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Research Article

Binormal Motion of Curves with Constant Torsion in 3-Spaces

Josu Arroyo, Óscar J. Garay, and Álvaro Pámpano

Department of Mathematics, Faculty of Science and Technology, University of the Basque Country, Apartado 644, 48014 Bilbao, Spain

Correspondence should be addressed to Óscar J. Garay; oscarj.garay@ehu.eus

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We study curve motion by the binormal flow with curvature and torsion depending velocity and sweeping out immersed surfaces. Using the Gauss-Codazzi equations, we obtain filaments evolving with constant torsion which arise from extremal curves of curvature energy functionals. They are "soliton" solutions in the sense that they evolve without changing shape.

1. Introduction

A large class of physical systems are modelled in terms of motion of curves and surfaces in Euclidean space $\mathbb{R}^3$. A remarkable example is the so-called localized induction equation (LIE)

$$x_t = x_s \times x_{ss}, \tag{1}$$

which is a soliton equation used to model the dynamics of a thin vortex filament in an incompressible, inviscid, homogeneous, 3-dimensional fluid [1–3]. Quite often, by resorting to the underlying geometry one can gain considerable insight into the dynamics of physical systems [3, 4]. Here, we use a geometrical approach to investigate an extension of (1) obtained by considering a smooth map $x : U \subset \mathbb{R}^2 \to M_r^3(\rho)$, $x(s,t)$, verifying

$$x_t = f\left(\left|\widetilde{\nabla}_{x_s} x_s\right|, \frac{\det\left(x_s, \widetilde{\nabla}_{x_s} x_s, \widetilde{\nabla}^2_{x_s} x_s\right)}{\left|\widetilde{\nabla}_{x_s} x_s\right|^2}\right) x_s \times \widetilde{\nabla}_{x_s} x_s, \tag{2}$$

where f is a suitable smooth function, $\widetilde{\nabla}$ denotes the Levi-Civita connection on $M_r^3(\rho)$, $r \in \{0,1\}$, and $M_r^3(\rho)$ is a Riemannian ($r = 0$) or Lorentzian ($r = 1$) 3-space form with constant curvature ρ; that is, $M_r^3(\rho)$ is one of the following: $\mathbb{R}^3$, the sphere $\mathbb{S}^3$, the hyperbolic space $\mathbb{H}^3$, the Minkowski space $\mathbb{R}^3_1$, the de Sitter space $\mathbb{S}^3_1$, or the anti de Sitter space $\mathbb{H}^3_1$.

Under mild conditions we will see that a curve motion following (2) describes a curve γ evolving under the binormal flow, with velocity depending on curvature and torsion (19), and determines an immersed surface, S_γ, in $M_r^3(\rho)$. Then, fundamental results of the theory of submanifolds can be applied and it will be seen that solving geometrically (2) amounts to solving the Gauss-Codazzi equations (40) and (41), since that would give us the curvature and torsion of a geodesic foliation of S_γ. Alternatively, one can determine the evolution by finding a single solution, working as initial condition $x(s,0) = \gamma(s)$, and then giving a geometrical description of the binormal flow.

If $M_r^3(\rho) = \mathbb{R}^3$ and $f \equiv 1$, (2) reduces to LIE (1) and it can be seen that Gauss-Codazzi equations boil down to Da Rios equations found in 1906 [2]. In Lorentzian backgrounds (1) has been studied in [1, 5], while long time existence of closed solutions in Riemannian ambient spaces are analyzed in [6]. If $f \equiv f(|\widetilde{\nabla}_{x_s} x_s|)$, travelling wave solutions of the Gauss-Codazzi equations have been investigated in [7]. In the second part of this work, we focus on curves evolving by (2) with constant torsion and use the Gauss-Codazzi equations to construct solutions by means of extremal curves for curvature dependent energies and associated 1-parameter groups of isometries.

2. Preliminaries

Consider the Euclidean semispace $\mathbb{E}^m_\nu$, that is, $\mathbb{R}^m$ endowed with the canonical metric of index ν, denoted by $\langle \cdot, \cdot \rangle$, and the Levi-Civita connection, denoted by $\overline{\nabla}$. Then, the semi-Riemannian 3-space forms $M_r^3(\rho)$ (the Riemannian case,

$r = 0$ will be simply denoted by $M^3(\rho)$) can be isometrically immersed in $\mathbb{E}^4_\nu$, the 4-dimensional Euclidean semispace, in a standard way [8]. The flat case, $M^3_r(\rho) = \mathbb{E}^3_r$, $\rho = 0$, $r = 0, 1$, corresponds to either $\mathbb{R}^3$ or the Minkowski space $\mathbb{R}^3_1 \equiv \mathbb{L}^3$. They can be isometrically immersed in $\mathbb{L}^4 = \mathbb{R}^4_1$ endorsed with the metric

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \tag{3}$$

in an obvious manner:

$$\begin{aligned} \mathbb{R}^3 &= \left\{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 \mid x_4 = 0\right\}, \\ \mathbb{L}^3 &= \left\{(x_1, x_2, x_3, x_4) \in \mathbb{L}^4 \mid x_1 = 0\right\}. \end{aligned} \tag{4}$$

When $\rho > 0$, $M^3_r(\rho)$ correspond to the 3-sphere, $\mathbb{S}^3(\rho)$ ($r = 0$), and the de Sitter 3-space, $\mathbb{S}^3_1(\rho)$ ($r = 1$), defined by

$$\mathbb{S}^3_r(\rho) = \left\{\mathbf{x} \in \mathbb{E}^4_r \mid \langle \mathbf{x}, \mathbf{x}\rangle = \frac{1}{\rho}\right\}, \tag{5}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Finally, for $\rho < 0$ we obtain the hyperbolic 3-space, $\mathbb{H}^3 (r = 0)$, and the anti de Sitter 3-space, $\mathbb{H}^3_1$ $(r = 1)$

$$\mathbb{H}^3_r(\rho) = \left\{\mathbf{x} \in \mathbb{E}^4_{r+1} \mid \langle \mathbf{x}, \mathbf{x}\rangle = \frac{1}{\rho}\right\}. \tag{6}$$

The standard isometric immersions of $M^3_r(\rho)$ into $\mathbb{E}^4_\nu$ ([8] p. 20) will be all denoted by i and the induced metrics also by $\langle\cdot,\cdot\rangle$, while the Levi-Civita connections on $\mathbb{E}^4_\nu$ and $M^3_r(\rho)$ are denoted by $\overline{\nabla}$ and $\widetilde{\nabla}$, respectively. As usual, the cross product of two vector fields X, Y in $M^3_r(\rho)$, denoted by $X \times Y$, is defined so that $\langle X \times Y, Z\rangle = \det(X, Y, Z)$ for any other vector field Z of $M^3_r(\rho)$, where $\det(X, Y, Z)$ stands for the determinant.

Now, for a given isometric immersion of a surface, $x : N^2_\nu \to M^3_r(\rho)$, $\nu \in \{0, 1\}$, we denote by ∇ the Levi-Civita connection of the immersion (N^2_ν, x). As it is also customary, for a surface N^2_ν in any 3-dimensional space form $M^3_r(\rho)$, we require the first fundamental form to be nondegenerate. Take X, Y, Z, W tangent vector fields to N^2_ν and choose ξ a normal vector field to N^2_ν in $M^3_r(\rho)$. Then the *formulas of Gauss and Weingarten* are, respectively, as follows [8]:

$$\begin{aligned} \overline{\nabla}_X Y &= \widetilde{\nabla}_X Y - \rho\langle X, Y\rangle x \\ &= \nabla_X Y + h(X, Y) - \rho\langle X, Y\rangle x, \end{aligned} \tag{7}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D^\perp_X \xi, \tag{8}$$

where $x = i \circ x$ is the position vector, h denotes the second fundamental form of N^2_ν in $M^3_r(\rho)$, A_ξ stands for the Weingarten map, and $D^\perp$ denotes the connection on the normal bundle of N^2_ν. By using (7) and (8) and denoting by R and $\widetilde{R}$ the Riemann curvature tensors associated with ∇ and $\widetilde{\nabla}$, respectively, the following relation holds:

$$\widetilde{R}(X, Y) Z = \rho(\langle Y, Z\rangle X - \langle X, Z\rangle Y), \tag{9}$$

while the *equations of Gauss and Codazzi* are given, respectively, by [8]

$$\begin{aligned} \left\langle \widetilde{R}(X, Y) Z, W\right\rangle = \langle R(X, Y) Z, W\rangle \\ - \langle h(X, W), h(Y, Z)\rangle \\ + \langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{10}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{11}$$

where ∇h is defined by

$$\begin{aligned} (\nabla h)(X, Y, Z) = D^\perp_X h(Y, Z) - h(\nabla_X Y, Z) \\ - h(Y, \nabla_X Z). \end{aligned} \tag{12}$$

Now choosing an adapted local orthonormal frame $\{e_1, e_2, e_3\}$ in $M^3_r(\rho)$ such that the vectors e_1, e_2 are tangent to N^2_ν and e_3 is normal to N^2_ν in $M^3_r(\rho)$, and denoting by $\{\omega^1, \omega^2, \omega^3\}$ the dual frame of $\{e_1, e_2, e_3\}$, the *Cartan connection forms* are defined by

$$\widetilde{\nabla}_X e_i = \sum_j \widetilde{\varepsilon}_j \omega^j_i(X) e_j, \tag{13}$$

for $i, j \in \{1, 2, 3\}$, where $\widetilde{\varepsilon}_j = \langle e_j, e_j\rangle$ is the causal character of e_j. Then, $\omega^j_i = -\omega^i_j$ and

$$\begin{aligned} h(e_i, e_j) &= \widetilde{\varepsilon}_3 h_{ij} e_3, \\ h_{ij} &= -\left\langle \widetilde{\nabla}_{e_i} e_3, e_j\right\rangle = \omega^3_j(e_i), \end{aligned} \tag{14}$$

$i, j \in \{1, 2\}$ [8]. We will often resort to the standard abuse of notation and identification tricks in submanifold theory.

3. Binormal Evolution Surfaces

The covariant derivative of a vector field X along a curve γ will be denoted by $(DX/ds)(s) := \widetilde{\nabla}_T X(s)$. Let $\gamma(s)$ be a unit speed nongeodesic curve immersed in $M^3_r(\rho)$ with nonnull velocity $(D\gamma/ds)(s) = T(s)$, $\forall s$; therefore, it is assumed to be either spacelike or timelike. If it also has nonnull acceleration $(DT/ds)(s)$, then $\gamma(s)$ is a *Frenet curve* of rank 2 or 3 and the standard *Frenet frame* along $\gamma(s)$ is given by $\{T, N, B\}(s)$, where B is chosen so that $\det(T, N, B) = 1$. Then the *Frenet equations* define the *curvature*, $\kappa(s)$, and *torsion*, $\tau(s)$, along $\gamma(s)$

$$\frac{DT}{ds}(s) = \varepsilon_2 \kappa(s) N(s), \tag{15}$$

$$\frac{DN}{ds}(s) = -\varepsilon_1 \kappa(s) T(s) + \varepsilon_3 \tau(s) B(s), \tag{16}$$

$$\frac{DB}{ds}(s) = -\varepsilon_2 \tau(s) N(s), \tag{17}$$

where ε_i, $1 \le i \le 3$, is the causal character of T, N, and B, respectively. Notice that the following relations hold:

$$T = \varepsilon_1 N \times B,$$
$$N = \varepsilon_2 B \times T, \quad (18)$$
$$B = \varepsilon_3 T \times N.$$

Curves for which both curvature and torsion are constant are called *Frenet helices*. In a semi-Riemannian space form any local geometrical scalar defined along Frenet curves can always be expressed as a function of their curvatures and derivatives.

Given a smooth map $x : U \subset \mathbb{R}^2 \to M_r^3(\rho)$, $x(s,t)$, satisfying (2), we usually identify x and $i \circ x$. Assume that the *initial condition* $\gamma(s) := x(s,0)$ is a unit speed Frenet curve of rank 2 or 3; then $\gamma^t(s) := x(s,t)$, which will be called *the filament at time t*, is also unit speed parametrized $\forall t$. In fact, we have $(\partial/\partial t)\langle x_s(s,t), x_s(s,t)\rangle = 2\langle \widetilde{\nabla}_{x_s} x_t, x_s\rangle = 0$, where the last equality is obtained from (2). So, since $\langle x_s(s,0), x_s(s,0)\rangle = \langle d\gamma/ds, d\gamma/ds\rangle = \langle T,T\rangle = \varepsilon_1$, then so is $\forall t$; that is, (2) is a length-preserving evolution. Assuming also that $\widetilde{\nabla}_{x_s} x_s(s,t)$ is nonnull everywhere, the associated Frenet frame will be defined for all γ^t and combining (2) and (15) we obtain

$$x_t = f(\kappa,\tau)\, x_s \times \widetilde{\nabla}_{x_s} x_s = f(\kappa,\tau)\, T \times \frac{DT}{ds}$$
$$= \varepsilon_2 \kappa f(\kappa,\tau)\, T \times N = \varepsilon_2\varepsilon_3 \kappa f(\kappa,\tau)\, B = \mathscr{F}(\kappa,\tau)\, B. \quad (19)$$

This means that $\gamma(s)$ evolves by the binormal flow with *velocity* $\mathscr{F}(\kappa,\tau)$. We are going to suppose also that f is never zero so that (U,x) defines an immersed surface in $M_r^3(\rho)$ swept out by $\gamma(s)$. It will be denoted by S_γ and called a *binormal evolution surface* (BES) *with initial condition* γ *and velocity* $\mathscr{F}$. The curves $\gamma^s(t) := x(s,t)$, perpendicular to the filaments, are called *fibers* of S_γ. The time variation of the Frenet frames is described in the following proposition (cf. (3.15) of [3] for surfaces in $\mathbb{R}^3$).

Proposition 1. *Let (U,x) be a BES of $M_r^3(\rho)$ with velocity $\mathscr{F}$ (19). Then*

$$\widetilde{\nabla}_{\partial/\partial s}\begin{pmatrix} T \\ N \\ B\end{pmatrix}(s,t) = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix} \varepsilon_1 T \\ \varepsilon_2 N \\ \varepsilon_3 B\end{pmatrix}(s,t), \quad (20)$$

$$\widetilde{\nabla}_{\partial/\partial t}\begin{pmatrix} T \\ N \\ B\end{pmatrix}(s,t)$$
$$= \begin{pmatrix} 0 & -\tau\mathscr{F} & \varepsilon_3\mathscr{F}_s \\ \tau\mathscr{F} & 0 & \varepsilon_2 h_{22}\mathscr{F} \\ -\varepsilon_3\mathscr{F}_s & -\varepsilon_2 h_{22}\mathscr{F} & 0\end{pmatrix}\begin{pmatrix} \varepsilon_1 T \\ \varepsilon_2 N \\ \varepsilon_3 B\end{pmatrix}(s,t), \quad (21)$$

where $h_{22} = (1/\kappa)\{\varepsilon_3(\mathscr{F}_{ss}/\mathscr{F}) - \varepsilon_2\tau^2 + \varepsilon_1\varepsilon_3\rho\}$ and $\kappa(s,t)$ and $\tau(s,t)$ denote the curvature and torsion of the curves $\gamma^t(s)$.

Proof. Under our assumptions, all γ^t are unit speed parametrized and they all have well defined Frenet frame satisfying (15)–(17), so (20) is clear. As for the second part, since $\gamma^t(s) = x(s,t)$ is not a geodesic in $M_r^3(\rho)$ and $\widetilde{\nabla}_{x_s} x_s$ is not null, then, for sufficiently small s, the unit Frenet normal to $\gamma^t(s)$, $N(s,t)$, is parallel to a (local) unit normal to S_γ. This means that $\gamma^t(s)$ are geodesics in S_γ for any t and the parametrization

$$x(s,t) = \gamma^t(s) \quad (22)$$

determines a *geodesic coordinate system* with respect to which the metric $\langle,\rangle \equiv g$ can be written as

$$g = \varepsilon_1 ds^2 + \varepsilon_3 \mathscr{F}^2 dt^2. \quad (23)$$

Now, the Gauss and Weingarten formulas (7) and (8), in combination with the Gauss and Codazzi equations (10) and (11), will give us all the relevant geometric information about the immersion (U,x). This requires bringing in some computational stuff and very long calculations whose details are omitted here. Thus, the Christoffel symbols of the Levi-Civita connection of (23) with respect to the parametrization (22) (see, e.g., [8], Proposition 1.1) can be computed from the metric coefficients g_{ij}. In our case, we have

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = 0,$$
$$\Gamma_{12}^2 = \frac{\mathscr{F}_s}{\mathscr{F}},$$
$$\Gamma_{22}^1 = -\varepsilon_1\varepsilon_3\mathscr{F}\mathscr{F}_s, \quad (24)$$
$$\Gamma_{22}^2 = \frac{\mathscr{F}_t}{\mathscr{F}},$$

where subscripts s and t mean partial derivative with respect to s and t, respectively. This makes it possible to know the expression for the Levi-Civita connection of S_γ ([8], §1.4), denoted here by ∇

$$\nabla_{\partial/\partial s}\frac{\partial}{\partial s} = 0,$$
$$\nabla_{\partial/\partial s}\frac{\partial}{\partial t} = \frac{\mathscr{F}_s}{\mathscr{F}}\frac{\partial}{\partial t}, \quad (25)$$
$$\nabla_{\partial/\partial t}\frac{\partial}{\partial t} = -\varepsilon_1\varepsilon_3\mathscr{F}\mathscr{F}_s\frac{\partial}{\partial s} + \frac{\mathscr{F}_t}{\mathscr{F}}\frac{\partial}{\partial t}.$$

As before, $\{T(s,t), N(s,t), B(s,t)\}$ represent the Frenet frames along $\gamma^t(s)$, and we choose the following local adapted frame on S_γ:

$$e_1 = x_s = T,$$
$$e_2 = \frac{x_t}{\mathscr{F}} = B, \quad (26)$$
$$e_3 = \xi = -\varepsilon_2 N,$$

where ξ is the unit normal to S_γ (locally defined). Then, combining (7), (8), (13), (14), and (15)–(17), one gets

$$\begin{aligned}
&\omega_1^2(e_1) = 0,\\
&\omega_1^2(e_2) = \varepsilon_3 \frac{\mathcal{F}_s}{\mathcal{F}},\\
&\omega_1^3(e_1) = h_{11} = -\varepsilon_2\kappa,\\
&\omega_1^3(e_2) = \omega_2^3(e_1) = h_{12} = \varepsilon_2\tau,\\
&\omega_2^3(e_2) \coloneqq h_{22},
\end{aligned} \tag{27}$$

where $\kappa(s,t)$ and $\tau(s,t)$ denote the curvature and torsion of the curves $\gamma^t(s)$.

The second fundamental form can be considered as a quadratic form given by $h(X) \coloneqq \langle A_\xi X, X\rangle$; therefore, we obtain from (27) that

$$h = -\varepsilon_2\kappa ds^2 + 2\varepsilon_2\tau\mathcal{F} ds dt + \mathcal{F}^2 h_{22} dt^2, \tag{28}$$

with respect to the parametrization (22), where h_{22} is the coefficient of the second fundamental form ([8], §2.3) of S_γ in $M_r^3(\rho)$ given by $h_{22} \coloneqq -\varepsilon_2\langle \widetilde{\nabla}_B B, N\rangle$. Since ∇ is determined by g, h_{22} can be computed with the aid of (25) and the Gauss formula (7) giving

$$h_{22} = \frac{1}{\kappa}\left\{\varepsilon_3 \frac{\mathcal{F}_{ss}}{\mathcal{F}} - \varepsilon_2\tau^2 + \varepsilon_1\varepsilon_3\rho\right\}. \tag{29}$$

Using again the Gauss and Weingarten formulas (7) and (8), it can be shown that $x : U \to M_r^3(\rho) \to \mathbb{E}_\nu^4$ satisfies the following PDE system:

$$x_{ss} = -\frac{\kappa}{\mathcal{F}} x_s \times x_t - \varepsilon_1\rho x, \tag{30}$$

$$x_{ts} = \frac{\mathcal{F}_s}{\mathcal{F}} x_t - \frac{\tau\mathcal{F}}{\kappa} x_{ss} - \varepsilon_1 \frac{\tau\mathcal{F}}{\kappa}\rho x, \tag{31}$$

$$\begin{aligned}
x_{tt} = &-\varepsilon_1\varepsilon_3\mathcal{F}\mathcal{F}_s x_s - \varepsilon_2 \frac{h_{22}\mathcal{F}^2}{\kappa} x_{ss}\\
&- \left(\frac{\varepsilon_1\varepsilon_2 h_{22}}{\kappa} + \varepsilon_3\right)\mathcal{F}^2\rho x + \frac{\mathcal{F}_t}{\mathcal{F}} x_t.
\end{aligned} \tag{32}$$

This system can be expressed equivalently in terms of the time variation of the Frenet frame. In fact, from the Gauss formula (10) and (30), we have

$$\frac{DT}{dt} = \widetilde{\nabla}_{\partial/\partial t} T = -\varepsilon_2\mathcal{F}\tau N + \mathcal{F}_s B. \tag{33}$$

So, differentiating $x_t = \mathcal{F}B$ and using once more Gauss and Weingarten formulas, we have

$$x_{tt} = \mathcal{F}_t B + \mathcal{F}\left(\widetilde{\nabla}_{\partial/\partial t} B - \varepsilon_3\rho\mathcal{F}x\right), \tag{34}$$

which combined with (32) gives

$$\frac{DB}{dt} = \widetilde{\nabla}_{\partial/\partial t} B = -\varepsilon_1\varepsilon_3\mathcal{F}_s T - h_{22}\mathcal{F}N. \tag{35}$$

Finally, from (33), (35), and the cross product relations (18), one has

$$\frac{DN}{dt} = \widetilde{\nabla}_{\partial/\partial t} N = \varepsilon_1\tau\mathcal{F}T + \varepsilon_2\varepsilon_3 h_{22}\mathcal{F}B, \tag{36}$$

which ends the proof. □

Now, combining (9) and Gauss and Codazzi equations (10) and (11) with (13), (25), (26), and (27), we obtain after a long computation

$$\varepsilon_2\varepsilon_3 \frac{\mathcal{F}_{ss}}{\mathcal{F}} = \varepsilon_2\kappa h_{22} + \tau^2 - \varepsilon_1\varepsilon_2\varepsilon_3\rho, \tag{37}$$

$$\kappa_t = -2\mathcal{F}_s\tau - \tau_s\mathcal{F}, \tag{38}$$

$$\tau_t = \varepsilon_1\varepsilon_3\kappa\mathcal{F}_s + \varepsilon_2\left(h_{22}\mathcal{F}\right)_s. \tag{39}$$

Notice that the Gauss equation (37) is equivalent to (29). By substitution of (29) in (39) we get that the Codazzi equations (38) and (39) for (U, x) boil down to

$$\kappa_t = -2\mathcal{F}_s\tau - \tau_s\mathcal{F}, \tag{40}$$

$$\tau_t = \varepsilon_1\varepsilon_3\kappa\mathcal{F}_s + \varepsilon_2\left(\frac{\mathcal{F}}{\kappa}\left(\varepsilon_3\frac{\mathcal{F}_{ss}}{\mathcal{F}} - \varepsilon_2\tau^2 + \varepsilon_1\varepsilon_3\rho\right)\right)_s. \tag{41}$$

Observe that if $\mathcal{F} = \kappa$, (40) and (41) are precisely Da Rios equations for the vortex filament [2]. Moreover, (40) and (41) are the compatibility conditions of the PDE system (30)–(32). Thus, from the fundamental theorem of submanifolds ([8], §2.7), given functions $\kappa(s,t)$, $\tau(s,t)$, and $\mathcal{F}(s,t)$, smoothly defined on a connected domain U and satisfying (40) and (41), there exists a solution of (30)–(32) (and, consequently, of (20)-(21)) determining a smooth isometric immersion (U, x) (unique up to rigid motions, if U is simply connected) of a surface in $M_r^3(\rho)$ whose metric and the second fundamental form are given, respectively, by $g = \varepsilon_1 ds^2 + \varepsilon_3\mathcal{F}^2 dt^2$ and $\varepsilon_2 h = -\kappa ds^2 + 2\tau\mathcal{F} ds dt + \varepsilon_2\mathcal{F}^2 h_{22} dt^2$, where h_{22} is obtained from (29).

Computing the Christoffel symbols from the metric coefficients for such an immersion, (U, x), we see that $\gamma^t(s)$ are arc-length parametrized geodesics $\forall t$. Then, a combination of the Gauss formula (7) and (15)–(17) for the Frenet frame along the coordinate curves $\gamma^t(s)$, $\{x_s = T(s,t), N(s,t), B(s,t)\}$, shows that the unit Frenet normals $N(s,t)$ are perpendicular to the surface (U, x). Hence, $x_t = \lambda(s,t)B(s,t)$, but then the second coefficient of g (23) implies that $\mathcal{F}(s,t) = \lambda(s,t)$ and (U, x) is a solution of (19). Since (U, x) is foliated by geodesics $\gamma^t(s)$ having $\kappa^t(s) \coloneqq \kappa(s,t)$ and $\tau^t(s) \coloneqq \tau(s,t)$ as curvature and torsion, respectively, the immersion itself, x, is geometrically determined by $\kappa^t(s) \coloneqq \kappa(s,t)$ and $\tau^t(s) \coloneqq \tau(s,t)$, because, from the fundamental theorem of curves, for any fixed t, there exists a unique curve $\gamma^t(s)$ (up to congruences and causal character of the Frenet frame) having $\kappa^t(s)$ and $\tau^t(s)$ as curvature and torsion. Then, smooth assembling of these curves $\gamma^t(s)$, $t \in J$, would give x. So, geometrically solving (19) amounts to solving system (40), (41).

Another consequence of the Gauss-Codazzi equations (40) and (41) is that, besides length, other geometric quantities may also be invariant for closed filaments. More precisely, we have the following.

Proposition 2. *Let (U, x) be a binormal evolution surface in $M_r^3(\rho)$ with velocity $\mathcal{F}$ and assume that $s \in [0,1]$ and all filament curves $\gamma^t(s) = x(s,t)$ are C^4-closed in $[0,1]$. If $\mathcal{F}(\kappa,\tau) = \mathcal{F}(\kappa)$, then $\int_0^1 \tau\, ds$ and $\int_0^1 (P(\kappa) + \eta\tau) ds$, with $dP/d\kappa = \mathcal{F}(\kappa)$, $\eta \in \mathbb{R}$, are independent of t. Furthermore, if $\mathcal{F}(\kappa)$ is constant, then also $\int_0^1 \kappa\, ds$ is invariant. Finally, if $\mathcal{F}(\kappa,\tau) = n\kappa + m\tau$, then $\int_o^1 (m\kappa - \varepsilon_1\varepsilon_3 n\tau) ds$ does not depend on t.*

Proof. Only first part is proved since the others are similar. If $\mathcal{F}(\kappa,\tau) = \mathcal{F}(\kappa)$, then, the invariance $\int_0^1 \tau\, ds$ is a direct consequence of $\kappa\mathcal{F}_s = (\kappa\mathcal{F} - P)_s$, because using (41) we have $(d/dt)\int_0^1 \tau\, ds = \int_0^1 \tau_t ds = 0$. Now, from (40)

$$\begin{aligned} \frac{d}{dt}\int_0^1 (P(\kappa) - \eta\tau)\, ds &= \int_0^1 \kappa_t \mathcal{F}\, ds - \eta \int_0^1 \tau_t ds \\ &= \int_0^1 \left(\mathcal{F}^2 \tau\right)_s ds = 0. \end{aligned} \tag{42}$$

□

4. Evolution with Constant Torsion

Now we study binormal evolution surfaces, whose filaments have the same constant torsion. Since $\tau = \tau_o \in \mathbb{R}$, $\mathcal{F}(s,t) = \mathcal{F}(\kappa(s,t),\tau_o) := \widetilde{\mathcal{F}}(\kappa(s,t))$. Choose $P(u)$ so that $\dot{P}(u) := dP/du = \widetilde{\mathcal{F}}(u)$. Assume first $\tau = 0$.

Proposition 3. *Let S_γ be a binormal evolution surface, whose filaments satisfy $\tau = 0$. Then γ is extremal for the energy $\Theta(\gamma) := \int_\gamma (P(\kappa) + \lambda) ds$ and $S_\gamma = \{\phi_t(\gamma),\ t \in \mathbb{R}\}$, where $\{\phi_t,\ t \in \mathbb{R}\}$ is a 1-parameter group of isometries of $M_r^3(\rho)$. Moreover, the fibers of S_γ have constant curvature and zero torsion (if they are not geodesics) in $M_r^3(\rho)$. In particular, if $\rho = 0$, S_γ are either ruled surfaces or rotational surfaces.*

Proof. By substituting $\tau = 0$ in (40) we have $\kappa(s,t) = \kappa(s)$ and the metric with respect to the chosen coordinate system is $g = \varepsilon_1 ds^2 + \varepsilon_3 \widetilde{\mathcal{F}}^2(s) dt^2$. This means that (U, x) is a *warped product* surface [8], and since $\partial g_{ij}/\partial t = 0$, we have that $x_t(s,t) = \widetilde{\mathcal{F}}(s) B(s,t)$ is a Killing field of (U, x). Now, integrating (41) we get

$$0 = \dot{P}_{ss} + \varepsilon_1\varepsilon_2 \dot{P}\left(\kappa^2 + \varepsilon_2\rho\right) - \varepsilon_1\varepsilon_2\kappa(P + \lambda), \tag{43}$$

for some $\lambda \in \mathbb{R}$. Moreover, since $\tau = 0$, we have that (43) is the Euler-Lagrange equation for $\Theta(\gamma)$ ([9, 10]) and γ^t must be an extremal of Θ in $M_r^2(\rho)$, $\forall t$. On the other hand, for a given field along γ, W, the following variation formulas for $v = |\dot{\gamma}|$, κ, and τ can be obtained using standard computations and the Frenet equations (see, [9, 10])

$$\begin{aligned} W(v) &= \varepsilon_1 v \left\langle \widetilde{\nabla}_T W, T \right\rangle, \\ W(\kappa) &= \left\langle \widetilde{\nabla}_T^2 W, N \right\rangle - 2\varepsilon_1\kappa \left\langle \widetilde{\nabla}_T W, T \right\rangle + \varepsilon_1\rho \langle W, N \rangle, \\ W(\tau) &= \varepsilon_2 \left(\frac{1}{\kappa}\left\langle \widetilde{\nabla}_T^2 W + \varepsilon_1\rho W, B \right\rangle\right)_s - \varepsilon_1\tau \left\langle \widetilde{\nabla}_T W, T \right\rangle \\ &\quad + \varepsilon_1\kappa \left\langle \widetilde{\nabla}_T W, B \right\rangle, \end{aligned} \tag{44}$$

where the subscript s denotes differentiation with respect to the arc-length. Now, combining (43), (44), and the Frenet equations, one can see that

$$\mathcal{J}(v) = \mathcal{J}(\kappa) = \mathcal{J}(\tau) = 0, \tag{45}$$

along γ, where $\mathcal{J} = \dot{P}B$. This means that $\mathcal{J} = \dot{P}B$ is a Killing field along γ ([9–11]), but this field is precisely $x_t = \widetilde{\mathcal{F}}B$. Now, (44) imply that the Killing vector fields along a curve γ form a six-dimensional linear space. Moreover, the Lie algebra of $M_r^3(\rho)$ is six-dimensional and the restriction of a Killing vector field in $M_r^3(\rho)$ to any curve γ gives a Killing vector field along γ. Hence, every Killing vector field along a curve, γ, is the restriction to γ of a Killing vector field of $M_r^3(\rho)$ [11]; in other words, x_t can be extended to a Killing field on $M_r^3(\rho)$ (denoted also by $\mathcal{J}$). Hence, the associated 1-parameter group $\{\phi_t,\ t \in \mathbb{R}\}$ is formed by isometries of $M_r^3(\rho)$ and $S_\gamma = x(U)$ is obtained as $S_\gamma = \{\phi_t(\gamma(s)),\ t \in \mathbb{R}\}$, where $\gamma(s) = x(s,0)$. Since $x_s = T(s,t)$, $x_t = \dot{P}(\kappa(s))B(s,t)$, and $\widetilde{\nabla}_{x_s} B(s,t) = -\varepsilon_2\tau N(s,t) = 0$, we get that $B(s,t)$ does not depend on s. Moreover, as fibers are orbits of a Killing field of $M_r^3(\rho)$, they have constant curvature. Now, for any s_o take an arc-length parametrization, $\delta(t)$, of the fiber of S_γ through s_o. With the subscript δ denoting the geometric elements associated with the curve δ, we have $T_\delta(t) = B(t/\dot{P}(\kappa(s_o)))$ and, using the last equation of (21), we obtain

$$\varepsilon_2^\delta \kappa_\delta N_\delta = -\varepsilon_1\varepsilon_3 \frac{\dot{P}_s}{\dot{P}} T - h_{22} N, \tag{46}$$

if δ has nonnull acceleration. Thus, differentiating (46) with respect to t and using again (21), we have that $\delta(t)$ must verify

$$\begin{aligned} \kappa_\delta(s,t) &= \kappa_\delta(s), \\ \kappa_\delta\tau_\delta &= 0, \\ \varepsilon_2^\delta \kappa_\delta^2 &= \varepsilon_1 \frac{\dot{P}_s^2}{\dot{P}^2} + \varepsilon_2 h_{22}^2, \end{aligned} \tag{47}$$

from which we see that either $\kappa_\delta = 0$ and $\delta(t)$ is a geodesic in $M_r^3(\rho)$ or $\kappa_\delta \neq 0$ and $\delta(t)$ is a planar circle.

On the other hand, if δ has null acceleration and is not a geodesic, then we can consider the following frame along γ. Define $N_\delta(t)$ as the lightlike field on δ given by $(DT_\delta/dt)(t) = N_\delta(t)$ and denote by $B_\delta(t)$ the only lightlike vector such that

$\langle N_\delta, B_\delta\rangle = 1$ and $\langle T_\delta, B_\delta\rangle = 0$. In this case, we have the following equations:

$$\frac{DT_\delta}{dt}(t) = N_\delta(t),$$
$$\frac{DN_\delta}{dt}(t) = \tau_\delta(t) N_\delta(t), \quad (48)$$
$$\frac{DN_\delta}{dt}(t) = -T(t) - \tau_\delta(t) B_\delta(t),$$

for certain function $\tau_\delta(t)$ which will be also called the *torsion* of γ (here, the "curvature" is considered to be 1). Then, from the second equation of (21), it is clear that $\tau_\delta(t) = 0$.

Finally, we restrict ourselves to flat ambient spaces $\mathbb{R}^3$ or $\mathbb{L}^3$. For simplicity, we take first $M_r^3(\rho) = \mathbb{R}^3$. If $P(\kappa) = \nu\kappa$, $\nu \in \mathbb{R}$, then any planar curve γ is critical for Θ [12] and S_γ must be a right cylinder shaped on γ. Assume then that $P(\kappa) \neq \nu\kappa$, $\nu \in \mathbb{R}$. Then the Killing field along γ, $\mathcal{I}$, can be written as $\mathcal{I} = \dot{P}B = \lambda_1(\gamma \times V) + \lambda_2 V$, for some $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq 0$, and a constant vector V in $\mathbb{R}^3$. By scalar multiplication of V with the covariant derivative of $\mathcal{I} = \dot{P}B = \lambda_1(\gamma \times V) + \lambda_2 V$ along γ, we obtain $\lambda_2 = 0$ and, then, S_γ is a rotational surface in $\mathbb{R}^3$ with profile curve γ. These facts can be extended to the Minkowski space $\mathbb{L}^3$ by similar arguments. □

Moreover, if $P(\kappa) \neq \nu\kappa$, $\nu \in \mathbb{R}$, it is not difficult to see that, identifying the plane containing γ and $\mathbb{R}^2$, choosing a coordinate system containing V, and using $\mathcal{I} = \dot{P}B$, it is possible to find a coordinate system in $\mathbb{R}^2$ where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_1 = \eta\dot{P}$, for some constant $\eta \in \mathbb{R}$. This also works in the Minkowski space $\mathbb{L}^3$. In particular, take an extremal curve for $\int_\gamma (P(\kappa) + \lambda) ds$ either in $\mathbb{R}^2$ or in $\mathbb{L}^2$ and choose a coordinate system where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_1 = \eta\dot{P} = \eta\widetilde{\mathcal{F}}$, for some constant $\eta \in \mathbb{R}$. Assuming $\eta \neq 0$, then

$$x(s,t) = \left(\eta\widetilde{\mathcal{F}} \sin(t), \eta\widetilde{\mathcal{F}} \cos(t), a(s)\right), \quad \text{if } \varepsilon_1^\delta \varepsilon_2^\delta = 1,$$
$$x(s,t) = \left(\eta\widetilde{\mathcal{F}} \sinh(t), a(s), \eta\widetilde{\mathcal{F}} \cosh(t)\right), \quad \text{if } \varepsilon_1^\delta = -1, \quad (49)$$
$$x(s,t) = \left(\eta\widetilde{\mathcal{F}} \cosh(t), a(s), \eta\widetilde{\mathcal{F}} \sinh(t)\right), \quad \text{if } \varepsilon_2^\delta = -1,$$

where $a(s)$ verifies that $\varepsilon_2^\delta((\eta\widetilde{\mathcal{F}})')^2 + \varepsilon_3^\delta(a'(s))^2 = \varepsilon_1$ are rotation surfaces with planar filaments evolving by (19). If the extremal curve lies in the Minkowski plane $\mathbb{L}^2$, then choosing the same coordinate system as before, one could also construct a surface with planar filaments evolving by (19) by rotating γ around a lightlike axis. In fact, suppose without loss of generality that the lightlike axis is determined by $\partial/\partial x_3 + \partial/\partial x_4$; then, the parametrization of S_γ is given by

$$x(s,t) = \left(\eta\widetilde{F} + \left(a(s) - \eta\widetilde{F}\right)\frac{t^2}{2}, \left(a(s) - \eta\widetilde{F}\right)t, a(s) + \left(a(s) - \eta\widetilde{F}\right)\frac{t^2}{2}\right), \quad (50)$$

where $a(s)$ verifies $((\eta\widetilde{\mathcal{F}})')^2 - (a'(s))^2 = \varepsilon_1$. In this case, fibers are spacelike curves with null acceleration.

We remark that a periodic solution $\kappa(s)$ of (43) does not determine a closed filament necessarily, and closure conditions have to be derived. For instance, if $\kappa(s)$ is a periodic solution of (19) with period ρ and $\gamma(s)$ is the corresponding curve in $\mathbb{R}^2$, then integrating (43) we have $d = \dot{P}_s^2 + (\kappa\dot{P} - P - \lambda)^2$, for some $d \in \mathbb{R}$. Choosing a coordinate system in $\mathbb{R}^2$ where $\gamma = (\gamma_1, \gamma_2)$ and $\gamma_1 = (1/\sqrt{d})\dot{P}$, we see that γ_1 is also periodic. Using the fact that γ is arc-length-parametrized, we have $\gamma_2'(s) = (1/\sqrt{d})(\kappa\dot{P} - P - \lambda)$. Hence, γ closes up, if and only if $\int_o^\rho (\kappa\dot{P} - P - \lambda) = 0$.

Take now $\tau_o \neq 0$. If $\kappa(s,t)$ is also constant, filaments are Frenet helices. The unit binormal B is a Killing vector field on Frenet helices; hence, their evolution under the B-flow satisfies $x_t = B$, so we assume $\kappa(s,t)$ is not constant. Then (40) suggests studying travelling wave solutions of the form $\kappa(s - \varepsilon_1\varepsilon_3\mu t)$, $\mu \in \mathbb{R}$, which implies $\widetilde{\mathcal{F}}(\kappa(s,t)) = (\varepsilon_1\varepsilon_3\mu/2\tau_0)\kappa + \lambda$, for some $\lambda \in \mathbb{R}$. Call $\iota = s - \varepsilon_1\varepsilon_3\mu t$. Then, by substitution in (41) we obtain

$$0 = \frac{\varepsilon_1\varepsilon_2\mu}{2\tau_o}\kappa_u + \frac{\mu}{4\tau_o}\kappa^3 + \frac{(-1)^r \mu}{2\tau_o}\kappa\left(\varepsilon_1\varepsilon_3\rho - \varepsilon_2\tau_o^2\right) + \lambda\left((-1)^r \rho - \tau_o^2\right). \quad (51)$$

Proposition 4. *Assume that $\gamma(\iota)$ is a curve in $M_r^3(\rho)$ with nonconstant curvature and constant torsion $\tau = \tau_o \neq 0$ which is an extremal of the energy $\widetilde{\Theta}(\gamma) = \int_\gamma ((\varepsilon_1\varepsilon_3\mu/4\tau_o)\kappa^2 + \lambda\kappa + \mu\tau + \eta)$, where $\mu \neq 0$ and $\lambda, \eta \in \mathbb{R}$. Then, there exists a 1-parameter group of isometries of $M_r^3(\rho)$, $\{\phi_t,\ t \in \mathbb{R}\}$, such that a suitable parametrization of the surface $S_\gamma := \phi_t(\gamma(\iota))$ is a solution of (19) with $\widetilde{\mathcal{F}}(\kappa(s,t)) = (\varepsilon_1\varepsilon_3\mu/2\tau_0)\kappa + \lambda$.*

Proof. It is easy to verify from (51) that $\kappa(\iota)$ and $\tau = \tau_o$ satisfy the Euler-Lagrange equations for the above energy $\widetilde{\Theta}$ ([9, 10]) for a suitable $\eta \in \mathbb{R}$. Consider $\kappa(\iota)$ a solution of (51) (observe that (51) can be explicitly solved with the aid of Jacobi elliptic functions); then $\kappa(\iota)$ and $\tau = \tau_o$ determine a curve $\gamma(\iota)$ in $M_r^3(\rho)$ which is an extremal for $\widetilde{\Theta}$. Now, the vector field $\mathcal{I}(\iota) := \varepsilon_1\varepsilon_3\mu T(\iota) + \widetilde{\mathcal{F}}(\iota)B(\iota)$ is a Killing field along $\gamma(\iota)$ ([9, 10]). As in previous proposition, $\mathcal{I}$ can be extended to a Killing field on $M_r^3(\rho)$ with 1-parameter group of isometries $\{\phi_t,\ t \in \mathbb{R}\}$. Consider the surface $S_\gamma := y(\iota, t) = \phi_t(\gamma(\iota))$, $t \in \mathbb{R}$. Then, the reparametrization $x(s,t) := y(s - \varepsilon_1\varepsilon_3\mu t, t)$ satisfies $x_t = ((\varepsilon_1\varepsilon_3\mu/2\tau_0)\kappa + \lambda)B$, all filaments having constant torsion $\tau = \tau_o$. □

If $\mu = 0$, then $\kappa(s,t) = \kappa(s)$ and $\widetilde{\mathcal{F}}(\kappa(s)) = \lambda$ and S_γ is flat since, combining (10) and (25), we obtain that the Gaussian curvature of S_γ is given by

$$K := \frac{R(\partial/\partial s, \partial/\partial t, \partial/\partial t, \partial/\partial s)}{g(\partial/\partial s, \partial/\partial s)\, g(\partial/\partial t, \partial/\partial t) - g(\partial/\partial s, \partial/\partial t)^2} = -\varepsilon_1 \frac{\mathcal{F}_{ss}}{\mathcal{F}} = 0. \quad (52)$$

Moreover, (51) implies that $\rho = (-1)^r \tau_o^2$. Thus $M_r^3(\rho)$ has to be either $\mathbb{S}^3(\rho)$ or $\mathbb{H}_1^3(\rho)$. Flat surfaces in $\mathbb{S}^3(\rho)$ can be locally described as the product, with respect to the Lie group structure of $\mathbb{S}^3(\rho)$, of two curves with torsions ρ and $-\rho$, respectively, [13]. In order to construct explicit parametrizations solving (19) in this case, we take the complex plane, $\mathbb{C}$, and consider the maps $\pi_\varepsilon : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $\pi_\varepsilon(z_1, z_2) = (1/2c)(\overline{z}_1 z_1 - \varepsilon \overline{z}_2 z_2, 2 z_2 \overline{z}_1)$, where $z_i \in \mathbb{C}$, $i \in \{1,2\}$, $\overline{z}_i$ denotes complex conjugate, $\varepsilon = \pm 1$, and $c \in \mathbb{R}$. Endow $\mathbb{C}^2$ with the semi-Riemannian metric $\langle z, w \rangle = \mathrm{Real}(z_1 \overline{w}_1 + \varepsilon z_2 \overline{w}_2)$. Then, the restriction of π_ε to the hyperquadrics $\langle z, z \rangle = \varepsilon c^2$, $\varepsilon = \pm 1$, gives two maps which are known as the *standard Hopf mappings*

$$\begin{aligned} \pi_+ &: \mathbb{S}^3\left(\frac{1}{c^2}\right) \longrightarrow \mathbb{S}^2\left(\frac{4}{c^2}\right), \\ \pi_- &: \mathbb{H}_1^3\left(\frac{-1}{c^2}\right) \longrightarrow \mathbb{H}^2\left(\frac{-4}{c^2}\right). \end{aligned} \quad (53)$$

Let γ be a curve in either $\mathbb{S}^2(4/c^2)$ or $\mathbb{H}^2(-4/c^2)$. Then, the complete lift $\mathbf{T}_\gamma^+ = \pi_+^{-1}(\gamma)$ (resp., $\mathbf{T}_\gamma^- = \pi_-^{-1}(\gamma)$) is a Riemannian (resp., Lorentzian) flat (zero Gaussian curvature) surface in $\mathbb{S}^3(1/c^2)$ (resp., in $\mathbb{H}_1^3(-1/c^2)$) which is called the *Hopf cylinder on* γ. The covering maps $\Psi^+ : \mathbb{R}^2 \to \mathbf{T}_\gamma^+$ and $\Psi^- : \mathbb{L}^2 \to \mathbf{T}_\gamma^-$

$$\Psi^\pm(t,s) = e^{it}\overline{\gamma}(s), \quad (54)$$

where $\overline{\gamma}(s)$ denotes a *horizontal lift* of γ, can be used to parametrize $\mathbf{T}_\gamma^\pm$. Assuming without loss of generality $c = 1$, that is, $\rho = \pm 1$, critical curves of $\int_\gamma \kappa$ in $\mathbb{S}^3(1)$ or $\mathbb{H}_1^3(-1)$ are characterized by having torsion $\tau^2 = 1$ [5, 12], and, therefore, they must be horizontal lifts via Hopf maps. Hence, we have the following.

Proposition 5. *Horizontal lifts via the Hopf map $\pi_\pm$ of arbitrary curves γ of $\mathbb{S}^2(4)$ or $\mathbb{H}^2(-4)$ parametrized by (54) evolve under $x_t = B$ by rigid motions and the corresponding binormal evolution surface is a Hopf cylinder of $\mathbb{S}^3(1)$ or $\mathbb{H}_1^3(-1)$ shaped on γ, $\mathbf{T}_\gamma^\pm$.*

Thus, explicit parametrizations of $\mathbf{T}_\gamma^\pm$ are obtained as follows. Take an arbitrary curve $\gamma(s) = (A_1(s), 0, A_2(s), A_3(s))$ in $\mathbb{S}^2(4)$ or $\mathbb{H}^2(-4)$; then horizontal lifts of γ via π_+ or π_- are given by

$$\begin{aligned} \overline{\gamma}(s) = (&M(s)\cos\alpha(s), M(s)\sin\alpha(s), D(s)\cos\alpha(s) \\ &- P(s)\sin\alpha(s), D(s)\sin\alpha(s) + P(s)\cos\alpha(s)), \end{aligned} \quad (55)$$

where

$$\begin{aligned} M(s) &= \frac{\sqrt{2A_1(s)+1}}{\sqrt{2}}, \\ D(s) &= \frac{\sqrt{2}A_2(s)}{\sqrt{2A_1(s)+1}}, \\ P(s) &= \frac{\sqrt{2}A_3(s)}{\sqrt{2A_1(s)+1}}, \\ \alpha(s) &= \pm 2 \int \frac{A_3(s)A_2'(s) - A_2(s)A_3'(s)}{2A_1(s)+1}. \end{aligned} \quad (56)$$

Hence, one uses (54) and (55) to obtain a solution of (19). Notice that if the curve γ is embedded in either $\mathbb{S}^2(4)$ or $\mathbb{H}^2(-4)$, then so is $\mathbf{T}_\gamma^\pm$ in $\mathbb{S}^3(1)$ or $\mathbb{H}_1^3(-1)$ and we have binormal Hopf cylinders with no self-intersections. Moreover, if γ is a closed curve then $\mathbf{T}_\gamma^\pm$ is a closed surface (a flat Hopf Tori) but the evolving filament $\overline{\gamma}(s)$ is not closed because of the nontrivial holonomy. However, if, in addition, the area enclosed by $\gamma(s)$ in either $\mathbb{S}^2(4)$ or $\mathbb{H}^2(-4)$ is a rational multiple of π, then there are $m \in \mathbb{Z}$ such that the horizontal lift of an m-cover of $\gamma(s)$ is a closed filament [12].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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