon}^u} \exp\left[S_{1\varepsilon}^u\left(1-\frac{S_{2\varepsilon}^u}{S_{1\varepsilon}^l}M_1\right)\right]. \tag{3.28}$$

Setting $\varepsilon \to 0$ in the above inequality, we have

$$\liminf_{k\to+\infty} x_1(k) \geq \frac{S_1^l}{S_2^u} \exp\left[S_1^u\left(1-\frac{S_2^u}{S_1^l}M_1\right)\right] =: m_1. \tag{3.29}$$

From (3.29) we know that there exists enough large $K_4 > K_3 + \tau$ such that for $k \geq K_4$,

$$x_1(k) \geq m_1 - \varepsilon. \tag{3.30}$$

From (3.30) and the second equation of system (1.4), it follows that

$$\begin{aligned} x_2(k+1) &\geq x_2(k)\exp\left[g^l - \frac{f^u(M_2+\varepsilon)}{m_1-\varepsilon} - p^u(N_2+\varepsilon)\right] \\ &=: x_2(k)\exp[D_{2\varepsilon}], \quad \text{for } k \geq K_4+\tau, \end{aligned} \tag{3.31}$$

where $D_{2\varepsilon} = g^l - f^u(M_2+\varepsilon)/(m_1-\varepsilon) - p^u(N_2+\varepsilon) \leq 0$.

This implies for any integer $\eta \leq k$

$$x_2(\eta) \leq x_2(k)\exp[-(k-\eta)D_{2\varepsilon}]. \tag{3.32}$$

From (3.32) and the fourth equation of system (1.4), by a procedure similar to the discussion of (3.21)–(3.24), we can verify that

$$u_2(k) \leq A_2^N(N_2+\varepsilon) + F_{2\varepsilon}x_2(k), \tag{3.33}$$

where

$$\begin{aligned} A_2 = 1-\alpha_2^l, \qquad F_{2\varepsilon} &= \sum_{j=0}^{N-1} A_2^j \beta_2^u \exp[-(j+1+\rho_2)D_{2\varepsilon}], \\ N = \max&\left\{1, \left[\log_{A_2}\left(\frac{g^l}{2p^u N_2}\right)\right]+1\right\}. \end{aligned} \tag{3.34}$$

Substituting (3.32) and (3.33) into the second equation of system (1.4), one has

$$\begin{aligned} x_2(k+1) &\geq x_2(k)\exp\left[g(k)-\frac{f(k)}{m_1-\varepsilon}x_2(k-\tau_2)-p(k)u_2(k-\sigma_2)\right] \\ &\geq x_2(k)\exp\left[g(k)-p(k)A_2^N(N_2+\varepsilon)\right. \\ &\qquad \left.-\left(\frac{f(k)}{m_1}\exp[-\tau_2 D_{2\varepsilon}]+p(k)F_{2\varepsilon}\exp[-\sigma_2 D_{2\varepsilon}]\right)x_2(k)\right] \\ &=: x_2(k)\exp\left[S_{3\varepsilon}(k)\left(1-\left(\frac{S_{4\varepsilon}(k)}{S_{3\varepsilon}(k)}\right)x_2(k)\right)\right], \end{aligned} \tag{3.35}$$

where

$$\begin{gathered} S_{3\varepsilon}(k)=g(k)-p(k)A_2^N(N_2+\varepsilon), \\ S_{4\varepsilon}(k)=\frac{f(k)}{m_1}\exp[-\tau_2 D_{2\varepsilon}]+p(k)F_{2\varepsilon}\exp[-\sigma_2 D_{2\varepsilon}]. \end{gathered} \tag{3.36}$$

In particular, we have

$$\begin{aligned} \frac{S_{4\varepsilon}^u}{S_{3\varepsilon}^l}M_2 &= \frac{(f^u/m_1)\exp[-\tau_2 D_{2\varepsilon}]+p^u F_2\exp[-\sigma_2 D_{2\varepsilon}]}{g^l-p^u A_2^N(N_2+\varepsilon)}M_2 \\ &\geq \frac{(f^u/m_1)\exp[-\tau_2 D_{2\varepsilon}]}{g^l}\,\frac{M_1\exp[g^u(\tau_2+1)-1]}{f^l} \\ &\geq \frac{(f^u/m_1)\exp[-\tau_2 g^l]}{g^l}\,\frac{M_1\exp[g^l(\tau_2+1)-1]}{f^u} \\ &\geq \frac{\exp[g^l-1]}{g^l} \\ &> 1. \end{aligned} \tag{3.37}$$

By applying Lemma 2.6 to (3.35) again, it follows that

$$\liminf_{k\to+\infty} x_2(k)\geq \frac{S_{3\varepsilon}^l}{S_{4\varepsilon}^u}\exp\left[S_{3\varepsilon}^u\left(1-\frac{S_{4\varepsilon}^u}{S_{3\varepsilon}^l}M_2\right)\right]. \tag{3.38}$$

Setting $\varepsilon \to 0$ in the above inequality, we have

$$\liminf_{k\to+\infty} x_2(k)\geq \frac{S_3^l}{S_4^u}\exp\left[S_3^u\left(1-\frac{S_4^u}{S_3^l}M_2\right)\right]=: m_2. \tag{3.39}$$

From (3.29) and (3.39) we know that there exists enough large $K_5 > K_4 + \tau$ such that for $k \geq K_5$,

$$x_i(k) \geq m_i - \varepsilon. \tag{3.40}$$

For $k > K_5 + \tau$, (3.40) combining with the third and fourth equations of system (1.4) produces

$$u_i(k+1) \geq (1 - \alpha_i^u) u_i(k) + \beta_i^l (m_i - \varepsilon), \quad i = 1, 2. \tag{3.41}$$

By applying Lemma 2.8, it follows from (3.41) that

$$\liminf_{k \to +\infty} u_i(k) \geq \frac{\beta_i^l (m_i - \varepsilon)}{\alpha_i^u}, \quad i = 1, 2. \tag{3.42}$$

Setting $\varepsilon \to 0$ in the above inequality, we have

$$\liminf_{k \to +\infty} u_i(k) \geq \frac{\beta_i^l m_i}{\alpha_i^u} =: n_i, \quad i = 1, 2. \tag{3.43}$$

Consequently, combining (3.5), (3.10), (3.15), (3.29), (3.39) with (3.43), system (1.4) is permanent. This completes the proof of Theorem 3.1. □

4. Global Attractivity

Firstly, we prove two lemmas which will be useful to our main result.

Lemma 4.1. *For any two positive solutions* $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ *and* $(y_1(k), y_2(k), v_1(k), v_2(k))^T$ *of system* (1.4), *one has*

$$\begin{aligned}
&\ln \frac{x_1(k+1)}{y_1(k+1)} \\
&\quad = \ln \frac{x_1(k)}{y_1(k)} - a(k)\left[x_1(k) - y_1(k)\right] \\
&\qquad - d(k)[u_1(k-\sigma_1) - v_1(k-\sigma_1)] + E_1(k)\left[x_1(k) - y_1(k)\right] + E_2(k)\left[x_2(k) - y_2(k)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ a(k)\sum_{s=k-\tau_1}^{k-1}\Bigg\{P_1(s)\Bigg[b(s)-a(s)y_1(s-\tau_1)\\
&\qquad\qquad -\frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s)+y_1^2(s)}-d(s)v_1(s-\sigma_1)\Bigg][x_1(s)-y_1(s)]\\
&\qquad\qquad + x_1(s)P_2(s)[-a(s)[x_1(s-\tau_1)-y_1(s-\tau_1)]\\
&\qquad\qquad\qquad - d(s)[u_1(s-\sigma_1)-v_1(s-\sigma_1)]\\
&\qquad\qquad\qquad + E_1(s)[x_1(s)-y_1(s)]+E_2(s)[x_2(s)-y_2(s)]]\Bigg\},
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
E_1(s) &= \frac{c(s)y_1(s)y_2(s)[x_1(s)+y_1(s)]}{[h^2x_2^2(s)+x_1^2(s)][h^2y_2^2(s)+y_1^2(s)]}-\frac{c(s)x_2(s)}{h^2x_2^2(s)+x_1^2(s)},\\
E_2(s) &= \frac{h^2c(s)y_1(s)y_2(s)[x_2(s)+y_2(s)]}{[h^2x_2^2(s)+x_1^2(s)][h^2y_2^2(s)+y_1^2(s)]}-\frac{c(s)y_1(s)}{h^2x_2^2(s)+x_1^2(s)},\\
P_1(s) &= \exp\Bigg\{\theta_1(s)\Bigg[b(s)-a(s)y_1(s-\tau_1)-\frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s)+y_1^2(s)}-d(s)v_1(s-\sigma_1)\Bigg]\Bigg\},\\
P_2(s) &= \exp\Bigg\{\theta_2(s)\Bigg[b(s)-a(s)x_1(s-\tau_1)-\frac{c(s)x_1(s)x_2(s)}{h^2x_2^2(s)+x_1^2(s)}-d(s)u_1(s-\sigma_1)\Bigg]\\
&\qquad + (1-\theta_2(s))\Bigg[b(s)-a(s)y_1(s-\tau_1)-\frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s)+y_1^2(s)}-d(s)v_1(s-\sigma_1)\Bigg]\Bigg\},\\
&\qquad\qquad \theta_1(s),\theta_2(s)\in(0,1).
\end{aligned}
\tag{4.2}
$$

Proof. It follows from the first equation of system (1.4) that

$$
\begin{aligned}
&\ln\frac{x_1(k+1)}{y_1(k+1)}-\ln\frac{x_1(k)}{y_1(k)}\\
&\quad=\ln\frac{x_1(k+1)}{x_1(k)}-\ln\frac{y_1(k+1)}{y_1(k)}\\
&\quad=\Bigg[b(k)-a(k)x_1(k-\tau_1)-\frac{c(k)x_1(k)x_2(k)}{h^2x_2^2(k)+x_1^2(k)}-d(k)u_1(k-\sigma_1)\Bigg]\\
&\qquad-\Bigg[b(k)-a(k)y_1(k-\tau_1)-\frac{c(k)y_1(k)y_2(k)}{h^2y_2^2(k)+y_1^2(k)}-d(k)v_1(k-\sigma_1)\Bigg]
\end{aligned}
$$

$$\begin{aligned}
&= -a(k)\left[x_1(k-\tau_1)-y_1(k-\tau_1)\right]-d(k)[u_1(k-\sigma_1)-v_1(k-\sigma_1)]\\
&\quad -\frac{c(k)x_1(k)x_2(k)}{h^2x_2^2(k)+x_1^2(k)}+\frac{c(k)y_1(k)y_2(k)}{h^2y_2^2(k)+y_1^2(k)}\\
&= -a(k)\left[x_1(k)-y_1(k)\right]-d(k)[u_1(k-\sigma_1)-v_1(k-\sigma_1)]\\
&\quad + E_1(k)\left[x_1(k)-y_1(k)\right]+E_2(k)\left[x_2(k)-y_2(k)\right]\\
&\quad + a(k)\left\{\left[x_1(k)-y_1(k)\right]-\left[x_1(k-\tau_1)-y_1(k-\tau_1)\right]\right\},
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
E_1(k) &= \frac{c(k)y_1(k)y_2(k)\left[x_1(k)+y_1(k)\right]}{\left[h^2x_2^2(k)+x_1^2(k)\right]\left[h^2y_2^2(k)+y_1^2(k)\right]}-\frac{c(k)x_2(k)}{h^2x_2^2(k)+x_1^2(k)},\\
E_2(k) &= \frac{h^2c(k)y_1(k)y_2(k)\left[x_2(k)+y_2(k)\right]}{\left[h^2x_2^2(k)+x_1^2(k)\right]\left[h^2y_2^2(k)+y_1^2(k)\right]}-\frac{c(k)y_1(k)}{h^2x_2^2(k)+x_1^2(k)}.
\end{aligned} \tag{4.4}$$

Hence

$$\begin{aligned}
\ln\frac{x_1(k+1)}{y_1(k+1)} = \ln\frac{x_1(k)}{y_1(k)} &- a(k)\left[x_1(k)-y_1(k)\right]-d(k)[u_1(k-\sigma_1)-v_1(k-\sigma_1)]\\
&+ E_1(k)\left[x_1(k)-y_1(k)\right]+E_2(k)\left[x_2(k)-y_2(k)\right]\\
&+ a(k)\left\{\left[x_1(k)-x_1(k-\tau_1)\right]-\left[y_1(k)-y_1(k-\tau_1)\right]\right\}.
\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}
&\left[x_1(k)-x_1(k-\tau_1)\right]-\left[y_1(k)-y_1(k-\tau_1)\right]\\
&\quad= \sum_{s=k-\tau_1}^{k-1}\left[x_1(s+1)-x_1(s)\right]-\sum_{s=k-\tau_1}^{k-1}\left[y_1(s+1)-y_1(s)\right]\\
&\quad= \sum_{s=k-\tau_1}^{k-1}\left\{\left[x_1(s+1)-y_1(s+1)\right]-\left[x_1(s)-y_1(s)\right]\right\},
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&\left[x_1(s+1)-y_1(s+1)\right]-\left[x_1(s)-y_1(s)\right]\\
&\quad= \left\{x_1(s)\exp\left[b(s)-a(s)x_1(s-\tau_1)-\frac{c(s)x_1(s)x_2(s)}{h^2x_2^2(s)+x_1^2(s)}-d(s)u_1(s-\sigma_1)\right]\right.\\
&\qquad \left.-y_1(s)\exp\left[b(s)-a(s)y_1(s-\tau_1)-\frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s)+y_1^2(s)}-d(s)v_1(s-\sigma_1)\right]\right\}\\
&\qquad -\left[x_1(s)-y_1(s)\right]
\end{aligned}$$

$$
\begin{aligned}
&= [x_1(s) - y_1(s)]\left\{\exp\left[b(s) - a(s)y_1(s-\tau_1) - \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)} - d(s)v_1(s-\sigma_1)\right] - 1\right\} \\
&\quad + x_1(s)\left\{\exp\left[b(s) - a(s)x_1(s-\tau_1) - \frac{c(s)x_1(s)x_2(s)}{h^2x_2^2(s) + x_1^2(s)} - d(s)u_1(s-\sigma_1)\right]\right. \\
&\qquad \left. - \exp\left[b(s) - a(s)y_1(s-\tau_1) - \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)} - d(s)v_1(s-\sigma_1)\right]\right\}.
\end{aligned}
\tag{4.7}
$$

By the mean value theorem, one has

$$
\begin{aligned}
&[x_1(s+1) - y_1(s+1)] - [x_1(s) - y_1(s)] \\
&\quad = [x_1(s) - y_1(s)]P_1(s)\left[b(s) - a(s)y_1(s-\tau_1) - \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)} - d(s)v_1(s-\sigma_1)\right] \\
&\qquad + x_1(s)P_2(s)\left[- a(s)[x_1(s-\tau_1) - y_1(s-\tau_1)] - d(s)[u_1(s-\sigma_1) - v_1(s-\sigma_1)]\right. \\
&\qquad\qquad \left. - \frac{c(s)x_1(s)x_2(s)}{h^2x_2^2(s) + x_1^2(s)} + \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)}\right] \\
&\quad = [x_1(s) - y_1(s)]P_1(s)\left[b(s) - a(s)y_1(s-\tau_1) - \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)} - d(s)v_1(s-\sigma_1)\right] \\
&\qquad + x_1(s)P_2(s)[-a(s)[x_1(s-\tau_1) - y_1(s-\tau_1)] - d(s)[u_1(s-\sigma_1) - v_1(s-\sigma_1)] \\
&\qquad\qquad + E_1(s)[x_1(s) - y_1(s)] + E_2(s)[x_2(s) - y_2(s)]].
\end{aligned}
\tag{4.8}
$$

Combining (4.6) with (4.8), we can easily obtain (4.1). The proof of Lemma 4.1 is completed. □

Lemma 4.2. *For any two positive solutions* $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ *and* $(y_1(k), y_2(k), v_1(k), v_2(k))^T$ *of system* (1.4), *one has*

$$
\begin{aligned}
&\ln\frac{x_2(k+1)}{y_2(k+1)} \\
&\quad = \ln\frac{x_2(k)}{y_2(k)} - \frac{f(k)}{x_1(k-\tau_2)}[x_2(k) - y_2(k)] + \frac{f(k)y_2(k-\tau_2)}{x_1(k-\tau_2)y_1(k-\tau_2)}[x_1(k-\tau_2) - y_1(k-\tau_2)] \\
&\qquad - p(k)[u_2(k-\sigma_2) - v_2(k-\sigma_2)]
\end{aligned}
$$

$$+\frac{f(k)}{x_1(k-\tau_2)}\sum_{s=k-\tau_2}^{k-1}\left\{Q_1(s)\left[g(s)-f(s)\frac{y_2(s-\tau_2)}{y_1(s-\tau_2)}-p(s)v_2(s-\sigma_2)\right][x_2(s)-y_2(s)]\right.$$
$$+x_2(s)Q_2(s)\left[-\frac{f(s)}{x_1(s-\tau_2)}[x_2(s-\tau_2)-y_2(s-\tau_2)]\right.$$
$$+\frac{f(s)y_2(s-\tau_2)}{x_1(s-\tau_2)y_1(s-\tau_2)}[x_1(s-\tau_2)-y_1(s-\tau_2)]$$
$$\left.\left.-p(k)[u_2(s-\sigma_2)-v_2(s-\sigma_2)]\right]\right\}, \tag{4.9}$$

where

$$Q_1(s)=\exp\left\{\varphi_1(s)\left[g(s)-f(s)\frac{y_2(s-\tau_2)}{y_1(s-\tau_2)}-p(s)v_2(s-\sigma_2)\right]\right\},$$
$$Q_2(s)=\exp\left\{\varphi_2(s)\left[g(s)-f(s)\frac{x_2(s-\tau_2)}{x_1(s-\tau_2)}-p(s)u_2(s-\sigma_2)\right]\right.$$
$$\left.+(1-\varphi_2(s))\left[g(s)-f(s)\frac{y_2(s-\tau_2)}{y_1(s-\tau_2)}-p(s)v_2(s-\sigma_2)\right]\right\},\quad \varphi_1(s),\ \varphi_2(s)\in(0,1). \tag{4.10}$$

Proof. The proofs of Lemma 4.2 are very similar to those of Lemma 4.1. So we omit the detail here. □

Now we are in the position of stating the main result on the global attractivity of system (1.4).

Theorem 4.3. *In addition to* (H_1)-(H_2), *assume further that*

(H_3) *there exist positive constants* s_i, ω_i, $i=1,2$ *such that*

$$\alpha=:\min\left(s_1\mu_1-s_2\mu_2-\omega_1\beta_1^u, s_2\nu_1-s_1\nu_2-\omega_2\beta_2^u, \omega_1\alpha_1^l-s_1\varrho_1, \omega_2\alpha_2^l-s_2\varrho_2\right)>0 \tag{4.11}$$

holds, where $\mu_1,\ \mu_2,\ \nu_1,\ \nu_2,\ \varrho_1,\ \varrho_2$ *are defined by* (4.37). *Then for any two positive solutions* $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ *and* $(y_1(k), y_2(k), v_1(k), v_2(k))^T$ *of system* (1.4), *one has*

$$\lim_{k\to+\infty}|x_i(k)-y_i(k)|=0,\qquad \lim_{k\to+\infty}|u_i(k)-v_i(k)|=0,\quad i=1,2. \tag{4.12}$$

Proof. For two arbitrary nontrivial solutions $(x_1(k), x_2(k), u_1(k), u_2(k))^T$ and $(y_1(k), y_2(k), v_1(k), v_2(k))^T$ of system (1.4), we have from Theorem 3.1 that there exist positive constants k_0 and M_i, N_i, m_i, n_i $(i = 1, 2)$ such that for all $k \geq k_0$ and $i = 1, 2$

$$\begin{aligned} m_i \leq x_i(k) \leq M_i, \qquad n_i \leq u_i(k) \leq N_i, \\ m_i \leq y_i(k) \leq M_i, \qquad n_i \leq v_i(k) \leq N_i. \end{aligned} \tag{4.13}$$

Firstly, we define

$$V_{11}(k) = \left|\ln x_1(k) - \ln y_1(k)\right|. \tag{4.14}$$

From (4.1), we have

$$\begin{aligned} \left|\ln \frac{x_1(k+1)}{y_1(k+1)}\right| \leq & \left|\ln \frac{x_1(k)}{y_1(k)} - a(k)\left[x_1(k) - y_1(k)\right]\right| + d(k)|u_1(k-\sigma_1) - v_1(k-\sigma_1)| \\ & + E_1(k)\left|x_1(k) - y_1(k)\right| + E_2(k)\left|x_2(k) - y_2(k)\right| \\ & + a(k) \sum_{s=k-\tau_1}^{k-1} \Big\{ [P_1(s)G_1(s) + x_1(s)P_2(s)E_1(s)]\left|x_1(s) - y_1(s)\right| \\ & \qquad + x_1(s)P_2(s)\Big[a(s)\left|x_1(s-\tau_1) - y_1(s-\tau_1)\right| \\ & \qquad\qquad + d(s)|u_1(s-\sigma_1) - v_1(s-\sigma_1)| \\ & \qquad\qquad + E_2(s)\left|x_2(s) - y_2(s)\right|\Big]\Big\}, \end{aligned} \tag{4.15}$$

where

$$G_1(s) = b(s) + a(s)y_1(s-\tau_1) + \frac{c(s)y_1(s)y_2(s)}{h^2y_2^2(s) + y_1^2(s)} + d(s)v_1(s-\sigma_1). \tag{4.16}$$

By the mean value theorem, we have

$$x_1(k) - y_1(k) = \exp[\ln x_1(k)] - \exp\left[\ln y_1(k)\right] = \xi_1(k) \ln \frac{x_1(k)}{y_1(k)}, \tag{4.17}$$

that is,

$$\ln \frac{x_1(k)}{y_1(k)} = \frac{1}{\xi_1(k)}\left[x_1(k) - y_1(k)\right], \tag{4.18}$$

where $\xi_1(k)$ lies between $x_1(k)$ and $y_1(k)$. So, we have

$$\begin{aligned}
&\left|\ln\frac{x_1(k)}{y_1(k)} - a(k)\left[x_1(k)-y_1(k)\right]\right| \\
&\quad= \left|\ln\frac{x_1(k)}{y_1(k)}\right| - \left|\ln\frac{x_1(k)}{y_1(k)}\right| + \left|\ln\frac{x_1(k)}{y_1(k)} - a(k)\left[x_1(k)-y_1(k)\right]\right| \\
&\quad= \left|\ln\frac{x_1(k)}{y_1(k)}\right| - \left[\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - a(k)\right|\right]\left|x_1(k)-y_1(k)\right|.
\end{aligned} \tag{4.19}$$

For $k > k_0 + \tau$, it follows that

$$\begin{aligned}
\Delta V_{11} \leq\ & -\left[\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - a(k)\right|\right]\left|x_1(k)-y_1(k)\right| \\
&+ d(k)|u_1(k-\sigma_1) - v_1(k-\sigma_1)| \\
&+ E_1(k)\left|x_1(k)-y_1(k)\right| + E_2(k)\left|x_2(k)-y_2(k)\right| \\
&+ a(k)\sum_{s=k-\tau_1}^{k-1}\Big\{\left[P_1(s)G_1(s) + M_1P_2(s)E_1(s)\right]\left|x_1(s)-y_1(s)\right| \\
&\qquad + M_1P_2(s)\Big[a(s)\left|x_1(s-\tau_1)-y_1(s-\tau_1)\right| \\
&\qquad\qquad + d(s)|u_1(s-\sigma_1) - v_1(s-\sigma_1)| \\
&\qquad\qquad + E_2(s)\left|x_2(s)-y_2(s)\right|\Big]\Big\}.
\end{aligned} \tag{4.20}$$

Secondly, we define

$$\begin{aligned}
V_{12}(k) =\ & \sum_{s=k-\sigma_1}^{k-1} d(s+\sigma_1)|u_1(s) - v_1(s)| \\
&+ \sum_{s=k}^{k-1+\tau_1} a(s)\sum_{u=s-\tau_1}^{k-1}\Big\{\left[P_1(u)G_1(u) + M_1P_2(u)E_1(u)\right]\left|x_1(u)-y_1(u)\right| \\
&\qquad + M_1P_2(u)\Big[a(u)\left|x_1(u-\tau_1)-y_1(u-\tau_1)\right| \\
&\qquad\qquad + d(u)|u_1(u-\sigma_1) - v_1(u-\sigma_1)| \\
&\qquad\qquad + E_2(u)\left|x_2(u)-y_2(u)\right|\Big]\Big\}.
\end{aligned} \tag{4.21}$$

Then

$$\begin{aligned}
\Delta V_{12} = {} & d(k+\sigma_1)|u_1(k)-v_1(k)| - d(k)|u_1(k-\sigma_1)-v_1(k-\sigma_1)| \\
& + \sum_{s=k+1}^{k+\tau_1} a(s)\Big\{\left[P_1(k)G_1(k)+M_1P_2(k)E_1(k)\right]\big|x_1(k)-y_1(k)\big| \\
& \qquad + M_1P_2(k)\Big[a(k)\big|x_1(k-\tau_1)-y_1(k-\tau_1)\big| \\
& \qquad\qquad + d(k)|u_1(k-\sigma_1)-v_1(k-\sigma_1)| + E_2(k)\big|x_2(k)-y_2(k)\big|\Big]\Big\} \\
& - a(k)\sum_{u=s-\tau_1}^{k-1}\Big\{\left[P_1(u)G_1(u)+M_1P_2(u)E_1(u)\right]\big|x_1(u)-y_1(u)\big| \\
& \qquad + M_1P_2(u)\Big[a(u)\big|x_1(u-\tau_1)-y_1(u-\tau_1)\big| \\
& \qquad\qquad + d(u)|u_1(u-\sigma_1)-v_1(u-\sigma_1)| + E_2(u)\big|x_2(u)-y_2(u)\big|\Big]\Big\}.
\end{aligned} \tag{4.22}$$

Thirdly, we define

$$\begin{aligned}
V_{13}(k) = {} & M_1\sum_{l=k-\tau_1}^{k-1} P_2(l+\tau_1)a(l+\tau_1)\big|x_1(l)-y_1(l)\big| \sum_{s=l+\tau_1+1}^{l+2\tau_1} a(s) \\
& + M_1\sum_{l=k-\sigma_1}^{k-1} P_2(l+\sigma_1)d(l+\sigma_1)|u_1(l)-v_1(l)| \sum_{s=l+\sigma_1+1}^{l+\sigma_1+\tau_1} a(s).
\end{aligned} \tag{4.23}$$

By a simple calculation, it follows that

$$\begin{aligned}
\Delta V_{13} = {} & \sum_{s=k+\tau_1+1}^{k+2\tau_1} a(s)M_1P_2(k+\tau_1)a(k+\tau_1)\big|x_1(k)-y_1(k)\big| \\
& - \sum_{s=k+1}^{k+\tau_1} a(s)M_1P_2(k)a(k)\big|x_1(k-\tau_1)-y_1(k-\tau_1)\big| \\
& + \sum_{s=k+\sigma_1+1}^{k+\sigma_1+\tau_1} a(s)M_1P_2(k+\sigma_1)d(k+\sigma_1)|u_1(k)-v_1(k)| \\
& - \sum_{s=k+1}^{k+\tau_1} a(s)M_1P_2(k)d(k)|u_1(k-\sigma_1)-v_1(k-\sigma_1)|.
\end{aligned} \tag{4.24}$$

We now define

$$V_1(k) = V_{11}(k) + V_{12}(k) + V_{13}(k). \tag{4.25}$$

Then for all $k > k_0 + \tau$, it follows from (4.14)–(4.24) that

$$\begin{aligned}
\Delta V_1 \le & -\left[\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - a(k)\right|\right]\left|x_1(k) - y_1(k)\right| + d(k+\sigma_1)|u_1(k) - v_1(k)| \\
& + E_1(k)\left|x_1(k) - y_1(k)\right| + E_2(k)\left|x_2(k) - y_2(k)\right| \\
& + \sum_{s=k+1}^{k+\tau_1} a(s)\left\{[P_1(k)G_1(k) + M_1P_2(k)E_1(k)]\left|x_1(k) - y_1(k)\right|\right. \\
& \qquad \left. + M_1P_2(k)E_2(k)\left|x_2(k) - y_2(k)\right|\right\} \\
& + \sum_{s=k+\tau_1+1}^{k+2\tau_1} a(s)M_1P_2(k+\tau_1)a(k+\tau_1)\left|x_1(k) - y_1(k)\right| \\
& + \sum_{s=k+\sigma_1+1}^{k+\sigma_1+\tau_1} a(s)M_1P_2(k+\sigma_1)d(k+\sigma_1)|u_1(k) - v_1(k)|.
\end{aligned} \tag{4.26}$$

Similarly, we define

$$V_2(k) = V_{21}(k) + V_{22}(k) + V_{23}(k), \tag{4.27}$$

where

$$V_{21}(k) = \left|\ln x_2(k) - \ln y_2(k)\right|, \tag{4.28}$$

$$\begin{aligned}
V_{22}(k) = & \sum_{s=k-\tau_2}^{k-1} \frac{M_2}{m_1^2} f(s+\tau_2)\left|x_1(s) - y_1(s)\right| + \sum_{s=k-\sigma_2}^{k-1} p(s+\sigma_2)|u_2(s) - v_2(s)| \\
& + \sum_{s=k}^{k-1+\tau_2} \frac{f(s)}{m_1} \sum_{u=s-\tau_2}^{k-1} \left\{Q_1(u)G_2(u)\left|x_2(u) - y_2(u)\right|\right. \\
& \qquad + M_2Q_2(u)\left[\frac{f(u)}{m_1}\left|x_2(u-\tau_2) - y_2(u-\tau_2)\right|\right. \\
& \qquad\qquad + \frac{M_2}{m_1^2} f(u)\left|x_1(u-\tau_2) - y_1(u-\tau_2)\right| \\
& \qquad\qquad \left.\left. + p(u)|u_2(u-\sigma_2) - v_2(u-\sigma_2)|\right]\right\},
\end{aligned} \tag{4.29}$$

$$\begin{aligned}V_{23}(k) = &\frac{M_2}{m_1}\sum_{l=k-\tau_2}^{k-1} Q_2(l+\tau_2)f(l+\tau_2)\left|x_2(l)-y_2(l)\right|\sum_{s=l+\tau_2+1}^{l+2\tau_2}\frac{f(s)}{m_1}\\ &+\frac{M_2^2}{m_1^2}\sum_{l=k-\tau_2}^{k-1} Q_2(l+\tau_2)f(l+\tau_2)\left|x_1(l)-y_1(l)\right|\sum_{s=l+\tau_2+1}^{l+2\tau_2}\frac{f(s)}{m_1}\\ &+M_2\sum_{l=k-\sigma_2}^{k-1} Q_2(l+\sigma_2)p(l+\sigma_2)|u_2(l)-v_2(l)|\sum_{s=l+\sigma_2+1}^{l+\sigma_2+\tau_2}\frac{f(s)}{m_1},\end{aligned} \tag{4.30}$$

$$G_2(s) = g(s)+f(s)\frac{y_2(s-\tau_2)}{y_1(s-\tau_2)}+p(s)v_2(s-\sigma_2). \tag{4.31}$$

Thus for all $k > k_0+\tau$, it follows from (4.27)–(4.30) that

$$\begin{aligned}\Delta V_2 \le &-\left[\frac{1}{\xi_2(k)}-\left|\frac{1}{\xi_2(k)}-\frac{f(k)}{x_1(k-\tau_2)}\right|\right]\left|x_2(k)-y_2(k)\right|\\ &+\frac{M_2}{m_1^2}f(k+\tau_2)\left|x_1(k)-y_1(k)\right|+p(k+\sigma_2)|u_2(k)-v_2(k)|\\ &+\sum_{s=k+1}^{k+\tau_2}\frac{f(s)}{m_1}Q_1(k)G_2(k)\left|x_2(k)-y_2(k)\right|\\ &+\sum_{s=k+\tau_2+1}^{k+2\tau_2}\frac{f(s)}{m_1}\frac{M_2}{m_1}Q_2(k+\tau_2)f(k+\tau_2)\left|x_2(k)-y_2(k)\right|\\ &+\sum_{s=k+\tau_2+1}^{k+2\tau_2}\frac{f(s)}{m_1}\frac{M_2^2}{m_1^2}Q_2(k+\tau_2)f(k+\tau_2)\left|x_1(k)-y_1(k)\right|\\ &+\sum_{s=k+\sigma_2+1}^{k+\sigma_2+\tau_2}\frac{f(s)}{m_1}M_2Q_2(k+\sigma_2)p(k+\sigma_2)|u_2(k)-v_2(k)|.\end{aligned} \tag{4.32}$$

For $i = 1, 2$, we define

$$V_i'(k) = |u_i(k)-v_i(k)|+\sum_{s=k-\rho_i}^{k-1}\beta_i(s+\rho_i)\left|x_i(s)-y_i(s)\right|. \tag{4.33}$$

By calculation, it derives that

$$\Delta V_i' \le -\alpha_i(k)|u_i(k)-v_i(k)|+\beta_i(k+\rho_i)\left|x_i(k)-y_i(k)\right|. \tag{4.34}$$

We now define a Lyapunov function as

$$V(k) = s_1V_1(k)+s_2V_2(k)+\omega_1V_1'(k)+\omega_2V_2'(k). \tag{4.35}$$

It is easy to see that $V(k) > 0$ and $V(k_0+\tau) < +\infty$. Calculating the difference of V along the solution of system (1.4), we have that for $k \ge k_0+\tau$,

$$
\begin{aligned}
\Delta V \le{}& \Bigg\{ s_1\left[-\min\left(a^l, \frac{2}{M_1} - a^u\right) + E_1^u + \tau_1 a^u\left[P_1^u G_1^u + M_1 P_2^u E_1^u + M_1 P_2^u a^u\right]\right] \\
&+ s_2 \frac{M_2}{m_1^2} f^u \left[1 + \tau_2 \frac{M_2}{m_1} f^u Q_2^u\right] + \omega_1 \beta_1^u \Bigg\} \left|x_1(k) - y_1(k)\right| \\
&+ \Bigg\{ s_2\left[-\min\left(\frac{f^l}{M_1}, \frac{2}{M_2} - \frac{f^u}{m_1}\right) + \tau_2 \frac{f^u}{m_1}\left[Q_1^u G_2^u + \frac{M_2}{m_1} Q_2^u f^u\right]\right] \\
&+ s_1 E_2^u \left[1 + \tau_1 a^u M_1 P_2^u\right] + \omega_2 \beta_2^u \Bigg\} \left|x_2(k) - y_2(k)\right| \\
&+ \left\{ -\omega_1 \alpha_1^l + s_1 d^u \left[1 + \tau_1 a^u M_1 P_2^u\right]\right\} |u_1(k) - v_1(k)| \\
&+ \left\{ -\omega_2 \alpha_2^l + s_2 p^u \left[1 + \tau_2 \frac{f^u}{m_1} M_2 Q_2^u\right]\right\} |u_2(k) - v_2(k)| \\
\le{}& -\left[s_1\mu_1 - s_2\mu_2 - \omega_1\beta_1^u\right]\left|x_1(k) - y_1(k)\right| - \left[s_2\nu_1 - s_1\nu_2 - \omega_2\beta_2^u\right]\left|x_2(k) - y_2(k)\right| \\
&- \left[\omega_1\alpha_1^l - s_1\varrho_1\right]|u_1(k) - v_1(k)| - \left[\omega_2\alpha_2^l - s_2\varrho_2\right]|u_2(k) - v_2(k)| \\
\le{}& -\alpha \sum_{i=1}^{2}\left(\left|x_i(k) - y_i(k)\right| + |u_i(k) - v_i(k)|\right),
\end{aligned}
\tag{4.36}
$$

where

$$
\begin{gathered}
\mu_1 = \min\left(a^l, \frac{2}{M_1} - a^u\right) - E_1^u - \tau_1 a^u\left[P_1^u G_1^u + M_1 P_2^u E_1^u + M_1 P_2^u a^u\right], \\
\mu_2 = \frac{M_2}{m_1^2} f^u \left[1 + \tau_2 \frac{M_2}{m_1} Q_2^u f^u\right], \\
\nu_1 = \min\left(\frac{f^l}{M_1}, \frac{2}{M_2} - \frac{f^u}{m_1}\right) - \tau_2 \frac{f^u}{m_1}\left[Q_1^u G_2^u + \frac{M_2}{m_1} Q_2^u f^u\right], \\
\nu_2 = E_2^u\left[1 + \tau_1 a^u M_1 P_2^u\right], \\
\varrho_1 = d^u\left[1 + \tau_1 a^u M_1 P_2^u\right], \\
\varrho_2 = p^u\left[1 + \tau_2 \frac{f^u}{m_1} M_2 Q_2^u\right].
\end{gathered}
\tag{4.37}
$$

Summing both sides of (4.36) from $k_0+\tau$ to k, it derives that

$$\sum_{s=k_0+\tau}^{k}[V(s+1)-V(s)] \le -\alpha \sum_{s=k_0+\tau}^{k}\sum_{i=1}^{2}\left(\left|x_i(s)-y_i(s)\right|+|u_i(s)-v_i(s)|\right). \tag{4.38}$$

It then follows from (4.38) that for $k > k_0+\tau$,

$$V(k+1)+\alpha \sum_{s=k_0+\tau}^{k}\sum_{i=1}^{2}\left(\left|x_i(s)-y_i(s)\right|+|u_i(s)-v_i(s)|\right) \le V(k_0+\tau), \tag{4.39}$$

that is

$$\sum_{s=k_0+\tau}^{k}\sum_{i=1}^{2}\left(\left|x_i(s)-y_i(s)\right|+|u_i(s)-v_i(s)|\right) \le \frac{V(k_0+\tau)}{\alpha}. \tag{4.40}$$

Then

$$\sum_{k=k_0+\tau}^{\infty}\sum_{i=1}^{2}\left(\left|x_i(s)-y_i(s)\right|+|u_i(s)-v_i(s)|\right) \le \frac{V(k_0+\tau)}{\alpha} < +\infty, \tag{4.41}$$

from which we see that

$$\lim_{k\to+\infty}\left|x_i(k)-y_i(k)\right| = 0, \qquad \lim_{k\to+\infty}|u_i(k)-v_i(k)| = 0, \quad i=1,2. \tag{4.42}$$

This completes the proof of Theorem 4.3. □

5. Almost Periodic Solutions

In this section, we consider the almost periodic property of system (1.4).

We assume that $a^*(k) \in H(a(k))$, $b^*(k) \in H(b(k))$, $c^*(k) \in H(c(k))$, $d^*(k) \in H(d(k))$, $f^*(k) \in H(f(k))$, $g^*(k) \in H(g(k))$, $p^*(k) \in H(p(k))$, $\alpha_i^*(k) \in H(\alpha_i(k))$, $\beta_i^*(k) \in H(\beta_i(k))$, $i=1,2$. By Definitions 2.3 and 2.4, there exists an integer valued sequence γ_n with $\gamma_n \to \infty$ as $n \to \infty$ such that for $i=1,2,\ k\in Z$

$$\begin{aligned}
&a(k+\gamma_n) \longrightarrow a^*(k), \qquad b(k+\gamma_n) \longrightarrow b^*(k), \qquad c(k+\gamma_n) \longrightarrow c^*(k),\\
&d(k+\gamma_n) \longrightarrow d^*(k), \qquad f(k+\gamma_n) \longrightarrow f^*(k), \qquad g(k+\gamma_n) \longrightarrow g^*(k),\\
&p(k+\gamma_n) \longrightarrow p^*(k), \qquad \alpha_i(k+\gamma_n) \longrightarrow \alpha_i^*(k), \qquad \beta_i(k+\gamma_n) \longrightarrow \beta_i^*(k).
\end{aligned} \tag{5.1}$$

Then we have the following Hull equation of system (1.4)

$$
\begin{aligned}
x_1(k+1) &= x_1(k)\exp\left[b^*(k) - a^*(k)x_1(k-\tau_1) - \frac{c^*(k)x_1(k)x_2(k)}{h^2x_2^2(k)+x_1^2(k)} - d^*(k)u_1(k-\sigma_1)\right],\\
x_2(k+1) &= x_2(k)\exp\left[g^*(k) - f^*(k)\frac{x_2(k-\tau_2)}{x_1(k-\tau_2)} - p^*(k)u_2(k-\sigma_2)\right],\\
\Delta u_i(k) &= -\alpha_i^*(k)u_i(k) + \beta_i^*(k)x_i(k-\rho_i), \quad i=1,2.
\end{aligned}
\tag{5.2}
$$

According to the almost periodic theory, we know that if system (1.4) satisfies (H_1)–(H_3), then Hull equation (5.2) also satisfies (H_1)–(H_3).

The following Lemma is Lemma 4.1 in [15].

Lemma 5.1. *If each hull equation of system* (1.4) *has a unique strictly positive solution, then the almost periodic difference system* (1.4) *has a unique strictly positive almost periodic solution.*

Theorem 5.2. *Assume that* (H_1)–(H_3) *holds. Then the almost periodic difference system* (1.4) *has a unique strictly positive almost periodic solution which is globally attractive.*

Proof. We divided the proof into two steps.

Step 1. We prove the existence of a strictly positive solution of each Hull equation (5.2).

By the almost periodicity of the parameters of system (1.4), there exists an integer-valued sequence γ_n with $\gamma_n \to \infty$ as $n \to \infty$ such that for $i = 1,2,\ k \in Z$

$$
\begin{aligned}
&a^*(k+\gamma_n) \longrightarrow a^*(k), \quad && b^*(k+\gamma_n) \longrightarrow b^*(k), \quad && c^*(k+\gamma_n) \longrightarrow c^*(k),\\
&d^*(k+\gamma_n) \longrightarrow d^*(k), && f^*(k+\gamma_n) \longrightarrow f^*(k), && g^*(k+\gamma_n) \longrightarrow g^*(k),\\
&p^*(k+\gamma_n) \longrightarrow p^*(k), && \alpha_i^*(k+\gamma_n) \longrightarrow \alpha_i^*(k), && \beta_i^*(k+\gamma_n) \longrightarrow \beta_i^*(k).
\end{aligned}
\tag{5.3}
$$

Suppose that $X(k) = (x_1(k), x_2(k), u_1(k), u_2(k))^T$ is any positive solution of Hull equation (5.2). Since (H_1), (H_2) hold, according to Theorem 3.1, we obtain

$$
\begin{aligned}
m_i &\le \liminf_{k\to+\infty} x_i(k) \le \limsup_{k\to+\infty} x_i(k) \le M_i,\\
n_i &\le \liminf_{k\to+\infty} u_i(k) \le \limsup_{k\to+\infty} u_i(k) \le N_i, \quad i=1,2.
\end{aligned}
\tag{5.4}
$$

Therefore, one has

$$
\begin{aligned}
0 &< \inf_{k\in Z^+} x_i(k) \le \sup_{k\in Z^+} x_i(k) < \infty,\\
0 &< \inf_{k\in Z^+} u_i(k) \le \sup_{k\in Z^+} u_i(k) < \infty, \quad i=1,2.
\end{aligned}
\tag{5.5}
$$

For any $\varepsilon > 0$ small enough, it follows from Theorem 3.1 that there exists enough large $K_0 > K_5 + \tau$ such that for $i = 1, 2$ and $k \geq K_0$

$$m_i - \varepsilon \leq x_i(k) \leq M_i + \varepsilon, \qquad n_i - \varepsilon \leq u_i(k) \leq N_i + \varepsilon. \tag{5.6}$$

Let $x_n(k) = x_n(k + \gamma_n)$ and $u_n(k) = u_n(k + \gamma_n)$ for $k \geq K_0 + \tau - \gamma_n$, $n = 1, 2, \ldots$. For any positive integer q and $i = 1, 2$, it is not difficult to show that there are sequences $\{x_{in}(k) : n \geq q\}$ and $\{u_{in}(k) : n \geq q\}$ such that the sequences $\{x_{in}(k)\}$ and $\{u_{in}(k)\}$ have subsequences, denoted by $\{x_{in}(k)\}$ and $\{u_{in}(k)\}$ again, converging on any finite interval of Z as $n \to \infty$, respectively. So we have sequences $\{y_i(k)\}$ and $\{v_i(k)\}$ $i = 1, 2$ such that for $k \in Z$, $n \to \infty$

$$\begin{aligned}
&x_{in}(k) \longrightarrow y_i(k), \qquad u_{in}(k) \longrightarrow v_i(k), \\
&x_1(k+1) = x_1(k) \exp\Bigg[b^*(k+\gamma_n) - a^*(k+\gamma_n) x_1(k-\tau_1) \\
&\qquad\qquad - \frac{c^*(k+\gamma_n) x_1(k) x_2(k)}{h^2 x_2^2(k) + x_1^2(k)} - d^*(k+\gamma_n) u_1(k-\sigma_1) \Bigg], \\
&x_2(k+1) = x_2(k) \exp\left[g^*(k+\gamma_n) - f^*(k+\gamma_n) \frac{x_2(k-\tau_2)}{x_1(k-\tau_2)} - p^*(k+\gamma_n) u_2(k-\sigma_2) \right], \\
&\Delta u_i(k) = -\alpha_i^*(k+\gamma_n) u_i(k) + \beta_i^*(k+\gamma_n) x_i(k-\rho_i), \quad i = 1, 2,
\end{aligned} \tag{5.7}$$

which implies

$$\begin{aligned}
y_1(k+1) &= y_1(k) \exp\left[b^*(k) - a^*(k) y_1(k-\tau_1) - \frac{c^*(k) y_1(k) y_2(k)}{h^2 y_2^2(k) + y_1^2(k)} - d^*(k) v_1(k-\sigma_1) \right], \\
y_2(k+1) &= y_2(k) \exp\left[g^*(k) - f^*(k) \frac{y_2(k-\tau_2)}{y_1(k-\tau_2)} - p^*(k) v_2(k-\sigma_2) \right], \\
\Delta v_i(k) &= -\alpha_i^*(k) v_i(k) + \beta_i^*(k) y_i(k-\rho_i), \quad i = 1, 2.
\end{aligned} \tag{5.8}$$

Thus we can easily see that $Y(k) = (y_1(k), y_2(k), v_1(k), v_2(k))^T$ is a solution of Hull equation (5.2) and $m_i - \varepsilon \leq y_i(k) \leq M_i + \varepsilon, n_i - \varepsilon \leq v_i(k) \leq N_i + \varepsilon$, for $i = 1, 2, k \in Z$. Since ε is an arbitrary small positive number, it follows that

$$\begin{aligned}
0 < \inf_{k \in Z^+} y_i(k) \leq \sup_{k \in Z^+} y_i(k) < \infty, \\
0 < \inf_{k \in Z^+} v_i(k) \leq \sup_{k \in Z^+} v_i(k) < \infty, \quad i = 1, 2.
\end{aligned} \tag{5.9}$$

This completes the proof of Step 1.

Step 2. We show the uniqueness of the strictly positive solution of each Hull equation (5.2).

Suppose that the Hull equation (5.2) has two arbitrary strictly positive solutions $X^*(k) = (x_1^*(k), x_2^*(k), u_1^*(k), u_2^*(k))^T$ and $Y^*(k) = (y_1^*(k), y_2^*(k), v_1^*(k), v_2^*(k))^T$. Now we define a Lyapunov function $V^*(k)$ on $k \in Z$ as follows:

$$V^*(k) = \sum_{i=1}^{2} s_i\left(V_{i1}^*(k) + V_{i2}^*(k) + V_{i3}^*(k)\right) + \sum_{i=1}^{2} \omega_i V_i^{'*}(k), \tag{5.10}$$

where

$$V_{i1}^*(k) = \left|\ln x_i^*(k) - \ln y_i^*(k)\right|, \quad i = 1, 2,$$

$$\begin{aligned}
V_{12}^*(k) = & \sum_{s=k-\sigma_1}^{k-1} d(s+\sigma_1)\left|u_1^*(s) - v_1^*(s)\right| \\
& + \sum_{s=k}^{k-1+\tau_1} a(s) \sum_{u=s-\tau_1}^{k-1} \{[P_1(u)G_1(u)M_1P_2(u)E_1(u)]\left|x_1^*(u) - y_1^*(u)\right| \\
& \qquad + M_1P_2(u)\left[a(u)\left|x_1^*(u-\tau_1) - y_1^*(u-\tau_1)\right|\right. \\
& \qquad\qquad + d(u)\left|u_1^*(u-\sigma_1) - v_1^*(u-\sigma_1)\right| \\
& \qquad\qquad \left.+ E_2(u)\left|x_2^*(u) - y_2^*(u)\right|\right]\},
\end{aligned}$$

$$\begin{aligned}
V_{13}^*(k) = & M_1 \sum_{l=k-\tau_1}^{k-1} P_2(l+\tau_1)a(l+\tau_1)\left|x_1^*(l) - y_1^*(l)\right| \sum_{s=l+\tau_1+1}^{l+2\tau_1} a(s) \\
& + M_1 \sum_{l=k-\sigma_1}^{k-1} P_2(l+\sigma_1)d(l+\sigma_1)\left|u_1^*(l) - v_1^*(l)\right| \sum_{s=l+\sigma_1+1}^{l+\sigma_1+\tau_1} a(s),
\end{aligned}$$

$$\begin{aligned}
V_{22}^*(k) = & \sum_{s=k-\tau_2}^{k-1} \frac{M_2}{m_1^2} f(s+\tau_2)\left|x_1^*(s) - y_1^*(s)\right| + \sum_{s=k-\sigma_2}^{k-1} p(s+\sigma_2)\left|u_2^*(s) - v_2^*(s)\right| \\
& + \sum_{s=k}^{k-1+\tau_2} \frac{f(s)}{m_1} \sum_{u=s-\tau_2}^{k-1} \left\{Q_1(u)G_2(u)\left|x_2^*(u) - y_2^*(u)\right|\right. \\
& \qquad + M_2Q_2(u)\left[\frac{f(u)}{m_1}\left|x_2^*(u-\tau_2) - y_2^*(u-\tau_2)\right|\right. \\
& \qquad\qquad + \frac{M_2}{m_1^2} f(u)\left|x_1^*(u-\tau_2) - y_1^*(u-\tau_2)\right| \\
& \qquad\qquad \left.\left.+ p(u)\left|u_2^*(u-\sigma_2) - v_2^*(u-\sigma_2)\right|\right]\right\},
\end{aligned}$$

$$V_{23}^{*}(k) = \frac{M_2}{m_1}\sum_{l=k-\tau_2}^{k-1} Q_2(l+\tau_2)f(l+\tau_2)\left|x_2^{*}(l)-y_2^{*}(l)\right|\sum_{s=l+\tau_2+1}^{l+2\tau_2}\frac{f(s)}{m_1}$$
$$+\frac{M_2^2}{m_1^2}\sum_{l=k-\tau_2}^{k-1} Q_2(l+\tau_2)f(l+\tau_2)\left|x_1^{*}(l)-y_1^{*}(l)\right|\sum_{s=l+\tau_2+1}^{l+2\tau_2}\frac{f(s)}{m_1}$$
$$+M_2\sum_{l=k-\sigma_2}^{k-1} Q_2(l+\sigma_2)p(l+\sigma_2)\left|u_2^{*}(l)-v_2^{*}(l)\right|\sum_{s=l+\sigma_2+1}^{l+\sigma_2+\tau_2}\frac{f(s)}{m_1},$$
$$V_i^{'*}(k) = \left|u_i^{*}(k)-v_i^{*}(k)\right| + \sum_{s=k-\rho_i}^{k-1}\beta_i(s+\rho_i)\left|x_i^{*}(s)-y_i^{*}(s)\right|, \tag{5.11}$$

where $E_i,\ G_i,\ P_i,\ Q_i,\ i=1,2$ are defined by Theorem 4.3.

Similar to the discussion of (3.32), calculating the difference of $V^{*}(k)$ along the solution of the Hull equation (5.2), one has

$$\Delta V^{*} \le -\alpha\sum_{i=1}^{2}\left(\left|x_i^{*}(k)-y_i^{*}(k)\right| + \left|u_i^{*}(k)-v_i^{*}(k)\right|\right),\quad \text{for } k\in Z. \tag{5.12}$$

It follows from (5.12) we know that $V^{*}(k)$ is a nonincreasing function on Z. Summing both sides of the above inequalities from k to 0, we get

$$\alpha\sum_{s=k}^{0}\sum_{i=1}^{2}\left(\left|x_i^{*}(s)-y_i^{*}(s)\right| + \left|u_i^{*}(s)-v_i^{*}(s)\right|\right) \le V^{*}(k)-V^{*}(0),\quad \text{for } k<0. \tag{5.13}$$

So we have $\sum_{s=k}^{0}\sum_{i=1}^{2}(|x_i^{*}(s)-y_i^{*}(s)| + |u_i^{*}(s)-v_i^{*}(s)|) < +\infty,\ k\to-\infty$, thus we obtain

$$\lim_{k\to-\infty}\left|x_i^{*}(k)-y_i^{*}(k)\right| = 0,\qquad \lim_{k\to-\infty}\left|u_i^{*}(k)-v_i^{*}(k)\right| = 0,\quad i=1,2. \tag{5.14}$$

From (5.14), for a positive integer K, we have

$$\left|x_i^{*}(k)-y_i^{*}(k)\right| < \varepsilon,\qquad \left|u_i^{*}(k)-v_i^{*}(k)\right| < \varepsilon,\quad \text{for } k<-K,\ i=1,2. \tag{5.15}$$

It follows from (5.10) that

$$
\begin{aligned}
V_{i1}^{*}(k) &\le \frac{1}{m_i}\left|x_i^{*}(k)-y_i^{*}(k)\right| \le \frac{1}{m_i}\varepsilon,\\
V_{12}^{*}(k) &= \sigma_1 d^u \max_{s\le k}\left|u_1^{*}(s)-v_1^{*}(s)\right|\\
&\quad + (\tau_1)^2 a^u\Bigg\{\left[P_1^u G_1^u + M_1 P_2^u E_1^u\right]\max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right|\\
&\quad + M_1 P_2^u\Bigg[a^u \max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right| + d^u \max_{s\le k}\left|u_1^{*}(s)-v_1^{*}(s)\right|\\
&\quad +E_2^u \max_{s\le k}\left|x_2^{*}(s)-y_2^{*}(s)\right|\Bigg]\Bigg\}\\
&\le \left\{\sigma_1 d^u + (\tau_1)^2 a^u\left[\left(P_1^u G_1^u + M_1 P_2^u E_1^u\right) + M_1 P_2^u\left(a^u + d^u + E_2^u\right)\right]\right\}\varepsilon,\\
V_{13}^{*}(k) &\le \tau_1 M_1 P_2^u a^u\left(\tau_1 a^u \max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right| + \sigma_1 d^u \max_{s\le k}\left|u_1^{*}(s)-v_1^{*}(s)\right|\right)\\
&\le \tau_1 M_1 P_2^u a^u(\tau_1 a^u + \sigma_1 d^u)\varepsilon,\\
V_{22}^{*}(k) &\le \tau_2\frac{M_2}{m_1^2} f^u \max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right| + \sigma_2 p^u \max_{s\le k}\left|u_2^{*}(s)-v_2^{*}(s)\right|\\
&\quad + (\tau_2)^2\frac{f^u}{m_1}\Bigg\{Q_1^u G_2^u \max_{s\le k}\left|x_2^{*}(s)-y_2^{*}(s)\right|\\
&\quad + M_2 Q_2^u\Bigg[\frac{f^u}{m_1}\max_{s\le k}\left|x_2^{*}(s)-y_2^{*}(s)\right| + \frac{M_2}{m_1^2} f^u \max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right|\\
&\quad +p^u \max_{s\le k}\left|u_2^{*}(s)-v_2^{*}(s)\right|\Bigg]\Bigg\},\\
&\le \left\{\tau_2\frac{M_2}{m_1^2} f^u + \sigma_2 p^u + (\tau_2)^2\frac{f^u}{m_1}\left[Q_1^u F_2^u + M_2 Q_2^u\left(\frac{f^u}{m_1} + \frac{M_2}{m_1^2} f^u + p^u\right)\right]\right\}\varepsilon,\\
V_{23}^{*}(k) &\le \tau_2\frac{f^u}{m_1} M_2 Q_2^u\Bigg(\tau_2\frac{f^u}{m_1}\max_{s\le k}\left|x_2^{*}(s)-y_2^{*}(s)\right| + \tau_2\frac{M_2}{m_1^2} f^u \max_{s\le k}\left|x_1^{*}(s)-y_1^{*}(s)\right|\\
&\quad + \sigma_2 p^u \max_{s\le k}\left|u_2^{*}(s)-v_2^{*}(s)\right|\Bigg)\\
&\le \tau_2\frac{f^u}{m_1} M_2 Q_2^u\left(\tau_2\frac{f^u}{m_1} + \tau_2\frac{M_2}{m_1^2} f^u + \sigma_2 p^u\right)\varepsilon,\\
V_i^{'*}(k) &\le \max_{s\le k}\left|u_i^{*}(s)-v_i^{*}(s)\right| + \rho_i\beta_i^u \max_{s\le k}\left|x_i^{*}(s)-y_i^{*}(s)\right| \le \left(1+\rho_i\beta_i^u\right)\varepsilon.
\end{aligned}
\tag{5.16}
$$

Let

$$
\begin{aligned}
Q = {} & s_1\left\{\frac{1}{m_1} + \sigma_1 d^u + (\tau_1)^2 a^u\left[\left(P_1^u G_1^u + M_1 P_2^u E_1^u\right) + M_1 P_2^u\left(a^u + d^u + E_2^u\right)\right]\right. \\
& \left. +\tau_1 M_1 P_2^u a^u\left(\tau_1 a^u + \sigma_1 d^u\right)\right\} \\
& + s_2\left\{\frac{1}{m_2} + \tau_2\frac{M_2}{m_1^2} f^u + \sigma_2 p^u + (\tau_2)^2\frac{f^u}{m_1}\left[Q_1^u G_2^u + M_2 Q_2^u\left(\frac{f^u}{m_1} + \frac{M_2}{m_1^2} f^u + p^u\right)\right]\right. \\
& \left. + \tau_2\frac{f^u}{m_1} M_2 Q_2^u\left(\tau_2\frac{f^u}{m_1} + \tau_2\frac{M_2}{m_1^2} f^u + \sigma_2 p^u\right)\right\} + \sum_{i=1}^{2}\omega_i\left(1 + \rho_i\beta_i^u\right).
\end{aligned}
\tag{5.17}
$$

It follows from (5.10) and the above inequalities that $V^*(k) \le Q\varepsilon$ for $k < -K$, hence $\lim_{k\to-\infty} V^*(k) = 0$. Furthermore, $V^*(k)$ is a nonincreasing function on Z, thus $V^*(k) \equiv 0$, that is, $x_i^*(k) = y_i^*(k),\ u_i^*(k) = v_i^*(k),\ i = 1,2$ for all $k \in Z$. This ends the proof of Step 2.

By the above discussion and Theorem 4.3, we conclude that any hull equation of system (1.4) has a unique strictly positive solution. Then the almost periodic system (1.4) has a unique strictly positive almost periodic solution which is globally attractive. The proof of Theorem 5.2 is completed. □

6. Concluding Remarks

In this paper, a discrete almost periodic ratio-dependent Leslie system with time delays and feedback controls is considered. By applying the difference inequality, some sufficient conditions are established, which are independent of feedback control variables, to ensure the permanence of system (1.4). By constructing the suitable Lyapunov function, we show that system (1.4) is globally attractive under some appropriate conditions. Furthermore, by using an almost periodic functional hull theory, we show that the almost periodic system (1.4) has a unique strictly positive almost periodic solution which is globally attractive.

We would like to mention here the question of whether the feedback control variables have the influence on the stability property of the system or not. It is, in fact, a very challenging problem, and we leave it for our future work.

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