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Research Article

The Equivalence of Datko and Lyapunov Properties for (h, k)-Trichotomic Linear Discrete-Time Systems

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The aim of this paper is to characterize a general property of (h, k)-trichotomy through some Lyapunov functions for linear discrete-time systems in infinite dimensional spaces. Also, we apply the results to illustrate necessary and sufficient conditions for nonuniform exponential trichotomy and nonuniform polynomial trichotomy.

1. Introduction

In the last few years an important development has been made in the field of the asymptotic behaviors of dynamical systems. Among the most important asymptotic behaviors studied, we mention the properties of stability, dichotomy, and trichotomy (see [1–14] and the references therein).

A remarkable characterization for the stability property of continuous dynamical systems was proved by Datko in 1972 (see [8]) and later, Przyłuski and Rolewicz obtain in [15] a similar result for discrete-time systems. This was a starting point for the development of the area and the results were extended to the dichotomy case in [16, 17].

An important generalization of the dichotomy concept (approached in various manners in [2, 3, 6, 18]) is the notion of trichotomy, the most complex asymptotic property of dynamical systems. The trichotomy supposes the splitting of the state space, at any moment, into three subspaces: the stable subspace, the unstable subspace, and the central subspace.

The concept of (exponential) trichotomy was introduced by Elaydi and Hajek (see [9, 10]) for nonlinear differential equations and later, the case of difference equations is treated by Elaydi and Janglajew in [11]. Also, important contributions on the line of trichotomy in discrete-time are due to Cuevas and Vidal [7], López-Fenner and Pinto [13], Megan and Stoica [19, 20], Papaschinopoulos [21], and Popa et al. [22].

In [23], A. L. Sasu and B. Sasu propose an interesting technique for exponential trichotomy of difference equations, the admissibility technique, and in [24] the authors obtain for the first time nonlinear conditions for the exponential trichotomy in infinite dimensional spaces.

The Lyapunov functions represent an important tool in the study of the asymptotic properties of dynamical systems (see, e.g., [4, 5, 25, 26]).

The objective of this paper is to approach the general concept of (h, k)-trichotomy, where h and k are growth rates, for linear discrete-time systems in Banach spaces and as particular cases we deduce the results for (nonuniform) exponential trichotomy and (nonuniform) polynomial trichotomy.

Also, we obtain necessary and sufficient conditions for a general concept of (h, k)-trichotomy (called (h, k)-trichotomy of Datko type) and the main result is the characterization of this concept of trichotomy in terms of Lyapunov functions.

The results are applied to illustrate criteria through the Lyapunov functions for nonuniform exponential trichotomy and nonuniform polynomial trichotomy.

2. Growth Rates

Definition 1. An increasing sequence $h : \mathbb{N} \to [1, +\infty)$, $h(n) = h_n$ is called a *growth rate* if $\lim_{n\to\infty} h_n = +\infty$.

Definition 2. One says that the growth rate (h_n) *satisfies hypothesis* $(\mathscr{H})$ if there exist a growth rate (f_n) and $M \in (1, +\infty)$ such that

(H_1)

$$\sum_{n=0}^{+\infty} \frac{f_n}{h_n} \le M; \tag{1}$$

(H_2)

$$\sum_{j=n}^{m-1} \frac{h_j}{f_j} \le M \frac{h_m^2}{f_m^2}, \tag{2}$$

for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, $m > n$.

Now, we present some examples of growth rates which satisfy hypothesis $(\mathscr{H})$.

Example 3. Let $h_n = e^{n\alpha}$, $\alpha > 0$, and $f_n = e^{n\beta}$; $\beta \in (0, \alpha)$ is a growth rate with

(H_1)

$$\sum_{n=0}^{+\infty} \frac{f_n}{h_n} = \sum_{n=0}^{+\infty} e^{n(\beta-\alpha)} = \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}} = M; \tag{3}$$

(H_2)

$$\begin{aligned} \sum_{j=n}^{m-1} \frac{h_j}{f_j} &= \sum_{j=n}^{m-1} e^{j(\alpha-\beta)} = \frac{e^{(\alpha-\beta)m} - e^{n(\alpha-\beta)}}{e^{\alpha-\beta} - 1} \\ &\le \frac{e^{\alpha}}{e^{\alpha} - e^{\beta}} e^{2m(\alpha-\beta)} = M \frac{h_m^2}{f_m^2}, \end{aligned} \tag{4}$$

for all $(m, n) \in \mathbb{N}^2$, $m > n$.

Example 4. If $h_n = (n+1)^{\alpha}$ with $\alpha > 1$, then $f_n = (n+1)^{\beta-1}$ with $\beta \in (1, \alpha)$ is a growth rate which satisfies the following:

(H_1)

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{f_n}{h_n} &= \sum_{n=0}^{+\infty} (n+1)^{\beta-1-\alpha} = \sum_{n=0}^{+\infty} \frac{1}{(n+1)^{\alpha-\beta+1}} \\ &= M \in (1, +\infty); \end{aligned} \tag{5}$$

(H_2)

$$\begin{aligned} \sum_{j=n}^{m-1} \frac{h_j}{f_j} &= \sum_{j=n}^{m-1} (j+1)^{\alpha-\beta+1} \le (m-n)(m+1)^{\alpha-\beta+1} \\ &\le (m+1)^{2(\alpha-\beta+1)} = \frac{h_m^2}{f_m^2} \le M \frac{h_m^2}{f_m^2}, \end{aligned} \tag{6}$$

for all $(m, n) \in \mathbb{N}^2$, $m > n$.

Example 5. Let $h_n = (n+1)^{\alpha} e^{n\gamma}$ with $\alpha > 1$ and $\gamma > 0$. Then $f_n = (n+1)^{\beta-1} e^{\delta n}$ with $\beta \in (1, \alpha)$, $\delta \in (0, \gamma)$ is a growth rate with the following properties:

(H_1)

$$\sum_{n=0}^{+\infty} \frac{f_n}{h_n} = \sum_{n=0}^{+\infty} \frac{e^{n(\delta-\gamma)}}{(n+1)^{\alpha-\beta+1}} \le \sum_{n=0}^{+\infty} e^{n(\delta-\gamma)} = \frac{e^{\gamma}}{e^{\gamma} - e^{\delta}} = M; \tag{7}$$

(H_2)

$$\begin{aligned} \sum_{j=n}^{m-1} \frac{h_j}{f_j} &= \sum_{j=n}^{m-1} (j+1)^{\alpha-\beta+1} e^{j(\gamma-\delta)} \\ &\le (m-n)(m+1)^{\alpha-\beta+1} e^{(\gamma-\delta)m} \\ &\le (m+1)^{2(\alpha-\beta+1)} e^{2(\gamma-\delta)m} = \frac{h_m^2}{f_m^2} \le M \frac{h_m^2}{f_m^2}, \end{aligned} \tag{8}$$

for all $(m, n) \in \mathbb{N}^2$, $m > n$.

3. (h, k)-Trichotomy

Let X be a real or complex Banach space and $\mathscr{B}(X)$ the Banach algebra of all bounded linear operators on X. I represents the identity operator on X and the norms on X and on $\mathscr{B}(X)$ will be denoted by $\|\cdot\|$. Also,

$$\Delta = \left\{(m, n) \in \mathbb{N}^2 : m \ge n\right\}, \tag{9}$$

where $\mathbb{N}$ is the set of nonnegative integers.

We consider the linear discrete-time system

$(\mathscr{A})$

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{N}, \tag{10}$$

with $A : \mathbb{N} \to \mathscr{B}(X)$, $A(n) = A_n$.

Every solution of $(\mathscr{A})$ is given by

$$x_m = A_m^n x_n, \tag{11}$$

for all $(m, n) \in \Delta$, where

$$A_m^n := \begin{cases} A_{m-1}, \ldots, A_n, & \text{if } m > n \\ I, & \text{if } m = n. \end{cases} \tag{12}$$

Remark 6. We observe that

$$A_m^n A_n^p = A_m^p, \tag{13}$$

for all $(m, n), (n, p) \in \Delta$.

Definition 7. A sequence $P : \mathbb{N} \to \mathscr{B}(X)$, $P(n) = P_n$ is called *a projections sequence on* X if

$$P_n^2 = P_n, \tag{14}$$

for all $n \in \mathbb{N}$.

Definition 8. One says that $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ is a family of

(i) *supplementary projections sequences* if

(s_1)

$$P_n^1 + P_n^2 + P_n^3 = I, \quad \forall n \in \mathbb{N}; \tag{15}$$

(s_2)

$$P_n^i P_n^j = 0, \quad \forall i \neq j,\ i, j \in \{1, 2, 3\}; \tag{16}$$

(ii) *invariant projections sequences* for $(\mathcal{A})$ if

$$P_{n+1}^i A_n = A_n P_n^i, \quad \forall n \in \mathbb{N},\ i \in \{1, 2, 3\}. \tag{17}$$

In what follows, we consider $h, k : \mathbb{N} \to [1, +\infty)$ two growth rates and a pair $(\mathcal{A}, \mathcal{P})$, where $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ is a family of supplementary and invariant projections sequences for $(\mathcal{A})$.

Definition 9. The pair $(\mathcal{A}, \mathcal{P})$ is called (h, k)*-trichotomic* if there exists a nondecreasing sequence (s_n), $s_n \geq 1$, such that

(ht_1)

$$h_m \left\| A_m^n P_n^1 x \right\| \leq h_n s_n \left\| P_n^1 x \right\|; \tag{18}$$

(ht_2)

$$h_m \left\| P_n^2 x \right\| \leq h_n s_m \left\| A_m^n P_n^2 x \right\|; \tag{19}$$

(kt_3)

$$k_n \left\| A_m^n P_n^3 x \right\| \leq k_m s_n \left\| P_n^3 x \right\|; \tag{20}$$

(kt_4)

$$k_n \left\| P_n^3 x \right\| \leq k_m s_m \left\| A_m^n P_n^3 x \right\|, \tag{21}$$

for all $(m, n, x) \in \Delta \times X$.

In the particular case when (s_n) is a constant sequence, $(\mathcal{A}, \mathcal{P})$ is called *uniformly* (h, k)*-trichotomic*.

As particular cases of (h, k)-trichotomy we remark the following:

(i) if $h_n = e^{n\alpha}$, $k_n = e^{n\beta}$ with $\alpha, \beta > 0$ we obtain the concept of *(nonuniform) exponential trichotomy* and if (s_n) is constant it results in the property of *uniform exponential trichotomy*;

(ii) if $h_n = (n+1)^{\alpha}$, $k_n = (n+1)^{\beta}$ with $\alpha, \beta > 1$ we recover the concept of *(nonuniform) polynomial trichotomy* and if (s_n) is constant it results in the property of *uniform polynomial trichotomy*;

(iii) if $P_n^3 = 0$ for all $n \in \mathbb{N}$ it results in the notion of *h-dichotomy*, nonuniform exponential dichotomy (for $h_n = e^{n\alpha}$, $\alpha > 0$), uniform exponential dichotomy (for $h_n = e^{n\alpha}$, $\alpha > 0$, and (s_n) constant), nonuniform polynomial dichotomy (for $h_n = (n+1)^{\alpha}$, $\alpha > 1$), and uniform polynomial dichotomy (for $h_n = (n+1)^{\alpha}$, $\alpha > 1$, and (s_n) constant).

We give a general example of a pair $(\mathcal{A}, \mathcal{P})$ which is (h, k)-trichotomic.

Example 10. Let (h_n) and (k_n) be two growth rates and (s_n) a nondecreasing sequence of positive real numbers, $s_n \geq 1$.

Let $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ be a family of supplementary projections sequences with

$$P_{n+1}^i P_n^i = P_n^i, \quad \forall n \in \mathbb{N},\ i = 1, 2, 3. \tag{22}$$

Linear discrete-time system $(\mathcal{A})$, defined by

$$A_n = \frac{h_n s_n}{h_{n+1} s_{n+1}} P_n^1 + \frac{h_{n+1} s_{n+1}}{h_n s_n} P_n^2 + \frac{k_{n+1} s_n}{k_n s_{n+1}} P_n^3, \tag{23}$$

verifies the relation

$$P_{n+1}^i A_n = A_n P_n^i, \quad i = 1, 2, 3. \tag{24}$$

Then

$$A_m^n = \frac{h_n s_n}{h_m s_m} P_n^1 + \frac{h_m s_m}{h_n s_n} P_n^2 + \frac{k_m s_n}{k_n s_m} P_n^3, \tag{25}$$

for all $(m, n) \in \Delta$.

For all $(m, n, x) \in \Delta \times X$ the following properties hold:

(i)

$$h_m \left\| A_m^n P_n^1 x \right\| = h_n \frac{s_n}{s_m} \left\| P_n^1 x \right\| \leq h_n s_n \left\| P_n^1 x \right\|; \tag{26}$$

(ii)

$$h_m \left\| P_n^2 x \right\| \leq h_m \frac{s_m^2}{s_n} \left\| P_n^2 x \right\| = h_n s_m \left\| A_m^n P_n^2 x \right\|; \tag{27}$$

(iii)

$$k_n \left\| A_m^n P_n^3 x \right\| = k_m \frac{s_n}{s_m} \left\| P_n^3 x \right\| \leq k_m s_n \left\| P_n^3 x \right\|; \tag{28}$$

(iv)

$$k_n \left\| P_n^3 x \right\| \leq \frac{k_m^2}{k_n} s_n \left\| P_n^3 x \right\| = k_m s_m \left\| A_m^n P_n^3 x \right\| \tag{29}$$

and we deduce that the pair $(\mathcal{A}, \mathcal{P})$ is (h, k)-trichotomic.

Remark 11. It is obvious that if the pair $(\mathcal{A}, \mathcal{P})$ is uniformly (h, k)-trichotomic then it is also (h, k)-trichotomic. In the following example we show that the converse implication is not valid.

Example 12. We consider $\mathcal{P} = \{P_n^1, P_n^2, P_n^3\}$ a family of supplementary projections sequences with the property

$$P_{n+1}^i P_n^i = P_n^i, \quad \forall n \in \mathbb{N},\ i = 1, 2, 3. \tag{30}$$

Linear discrete-time system $(\mathcal{A})$ is given by

$$A_n = \frac{u_n}{u_{n+1}} P_n^1 + \frac{u_{n+1}}{u_n} P_n^2 + \frac{v_n}{v_{n+1}} P_n^3, \tag{31}$$

where $u_n = e^{5n/(2+w_n)}$ and $v_n = e^{4n/(4+w_n)}$ and $w_n = \{6, 7, 8, 6, 7, 8, \ldots\}$ is a periodic sequence.

We have that $\mathscr{P}$ is invariant for $(\mathscr{A})$ and

$$A_m^n = \frac{u_n}{u_m} P_n^1 + \frac{u_m}{u_n} P_n^2 + \frac{v_n}{v_m} P_n^3, \quad (32)$$

for all $(m, n) \in \Delta$.

A simple computation shows that for $h_n = e^{n\alpha}$, $k_n = e^{n\beta}$ with $\alpha \in (0, 1/2)$, $\beta \in (0, 1/3)$, and $s_n = e^{n/2}$ the pair $(\mathscr{A}, \mathscr{P})$ is (h, k)-trichotomic.

If we suppose that $(\mathscr{A}, \mathscr{P})$ is uniformly (h, k)-trichotomic, then, for $h_n = e^{n\alpha}$, $m = 4k + 1$, $n = 4k$, $w_n = 6$, and $w_m = 8$, we obtain

$$e^{\alpha} e^{k+1} \leq M, \quad (33)$$

which is a contradiction.

4. (h, k)-Trichotomy of Datko Type

Let (h_n) be a growth rate which satisfies hypothesis $(\mathscr{H})$ and let (f_n) be a growth rate given by Definition 2.

We consider a pair $(\mathscr{A}, \mathscr{P})$, where $\mathscr{P} = \{P_n^1, P_n^2, P_n^3\}$ is a family of supplementary and invariant projections sequences for $(\mathscr{A})$.

The following result emphasizes a necessary condition for (h, k)-trichotomy.

Theorem 13. *If $(\mathscr{A}, \mathscr{P})$ is (h, k)-trichotomic and (h_n) satisfies hypothesis $(\mathscr{H})$ then there exist a growth rate (g_n) and a nondecreasing sequence (d_n), $d_n \geq 1$, such that*

(fD_1)

$$\sum_{j=n+1}^{+\infty} f_j \left\| A_j^n P_n^1 x \right\| \leq d_n f_n \left\| P_n^1 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (34)$$

(fD_2)

$$\sum_{j=n}^{m-1} \frac{1}{f_j} \left\| A_j^n P_n^2 x \right\| \leq \frac{d_m}{f_m} \left\| A_m^n P_n^2 x \right\|, \quad \forall (m, n, x) \in \Delta \times X, \ m > n; \quad (35)$$

(gD_3)

$$\sum_{j=n+1}^{+\infty} \frac{1}{g_j} \left\| A_j^n P_n^3 x \right\| \leq \frac{d_n}{g_n} \left\| P_n^3 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (36)$$

(gD_4)

$$\sum_{j=n}^{m-1} g_j \left\| A_j^n P_n^3 x \right\| \leq g_m d_m \left\| A_m^n P_n^3 x \right\|, \quad \forall (m, n, x) \in \Delta \times X, \ m > n. \quad (37)$$

Proof. It is easy to see that, for

$$d_n = M \frac{h_n s_n}{f_n}, \qquad g_n = \frac{h_n k_n}{f_n}, \quad (38)$$

where $M \in (1, +\infty)$, (f_n) are given by Definition 2, and (s_n), (k_n) are given by Definition 9, relations (fD_1), (fD_2), (gD_3), and (gD_4) are satisfied. □

A necessary condition for polynomial trichotomy is represented by the following.

Corollary 14. *If the pair $(\mathscr{A}, \mathscr{P})$ is polynomially trichotomic, then there are a nondecreasing sequence (d_n), $d_n \geq 1$, and two constants $a, b > 0$ such that*

(pD_1)

$$\sum_{j=n+1}^{+\infty} (j+1)^a \left\| A_j^n P_n^1 x \right\| \leq d_n (n+1)^a \left\| P_n^1 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (39)$$

(pD_2)

$$\sum_{j=n}^{m-1} (j+1)^{-a} \left\| A_j^n P_n^2 x \right\| \leq d_m (m+1)^{-a} \left\| A_m^n P_n^2 x \right\|, \quad \forall (m, n, x) \in \Delta \times X, \ m > n; \quad (40)$$

(pD_3)

$$\sum_{j=n+1}^{+\infty} (j+1)^{-b} \left\| A_j^n P_n^3 x \right\| \leq d_n (n+1)^{-b} \left\| P_n^3 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (41)$$

(pD_4)

$$\sum_{j=n}^{m-1} (j+1)^{b} \left\| A_j^n P_n^3 x \right\| \leq d_m (m+1)^{b} \left\| A_m^n P_n^3 x \right\|, \quad \forall (m, n, x) \in \Delta \times X, \ m > n. \quad (42)$$

Proof. It results from Theorem 13. □

Definition 15. One says that the pair $(\mathscr{A}, \mathscr{P})$ admits a *(h, k)-trichotomy of Datko type* if there exists a nondecreasing sequence (d_n) with $d_n \geq 1$ such that

(hD_1)

$$\sum_{j=n+1}^{+\infty} h_j \left\| A_j^n P_n^1 x \right\| \leq d_n h_n \left\| P_n^1 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (43)$$

(hD_2)

$$\sum_{j=n}^{m-1} \frac{1}{h_j} \left\| A_j^n P_n^2 x \right\| \leq \frac{d_m}{h_m} \left\| A_m^n P_n^2 x \right\|, \quad \forall (m, n, x) \in \Delta \times X, \ m > n; \quad (44)$$

(kD_3)

$$\sum_{j=n+1}^{+\infty} \frac{1}{k_j} \left\| A_j^n P_n^3 x \right\| \leq \frac{d_n}{k_n} \left\| P_n^3 x \right\|, \quad \forall (n, x) \in \mathbb{N} \times X; \quad (45)$$

(kD_4)

$$\sum_{j=n}^{m-1} k_j \left\| A_j^n P_n^3 x \right\| \le k_m d_m \left\| A_m^n P_n^3 x \right\|, \quad \forall (m,n,x) \in \Delta \times X,\ m > n. \tag{46}$$

As particular cases, we mention the following:

(i) if $h_n = e^{n\alpha}$ with $\alpha > 0$ and $k_n = e^{n\beta}$ with $\beta > 0$ we obtain the notion of *exponential trichotomy of Datko type*;

(ii) if $h_n = (n+1)^\alpha$ with $\alpha > 1$ and $k_n = (n+1)^\beta$ with $\beta > 1$ we recover the concept of *polynomial trichotomy of Datko type*.

Remark 16. Theorem 13 emphasizes that if (h_n), (k_n) are two growth rates, (h_n) satisfies hypothesis $(\mathscr{H})$, and the pair $(\mathscr{A}, \mathscr{P})$ is (h,k)-trichotomic, then $(\mathscr{A}, \mathscr{P})$ admits a (f,g)-trichotomy of Datko type.

Theorem 17. *If the pair $(\mathscr{A}, \mathscr{P})$ admits a (h,k)-trichotomy of Datko type, then $(\mathscr{A}, \mathscr{P})$ is (h,k)-trichotomic.*

Proof. (ht_1) Using condition (hD_1) we obtain

$$h_m \left\| A_m^n P_n^1 x \right\| \le h_n d_n \left\| P_n^1 x \right\|, \quad \forall (m,n,x) \in \Delta \times X. \tag{47}$$

(ht_2) Similarly, by (hD_2) for $j = n$ we have

$$h_m \left\| P_n^2 x \right\| \le h_n d_m \left\| A_m^n P_n^2 x \right\|, \quad \forall (m,n,x) \in \Delta \times X,\ m > n. \tag{48}$$

Obviously, the relation is valid for $m = n$.

(kt_3) Inequality (kD_3) implies that

$$k_n \left\| A_m^n P_n^3 x \right\| \le d_n k_m \left\| P_n^3 x \right\|, \quad \forall (m,n,x) \in \Delta \times X. \tag{49}$$

(kt_4) By (kD_4) (for $j = n$) we deduce

$$k_n \left\| P_n^3 x \right\| \le k_m d_m \left\| A_m^n P_n^3 x \right\|, \quad \forall (m,n,x) \in \Delta \times X,\ m > n, \tag{50}$$

and the inequality is verified for $m = n$.

So, the pair $(\mathscr{A}, \mathscr{P})$ is (h,k)-trichotomic. □

Corollary 18. *If there exist two constants $a, b > 1$ and a nondecreasing sequence (d_n), $d_n \ge 1$, such that conditions (pD_1) and (pD_3) ((pD_2) and (pD_4), resp.) from Corollary 14 are fulfilled for all $(n,x) \in \mathbb{N} \times X$ (for all $(m,n,x) \in \Delta \times X$, resp.) with $m > n$ then $(\mathscr{A}, \mathscr{P})$ is polynomially trichotomic.*

Proof. It results from Theorem 17 for $h_n = (n+1)^a$ with $a > 1$ and $k_n = (n+1)^b$, with $b > 1$. □

The following result represents a characterization for the exponential trichotomy.

Corollary 19. *The pair $(\mathscr{A}, \mathscr{P})$ is exponentially trichotomic if and only if there exist the constants $a, b > 0$ and a nondecreasing sequence (d_n), $d_n \ge 1$, with the following properties:*

(eD_1)

$$\sum_{j=n+1}^{+\infty} e^{ja} \left\| A_j^n P_n^1 x \right\| \le d_n e^{na} \left\| P_n^1 x \right\|, \quad \forall (n,x) \in \mathbb{N} \times X; \tag{51}$$

(eD_2)

$$\sum_{j=n}^{m-1} e^{-ja} \left\| A_j^n P_n^2 x \right\| \le d_m e^{-ma} \left\| A_m^n P_n^2 x \right\|, \quad \forall (m,n,x) \in \Delta \times X,\ m > n; \tag{52}$$

(eD_3)

$$\sum_{j=n+1}^{+\infty} e^{-bj} \left\| A_j^n P_n^3 x \right\| \le d_n e^{-nb} \left\| P_n^3 x \right\|, \quad \forall (n,x) \in \mathbb{N} \times X; \tag{53}$$

(eD_4)

$$\sum_{j=n}^{m-1} e^{jb} \left\| A_j^n P_n^3 x \right\| \le d_m e^{mb} \left\| A_m^n P_n^3 x \right\|, \quad \forall (m,n,x) \in \Delta \times X,\ m > n. \tag{54}$$

Proof.

Necessity. It is a particular case of Theorem 13 for

$$\begin{aligned} h_n &= e^{n\alpha}, \\ f_n &= e^{na} \quad \text{with } a \in (0, \alpha), \\ k_n &= e^{n\beta} \quad \text{with } \beta > 0, \\ g_n &= e^{nb}, \quad b = \beta + \alpha - a. \end{aligned} \tag{55}$$

Sufficiency. Using Theorem 17, for $h_j = e^{ja}$ and $k_j = e^{jb}$ we obtain that $(\mathscr{A}, \mathscr{P})$ is exponentially trichotomic. □

Remark 20. The previous result shows that the exponential trichotomy and the exponential trichotomy of Datko type are equivalent.

5. Lyapunov Functions for (h,k)-Trichotomy

Throughout this section, (h_n), (k_n) represent two growth rates, $(\mathscr{A})$ is a linear discrete-time system, and $\mathscr{P} = \{P_n^1, P_n^2, P_n^3\}$ is a family of supplementary and invariant projections sequences for $(\mathscr{A})$.

Definition 21. Two mappings $L_1, L_2 : \Delta \times X \to \mathbb{R}_+$ are called (h,k)-*Lyapunov functions* for the pair $(\mathscr{A}, \mathscr{P})$ if there exists a nondecreasing sequence (t_n), $t_n \geq 1$, such that

(hL_1^1)

$$L_1\left(m, p, P_p^1 x\right) + \sum_{j=n+1}^{m} \frac{h_j}{h_n} \left\| A_j^p P_p^1 x \right\| \leq L_1\left(n, p, P_p^1 x\right) \leq t_p \left\| P_p^1 x \right\|; \tag{56}$$

(kL_1^3)

$$\sum_{j=n+1}^{m} \frac{k_p}{k_j} \left\| A_j^p P_p^3 x \right\| \leq L_1\left(n, p, P_p^3 x\right) \leq t_p \left\| P_p^3 x \right\|; \tag{57}$$

(hL_2^2)

$$L_2\left(n, p, P_p^2 x\right) + \sum_{j=n}^{m-1} \frac{h_m}{h_j} \left\| A_j^p P_p^2 x \right\| \leq L_2\left(m, p, P_p^2 x\right) \leq t_m \left\| A_m^p P_p^2 x \right\|; \tag{58}$$

(kL_2^3)

$$\sum_{j=n}^{m-1} \frac{k_j}{k_m} \left\| A_j^p P_p^3 x \right\| \leq L_2\left(m, p, P_p^3 x\right) \leq t_m \left\| A_m^p P_p^3 x \right\|, \tag{59}$$

for all $(m, n, p) \in \mathbb{N}^3$ with $m > n \geq p$ and for all $x \in X$.

In particular, if

(i) $h_n = e^{n\alpha}$, with $\alpha > 0$, $k_n = e^{n\beta}$, with $\beta > 0$ then the (h,k)-Lyapunov functions are called *exponential Lyapunov functions;*

(ii) $h_n = (n+1)^\alpha$, with $\alpha > 0$, $k_n = (n+1)^\beta$, with $\beta > 0$ then the (h,k)-Lyapunov functions are called *polynomial Lyapunov functions.*

Example 22. On $X = l^\infty(\mathbb{N}, \mathbb{R})$, the Banach space of bounded real-valued sequences, endowed with the norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|, \quad x = (x_n)_{n\in\mathbb{N}} \in X, \tag{60}$$

we consider $\mathscr{P} = \{P_n^1, P_n^2, P_n^3\}$, with

$$\begin{aligned} P_n^1 x &= \chi_{3\mathbb{N}} x, \\ P_n^2 x &= \chi_{3\mathbb{N}+1} x, \\ P_n^3 x &= \chi_{3\mathbb{N}+2} x, \end{aligned} \tag{61}$$

where χ_A represents the characteristic function of set A.

Also, linear discrete-time system $(\mathscr{A})$ is defined by

$$A_n = e^{3n-3(n+1)} P_n^1 + e^{3(n+1)-3n} P_n^2 + P_n^3 \tag{62}$$

and we have that

$$A_m^n = e^{3n-3m} P_n^1 + e^{3m-3n} P_n^2 + P_n^3, \tag{63}$$

for all $(m, n) \in \Delta$.

For the growth rates $h_n = e^{2n}$ and $k_n = e^{4n}$ we define the exponential Lyapunov functions:

$$L_1(m,n,x) = \sum_{j=m+1}^{+\infty} e^{2j-2n} \left\| A_j^n P_n^1 x \right\| + \sum_{j=m+1}^{+\infty} e^{4n-4j} \left\| A_j^n P_n^3 x \right\|,$$

$$L_2(m,n,x) = \begin{cases} \displaystyle\sum_{j=n}^{m-1} e^{2m-2j} \left\| A_j^n P_n^2 x \right\| + \sum_{j=n}^{m-1} e^{4j-4m} \left\| A_j^n P_n^3 x \right\|, & \text{if } m > n \\ 0, & \text{if } m = n. \end{cases} \tag{64}$$

After some computations, we obtain that for $t_n = e^{4(n+1)}/(e^4 - 1)$ the mappings L_1 and L_2 are exponential Lyapunov functions for the pair $(\mathscr{A}, \mathscr{P})$.

In the following, we give a characterization for the (h,k)-trichotomy of Datko type in terms of (h,k)-Lyapunov functions.

Theorem 23. *The pair $(\mathscr{A}, \mathscr{P})$ admits a (h,k)-trichotomy of Datko type if and only if there exist two (h,k)-Lyapunov functions for $(\mathscr{A}, \mathscr{P})$.*

Proof.

Necessity. Let $L_1, L_2 : \Delta \times X \to \mathbb{R}_+$, defined by

$$L_1(m,n,x) = \sum_{j=m+1}^{+\infty} \frac{h_j}{h_n} \left\| A_j^n P_n^1 x \right\| + \sum_{j=m+1}^{+\infty} \frac{k_n}{k_j} \left\| A_j^n P_n^3 x \right\|, \tag{65}$$

$$L_2(m,n,x) = \begin{cases} \displaystyle\sum_{j=n}^{m-1} \frac{h_m}{h_j} \left\| A_j^n P_n^2 x \right\| + \sum_{j=n}^{m-1} \frac{k_j}{k_m} \left\| A_j^n P_n^3 x \right\|, & \text{if } m > n \\ 0, & \text{if } m = n, \end{cases} \tag{66}$$

respectively. Then

(hL_1^1)

$$\begin{aligned} &L_1\left(m, p, P_p^1 x\right) + \sum_{j=n+1}^{m} \frac{h_j}{h_n} \left\| A_j^p P_p^1 x \right\| \\ &= \sum_{j=m+1}^{+\infty} \frac{h_j}{h_p} \left\| A_j^p P_p^1 x \right\| + \sum_{j=n+1}^{m} \frac{h_j}{h_n} \left\| A_j^p P_p^1 x \right\| \\ &\leq \sum_{j=n+1}^{+\infty} \frac{h_j}{h_p} \left\| A_j^p P_p^1 x \right\| = L_1\left(n, p, P_p^1 x\right) \leq d_p \left\| P_p^1 x \right\|; \end{aligned} \tag{67}$$

(kL_1^3)

$$\begin{aligned} &\sum_{j=n+1}^{m} \frac{k_p}{k_j} \left\| A_j^p P_p^3 x \right\| \leq \sum_{j=n+1}^{+\infty} \frac{k_p}{k_j} \left\| A_j^p P_p^3 x \right\| \\ &= L_1\left(n, p, P_p^3 x\right) \leq d_p \left\| P_p^3 x \right\|; \end{aligned} \tag{68}$$

(hL_2^2)

$$\begin{aligned}L_2\left(n,p,P_p^2x\right)+\sum_{j=n}^{m-1}\frac{h_m}{h_j}\left\|A_j^pP_p^2x\right\| &= \sum_{j=p}^{n-1}\frac{h_n}{h_j}\left\|A_j^pP_p^2x\right\|+\sum_{j=n}^{m-1}\frac{h_m}{h_j}\left\|A_j^pP_p^2x\right\| \\ &\le \sum_{j=p}^{m-1}\frac{h_m}{h_j}\left\|A_j^pP_p^2x\right\| = L_2\left(m,p,P_p^2x\right) \\ &\le d_m\left\|A_m^pP_p^2x\right\|;\end{aligned} \tag{69}$$

(kL_2^3)

$$\begin{aligned}\sum_{j=n}^{m-1}\frac{k_j}{k_m}\left\|A_j^pP_p^3x\right\| \le \sum_{j=p}^{m-1}\frac{k_j}{k_m}\left\|A_j^pP_p^3x\right\| = L_2\left(m,p,P_p^3x\right) \\ \le d_m\left\|A_m^pP_p^3x\right\|,\end{aligned} \tag{70}$$

for all $(m,n,p)\in\mathbb{N}^3$ with $m>n\ge p$ and for all $x\in X$.

Sufficiency. (hD_1) From (hL_1^1), for $p=n$ we obtain

$$\sum_{j=n+1}^{m}\frac{h_j}{h_n}\left\|A_j^nP_n^1x\right\|\le t_n\left\|P_n^1x\right\|, \tag{71}$$

and for $m\to+\infty$ it results in condition (hD_1).

(hD_2) For $p=n$ in (hL_2^2) we deduce

$$\sum_{j=n}^{m-1}\frac{h_m}{h_j}\left\|A_j^nP_n^2x\right\|\le t_m\left\|A_m^nP_n^2x\right\|, \tag{72}$$

for all $(m,n)\in\Delta,\ m>n,\ x\in X$.

(kD_3) Putting $p=n$ in (kL_1^3) we obtain

$$\sum_{j=n+1}^{m}\frac{k_n}{k_j}\left\|A_j^nP_n^3x\right\|\le t_n\left\|P_n^3x\right\|, \tag{73}$$

and for $m\to+\infty$ it results in (kD_3).

(kD_4) By condition (kL_2^3) it follows

$$\sum_{j=n}^{m-1}\frac{k_j}{k_m}\left\|A_j^nP_n^3x\right\|\le t_m\left\|A_m^nP_n^3x\right\|, \tag{74}$$

for all $(m,n)\in\Delta,\ m>n,\ x\in X$.

Thus, it results in that $(\mathscr{A},\mathscr{P})$ admits a (h,k)-trichotomy of Datko type. □

A sufficient condition for (h,k)-trichotomy given through the Lyapunov functions is as follows.

Corollary 24. *If there exist two (h,k)-Lyapunov functions L_1, L_2 for the pair $(\mathscr{A},\mathscr{P})$, then $(\mathscr{A},\mathscr{P})$ is (h,k)-trichotomic.*

Proof. It is obtained from Theorems 17 and 23. □

Corollary 25. *Let (h_n), (k_n) be two growth rates such that (h_n) satisfies hypothesis $(\mathscr{H})$. If $(\mathscr{A},\mathscr{P})$ is (h,k)-trichotomic then there exist two (f,g)-Lyapunov functions for $(\mathscr{A},\mathscr{P})$, where (f_n), (g_n) are the growth rates given by Definition 2 (Theorem 13, resp.).*

Proof. It is immediate by Remark 16 and Theorem 23. □

An important characterization for the exponential trichotomy in terms of Lyapunov functions is represented by the following.

Corollary 26. *The pair $(\mathscr{A},\mathscr{P})$ is exponentially trichotomic if and only if there exist L_1, L_2 two exponential Lyapunov functions for $(\mathscr{A},\mathscr{P})$.*

Proof. It results by Remark 20 and Theorem 23. □

Corollary 27. *If $(\mathscr{A},\mathscr{P})$ is polynomially trichotomic, then there exist L_1, L_2 polynomial Lyapunov functions for $(\mathscr{A},\mathscr{P})$.*

Proof. It is a consequence of Corollary 25. □

Corollary 28. *If there exist two (h,k)-Lyapunov functions for the pair $(\mathscr{A},\mathscr{P})$, where*

$$\begin{aligned}h_n&=(n+1)^\alpha, \quad \textit{with } \alpha>1,\\ k_n&=(n+1)^\beta, \quad \textit{with } \beta>1,\end{aligned} \tag{75}$$

then $(\mathscr{A},\mathscr{P})$ is polynomially trichotomic.

Proof. It results from Theorems 17 and 23 for

$$\begin{aligned}h_n&=(n+1)^\alpha, \quad \text{with } \alpha>1,\\ k_n&=(n+1)^\beta, \quad \text{with } \beta>1.\end{aligned} \tag{76}$$

□

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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