

UR-1278

ER-40685-730

hep-th/9209129

THREE DIMENSIONAL QUANTUM CHROMODYNAMICS†

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ABSTRACT

The subject of this talk was the review of our study of three $(2 + 1)$ dimensional Quantum Chromodynamics. In our previous works, we showed the existence of a phase where parity is unbroken and the flavor group $U(2n)$ is broken to a subgroup $U(n) \times U(n)$. We derived the low energy effective action for the theory and showed that it has solitonic excitations with Fermi statistic, to be identified with the three dimensional “baryon”. Finally, we studied the current algebra for this effective action and we found a co-homologically non trivial generalization of Kac-Moody algebras to three dimensions.

† This work was supported in part by DOE Grant DE-FG02-91ER40685.

‡ Talk presented by S.G. Rajeev at the XXVI International Conference on High Energy Physics, Dallas TX Aug. 1992

INTRODUCTION

Quantum Chromodynamics (QCD) is the universally accepted theory of strong interactions. In spite of this fact, there are still many unresolved issues that need to be addressed before we can declare our understanding of QCD complete. Among the least understood problems are those that cannot be studied by perturbation theory, such as chiral symmetry breaking and quark confinement. It is therefore natural to look for other models that retain the basic features of QCD but allow one to study these issues in a simpler setting. One way to construct such models is to lower the dimensionality of the system.

Two $(1 + 1)$ dimensional QCD has been extensively studied but it fails to be a good analogue for some purposes: gauge symmetries in two dimensions are somewhat trivial and no spontaneous breaking of continuous symmetries can occur. We chose to consider three dimensional QCD and to study the breaking of the global symmetries in this context. We find a remarkable similarity between this model and what is known about four dimensional QCD from the study of the Skyrme model. Working in lower dimensions we have the extra bonus of obtaining an effective theory that is more tractable. This theory can be regarded as a limiting case of coset models that are renormalizable in the $1/N$ expansions. Furthermore, its current algebra is a nontrivial abelian extension of the naive algebra and it might give rise to interesting representation theory.

THREE DIMENSIONAL QCD

Let us begin by writing down the Lagrangian density for three dimensional QCD. The gauge group is $SU(N_c)$, the tensor $F_{\mu\nu}$ is the curvature associated to the gauge field and q_i represent the quark fields. The flavor index i runs from 1 to N and we will always assume N to be even ($N = 2n$). Color and spinor indices are suppressed but it should be kept in mind that, as spinor, q_i is a two component complex Grassmann field.

In this notation, the Lagrangian reads

$$L = -\frac{1}{\alpha} \text{tr} F_{\mu\nu} F^{\mu\nu} + \sum_i \bar{q}_i (\gamma \cdot \nabla + m_i) q_i \quad (1)$$

In the massless limit, this Lagrangian possess a global $Z_2 \times U(2n)$ symmetry. Since the number of flavor is even, it is possible to make all quarks massive without breaking parity

explicitly. This is done by pairing them into doublets (q_i, q_{-i}) of equal and opposite masses, $m_i = -m_{-i}$, $i = 1 \cdots n$.

Our two main results regarding the spontaneous breaking of the global symmetries are the following. First, parity is not spontaneously broken if N is even. No Chern-Simons term for the color gauge field can arise and the theory should be in the confining phase. The proof of this statement is an adaptation of the argument given by Vafa and Witten for four dimensional QCD². It relies on the possibility of defining a positive fermionic measure for the path integral. For completeness, let us remark that this argument fails when the number of flavors is odd³, indicating that the theory is in a parity violating phase and therefore it is not a good analogue for QCD.

Second, in the massless limit, the flavor symmetry $U(2n)$ is spontaneously broken to $U(n) \times U(n)$. By arguments similar to those given by Coleman and Witten⁴, one can show that if the symmetry is broken, then it must break to this specific subgroup. One is then left with the task of finding at least one correlation function of the theory that breaks the $U(2n)$ symmetry in the limit of vanishing quark masses. This function can be chosen to be the two point correlation function of the flavor currents:

$$\langle J_\mu^i(k) J_\nu^k(-k) \rangle = \frac{iN_c}{4\pi} \epsilon^{ik} \epsilon_{\mu\nu\rho} k^\rho + O(k^2). \quad (2)$$

In the massless limit, the leading term of this correlation function is proportional to $\epsilon^{ik} = \pm\delta_{ik}$, the plus sign being present for $i = k > 0$, the minus sign being present in the opposite case. The coefficient in (2) can be calculated exactly from some no-renormalization theorems⁵.

EFFECTIVE LAGRANGIAN

Having understood the symmetry breaking pattern, we can now write down the effective Lagrangian¹ for the light particles of the theory, i.e. the Goldstone bosons “pions” associated to the broken generators. These are described by a field Φ valued in the Grassmannian manifold $\text{Gr}_{2n,n} = U(2n)/U(n) \times U(n)$. Earlier, a different coset model with a non trivial topological term had been proposed by Rabinovici et al.⁶, where the target manifold was not the complex Grassmannian but the projective quaternionic space.

By the standard properties of the Grassmannian, Φ can be regarded as a $2n \times 2n$

traceless hermitian matrix satisfying the condition $\Phi^2 = 1$. The odd dimensional analogue of the Wess-Zumino-Witten term (WZW) can also be found by studying the generators of $H^4(\text{Gr}_{2n,n})$. As in the four dimensional case, the WZW term cannot be written as the integral of a local density but one can regard space-time M_3 as the boundary of a four dimensional manifold with boundary M_4 .

The form of the effective action is, in differential geometry notation,

$$S = \frac{F_\pi}{2} \int_{M_3} d\Phi * d\Phi + \frac{N_c}{64\pi} \int_{M_4} \text{tr} \Phi (d\Phi)^4. \quad (3)$$

The constant F_π has the dimension of a mass in natural units and can be regarded as the ‘‘pion decay constant’’. The particular form for the WZW term (second term in eq. (3)), has been chosen so that the equation of motion derived from it possess the same discrete symmetry as three dimensional QCD. In particular, the equation of motion must violate both internal $((t, x, y) \rightarrow (t, -x, y))$ and external $(\Phi \rightarrow -\Phi)$ parity but it must preserve the combination of both. The coefficient $N_c/64\pi$ in front of the WZW term is fixed by requiring $\exp(-S)$ to be independent on the extension to M_4 and by comparison with (2).

One can relax the assumption that led to eq. (3) and include the vector mesons as dynamical degrees of freedom into the Lagrangian. It turns out that this is the correct way to study the solitonic excitations (‘‘baryons’’) of the theory. The vector mesons provide the short range repulsion needed to stabilize the soliton.

In our theory, a Chern-Simons term for these vector mesons arises from the WZW term. This is the Chern-Simons term for the unbroken subgroup $U(n) \times U(n)$, not to be confused with the (unexisting) color Chern-Simons term. Each $U(n)$ factor in the unbroken subgroup gives rise to its own Chern-Simons term. The relative coefficients of the two terms must be chosen to be equal and opposite in order to preserve parity as a symmetry of the effective theory. By writing $\Phi = \chi\epsilon\chi^\dagger$, $\chi \in U(2n)$, ϵ as in (2) and denoting by A_1 and A_2 the vector fields associated to each $U(n)$, the new effective action becomes

$$\tilde{S} = \frac{F_\pi}{2} \int \text{tr} (\nabla_\mu \chi^\dagger \nabla^\mu \chi + m_\pi^2 \epsilon \chi \epsilon \chi^\dagger) dx^3 + \frac{k}{4\pi} \sum_{i=1}^2 (-1)^i \int \text{tr} (A_i dA_i + \frac{2}{3} A_i^3) \quad (4)$$

The two models (3) and (4) coincide in the limit of infinite vector meson mass.

BARYONS AS SOLITONS

The “baryons ” of three dimensional QCD are described by the static solitonic solutions of eq (4) with winding number one⁷. (Recall that $\pi_2(\text{Gr}_{2n,n}) = Z$, the baryon number B is always identified with the winding number.) The low lying baryons transform under flavor symmetry just as predicted by the quark model. The only important difference is in the size of the baryon. The size predicted by the sigma model is larger by a factor $\log(F_\pi/N_c m_\pi)$ than the prediction from the naive quark model. This seems to indicate that three is the lowest critical dimension for the applicability of the Skyrme model.

There are also higher baryon number solutions. Particularly pleasing is the existence of a cylindrically symmetric $B = 2$ “di-baryon” solution. The existence of these solutions, in particular the stability of the di-baryon solution has been proven by numerical methods, since the equation of motion that arises for the cylindrically symmetric ansatz are not exactly solvable. By use of relaxation methods we have obtained the radial profile for the baryon density in both the $B = 1$ and the $B = 2$ case⁷.

CURRENT ALGEBRA

The effective action (3) admits an interesting current algebra. The canonical formulation of the WZW model yields the Kac-Moody algebra in two dimension but fails to give a Lie algebra in four. In our intermediate case we find that the current algebra (more precisely the current-field algebra) is still a Lie algebra⁸. Care must be taken in choosing the correct expression for the current J in a way that makes the Poisson brackets linear in Φ and J . If the two dimensional space-like surface Σ is a torus, we can write these relations in a plane wave basis:

$$\begin{aligned} \{\Phi_m^a, \Phi_n^b\} &= 0, & \{J_m^a, \Phi_n^b\} &= f^{abc} \Phi_{m+n}^c \\ \{J_m^a, J_n^b\} &= f^{abc} J_{m+n}^c - \frac{k}{16\pi} d^{abc} \epsilon^{ij} m_i n_j \Phi_{m+n}^c \end{aligned} \tag{5}$$

Above, m, n are two-dimensional vectors with integer components. Also, d^{abc} is the usual symmetric cubic invariant of $U(2n)$ and f^{abc} the structure constants.

The canonical formulation is completed by two first class constraints

$$\Phi^2 - 1 = 0 \quad [J, \Phi]_+ + \frac{k}{16\pi} \epsilon^{ij} (\partial_i \Phi \partial_j \Phi) = 0, \tag{6}$$

and by the Hamiltonian function

$$H = \frac{1}{2} \int_{\Sigma} \text{tr} \left(-\frac{1}{F_{\pi}\sqrt{g}} \left(J + \frac{k}{32\pi} \epsilon^{ij} \partial_i \Phi \partial_j \Phi \right)^2 + \frac{F_{\pi}\sqrt{g}}{4} g^{ij} \partial_i \Phi \partial_j \Phi \right) d^2x. \quad (7)$$

It should be stressed that the quantities J and Φ are invariant under the action of “gauge transformations” generated by the constraints. The constraints themselves express the fact that the co-adjoint orbit of this algebra is the cotangent bundle of the Grassmannian.

We could quantize the action (3) if we could find a unitary, highest weight representation for this algebra. At first sight this seems rather unphysical because (3) is not perturbatively renormalizable. However, (3) belongs to a class of models that are renormalizable in the $1/N$ expansion. This is more clear if we consider a less symmetrical Grassmannian, where Φ takes values in $\text{Gr}_{N,n}U(N)/U(N-n) \times U(n)$. For $n = 1$ this is just the usual CP^{N-1} model, known to be renormalizable in the large N limit. For $n > 1$, this model is still renormalizable in the limit $N \rightarrow \infty$, n finite. Its current algebra is still given by (5) and everything is left unchanged, except that now $\text{tr} \Phi = N - 2n \neq 0$. What fails in this case is the connection with three dimensional QCD because for these “asymmetrical” models it is not possible to preserve parity. We expect these models to yield a good quantum theory and this would be very exciting by itself. The original model on $\text{Gr}_{2n,n}$ can probably be approached as limiting case.

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