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# A NOTE ON SEMIPRIME RINGS WITH DERIVATION

Dedicated to the memory of Professor H. Tominaga

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**ABSTRACT.** Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R, Z the center of R and  $d:R \to R$  a derivation. If  $d[x,y] + [x,y] \in Z$  or  $d[x,y] - [x,y] \in Z$  for all  $x, y \in I$ , then R is commutative.

KEY WORDS AND PHRASES: Derivation, semiprime ring, 2-torsion free ring. 1991 AMS SUBJECT CLASSIFICATION CODES: 16W25, 16N60.

## 1 INTRODUCTION.

Throughout, R will represent a ring, Z the center of R, I a nonzero ideal of R, and  $d:R\to R$  a derivation. As usual, for  $x,\ y\in R$ , we write [x,y]=xy-yx and  $x\circ y=xy+yx$ . Given a subset S of R, we put  $V_R(S)=\{x\in R\mid [x,s]=0 \text{ for all }s\in S\}$ . In [1], Daif and Bell showed that a semiprime ring R must be commutative if it admits a derivation d such that (i) d[x,y]=[x,y] for all  $x,\ y\in R$ , or (ii) d[x,y]+[x,y]=0 for all  $x,\ y\in R$ . Our present objective is to generalize this result.

### 2 THE RESULTS.

As mentioned in §1, our present objective is to prove the following theorem which generalizes [1, Theorem 3].

**THEOREM 1.** Let R be a 2-torsion free semiprime ring, and let I be a nonzero ideal of R. Then the following conditions are equivalent:

- (1) R admits a derivation d such that  $d[x,y] [x,y] \in Z$  for all  $x, y \in I$ .
- (2) R admits a derivation d such that  $d[x,y] + [x,y] \in Z$  for all  $x, y \in I$ .
- (3) R admits a derivation d such that  $d[x,y] + [x,y] \in Z$  or  $d[x,y] [x,y] \in Z$  for all  $x, y \in I$ .
  - (4)  $I \subseteq Z$ .

In preparation for proving our theorem, we state the following two lemmas.

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**LEMMA 1.** Let R be a semiprime ring, I a nonzero ideal of R, and  $a \in R$ .

- (1) Let  $b \in I$ . If [b,x] = 0 for all  $x \in I$ , then  $b \in Z$ . Therefore, if I is commutative, then  $I \subseteq Z$ .
  - (2) If  $[a,x] \in Z$  for all  $x \in I$ , then  $a \in V_R(I)$ .
  - (3) Let R be a 2-torsion free ring and  $[a,[x,y]] \in Z$  for all  $x, y \in I$ , then  $a \in V_R(I)$ . **PROOF.** (1) is well known.
- (2) For any  $x \in I$ , we have  $a[a,x] = [a,ax] \in Z$ , and so we get  $0 = [a[a,x],x] = [a,x]^2$ . Since R is semiprime and  $[a,x] \in Z$ , we obtain that [a,x] = 0 for all  $x \in I$ . Hence  $a \in V_R(I)$ .
- (3) Since  $Z \ni [a, [x, xy]] = [a, x[x, y]] = x[a, [x, y]] + [a, x][x, y]$  for all  $x, y \in I$ , we have 0 = [a, x[a, [x, y]] + [a, x][x, y]] + [a, x][x, y]] + [a, [a, x]][x, y]. Now, substituting ax for y, we get  $0 = 2[a, x][a, [x, ax]] + [a, [a, x]][x, ax] = 2[a, x][a, [x, a]x] + [a, [a, x]][x, a]x = -2[a, x]^3 2[a, x][a, [a, x]]x [a, [a, x]][a, x]x$ . Substituting [x, y] for  $x \in I$ , we have  $x \in I$ . Since  $x \in I$  is a 2-torsion free semiprime ring and  $x \in I$ , we get  $x \in I$ . Hence we have  $x \in I$  by  $x \in I$ . Hence we have  $x \in I$  by  $x \in I$ . Lemma 1].
- **LEMMA 2.** Let R be a semiprime ring, I a nonzero ideal of R, and  $d: R \to R$  a nonzero derivation such that  $d[x,y]+[x,y] \in Z$  or  $d[x,y]-[x,y] \in Z$  for all  $x, y \in I$ . If  $d(I) \subseteq V_R(I)$ , then I is commutative, and so  $I \subseteq Z$ .
- **PROOF.** Let  $a \in I$ . For any  $x, y \in I$ , we have  $0 = [a, d[x, y] \pm [x, y]] = \pm [a, [x, y]]$ , and so we get  $a \in V_R(I)$  by [1, Lemma 1]. Therefore, I is commutative, and so we obtain that  $I \subseteq Z$  by Lemma 1 (1).

We are now ready to complete the proof of Theorem 1.

**PROOF OF THEOREM 1.** (1)  $\Rightarrow$  (4). Let d be a derivation such that  $d[x,y]-[x,y] \in Z$  for all  $x, y \in I$ . If d=0, then  $I \subseteq Z$  by Lemma 1 (1) and (2). Now we suppose that  $d \neq 0$ . For any  $x, y, z \in I$ , we have  $Z \ni d[x,[y,z]]-[x,[y,z]]=[d(x),[y,z]]+[x,d[y,z]]-[x,[y,z]]=[d(x),[y,z]]+[x,d[y,z]-[y,z]]=[d(x),[y,z]]$ , and so we have  $d(x) \in V_R(I)$  by Lemma 1 (3), that is,  $d(I) \subseteq V_R(I)$ . Therefore we have  $I \subseteq Z$  by Lemma 2.

- $(2) \Rightarrow (4)$ . Let d be a derivation such that  $d[x,y] + [x,y] \in Z$  for all  $x, y \in I$ . Then the derivation (-d) satisfies the condition  $(-d)[x,y] [x,y] \in Z$ . And so we have  $I \subseteq Z$  by (1).
- (3)  $\Rightarrow$  (4). For each  $x \in I$ , we put  $I_x = \{y \in I \mid d[x,y] [x,y] \in Z\}$  and  $I_x^* = \{y \in I \mid d[x,y] + [x,y] \in Z\}$ . Then  $I = I_x \cup I_x^*$ . By Brauer's Trick, we have  $I = I_x$  or  $I = I_x^*$ . By the same method, we can see that  $I = \{x \in I \mid I = I_x\}$  or  $I = \{x \in I \mid I = I_x^*\}$ . Therefore, by (1) and (2) we have  $I \subseteq Z$ .
  - $(4) \Rightarrow (1), (4) \Rightarrow (2)$  and  $(4) \Rightarrow (3)$  are clear.

The next is a generalization of [1, Theorem 2].

**COROLLARY 1.** Let R be a 2-torsion free semiprime ring, Z the center of R and  $d: R \to R$  a derivation. If  $d[x,y] + [x,y] \in Z$  or  $d[x,y] - [x,y] \in Z$  for all  $x, y \in R$ , then R is commutative.

**PROPOSITION 1.** Let R be a 2-torsion free ring with identity 1. Then there is no derivation  $d: R \to R$  such that  $d(x \circ y) = x \circ y$  for all  $x, y \in R$  or  $d(x \circ y) + (x \circ y) = 0$  for all  $x, y \in R$ .

**PROOF.** If there exists a nonzero derivation  $d: R \to R$  such that  $d(x \circ y) = x \circ y$  or  $d(x \circ y) + (x \circ y) = 0$  for  $x, y \in R$ , then we have  $2x = x \circ 1 = \pm d(x \circ 1) = \pm 2d(x)$  for all  $x \in R$ . Since R is 2-torsion free, we get  $d(x) = \pm x$  for all  $x \in R$ . For any  $x, y \in R$ , we have  $xy + yx = x \circ y = \pm d(x \circ y) = \pm d(xy + yx) = 2(xy + yx)$ , and so we get  $x \circ y = xy + yx = 0$ . Since R is 2-torsion free, we have  $x^2 = 0$ . Hence we have  $x \circ y = xy + yx = 0$ .

get x=0 for all  $x \in R$ ; a contradiction. If there exists a zero derivation  $d:R \to R$  such that  $d(x \circ y) = x \circ y$  or  $d(x \circ y) + (x \circ y) = 0$  for all  $x, y \in R$ , then we can easily see that x=0 for all  $x \in R$ ; a contradiction.

**REMARK.** In Theorem 1 and Corollary 1, we can not exclude the condition "2-torsion free" as below.

**EXAMPLE.** We denote by Z the integer system. Let  $R = \begin{pmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{pmatrix}$ ,  $a = \begin{pmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{pmatrix}$ 

 $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ , and d the inner derivation induced by a, that is, d(x) = [a, x] for all  $x \in R$ . Then R is a non-commutative prime ring with char R = 2, and  $d[x, y] \pm [x, y] \in Z$  for all  $x, y \in R$ . Finally, we state two questions.

Let R be a 2-torsion free semiprime ring,  $d: R \to R$  a nonzero derivation, and I a nonzero ideal of R. And let n be a fixed positive integer.

**QUESTION 1.** Does the condition that  $d^n[x,y] + [x,y] \in Z$  or  $d^n[x,y] - [x,y] \in Z$  for all  $x, y \in I$  imply that  $I \subseteq Z$ ?

**QUESTION 2.** Does the condition that  $d^m[x,y]+d^p[x,y] \in Z$  or  $d^m[x,y]-d^p[x,y] \in Z$  for some positive integers m=m(x,y) and p=p(x,y), and for all  $x,y \in I$  imply that  $I \subseteq Z$ ?

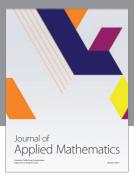
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### REFERENCE

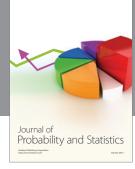
[1] DAIF, M.N. and BELL, H.E., "Remarks on derivations on semiprime rings," Internat. J. Math. & Math. Sci. 15 (1992), 205-206.











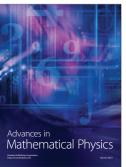


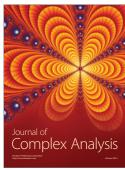




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