Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 467929, 7 pages
http://dx.doi.org/10.1155/2014/467929

Research Article

On Certain Subclass of Harmonic Starlike Functions

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Received 13 February 2014; Accepted 19 March 2014; Published 9 April 2014

Academic Editor: V. Ravichandran

Coefficient conditions, distortion bounds, extreme points, convolution, convex combinations, and neighborhoods for a new class of harmonic univalent functions in the open unit disc are investigated. Further, a class preserving integral operator and connections with various previously known results are briefly discussed.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. There is a close interrelation between analytic functions and harmonic functions. For example, for real harmonic functions u and v, there exist analytic functions U and V so that $u = \mathrm{Re}(U)$ and $v = \mathrm{Im}(V)$. Then $f(z) = h(z) + \overline{g(z)}$, where h and g are, respectively, the analytic functions $(U + V)/2$ and $(U - V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by $J_f = |h'(z)|^2 - |g'(z)|^2$. The mapping $z \to f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f > 0$ in D. The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \to f(z)$ is orientation preserving, harmonic, and one-to-one in D. We call h the analytic part and g the coanalytic part of f (see Clunie and Sheil-Small [1]).

Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

Note that $\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the coanalytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

A function $f = h + \overline{g}$ with h and g given by (1) is said to be harmonic starlike of order β $(0 \le \beta < 1)$ for $|z| = r < 1$, if

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) = \mathrm{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) \ge \beta. \tag{3}$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}_H^*(\beta)$ and extensively studied by Jahangiri [2]. The cases $\beta = 0$ and $b_1 = 0$ were studied by Silverman and Silvia [3] and Silverman [4]. Other related works of the class $\mathcal{H}$ also appeared in [5–16].

Definition 1. Let $f = h + \overline{g}$ where h and g are given by (1). Let $0 \le \beta < 1$ and $\alpha \ge 0$. Then $f \in \mathcal{S}_H^*(\alpha, \beta)$ if and only if

$$\mathrm{Re}\left(\frac{\alpha z^2 h''(z) + zh'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) zg'(z)}}{h(z) + \overline{g(z)}}\right) \ge \beta. \tag{4}$$

We note that for $\alpha = 0$, the class $\mathcal{S}_H^*(\alpha, \beta)$ reduces to the class $\mathcal{S}_H^*(\beta)$. Further, if the coanalytic part $g(z)$ is zero, the class $\mathcal{S}_H^*(\alpha,\beta)$ reduces to the class $\mathcal{P}(\alpha,\beta)$ of functions $f\in\mathcal{S}$ which satisfy the condition

$$\operatorname{Re}\left(\frac{\alpha z^2 f''(z)+zf'(z)}{f(z)}\right)\geq\beta \tag{5}$$

for some β $(0\leq\beta<1)$, $\alpha\geq 0$, $f(z)/z\neq 0$ and $z\in\mathbb{U}$. Observe that the classes $\mathcal{P}(\alpha,\beta)$ and $\mathcal{P}(\alpha,0)$ were introduced and studied by many authors and these include, for example, by Obradovic and Joshi [17], Padmanabhan [18], Li and Owa [19], Xu and Yang [20], Singh and Gupta [21], and Lashin [22]. We also note that for $\alpha=0$, the class $\mathcal{P}(0,\beta)$ was studied by Silverman [23].

2. Main Results

The first theorem of this section determines the sufficient coefficient condition for functions $f=h+\overline{g}$ to belong to the class $\mathcal{S}_H^*(\alpha,\beta)$. The following lemma obtained by Jahangiri is needed.

Lemma 2 (see [2, Theorem 1]). *Let $f=h+\overline{g}$ with h and g of the form* (1) *and let*

$$\sum_{n=2}^{\infty}\frac{n-\beta}{1-\beta}|a_n|+\sum_{n=1}^{\infty}\frac{n+\beta}{1-\beta}|b_n|\leq 1, \tag{6}$$

where $0\leq\beta<1$. Then f is harmonic, orientation preserving, and univalent in $\mathbb{U}$, and $f\in\mathcal{S}_H^*(\beta)$.

Theorem 3. *Let $f=h+\overline{g}$ where h and g are of the form* (1). *If*

$$\sum_{n=2}^{\infty}\frac{\alpha n(n-1)+n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty}\frac{\alpha n(n+1)+n+\beta}{1-\beta}|b_n|\leq 1 \tag{7}$$

for some β, $(0\leq\beta<1)$ and $\alpha\geq 0$, then f is harmonic, orientation preserving, and univalent in $\mathbb{U}$ and $f\in\mathcal{S}_H^*(\alpha,\beta)$.

Proof. Since $(n-\beta)\leq\alpha n(n-1)+n-\beta$ and $n+\beta\leq\alpha n(n+1)+n+\beta$, $(n\geq 1)$, it follows from Lemma 2 that $f\in\mathcal{S}_H^*(\beta)$ and hence f is harmonic, orientation preserving, and univalent in $\mathbb{U}$. Now, we only need to show that if (7) holds then

$$\operatorname{Re}\left(\frac{\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)}}{h(z)+\overline{g(z)}}\right) = \operatorname{Re}\frac{A(z)}{B(z)}\geq\beta. \tag{8}$$

Using the fact that $\operatorname{Re}(w)\geq\beta$ if and only if $|1-\beta+w|\geq|1+\beta-w|$, it suffices to show that

$$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\geq 0, \tag{9}$$

where

$$A(z)=\alpha z^2h''(z)+zh'(z)+\overline{\alpha z^2g''(z)+(2\alpha-1)zg'(z)},$$
$$B(z)=h(z)+\overline{g(z)}. \tag{10}$$

Substituting for $A(z)$ and $B(z)$ in (9), we obtain

$$\begin{aligned}
&|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|\\
&=\left|(2-\beta)z+\sum_{n=2}^{\infty}((\alpha n+1)(n-1)+2-\beta)a_nz^n\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}(\alpha n(n+1)-(n-1+\beta))\overline{b_n z^n}\right|\\
&\quad-\left|-\beta z+\sum_{n=2}^{\infty}((\alpha n+1)(n-1)-\beta)a_nz^n\right.\\
&\qquad\left.+\sum_{n=1}^{\infty}(\alpha n(n+1)-(n+1+\beta))\overline{b}_n\overline{z}^n\right|\\
&\geq(2-\beta)|z|\\
&\quad-\sum_{n=2}^{\infty}((\alpha n+1)(n-1)+2-\beta)|a_n||z|^n\\
&\quad-\sum_{n=1}^{\infty}|\alpha n(n+1)-(n-1+\beta)||b_n||z|^n-\beta|z|\\
&\quad-\sum_{n=2}^{\infty}((\alpha n+1)(n-1)-\beta)|a_n||z|^n\\
&\quad-\sum_{n=1}^{\infty}|\alpha n(n+1)-(n+1+\beta)||b_n||z|^n\\
&\geq 2(1-\beta)|z|\\
&\quad-2\sum_{n=2}^{\infty}(\alpha n(n-1)+n-\beta)|a_n||z|^n\\
&\quad-2\sum_{n=1}^{\infty}(\alpha n(n+1)+(n+\beta))|b_n||z|^n\\
&\geq 2(1-\beta)|z|\\
&\quad\times\left(1-\sum_{n=2}^{\infty}\frac{\alpha n(n-1)+n-\beta}{1-\beta}|a_n|\right.\\
&\qquad\left.-\sum_{n=1}^{\infty}\frac{\alpha n(n+1)+(n+\beta)}{1-\beta}|b_n|\right)\geq 0,
\end{aligned} \tag{11}$$

by the given condition (7).

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1)+n-\beta} x_n z^n + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n+1)+(n+\beta)} \overline{y}_n \overline{z}^n, \tag{12}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$ shows that the coefficient bound given in (7) is sharp. The functions of the form (12) are in $\mathcal{S}_H^*(\alpha,\beta)$ since

$$\sum_{n=2}^{\infty} \frac{\alpha n(n-1)+n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}|b_n| = \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1. \tag{13}$$

□

Remark 4. Setting $g(z) = 0$ in Theorem 3 yields the result obtained by Lashin [22, Theorem 2.1].

We denote by $\overline{\mathcal{S}}_H^*(\alpha,\beta)$ the class of functions $f \in \mathcal{S}_H^*(\alpha,\beta)$ whose coefficients satisfy the condition (7).

Theorem 5. *Let* $0 \le \alpha_1 < \alpha_2$ *and* $0 \le \beta < 1$. *Then* $\overline{\mathcal{S}}_H^*(\alpha_2,\beta) \subset \overline{\mathcal{S}}_H^*(\alpha_1,\beta)$.

Proof. For $f \in \overline{\mathcal{S}}_H^*(\alpha_2,\beta)$, it follows from (7) that

$$\begin{aligned}&\sum_{n=2}^{\infty} \frac{\alpha_1 n(n-1)+n-\beta}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{\alpha_1 n(n+1)+n+\beta}{1-\beta}|b_n| \\ &\quad < \sum_{n=2}^{\infty} \frac{\alpha_2 n(n-1)+n-\beta}{1-\beta}|a_n| \\ &\qquad + \sum_{n=1}^{\infty} \frac{\alpha_2 n(n+1)+n+\beta}{1-\beta}|b_n| \le 1.\end{aligned} \tag{14}$$

Hence $f \in \overline{\mathcal{S}}_H^*(\alpha_1,\beta)$. □

As a consequence of Theorem 5, the functions in $\overline{\mathcal{S}}_H^*(\alpha,\beta)$ are starlike harmonic in $\mathbb{U}$.

Corollary 6. *For* $\alpha \ge 0$ *and* $0 \le \beta < 1$, $\overline{\mathcal{S}}_H^*(\alpha,\beta) \subset \mathcal{S}_H^*(\beta)$.

3. Distortion Bounds and Extreme Points

In this section, we obtain the distortion bounds and extreme points for functions in the class $\overline{\mathcal{S}}_H^*(\alpha,\beta)$.

Theorem 7. *Let* $f = h + \overline{g}$ *where* h *and* g *are of the form* (1) *and* $f \in \overline{\mathcal{S}}_H^*(\alpha,\beta)$. *Then for* $|z| = r < 1$, *we have*

$$|f(z)| \le (1+|b_1|)r + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right)r^2, \tag{15}$$

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right)r^2, \tag{16}$$

where

$$|b_1| \le \frac{1-\beta}{2\alpha+1+\beta}. \tag{17}$$

The result is sharp.

Proof. We shall prove the first inequality. Let $f \in \overline{\mathcal{S}}_H^*(\alpha,\beta)$. Then we have

$$\begin{aligned}|f(z)| &\le (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\ &\le (1+|b_1|)r + r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|) \\ &= (1+|b_1|)r + \frac{1-\beta}{2\alpha+2-\beta} \\ &\quad \times \sum_{n=2}^{\infty}\frac{2\alpha+2-\beta}{1-\beta}(|a_n|+|b_n|)r^2\end{aligned} \tag{18}$$

and so

$$\begin{aligned}|f(z)| &\le (1+|b_1|)r \\ &\quad + \frac{1-\beta}{2\alpha+2-\beta} \\ &\quad \times \sum_{n=2}^{\infty}\left(\frac{\alpha n(n-1)+n-\beta}{1-\beta}|a_n| + \frac{\alpha n(n+1)+n+\beta}{1-\beta}|b_n|\right)r^2 \\ &\le (1+|b_1|)r \\ &\quad + \frac{1-\beta}{2\alpha+2-\beta}\left(1 - \frac{2\alpha+1+\beta}{1-\beta}|b_1|\right)r^2 \\ &= (1+|b_1|)r \\ &\quad + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta}|b_1|\right)r^2.\end{aligned} \tag{19}$$

The proof of the inequality (16) is similar to the proof of the inequality (15), thus we omit it.

The upper bound given for $f \in \overline{\mathcal{S}}^*_H(\alpha,\beta)$ is sharp and the equality occurs for the function

$$\begin{aligned} f(z) &= z + |b_1| \overline{z} \\ &\quad + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta} |b_1| \right) \overline{z}^2 \quad (z = r), \end{aligned} \tag{20}$$

where $|b_1| \leq (1-\beta)/(2\alpha+1+\beta)$. This completes the proof of Theorem 7. □

Now, we determine the extreme points of the closed convex hull of the class $\overline{\mathcal{S}}^*_H(\alpha,\beta)$ denoted by clco $\overline{\mathcal{S}}^*_H(\alpha,\beta)$.

Theorem 8. *Let $f = h + \overline{g}$ where h and g are given by* (1)*. Then $f \in clco\overline{\mathcal{S}}^*_H(\alpha,\beta)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n), \tag{21}$$

where

$$\begin{gathered} h_1(z) = z, \\ h_n(z) = z + \frac{1-\beta}{\alpha n(n-1)+n-\beta} z^n \quad (n = 2,3,\ldots); \\ g_n(z) = z + \frac{1-\beta}{\alpha n(n+1)+n+\beta} \overline{z}^n \quad (n = 1,2,3,\ldots), \\ \sum_{n=1}^{\infty} (X_n + Y_n) = 1, \quad X_n \geq 0, \; Y_n \geq 0. \end{gathered} \tag{22}$$

In particular, the extreme points of the class $\overline{\mathcal{S}}^*_H(\alpha,\beta)$ are $\{h_n\}$ and $\{g_n\}$, respectively.

Proof. For a function f of the form (21), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z \\ &\quad + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1)+n-\beta} X_n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1)+n+\beta} Y_n \overline{z}^n \\ &= z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1)+n-\beta} X_n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1)+n+\beta} Y_n \overline{z}^n. \end{aligned} \tag{23}$$

But

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\alpha n(n-1)+n-\beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n-1)+n-\beta} X_n \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n+1)+n+\beta} Y_n \right) \\ &\quad = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1. \end{aligned} \tag{24}$$

Thus $f \in$ clco $\overline{\mathcal{S}}^*_H(\alpha,\beta)$.

Conversely, suppose that $f \in$ clco $\overline{\mathcal{S}}^*_H(\alpha,\beta)$. Set

$$\begin{aligned} X_n &= \frac{\alpha n(n-1)+n-\beta}{1-\beta} |a_n| \quad (n = 2,3,\ldots), \\ Y_n &= \frac{\alpha n(n+1)+n+\beta}{1-\beta} |b_n| \quad (n = 1,2,3,\ldots). \end{aligned} \tag{25}$$

Then by the inequality (7), we have $0 \leq X_n \leq 1$ $(n = 2,3,\ldots)$ and $0 \leq Y_n \leq 1$ $(n = 1,2,\ldots)$. Define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that $X_1 \geq 0$. Thus we obtain $f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$. This completes the proof of the theorem. □

4. Convolution and Convex Combinations

For two harmonic functions

$$\begin{aligned} f(z) &= z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \overline{z}^n, \\ F(z) &= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n \end{aligned} \tag{26}$$

we define their convolution

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n. \tag{27}$$

Using this definition, we show that the class $\overline{\mathcal{S}}^*_H(\alpha,\beta)$ is closed under convolution.

Theorem 9. *For $0 \leq \beta < 1$ and $\alpha \geq 0$, let $f, F \in \overline{\mathcal{S}}^*_H(\alpha,\beta)$. Then $f * F \in \overline{\mathcal{S}}^*_H(\alpha,\beta)$.*

Proof. We note that $|A_n| \le 1$ and $|B_n| \le 1$. For the convolution $(f * F)$, we have

$$\begin{aligned}\sum_{n=2}^{\infty} & \frac{\alpha n(n-1)+n-\beta}{1-\beta}\left|A_n a_n\right| \\ & +\sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}\left|B_n b_n\right| \\ \le & \sum_{n=2}^{\infty} \frac{\alpha n(n-1)+n-\beta}{1-\beta}\left|a_n\right| \\ & +\sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}\left|b_n\right| \le 1 .\end{aligned} \tag{28}$$

Therefore $f * F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. □

We show that the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination of its members.

Theorem 10. *The class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$ where $f_i(z)$ is given by

$$f_i(z)=z+\sum_{n=2}^{\infty} a_{ni} z^n+\sum_{n=1}^{\infty} \overline{b}_{ni} \overline{z}^n . \tag{29}$$

Then by (7), we have

$$\begin{aligned}\sum_{n=2}^{\infty} & \frac{\alpha n(n-1)+n-\beta}{1-\beta}\left|a_{ni}\right| \\ & +\sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}\left|b_{ni}\right| \le 1 .\end{aligned} \tag{30}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i=z+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{ni}\right) z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i \overline{b_{ni}}\right) \overline{z}^n . \tag{31}$$

Then by (7), we have

$$\begin{aligned}\sum_{n=2}^{\infty} & \frac{\alpha n(n-1)+n-\beta}{1-\beta}\left|\sum_{i=1}^{\infty} t_i a_{ni}\right| \\ & +\sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}\left|\sum_{i=1}^{\infty} t_i \overline{b}_{ni}\right| \\ \le & \sum_{i=1}^{\infty} t_i\left(\sum_{n=2}^{\infty} \frac{\alpha n(n-1)+n-\beta}{1-\beta}\left|a_{ni}\right|\right. \\ & \left.+\sum_{n=1}^{\infty} \frac{\alpha n(n+1)+n+\beta}{1-\beta}\left|b_{ni}\right|\right) \\ \le & \sum_{i=1}^{\infty} t_i=1 .\end{aligned} \tag{32}$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. □

5. Neighborhood Results

Following the earlier investigations by Goodman [24], Ruscheweyh [25], Altintas et al. [26], and Porwal and Aouf [27], we define the δ-neighborhood of function $f(z) \in \mathscr{H}$ by

$$\begin{aligned}N_\delta(f)= & \left\{F \in \mathscr{H}: F(z)=z+\sum_{n=2}^{\infty} A_n z^n+\sum_{n=1}^{\infty} \overline{B_n} \overline{z}^n,\right. \\ & \left.\sum_{n=2}^{\infty} n\left|a_n-A_n\right|+\sum_{n=1}^{\infty} n\left|b_n-B_n\right| \le \delta\right\} .\end{aligned} \tag{33}$$

In particular, for the identity function $e(z) = z$, we immediately have

$$\begin{aligned}N_\delta(e)= & \left\{F \in \mathscr{H}: F(z)=z+\sum_{n=2}^{\infty} A_n z^n+\sum_{n=1}^{\infty} \overline{B}_n \overline{z}^n,\right. \\ & \left.\sum_{n=2}^{\infty} n\left|A_n\right|+\sum_{n=1}^{\infty} n\left|B_n\right| \le \delta\right\} .\end{aligned} \tag{34}$$

Theorem 11. *$\overline{\mathcal{S}}_H^*(\alpha, \beta) \subseteq N_\delta(e)$, where*

$$\delta=\frac{(1-\beta)(2 \alpha+1)}{(\alpha+1)(2 \alpha+1-\beta)} . \tag{35}$$

Proof. Let $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then, in view of (7), since $\alpha n(n-1)+n-\beta$ and $\alpha n(n+1)+n+\beta$ are increasing functions of $n\,(n \ge 1)$, we have

$$\begin{aligned}(2 \alpha+1-\beta) & \left(\sum_{n=2}^{\infty}\left|a_n\right|+\sum_{n=1}^{\infty}\left|b_n\right|\right) \\ & \le(2 \alpha+2-\beta) \sum_{n=2}^{\infty}\left|a_n\right|+(2 \alpha+1+\beta) \sum_{n=1}^{\infty}\left|b_n\right| \\ & \le \sum_{n=2}^{\infty}(\alpha n(n-1)+n-\beta)\left|a_n\right| \\ & \quad+\sum_{n=1}^{\infty}(\alpha n(n+1)+n+\beta)\left|b_n\right| \le(1-\beta),\end{aligned} \tag{36}$$

which yields

$$\sum_{n=2}^{\infty}\left|a_n\right|+\sum_{n=1}^{\infty}\left|b_n\right| \le \frac{1-\beta}{(2 \alpha+1-\beta)} . \tag{37}$$

On the other hand, we also find from (7)

$$\begin{aligned}&\left((\alpha+1)\sum_{n=2}^{\infty}n|a_n|-\beta\sum_{n=2}^{\infty}|a_n|\right)\\&\quad+\left((\alpha+1)\sum_{n=1}^{\infty}n|b_n|-\beta\sum_{n=1}^{\infty}|b_n|\right)\\&\leq\sum_{n=2}^{\infty}\left((\alpha(n-1)+1)n-\beta\right)|a_n|\\&\quad+\sum_{n=1}^{\infty}\left((\alpha(n+1)+1)n+\beta\right)|b_n|\\&\leq(1-\beta).\end{aligned}\tag{38}$$

From (37) and (38), we obtain

$$\begin{aligned}&(\alpha+1)\left(\sum_{n=2}^{\infty}n|a_n|+\sum_{n=1}^{\infty}n|b_n|\right)\\&\quad\leq(1-\beta)+\beta\left(\sum_{n=2}^{\infty}|a_n|+\sum_{n=1}^{\infty}|b_n|\right)\\&\quad=(1-\beta)+\beta\frac{1-\beta}{(2\alpha+1-\beta)},\end{aligned}\tag{39}$$

which is equivalent to

$$\sum_{n=2}^{\infty}n|a_n|+\sum_{n=1}^{\infty}n|b_n|\leq\frac{(1-\beta)(2\alpha+1)}{(\alpha+1)(2\alpha+1-\beta)}=\delta.\tag{40}$$

□

6. A Family of Class Preserving Integral Operator

In this section, we consider the closure property of the class $\overline{\mathcal{S}}_H^*(\alpha,\beta)$ under the Bernardi integral operator $F(z)$, which is defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)\,dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}g(t)\,dt}\qquad(c>-1).\tag{41}$$

Theorem 12. *Let $f=h+\overline{g}$ be in the class $\overline{\mathcal{S}}_H^*(\alpha,\beta)$, where h and g are given by* (1). *Then $F(z)$ defined by* (41) *also belongs to the class $\overline{\mathcal{S}}_H^*(\alpha,\beta)$.*

Proof. From the representation of F, it follows that

$$F(z)=z+\sum_{n=2}^{\infty}\frac{c+1}{c+n}a_nz^n+\sum_{n=1}^{\infty}\frac{c+1}{c+n}\overline{b_n}\overline{z}^n.\tag{42}$$

Now

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\alpha n(n-1)+n-\beta}{1-\beta}\left(\frac{c+1}{c+n}|a_n|\right)\\&\quad+\sum_{n=1}^{\infty}\frac{\alpha n(n+1)+n+\beta}{1-\beta}\left(\frac{c+1}{c+n}|b_n|\right)\\&\leq\sum_{n=2}^{\infty}\frac{\alpha n(n-1)+n-\beta}{1-\beta}|a_n|\\&\quad+\sum_{n=1}^{\infty}\frac{\alpha n(n+1)+n+\beta}{1-\beta}|b_n|\leq1\end{aligned}\tag{43}$$

by (7). Thus $F(z)\in\overline{\mathcal{S}}_H^*(\alpha,\beta)$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author would like to express his gratitude to Professor Dr. V. Ravichandran and the referees for their valuable comments which have essentially improved the presentation of this paper.

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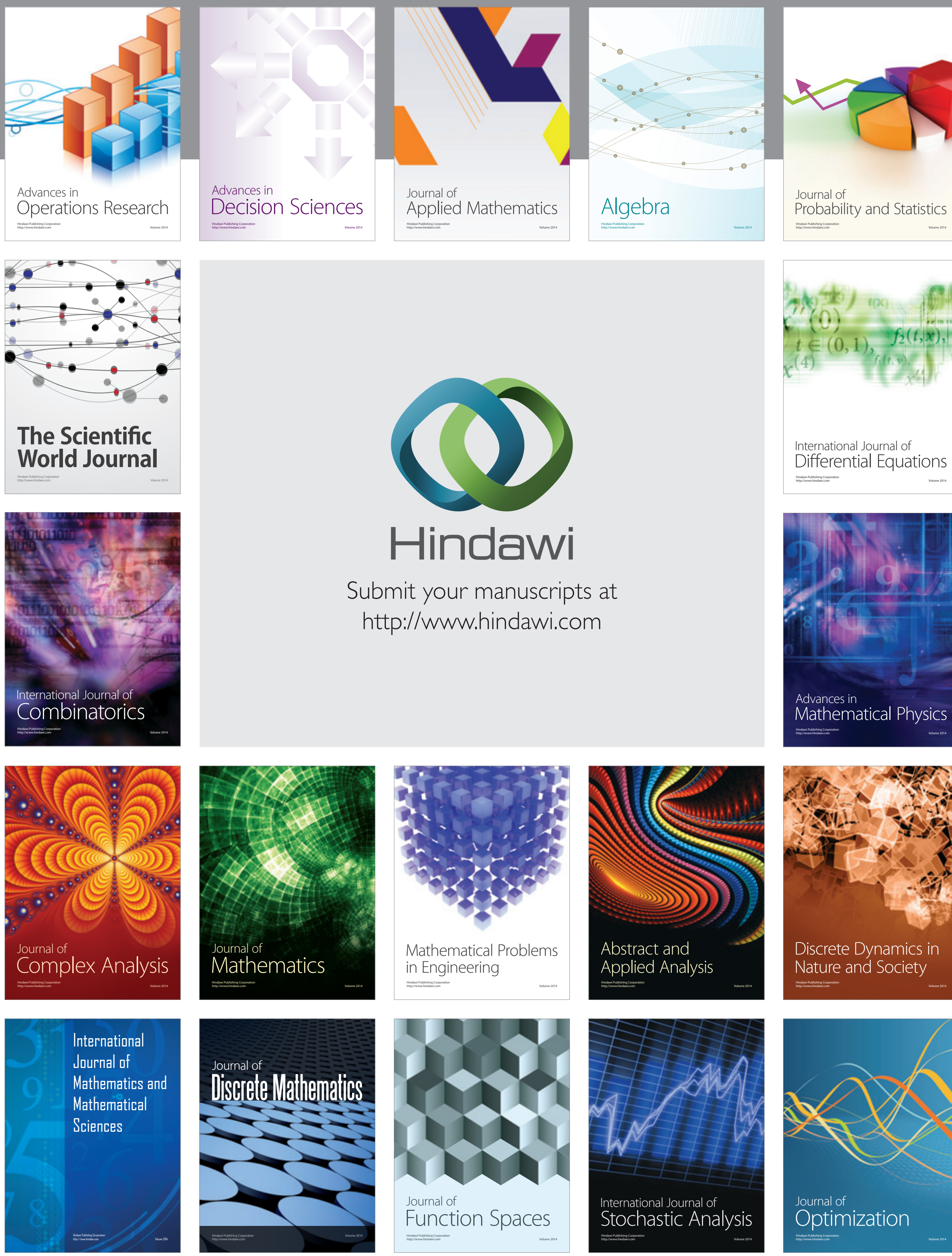
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