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A NOTE ON SEMIPRIME RINGS WITH DERIVATION

Dedicated to the memory of Professor H. Tominaga

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ABSTRACT. Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R, Z the center of R and $d: R \to R$ a derivation. If $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all x, $y \in I$, then R is commutative.

KEY WORDS AND PHRASES: Derivation, semiprime ring, 2-torsion free ring.
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1 INTRODUCTION.

Throughout, R will represent a ring, Z the center of R, I a nonzero ideal of R, and $d: R \to R$ a derivation. As usual, for x, $y \in R$, we write $[x,y] = xy - yx$ and $x \circ y = xy + yx$. Given a subset S of R, we put $V_R(S) = \{ x \in R \mid [x,s] = 0 \text{ for all } s \in S \}$. In [1], Daif and Bell showed that a semiprime ring R must be commutative if it admits a derivation d such that (i) $d[x,y] = [x,y]$ for all x, $y \in R$, or (ii) $d[x,y] + [x,y] = 0$ for all x, $y \in R$. Our present objective is to generalize this result.

2 THE RESULTS.

As mentioned in §1, our present objective is to prove the following theorem which generalizes [1, Theorem 3].

THEOREM 1. Let R be a 2-torsion free semiprime ring, and let I be a nonzero ideal of R. Then the following conditions are equivalent:

(1) R admits a derivation d such that $d[x,y] - [x,y] \in Z$ for all x, $y \in I$.

(2) R admits a derivation d such that $d[x,y] + [x,y] \in Z$ for all x, $y \in I$.

(3) R admits a derivation d such that $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all x, $y \in I$.

(4) $I \subseteq Z$.

In preparation for proving our theorem, we state the following two lemmas.

LEMMA 1. Let R be a semiprime ring, I a nonzero ideal of R, and $a \in R$.

(1) Let $b \in I$. If $[b,x] = 0$ for all $x \in I$, then $b \in Z$. Therefore, if I is commutative, then $I \subseteq Z$.

(2) If $[a,x] \in Z$ for all $x \in I$, then $a \in V_R(I)$.

(3) Let R be a 2-torsion free ring and $[a,[x,y]] \in Z$ for all $x,\ y \in I$, then $a \in V_R(I)$.

PROOF. (1) is well known.

(2) For any $x \in I$, we have $a[a,x] = [a,ax] \in Z$, and so we get $0 = [a[a,x],x] = [a,x]^2$. Since R is semiprime and $[a,x] \in Z$, we obtain that $[a,x] = 0$ for all $x \in I$. Hence $a \in V_R(I)$.

(3) Since $Z \ni [a,[x,xy]] = [a,x[x,y]] = x[a,[x,y]] + [a,x][x,y]$ for all $x,\ y \in I$, we have $0 = [a, x[a,[x,y]] + [a,x][x,y]] = 2[a,x][a,[x,y]] + [a,[a,x]][x,y]$. Now, substituting ax for y, we get $0 = 2[a,x][a,[x,ax]] + [a,[a,x]][x,ax] = 2[a,x][a,[x,a]x] + [a,[a,x]][x,a]x = -2[a,x]^3 - 2[a,x][a,[a,x]]x - [a,[a,x]][a,x]x$. Substituting $[x,y]$ for x ($y \in I$), we have $2[a,[x,y]]^3 = 0$. Since R is a 2-torsion free semiprime ring and $[a,[x,y]] \in Z$, we get $[a,[x,y]] = 0$ for all $x,\ y \in I$. Hence we have $a \in V_R(I)$ by [1, Lemma 1].

LEMMA 2. Let R be a semiprime ring, I a nonzero ideal of R, and $d\colon R \to R$ a nonzero derivation such that $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$. If $d(I) \subseteq V_R(I)$, then I is commutative, and so $I \subseteq Z$.

PROOF. Let $a \in I$. For any $x,\ y \in I$, we have $0 = [a, d[x,y] \pm [x,y]] = \pm[a,[x,y]]$, and so we get $a \in V_R(I)$ by [1, Lemma 1]. Therefore, I is commutative, and so we obtain that $I \subseteq Z$ by Lemma 1 (1).

We are now ready to complete the proof of Theorem 1.

PROOF OF THEOREM 1. (1)$\Rightarrow$(4). Let d be a derivation such that $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$. If $d = 0$, then $I \subseteq Z$ by Lemma 1 (1) and (2). Now we suppose that $d \neq 0$. For any $x,\ y,\ z \in I$, we have $Z \ni d[x,[y,z]] - [x,[y,z]] = [d(x),[y,z]] + [x, d[y,z]] - [x,[y,z]] = [d(x),[y,z]] + [x, d[y,z] - [y,z]] = [d(x),[y,z]]$, and so we have $d(x) \in V_R(I)$ by Lemma 1 (3), that is, $d(I) \subseteq V_R(I)$. Therefore we have $I \subseteq Z$ by Lemma 2.

(2)$\Rightarrow$ (4). Let d be a derivation such that $d[x,y] + [x,y] \in Z$ for all $x,\ y \in I$. Then the derivation $(-d)$ satisfies the condition $(-d)[x,y] - [x,y] \in Z$. And so we have $I \subseteq Z$ by (1).

(3)$\Rightarrow$(4). For each $x \in I$, we put $I_x = \{ y \in I \mid d[x,y] - [x,y] \in Z\}$ and $I_x^* = \{ y \in I \mid d[x,y] + [x,y] \in Z\}$. Then $I = I_x \cup I_x^*$. By Brauer's Trick, we have $I = I_x$ or $I = I_x^*$. By the same method, we can see that $I = \{ x \in I \mid I = I_x\}$ or $I = \{ x \in I \mid I = I_x^*\}$. Therefore, by (1) and (2) we have $I \subseteq Z$.

(4) $\Rightarrow$ (1), (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are clear.

The next is a generalization of [1, Theorem 2].

COROLLARY 1. Let R be a 2-torsion free semiprime ring, Z the center of R and $d\colon R \to R$ a derivation. If $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x,\ y \in R$, then R is commutative.

PROPOSITION 1. Let R be a 2-torsion free ring with identity 1. Then there is no derivation $d\colon R \to R$ such that $d(x \circ y) = x \circ y$ for all $x,\ y \in R$ or $d(x \circ y) + (x \circ y) = 0$ for all $x,\ y \in R$.

PROOF. If there exists a nonzero derivation $d\colon R \to R$ such that $d(x \circ y) = x \circ y$ or $d(x \circ y) + (x \circ y) = 0$ for $x,\ y \in R$, then we have $2x = x \circ 1 = \pm d(x \circ 1) = \pm 2d(x)$ for all $x \in R$. Since R is 2-torsion free, we get $d(x) = \pm x$ for all $x \in R$. For any $x,\ y \in R$, we have $xy + yx = x \circ y = \pm d(x \circ y) = \pm d(xy + yx) = 2(xy + yx)$, and so we get $x \circ y = xy + yx = 0$. Since R is 2-torsion free, we have $x^2 = 0$. Hence we have $0 = x \circ (x+1) = 2x$, and so we get $x = 0$ for all $x \in R$; a contradiction. If there exists a zero derivation $d: R \to R$ such that $d(x \circ y) = x \circ y$ or $d(x \circ y) + (x \circ y) = 0$ for all x, $y \in R$, then we can easily see that $x = 0$ for all $x \in R$; a contradiction.

REMARK. In Theorem 1 and Corollary 1, we can not exclude the condition "2-torsion free" as below.

EXAMPLE. We denote by Z the integer system. Let $R = \begin{pmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and d the inner derivation induced by a, that is, $d(x) = [a, x]$ for all $x \in R$. Then R is a non-commutative prime ring with $char\, R = 2$, and $d[x,y] \pm [x,y] \in Z$ for all x, $y \in R$.

Finally, we state two questions.

Let R be a 2-torsion free semiprime ring, $d: R \to R$ a nonzero derivation, and I a nonzero ideal of R. And let n be a fixed positive integer.

QUESTION 1. Does the condition that $d^n[x,y] + [x,y] \in Z$ or $d^n[x,y] - [x,y] \in Z$ for all x, $y \in I$ imply that $I \subseteq Z$?

QUESTION 2. Does the condition that $d^m[x,y] + d^p[x,y] \in Z$ or $d^m[x,y] - d^p[x,y] \in Z$ for some positive integers $m = m(x,y)$ and $p = p(x,y)$, and for all $x, y \in I$ imply that $I \subseteq Z$?

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REFERENCE

[1] DAIF, M.N. and BELL, H.E., "Remarks on derivations on semiprime rings," *Internat. J. Math. & Math. Sci.* **15** (1992), 205–206.