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A new Einstein-nonlinear electrodynamics solution in $2+1$ dimensions

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Abstract We introduce a class of solutions in $2+1$-dimensional Einstein–Power–Maxwell theory for a circularly symmetric electric field. The electromagnetic field is considered with an angular component given by $F_{\mu\nu} = E_0\delta^t_\mu\delta^\theta_\nu$ for $E_0 =$ constant. First, we show that the metric for zero cosmological constant and the Power–Maxwell Lagrangian of the form of $\sqrt{|F_{\mu\nu}F^{\mu\nu}|}$ coincides with the solution given in $2+1$-dimensional gravity coupled with a massless, self-interacting real scalar field. With the same Lagrangian and a non-zero cosmological constant we obtain a non-asymptotically flat wormhole solution in $2+1$ dimensions. The confining motions of massive charged and chargeless particles are investigated too. Secondly, another interesting solution is given for zero cosmological constant together with the conformal invariant condition. The formation of a timelike naked singularity for this particular case is investigated within the framework of the quantum mechanics. Quantum fields obeying the Klein–Gordon and Dirac equations are used to probe the singularity and test the quantum mechanical status of the singularity.

1 Introduction

There have always been benefits in studying lower-dimensional field theoretical spacetimes such as $2+1$ dimensions in general relativity. This is believed to be the projection of higher-dimensional cases to the more tractable situations that may inherit the physics of the intricate higher dimensions. In recent decades one proved that the cases of lower dimensions are still far from being easily understandable and this in fact entails its own characteristics. The absence of a gravitational degree of freedom such as the Weyl tensor or pure gravitational waves necessitates endowment of physical sources to fill the blank and create its own curvatures. Among these the most popular addition has been a negative cosmological constant, which affects anti-de Sitter spacetimes to the extent that it makes possible even black holes [1–3]. The addition of electromagnetic [4] and scalar fields [5–8] also are potential candidates to be considered in the same context. Beside minimally coupled massless scalar fields, which have little significance to add, non-minimally self-coupled scalar fields have also been considered. In particular, the real, radial, self-interacting scalar field with a Liouville potential among others seems promising [8]. The distinctive feature of the source in such a study is that the radial pressure turns out to be the only non-zero (i.e., $T^r_r \neq 0$) component of the energy-momentum tensor [8]. In effect, such a radial pressure turns out to make a naked singularity but not a black hole. Being motivated by the self-interacting scalar field in $2+1$-dimensional gravity we attempt in this paper to study similar physics with a nonlinear electromagnetic field, which naturally adds its own nonlinearity. Our choice for the Lagrangian in nonlinear electrodynamics (NED) is the square root of the Maxwell invariant, i.e. $\sqrt{|F_{\mu\nu}F^{\mu\nu}|}$, where, as usual, the field tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian belongs to the class of NED with k-power law Maxwell invariant $|F_{\mu\nu}F^{\mu\nu}|^k$ [9–22]. A similar Lagrangian in $3+1$ dimensions with magnetic field source has been considered in [23–25], and with an electric field as well as both electric and magnetic fields in [26]. A combination of a linear Maxwell invariant with the square-root term has been investigated in [27–34].

Such a Lagrangian naturally breaks scale invariance, i.e. $x_\mu \to \lambda x_\mu$, $A_\mu \to \frac{1}{\lambda}A_\mu$ for $\lambda =$ constant, even in $3+1$ dimensions, so that interesting results are expected to ensue. Let us add that such a choice of NED has the feature that it does not attain the familiar linear Maxwell limit unless $k=1$. One physical consequence beside others of the square-root Maxwell Lagrangian is that it gives rise to confinement for

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geodesic particles. For a general discussion of confinement in general relativistic field theory the reader may consult [27–34].

Contrary to the previous considerations [8] in this study our electric field is not radial; instead our field tensor is expressed in the form $F_{\mu\nu} = E_0 \delta^t_\mu \delta^\theta_\nu$ for E_0 = constant. This amounts to the choice for the electromagnetic vector potential $A_\mu = E_0 (a_0\theta, 0, b_0 t)$, where our spacetime coordinates are labeled as $x^\mu = \{t, r, \theta\}$ and the constants a_0 and b_0 satisfy $a_0 + b_0 = 1$. The particular choice $a_0 = 0$, $b_0 = 1$ leaves us with the vector potential $A_\mu = \delta^\theta_\mu E_0 t$, which yields a uniform field in the angular direction. The only non-vanishing energy-momentum tensor component is T^r_r, which accounts for the radial pressure. Our ansatz electromagnetic field in the circularly symmetric static metric gives a solution with zero cosmological constant that is identical with the spacetime obtained from an entirely different source, namely the self-interacting real scalar field [8]. This is a conformally flat anti-de Sitter solution in 2 + 1 dimensions without formation of a black hole. The uniform electric field being self-interacting is strong enough to make a naked singularity at the circular center. Another solution for a non-zero cosmological constant can be interpreted as a non-asymptotically flat wormhole solution. In analogy with the case of 3 + 1 dimensions [27–34] we search for possible particle confinement in this 2 + 1-dimensional model with $\Lambda = 0$. Truly the geodesics for both neutral and charged particles are confined.

In this paper, in addition to providing a new solution in NED theory, we investigate the resulting spacetime structure for a specific value of $k = 3/4$, which arises by imposing the tracelessness condition on the Maxwell energy-momentum, which is known to satisfy the conformal invariance condition. In this particular case the character of the singularity is timelike. For specific values of the parameters, the timelike character of the naked singularity at $r = 0$ is also encountered in [40], in which a radial electric field is assumed within the context of NED with a power parameter $k = 3/4$. This singularity is investigated within the framework of quantum mechanics in [41]. Therein a timelike naked singularity is probed with quantum fields obeying the Klein–Gordon and Dirac equations.

We investigate the timelike naked singularity, developed at $r = 0$, for the new solution, which incorporates a power parameter $k = 3/4$, from the quantum mechanical point of view. The main motivation to study the singularity is to clarify whether the uniform electric field in the angular direction has an effect on the resolution of this singularity or not. In order to compare the present study with the results obtained in [41], the singularity will be probed with two different quantum waves having spin structures 0 and 1/2, namely, bosonic waves and fermionic waves, respectively. The result of this investigation is that, with respect to the bosonic wave probe, the singularity remains quantum singular, whereas with respect to the fermionic wave probe the singularity is shown to be healed.

The organization of the paper is as follows. In Sect. 2 we introduce our formalism and derive the field equations. New solutions for $k = \frac{1}{2}$ are presented in Sect. 3 as naked singular / wormhole and we discuss its geodesic confining properties. Section 4 considers the case with $k = \frac{1}{2}$ further. The paper ends with our Conclusion in Sect. 5.

2 Field equations and the new solution

We start with the following action for the Einstein theory of gravity coupled with a NED Lagrangian:

$$I = \frac{1}{2} \int \mathrm{d}x^3 \sqrt{-g} \left(R - 2\Lambda + \alpha \left|\mathcal{F}\right|^k \right). \tag{1}$$

Here $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, α is a real coupling constant, k is a rational number, and Λ is the cosmological constant. Our line element is circularly symmetric. It is given by

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2, \tag{2}$$

where $A(r)$ and $B(r)$ are unknown functions of r and $0 \leq \theta \leq 2\pi$. Also we choose the field ansatz as

$$\mathbf{F} = E_0 \mathrm{d}t \wedge \mathrm{d}\theta \tag{3}$$

in which E_0 =constant is a uniform electric field and its dual can be found as ${}^\star\mathbf{F} = \frac{E_0}{r}\sqrt{\frac{B}{A}}\mathrm{d}r$. Naturally, the integral of ${}^\star\mathbf{F}$ gives the total charge. This electric field is derived from an electric potential one-form given by

$$\mathbf{A} = E_0 (a_0 t \mathrm{d}\theta - b_0 \theta \mathrm{d}t) \tag{4}$$

in which a_0 and b_0 are constants satisfying $a_0 + b_0 = 1$. The nonlinear Maxwell equation reads

$$d\left({}^\star\mathbf{F} \frac{|\mathcal{F}|^k}{\mathcal{F}} \right) = 0, \tag{5}$$

which upon the substitution

$$\mathcal{F} = 2F_{t\theta} F^{t\theta} = \frac{-2E_0^2}{A(r)\, r^2} \tag{6}$$

is trivially satisfied. Next, the Einstein–NED equations are given by

$$G^\nu_\mu + \frac{1}{3}\Lambda \delta^\nu_\mu = T^\nu_\mu \tag{7}$$

in which

$$T_{\mu}^{\nu}=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k}\left(\delta_{\mu}^{\nu}-\frac{4k\left(F_{\mu\lambda}F^{\nu\lambda}\right)}{\mathcal{F}}\right). \tag{8}$$

With $\mathcal{F}$ known one finds

$$T_{t}^{t}=T_{\theta}^{\theta}=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k}(1-2k) \tag{9}$$

and

$$T_{r}^{r}=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k} \tag{10}$$

as the only non-vanishing energy-momentum components. To proceed, we must have the exact form of the Einstein tensor components given by

$$G_{t}^{t}=\frac{B'}{2r} \tag{11}$$

$$G_{r}^{r}=\frac{BA'}{2rA} \tag{12}$$

and

$$G_{\theta}^{\theta}=\frac{2A''AB-A'^{2}B+A'B'A}{4A^{2}}, \tag{13}$$

in which a prime means $\frac{\mathrm{d}}{\mathrm{d}r}$. The field equations then read as follows:

$$\frac{B'}{2r}+\frac{1}{3}\Lambda=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k}(1-2k), \tag{14}$$

$$\frac{BA'}{2rA}+\frac{1}{3}\Lambda=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k}, \tag{15}$$

and

$$\frac{2A''AB-A'^{2}B+A'B'A}{4A^{2}}+\frac{1}{3}\Lambda=\frac{\alpha}{2}\left|\mathcal{F}\right|^{k}(1-2k). \tag{16}$$

3 An exact solution for $k=\frac{1}{2}$

Among the values for k that may have most interest is $k=\frac{1}{2}$. In this specific case $T_{t}^{t}=T_{\theta}^{\theta}=0$ and $T_{r}^{r}=\frac{\alpha}{2}\sqrt{|\mathcal{F}|}$ is the only non-zero component of the energy-momentum tensor. The field equations admit the general solutions for $A(r)$ and $B(r)$ given by

$$B(r)=D-\frac{\Lambda}{3}r^{2} \tag{17}$$

and

$$A(r)=\left(D-\frac{\Lambda}{3}r^{2}\right)\left(C+\frac{r\alpha\left|E_{0}\right|}{D\sqrt{2}\sqrt{D-\frac{\Lambda}{3}r^{2}}}\right)^{2}, \tag{18}$$

in which D and C are two integration constants. One observes that setting $E_0=0$ gives the correct limit of a BTZ black hole up to a constant C^2, which can be absorbed in time. The other interesting limit of the above solution is found when we set $C=0$, which yields

$$A(r)=\left(\frac{r\alpha\left|E_{0}\right|}{D\sqrt{2}}\right)^{2}. \tag{19}$$

The reduced line element, therefore, becomes

$$\mathrm{d}s^{2}=-\left(\frac{\alpha\left|E_{0}\right|}{D\sqrt{2}}\right)^{2}r^{2}\mathrm{d}t^{2}+\frac{1}{D-\frac{\Lambda}{3}r^{2}}\mathrm{d}r^{2}+r^{2}\mathrm{d}\theta^{2} \tag{20}$$

which for three different cases admits different geometries.

(i) $\Lambda>0$: When the cosmological constant is positive the solution becomes non-physical for $r^2>\frac{3D}{\Lambda}$. For $D<0$ the signature of the spacetime is openly violated.

(ii) $\Lambda<0$: In this case the solution is a black string for $D>0$ whose Kretschmann scalar is given by

$$\mathcal{K}=\frac{4\left(r^{4}\Lambda^{2}-2\Lambda r^{2}D+3D^{2}\right)}{3r^{4}}, \tag{21}$$

with the singularity at the origin which is also the horizon. However, for $D<0$ with negative cosmological constant the solution becomes a wormhole with the throat located at $r=r_0=\sqrt{\frac{3|D|}{|\Lambda|}}$ and

$$\mathrm{d}s^{2}=-\left(\frac{\alpha\left|E_{0}\right|}{D\sqrt{2}}\right)^{2}r^{2}\mathrm{d}t^{2}+\frac{1}{|D|\left(\frac{r^{2}}{r_{0}^{2}}-1\right)}\mathrm{d}r^{2}+r^{2}\mathrm{d}\theta^{2}. \tag{22}$$

In order to conceive the geometry of the wormhole in this case we introduce a new coordinate, $z=z(r)$, such that

$$\frac{\mathrm{d}r^{2}}{|D|\left(\frac{r^{2}}{r_{0}^{2}}-1\right)}=\mathrm{d}r^{2}+\mathrm{d}z^{2} \tag{23}$$

where

$$z(r)=\pm\int\left(\sqrt{\frac{1}{|D|\left(\frac{r^{2}}{r_{0}^{2}}-1\right)}-1}\right)\mathrm{d}r. \tag{24}$$

It should be added that the ranges of r must satisfy

$$r_0<r. \tag{25}$$

To cast our spacetime into the standard wormhole metric we express it as

$$\mathrm{d}s^2 = -e^{2f}\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1-\frac{b(r)}{r}\right)} + r^2\mathrm{d}\theta^2. \tag{26}$$

Here $f(r) \sim \ln r$ and $b(r) = r(1 + |D| - \frac{|D|}{r_0^2}r^2)$ are known as the redshift and shape functions, respectively. The throat of our wormhole is at r_0 where $b(r_0) = r_0$, and the flare-out condition (i.e., $b'(r_0) < 1$) is satisfied by the choice of our parameters. We have also $\frac{b(r)}{r} < 1$ for $r > r_0$. We must add that, distinct from an asymptotically flat wormhole, here we have a range for r, given in (25).

(iii) $\Lambda = 0$: for the case when the cosmological constant is zero one finds (only $D > 0$ is physical)

$$\mathrm{d}s^2 = -\left(\frac{\alpha\,|E_0|}{D\sqrt{2}}\right)^2 r^2\mathrm{d}t^2 + \frac{1}{D}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{27}$$

which after a simple rescaling of time and setting $D = 1$ leads to

$$\mathrm{d}s^2 = -r^2\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2. \tag{28}$$

We notice that although our solution is not a standard black hole; there still exists a horizon at $r = 0$, which makes our solution a black point [35–38]. In [35] such black points appeared in $3+1$-dimensional gravity coupled to the logarithmic $U(1)$ gauge theory and in [36–38] coupled to charged dilatonic fields. This is the conformally flat $2+1$-dimensional line element, and through the transformation $r = e^R$, which entails

$$\mathrm{d}s^2 = e^{2R}\left(-\mathrm{d}\tilde{t}^2 + \mathrm{d}R^2 + \mathrm{d}\tilde{\theta}^2\right), \tag{29}$$

it is obtained also in the self-interacting scalar field model [8].

3.1 Geodesic motion for $\Lambda = 0$

3.1.1 Chargeless particle

To know more about the solution found above one may study the geodesic motion of a massive particle (timelike). The Lagrangian of the motion of a unit mass particle within the spacetime (28) is given by (for simplicity we remove tildes over the coordinates)

$$L = -\frac{1}{2}r^2\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 \tag{30}$$

where a 'dot' denotes the derivative $\frac{\mathrm{d}}{\mathrm{d}s}$ with s an affine parameter. The conserved quantities are

$$\frac{\partial L}{\partial \dot{t}} = -r^2\dot{t} = -\alpha_0 \tag{31}$$

$$\frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} = \beta_0 \tag{32}$$

with α_0 and β_0 as constants of the energy and angular momentum. The metric condition reads

$$-1 = -r^2\dot{t}^2 + \dot{r}^2 + r^2\dot{\theta}^2, \tag{33}$$

which upon using (31) and (32) yields

$$\dot{r}^2 = \left(\frac{\alpha_0^2 - \beta_0^2}{r^2} - 1\right). \tag{34}$$

This equation clearly shows a confinement in the motion for the particle geodesics in the form

$$r^2 \leq \alpha_0^2 - \beta_0^2. \tag{35}$$

Considering the affine parameter as the proper distance one finds, from (41),

$$r = \sqrt{\alpha_0^2 - \beta_0^2 - (s - s_0)^2}, \tag{36}$$

leading to manifest confinement.

3.1.2 Charged particle geodesics

For a massive charged particle with unit mass and charge q_0 the Lagrangian is given by

$$L = -\frac{1}{2}r^2\dot{t}^2 + \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + q_0 A_\mu \dot{x}^\mu, \tag{37}$$

in which $A_\mu\dot{x}^\mu = A_\theta\dot{x}^\theta = E_0 t\dot{\theta}$ i.e. the choice $a_0 = 1$, $b_0 = 0$ in Eq. (4). The metric condition is as in Eq. (33) and therefore the Lagrange equations yield

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\dot{\theta} + q_0 E_0 t\right) = 0, \tag{38}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(r^2\dot{t}\right) = -q_0 E_0\dot{\theta}, \tag{39}$$

and

$$\ddot{r} = -r\dot{t}^2 + r\dot{\theta}^2. \tag{40}$$

The first equation implies

$$r^2\dot{\theta} + q_0 E_0 t = \gamma_0 = \text{const}. \tag{41}$$

while the second equation, with a change of variable, $r^2 \frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}z}$, and imposing (41), yields

$$\frac{\mathrm{d}^2 t}{\mathrm{d}z^2} = -q_0 E_0 (\gamma_0 - q_0 E_0 t) . \tag{42}$$

This equation admits an exact solution for $t(z)$

$$t(z) = \frac{\gamma_0}{\omega} + C_1 e^{\omega z} + C_2 e^{-\omega z}, \tag{43}$$

in which C_1 and C_2 are two integration constants and $\omega = q_0 E_0$. Next, the radial equation upon imposing the metric condition (33) becomes decoupled as

$$r\ddot{r} + \dot{r}^2 + 1 = 0. \tag{44}$$

The general solution for this equation is given by

$$r = \pm\sqrt{\tilde{C}_1 + 2\tilde{C}_2 s - s^2}, \tag{45}$$

where we consider the positive root. Once more, one finds

$$r^2 \frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}z}, \tag{46}$$

which in turn becomes

$$\left(\tilde{C}_1 + 2\tilde{C}_2 s - s^2\right) \frac{\mathrm{d}t}{\mathrm{d}s} = \omega \left(C_1 e^{\omega z} - C_2 e^{-\omega z}\right) . \tag{47}$$

To proceed further we set $\tilde{C}_2 = 0$, $\tilde{C}_1 = b_0^2$, $C_2 = 0$, and $C_1 = 1$ so that

$$\left(b_0^2 - s^2\right) \frac{\mathrm{d}t}{\mathrm{d}s} = \omega \left(t - \frac{\gamma_0}{\omega}\right) . \tag{48}$$

One may note that with this choice of integration constants $b_0^2 - s^2 \geq 0$. This leads to

$$\left| t - \frac{\beta_0}{\omega} \right| = \left(\frac{b_0 + s}{b_0 - s}\right)^{\frac{\omega}{2b_0}} + \zeta, \tag{49}$$

in which ζ is an integration constant to be set to zero for simplicity. From the latter relation one finds

$$r(t) = \frac{2b_0 \left| t - \frac{\gamma_0}{\omega} \right|^{\frac{b_0}{\omega}}}{\left| t - \frac{\gamma_0}{\omega} \right|^{\frac{2b_0}{\omega}} + 1}, \tag{50}$$

which clearly shows the confinement of the motion for the charged particles. This conclusion could also be obtained from Eq. (45), which implies $\tilde{C}_1 + 2\tilde{C}_2 s - s^2 \geq 0$ and consequently $r \leq \sqrt{\tilde{C}_1 + \tilde{C}_2^2}$. The angular variable $\theta(t)$ can also be reduced to an integral expression.

4 A brief account for a conformally invariant Maxwell source with $\Lambda = 0$

In the first paper of Hassaïne and Martínez [9–22] it was shown that the action given in (1) is conformally invariant if $k = \frac{3}{4}$. In this section we set $k = \frac{3}{4}$ and $\Lambda = 0$ so that the following solution is obtained for the field equations (14)–(16):

$$A(r) = \left(1 + \frac{\alpha \left(E_0^2\right)^{3/4}}{2^{1/4}} \sqrt{r}\right)^4 \tag{51}$$

and

$$B(r) = \frac{1}{\left(1 + \frac{\alpha \left(E_0^2\right)^{3/4}}{2^{1/4}} \sqrt{r}\right)^2}. \tag{52}$$

To find these solutions we have to set the integration constant in such a way that the flat limit easily comes after one imposes $E_0 = 0$. The Kretschmann scalar of the spacetime is given by

$$K = \frac{3 \left|E_0\right|^3 \alpha^2 \left(r\alpha^2 \left|E_0\right|^3 + 2\sqrt[4]{2} \left(E_0^2\right)^{3/4} \sqrt{r}\alpha + \sqrt{2}\right)}{r^3 \left(1 + \frac{\alpha \left(E_0^2\right)^{3/4}}{2^{1/4}} \sqrt{r}\right)^8}. \tag{53}$$

As one can see from the action (1), $\alpha < 0$ implies a ghost field which is not physical. Therefore we assume $\alpha > 0$ and consequently the only singularity of the spacetime is located at the origin $r = 0$. For $\alpha > 0$ the solution represents a non-black hole and non-asymptotically flat spacetime with a naked singularity at $r = 0$. In order to find the nature of the singularity at $r = 0$, we perform a conformal compactification. The conformal radial / tortoise coordinate,

$$r_* = \int \frac{\mathrm{d}r}{1 + a\sqrt{r}} = \frac{2}{a^2} \left\{a\sqrt{r} - \ln\left|1 + a\sqrt{r}\right|\right\}, \tag{54}$$

with $a = \frac{\alpha \left(E_0^2\right)^{3/4}}{2^{1/4}}$ helps us to introduce the retarded and advanced coordinates, i.e. $u = t - r_*$ and $v = t + r_*$. The Kruskal coordinates are defined, using the coordinates u and v, by

$$u' = -e^{a^2 u}, \quad v' = e^{-a^2 v} \tag{55}$$

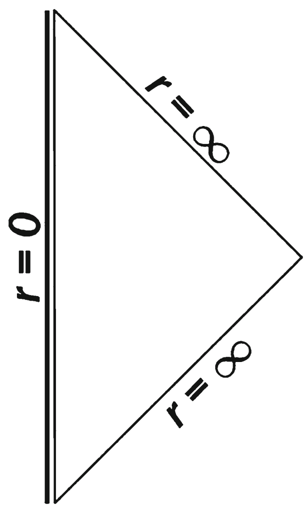

Fig. 1 Carter–Penrose diagram for $k = \frac{3}{4}$ and $\Lambda = 0$, i.e. Eqs. (51) and (52). We see that $r = 0$ is a naked timelike singularity

and consequently the line element in u' and v' coordinates becomes

$$ds^2 = -\frac{e^{4a\sqrt{r}}}{a^4}du^{'}dv^{'} + r^2 d\theta^2, \tag{56}$$

with the constraint

$$u^{'}v^{'} = -e^{-4a\sqrt{r}}\left(1 + a\sqrt{r}\right)^4. \tag{57}$$

The final change of the coordinate

$$\begin{aligned} u^{''} &= \arctan u^{'}, \quad 0 < u^{''} < \pi/2, \\ v^{''} &= \arctan v^{'}, \quad 0 < v^{''} < \pi/2 \end{aligned} \tag{58}$$

brings infinity into a finite coordinate. In Fig. 1 we plot Carter–Penrose diagrams of the solutions (51) and (52). One observes in this diagram that the singularity located at $r = 0$ is timelike. The corresponding energy-momentum tensor

$$T_{\mu}^{\nu} = -\frac{1}{2}\alpha\left(\frac{2E_0^2}{A(r)r^2}\right)^{3/4}(1, -1, 1) \tag{59}$$

implies that

$$\rho = -T_t^t = \frac{1}{2}\alpha\left(\frac{2E_0^2}{A(r)r^2}\right)^{3/4}, \tag{60}$$

$$p = T_r^r = \rho = \frac{1}{2}\alpha\left(\frac{2E_0^2}{A(r)r^2}\right)^{3/4}, \tag{61}$$

$$q = T_\theta^\theta = -\rho = -\frac{1}{2}\alpha\left(\frac{2E_0^2}{A(r)r^2}\right)^{3/4}, \tag{62}$$

and therefore all energy conditions, including the weak, strong, and dominant versions, are satisfied.

5 Singularity analysis

5.1 Quantum singularities

One of the important predictions of the Einstein theory of relativity is the formation of the spacetime singularities in which the evolution of timelike or null geodesics is not defined after a proper time. The deterministic nature of general relativity requires that the spacetime singularities must be hidden by horizon(s), as conjectured by Penrose's weak cosmic censorship hypothesis (CCH). However, there are some cases where the spacetime singularity is not covered by horizon(s), and then it is called a naked singularity. Hence, naked singularities violates the CCH and their resolution becomes extremely important. The most powerful candidate theory in resolving the singularities is the quantum theory of gravity. However, there is no consistent quantum theory of gravity yet. String theory [42,43] and loop quantum gravity [44] are the two fields of study for the resolution of singularities. Yet another method that we shall employ in this paper is the criterion proposed by Horowitz and Marolf (HM) [45], which incorporates "self-adjointness" of the spatial part of the wave operator. Invoking this criterion, the classical notion of *geodesics incompleteness* with respect to the point-particle probe will be replaced by the notion of the *quantum singularity* with respect to the wave probe. This criterion can be applied only to static spacetimes having timelike singularities. To understand better what is going on, let us consider the Klein–Gordon equation for a free particle that satisfies $i\frac{d\psi}{dt} = \sqrt{\mathcal{A}_E}\psi$, whose solution is $\psi(t) = \exp\left[-it\sqrt{\mathcal{A}_E}\right]\psi(0)$ in which $\mathcal{A}_E$ denotes the extension of the spatial part of the wave operator.

If $\mathcal{A}$ is not essentially self-adjoint, in other words, if $\mathcal{A}$ has an extension, the future time evolution of the wave function $\psi(t)$ is ambiguous. Then the HM criterion defines the spacetime as quantum mechanically singular. However, if there is only a single self-adjoint extension, the operator $\mathcal{A}$ is said to be essentially self-adjoint and the quantum evolution described by $\psi(t)$ is uniquely determined by the initial conditions. According to the HM criterion, this spacetime is said to be quantum mechanically non-singular. The essential self-adjointness of the operator $\mathcal{A}$ can be verified by considering solutions of the equation

$$\mathcal{A}^*\psi \pm i\psi = 0 \tag{63}$$

and showing that the solutions of Eq. (63) do not belong to the Hilbert space $\mathcal{H}$ (we refer to the references; [46] for detailed mathematical analysis and [47–60] for applications of the HM approach in different spacetimes). This will be achieved by defining the function space on each $t =$constant hypersurface Σ as $\mathcal{H} = \{R | \, \|R\| < \infty\}$ with the following norm given for the metric (2):

$$\|R\|^2 = \int\limits_0^{\text{constant}} \frac{r}{\sqrt{A(r)B(r)}} |R|^2 \, \mathrm{d}r. \tag{64}$$

5.1.1 Klein–Gordon fields

The Klein–Gordon equation for the metric (2) can be written by splitting temporal and spatial parts and it is given by

$$\frac{\partial^2 \psi}{\partial t^2} = -\mathcal{A}\psi, \tag{65}$$

in which $\mathcal{A}$ denotes the spatial operator of the massless scalar wave given by

$$\mathcal{A} = -\left(1 + a\sqrt{r}\right)^2 \frac{\partial^2}{\partial r^2} - \frac{\left(1 + a\sqrt{r}\right)\left(1 + \frac{3a\sqrt{r}}{2}\right)}{r} \frac{\partial}{\partial r} - \frac{\left(1 + a\sqrt{r}\right)^4}{r^2} \frac{\partial^2}{\partial \theta^2}. \tag{66}$$

Applying a separation of variables, $\psi = R(r)Y(\theta)$, the radial part of Eq. (63) becomes

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{\left(1 + \frac{3a\sqrt{r}}{2}\right)}{r\left(1 + a\sqrt{r}\right)} \frac{\partial R(r)}{\partial r} + \left(\frac{c\left(1 + a\sqrt{r}\right)^2}{r^2} \pm \frac{i}{\left(1 + a\sqrt{r}\right)}\right) R(r) = 0, \tag{67}$$

where c stands for the separation constant. The spatial operator $\mathcal{A}$ is essentially self-adjoint if neither of the two solutions of Eq. (67) is square integrable over all space $L^2(0, \infty)$. Because of the complexity in finding an exact analytic solution to Eq. (67), we study the behavior of $R\,(r)$ near $r \to 0$ and $r \to \infty$. The behavior of Eq. (67) near $r = 0$ is given by

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \frac{c}{r^2} R(r) = 0, \tag{68}$$

whose solution is

$$R\,(r) = C_1 \sin\left(\sqrt{c}\ln(r)\right) + C_2 \cos\left(\sqrt{c}\ln(r)\right). \tag{69}$$

The square integrability is checked by calculating the norm given in Eq. (64). Calculation has shown that $R\,(r)$ is square integrable near $r = 0$, and, hence, it belongs to the Hilbert space and the operator $\mathcal{A}$ is not essentially self-adjoint. As a result, the timelike naked singularity remains quantum mechanically singular with respect to the spinless wave probe. This result is in conformance with the analysis in [41]. This result seems to show that irrespective of the direction of the electric field, the singularity remains quantum singular with respect to waves obeying the Klein–Gordon equation.

5.1.2 Dirac fields

The Dirac equation in $2+1$-dimensional curved spacetime for a free particle with mass m is given by

$$i\sigma^\mu(x)\left[\partial_\mu - \Gamma_\mu(x)\right]\Psi(x) = m\Psi(x), \tag{70}$$

where $\Gamma_\mu(x)$ is the spinorial affine connection given by

$$\Gamma_\mu(x) = \frac{1}{4} g_{\lambda\alpha}\left[e^{(i)}_{\nu,\mu}(x) e^{\alpha}_{(i)}(x) - \Gamma^{\alpha}_{\nu\mu}(x)\right] s^{\lambda\nu}(x), \tag{71}$$

with

$$s^{\lambda\nu}(x) = \frac{1}{2}\left[\sigma^\lambda(x), \sigma^\nu(x)\right]. \tag{72}$$

Since the fermions have only one spin polarization in $2+1$ dimensions [61], the Dirac matrices $\gamma^{(j)}$ can be expressed in terms of the Pauli spin matrices $\sigma^{(i)}$ [62] so that

$$\gamma^{(j)} = \left(\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}\right), \tag{73}$$

where the Latin indices represent an internal (local) frame. In this way,

$$\left\{\gamma^{(i)}, \gamma^{(j)}\right\} = 2\eta^{(ij)} I_{2\times 2}, \tag{74}$$

where $\eta^{(ij)}$ is the Minkowski metric in $2+1$ dimension and $I_{2\times 2}$ is the identity matrix. The coordinate dependent metric tensor $g_{\mu\nu}(x)$ and the matrices $\sigma^\mu(x)$ are related to the triads $e^{(i)}_\mu(x)$ by

$$\begin{aligned} g_{\mu\nu}(x) &= e^{(i)}_\mu(x)\, e^{(j)}_\nu(x)\, \eta_{(ij)}, \\ \sigma^\mu(x) &= e^\mu_{(i)} \gamma^{(i)}, \end{aligned} \tag{75}$$

where μ and ν stand for the external (global) indices. The suitable triads for the metric are given by

$$e^{(i)}_\mu(t, r, \theta) = \operatorname{diag}\left(\left(1 + a\sqrt{r}\right)^2, \left(1 + a\sqrt{r}\right), r\right). \tag{76}$$

The coordinate dependent gamma matrices and the spinorial affine connection are given by

$$\begin{aligned} \sigma^\mu(x) &= \left(\sigma^{(3)}\left(1+a\sqrt{r}\right)^{-2}, i\left(1+a\sqrt{r}\right)^{-1}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \\ \Gamma_\mu(x) &= \left(\frac{a\sigma^{(2)}}{2\sqrt{r}}, 0, 0\right). \end{aligned} \tag{77}$$

Now, for the spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{78}$$

the Dirac equation can be written as

$$\frac{i}{\left(1+a\sqrt{r}\right)^2}\frac{\partial\psi_1}{\partial t}-\frac{ai}{2\sqrt{r}\left(1+a\sqrt{r}\right)^2}\psi_2 -\frac{1}{\left(1+a\sqrt{r}\right)}\frac{\partial\psi_2}{\partial r}+\frac{1}{r}\frac{\partial\psi_2}{\partial\theta}-m\psi_1=0 \tag{79}$$

$$\frac{-i}{\left(1+a\sqrt{r}\right)^2}\frac{\partial\psi_2}{\partial t}-\frac{ai}{2\sqrt{r}\left(1+a\sqrt{r}\right)^2}\psi_1 -\frac{1}{\left(1+a\sqrt{r}\right)}\frac{\partial\psi_1}{\partial r}-\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}-m\psi_2=0. \tag{80}$$

The following ansatz will be employed for the positive frequency solutions:

$$\Psi_{n,E}\left(t,x\right)=\begin{pmatrix}R_{1n}(r)\\ R_{2n}(r)e^{i\theta}\end{pmatrix}e^{in\theta}e^{-iEt}. \tag{81}$$

The radial part of the Dirac equation becomes

$$\frac{\partial R_{2n}(r)}{\partial r}-e^{-i\theta}\left(\frac{E}{1+a\sqrt{r}}-m\left(1+a\sqrt{r}\right)\right)R_{1n}(r) -i\left[\frac{(n+1)\left(1+a\sqrt{r}\right)}{r}-\frac{c}{2\sqrt{r}\left(1+a\sqrt{r}\right)}\right]R_{2n}(r)=0, \tag{82}$$

$$\frac{\partial R_{1n}(r)}{\partial r}-e^{i\theta}\left(\frac{1}{1+a\sqrt{r}}-m\left(1+a\sqrt{r}\right)\right)R_{2n}(r) +i\left[\frac{n\left(1+a\sqrt{r}\right)}{r}+\frac{c}{2\sqrt{r}\left(1+a\sqrt{r}\right)}\right]R_{1n}(r)=0. \tag{83}$$

The behavior of the Dirac equations near $r=0$ reduces to

$$\frac{\partial^2 R_{2n}(r)}{\partial r^2}+\frac{i}{r}\frac{\partial R_{2n}(r)}{\partial r}+\left\{\frac{n+1}{r^2}\left(n+i\right)-\beta\right\}R_{2n}(r)=0, \tag{84}$$

$$\frac{\partial^2 R_{1n}(r)}{\partial r^2}+\frac{i}{r}\frac{\partial R_{1n}(r)}{\partial r}+\left\{\frac{n}{r^2}\left(n+1-i\right)-\beta\right\}R_{1n}(r)=0, \tag{85}$$

in which $\beta=m^2-m(E+1)+E$. The two Eqs. (84) and (85) must be investigated for essential self-adjointness by using Eq. (63). Hence, we have

$$\frac{\partial^2 R_{2n}(r)}{\partial r^2}+\frac{i}{r}\frac{\partial R_{2n}(r)}{\partial r}+\left\{\frac{n+1}{r^2}\left(n+i\right)-\beta\pm i\right\}R_{2n}(r)=0, \tag{86}$$

$$\frac{\partial^2 R_{1n}(r)}{\partial r^2}+\frac{i}{r}\frac{\partial R_{1n}(r)}{\partial r}+\left\{\frac{n}{r^2}\left(n+1-i\right)-\beta\pm i\right\}R_{1n}(r)=0. \tag{87}$$

Since we are looking for a solution near $r=0$, the constant terms inside the curly brackets can be ignored and the solutions are given by

$$R_{2n}(r)=r^{\frac{1-I}{2}}\left[C_{1n}r^{\frac{\sqrt{-2I-4\chi_1}}{2}}+C_{2n}r^{\frac{-\sqrt{-2I-4\chi_1}}{2}}\right], \tag{88}$$

$$R_{1n}(r)=r^{\frac{1-I}{2}}\left[C_{3n}r^{\frac{\sqrt{-2I-4\chi_2}}{2}}+C_{4n}r^{\frac{-\sqrt{-2I-4\chi_2}}{2}}\right], \tag{89}$$

in which the C_{in} are arbitrary integration constants

$$\chi_1=(n+1)\left(n+i\right), \tag{90}$$

and

$$\chi_2=n\left(n+1-i\right). \tag{91}$$

The square integrability of these solutions is checked by using the norm defined in Eq. (64). Based on the numerical calculation, both $R_{1n}(r)$ and $R_{2n}(r)$ are square integrable near $r=0$. As a result, the arbitrary wave packet can be written as

$$\Psi\left(t,x\right)=\sum_{n=-\infty}^{+\infty}\begin{pmatrix}R_{1n}(r)\\ R_{2n}(r)e^{i\theta}\end{pmatrix}e^{in\theta}e^{-iEt}, \tag{92}$$

and the initial condition $\Psi\left(0,x\right)$ is enough to determine the time evolution of the wave. Hence, the initial value problem is well-posed and the spacetime remains non-singular when probed with spinorial waves obeying the Dirac equation.

6 Conclusion

We considered a specific form of NED Lagrangian in the form of a power law Maxwell invariant $\left|F_{\mu\nu}F^{\mu\nu}\right|^k$ with $k=\frac{1}{2}$. It is well known that a pure radial electric field with $k=\frac{1}{2}$ does not satisfy the energy conditions [39] so our field ansatz has been chosen differently to be a uniform angular electric field. One direct feature of this form of field ansatz is that the only non-zero component of the energy-momentum is T_r^r. This indeed means that the energy density $\rho=-T_t^t$ and the angular pressure $p_\theta=T_\theta^\theta$ are both zero, while the radial pressure is $p_r=T_r^r=\frac{\xi}{r}$ with $\xi=\sqrt{\frac{\sqrt{2}\alpha E_0}{8}}$. Because of these features the weak energy conditions (WECs) are satisfied, i.e., $\rho\geq 0$, $\rho+p_r\geq 0$ and $\rho+p_\theta\geq 0$. Even more, the strong energy conditions (SECs) are also satisfied; these are WECs together with $\rho+p_r+p_\theta\geq 0$. Having a radial pressure that is non-zero and a divergence at $r=0$ are features that can be seen from the nature of the resulting spacetime. For $k=\frac{1}{2}$ we give two different classes of solutions, for $\Lambda=0$ and $\Lambda\neq 0$. The one for $\Lambda=0$ is not a black-hole solution but is singular at $r=0$, since the spacetime invariants are $R \sim \frac{1}{r^2}$, $R_{\mu\nu}R^{\mu\nu} \sim \frac{1}{r^4}$ and the Kretschmann scalar is $\sim \frac{1}{r^4}$. This singularity is of the same order of divergence as the charged BTZ black hole with a radial singular electric field. As has been found in this work, the singularity at the origin confines the radial motion of a massive particle (both charged and uncharged). ($r = 0$ is also a zero for g_{tt}, which makes our solution a black point.) This confinement means that the particle cannot go beyond a maximum radius. In the work of Schmidt and Singleton [8] where a different matter source, namely a self-interacting scalar field, is employed, the same solution has been found. This is perhaps an indication that the geometry found in the context of a real scalar field, sharing a common metric with different sources in $2 + 1$ dimensions, may apply to higher-dimensional spacetimes. The case for $\Lambda < 0$ ($k = \frac{1}{2}$) corresponds to a non-asymptotically flat wormhole built entirely from the cosmological constant. The conformally invariant case ($k = \frac{3}{4}$) for the Maxwell field has also been considered briefly. Our timelike null singularity turns out to be quantum regular against the Dirac probe and singular against the Klein–Gordon probe.

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