Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 213569, 13 pages
http://dx.doi.org/10.1155/2014/213569

Research Article

Global Existence and Large Time Behavior of Solutions to the Bipolar Nonisentropic Euler-Poisson Equations

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Received 6 November 2013; Accepted 22 December 2013; Published 27 January 2014

Academic Editor: Rita Tracinà

We study the one-dimensional bipolar nonisentropic Euler-Poisson equations which can model various physical phenomena, such as the propagation of electron and hole in submicron semiconductor devices, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. We show the existence and large time behavior of global smooth solutions for the initial value problem, when the difference of two particles' initial mass is nonzero, and the far field of two particles' initial temperatures is not the ambient device temperature. This result improves that of Y.-P. Li, for the case that the difference of two particles' initial mass is zero, and the far field of the initial temperature is the ambient device temperature.

1. Introduction

In this paper we study the following 1D bipolar nonisentropic Euler-Poisson equations:

$$
\begin{aligned}
& n_{1t} + j_{1x} = 0, \\
& j_{1t} + \left(\frac{j_1^2}{n_1}\right)_x + (n_1 T_1)_x = n_1 E - j_1, \\
& T_{1t} + \frac{j_1}{n_1} T_{1x} + \frac{2}{3} T_1 \left(\frac{j_1}{n_1}\right)_x - \frac{2}{3n_1} T_{1xx} = \frac{1}{3}\left(\frac{j_1}{n_1}\right)^2 - (T_1 - T^*), \\
& n_{2t} + j_{2x} = 0, \\
& j_{2t} + \left(\frac{j_2^2}{n_2}\right)_x + (n_2 T_2)_x = -n_2 E - j_2, \\
& T_{2t} + \frac{j_2}{n_2} T_{2x} + \frac{2}{3} T_2 \left(\frac{j_2}{n_2}\right)_x - \frac{2}{3n_2} T_{2xx} = \frac{1}{3}\left(\frac{j_2}{n_2}\right)^2 - (T_2 - T^*), \\
& E_x = n_1 - n_2,
\end{aligned}
\tag{1}
$$

where $n_i > 0, j_i, T_i, (i = 1, 2)$, and E denote the particle densities, current densities, temperatures, and the electric field, respectively, and $T^* > 0$ stands for the ambient device temperature. The system (1) models various physical phenomena, such as the propagation of electron and hole in submicron semiconductor derives, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. When the temperature T_i $(i = 1, 2)$ is the function of the density n_i $(i = 1, 2)$, the system (1) reduces to the isentropic bipolar Euler-Poisson equations. For more details on the bipolar isentropic and nonisentropic Euler-Poisson equations (hydrodynamic models), we can see [1–3] and so forth.

Due to their physical importance, mathematical complexity, and wide rang, of applications, many results concerning the existence and uniqueness of (weak, strong, or smooth) solutions for the bipolar Euler-Poisson equations can be found in [4–14] and the references cited therein. However, the study of the corresponding nonisentropic bipolar Euler-Poisson equation is very limited in the literature. In [15] Li investigated the global existence and nonlinear diffusive waves of smooth solutions for the initial value problem of the one-dimensional nonisentropic bipolar hydrodynamic model when the difference of two particles' initial mass is zero, and the far field of two particles' initial temperatures is the ambient device temperature. We also mention that there are some results about the relaxation limit and quasineutral limit of the bipolar Euler-Poisson system see [16–19]. In this paper, we will show the existence and large time behavior of global smooth solutions for the initial value problem of (1), when the difference of two particles' initial mass is nonzero and the far field of the initial temperatures is not the ambient device temperature. We now prescribe the following initial data:

$$\begin{gathered}(n_i, j_i, T_i)(x, t=0) = (n_{i0}, j_{i0}, T_{i0})(x), \\ n_{i0} > 0, \quad i = 1, 2, \\ \lim_{x \to \pm\infty} (n_{i0}, j_{i0}, T_{i0})(x) = (n_\pm, j_{i\pm}, T_{i\pm}), \\ n_\pm > 0, T_{i\pm} > 0, \quad i = 1, 2,\end{gathered} \tag{2}$$

and $(n_\pm, j_{i\pm}, T_{i\pm})$ are the state constants. We also give the electric field as $x = -\infty$; that is,

$$E(-\infty, t) = 0. \tag{3}$$

The nonlinear diffusive phenomena both in smooth and weak senses were also observed for the bipolar isentropic and nonisentropic by Gasser et al. [4], Huang and Li [5], and Li [15], respectively. Namely, according to the Darcy's law, it is expected that the solutions $(n_1, j_1, T_1, n_2, j_2, T_2, E)(x, t)$ converge in L^∞-sense to $(\bar{n}, \bar{j}, T^*, \bar{n}, \bar{j}, T^*, 0)(x, t)$; here $(\bar{n}, \bar{j}) = (\bar{n}, \bar{j})((x+x_0)/\sqrt{1+t})$ (x_0 is a shift constants) is the nonlinear diffusion waves, which is self-similar solutions to the following equations:

$$\begin{gathered}\bar{n}_t - (\bar{n}T^*)_{xx} = 0, \\ \bar{j} := -(\bar{n}T^*)_x, \\ (\bar{n}, \bar{j}) \longrightarrow (n_\pm, 0), \quad \text{as } x \longrightarrow \pm\infty.\end{gathered} \tag{4}$$

Note that in [15], the author assumed that

$$j_{i+} = j_{i-}, \quad T_{i\pm} = T^*, \quad i = 1, 2 \tag{5}$$

which lead to the difference of two particles' initial mass to be zero; that is,

$$\int_{\mathbb{R}} [n_{10}(x) - n_{20}(x)]\, dx = 0, \quad i = 1, 2. \tag{6}$$

This implies, from the last equation of (1), that

$$E(+\infty, t) - E(-\infty, t) = 0. \tag{7}$$

In this paper, we try to drop off these too stiff conditions. That is, $j_{i+} \neq j_{i-}, T_{i\pm} \neq T^*$ $(i = 1, 2)$. Moreover, for stating our results, set for $i = 1, 2$,

$$\begin{aligned}&(\varphi_{i0}, \psi_{i0}, \theta_{i0})(x) \\ &= \Big(\int_{-\infty}^{x} [n_{i0}(\xi) - \hat{n}_i(\xi, t) - \bar{n}(\xi + x_0, t = 0)]\, d\xi, \\ &\quad j_{i0}(x) - \hat{j}_{i0}(x) - \bar{j}(x + x_0, t = 0), T_{i0}(x) - \hat{T}_{i0}(x) - T^* \Big),\end{aligned} \tag{8}$$

where $(\hat{n}_1, \hat{j}_1, \hat{T}_1, \hat{n}_2, \hat{j}_2, \hat{T}_2, \hat{E})$ are the gap functions (or say correction functions) which will be given in Section 2, and $(\bar{n}, \bar{j}) = (\bar{n}, \bar{j})(x + x_0, t)$ are the shifted diffusion waves with $x_0 := (1/(n_+ - n_-)) \int_{\mathbb{R}} [n_{i0}(x) - \hat{n}_{i0}(x) - \bar{n}(x)]\, dx$ for $i = 1, 2$.

Throughout this paper, the diffusion waves are always denoted by $(\bar{n}, \bar{j})(x/\sqrt{1+t})$. C denotes the generic positive constant. $L^p(\mathbb{R})$ $(1 \le p < \infty)$ denotes the space of measurable functions whose p-powers are integrable on $\mathbb{R}$, with the norm $\|\cdot\|_{L^p} = (\int_{\mathbb{R}} |\cdot|^p dx)^{1/p}$, and L^∞ is the space of bounded measurable functions on $\mathbb{R}$, with the norm $\|\cdot\|_{L^\infty} = \operatorname{esssup}_x |\cdot|$. Without confusion, we also denote the norm of $L^2(\mathbb{R})$ by $\|\cdot\|$ for brevity. $H^k(\mathbb{R})$ (H^k without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_k$, especially $\|\cdot\|_0 = \|\cdot\|$.

Now we state our main results as follows.

Theorem 1. *Let $(\phi_{i0}, \psi_{i0}, \theta_{i0})(i = 1, 2) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^3(\mathbb{R})$, and set $\delta := |j_{1+}| + |j_{1-}| + |j_{2+}| + |j_{2-}| + |T_{1+} - T^*| + |T_{1-} - T^*| + |T_{2+} - T^*| + |T_{2-} - T^*| + |n_+ - n_-|$ and $\Phi_0 := \|(\phi_{10}, \phi_{20})\|_3 + \|(\psi_{10}, \psi_{20})\|_2 + \|(\theta_{10}, \theta_{20})\|_3$. Then, there is a $\delta_0 > 0$ such that if $\Phi_0 + \delta \le \delta_0$ the solutions $(n_1, n_2, j_1, j_2, \theta_1, \theta_2, E)$ of IVP* (1)–(3) *uniquely and globally exist and satisfy*

$$\begin{aligned}&n_1 - \hat{n}_1 - \bar{n}, n_2 - \hat{n}_2 - \bar{n} \\ &\quad \in C([0, +\infty), H^2(\mathbb{R})) \cap C^1([0, +\infty), H^1(\mathbb{R})), \\ &j_1 - \hat{j}_1 - \bar{j}, j_2 - \hat{j}_2 - \bar{j} \\ &\quad \in C([0, +\infty), H^2(\mathbb{R})) \cap C^1([0, +\infty), H^1(\mathbb{R})), \\ &T_1 - \hat{T}_1 - T^*, T_2 - \hat{T}_2 - T^* \\ &\quad \in C([0, +\infty), H^3(\mathbb{R})) \cap C^1([0, +\infty), H^1(\mathbb{R})), \\ &E - \hat{E} \in C([0, +\infty), H^3(\mathbb{R})) \\ &\qquad \cap C^1([0, +\infty), H^2(\mathbb{R})) \cap C^2([0, +\infty), H^1(\mathbb{R})).\end{aligned} \tag{9}$$

Moreover, it holds that

$$\begin{aligned}&\sum_{k=0}^{2} (1+t)^{k+1} \Big\| \partial_x^k \Big(n_1 - \hat{n}_1 - \bar{n}, T_1 - \hat{T}_1 - T^*, \\ &\qquad n_2 - \hat{n}_2 - \bar{n}, T_2 - \hat{T}_2 - T^* \Big)(t) \Big\|^2 \\ &\quad + \sum_{k=0}^{2} (1+t)^{k+2} \Big\| \partial_x^k (j_1 - \hat{j}_1 - \bar{j}, j_2 - \hat{j}_2 - \bar{j})(t) \Big\|^2 \\ &\quad + (1+t)^3 \Big\| \partial_x^3 (T_1 - \hat{T}_1 - T^*, T_2 - \hat{T}_2 - T^*)(t) \Big\|^2 \\ &\le C(\delta + \Phi_0),\end{aligned}$$

$$\begin{aligned}&\left\|\left(n_1-\widehat{n}_1-n_2+\widehat{n}_2\right)(t)\right\|_1^2\\&\quad+\left\|\left(j_1-\widehat{j}_1-j_2+\widehat{j}_2\right)(t)\right\|_1^2\\&\quad+\left\|\left(T_1-\widehat{T}_1-T_2+\widehat{T}_2\right)(t)\right\|_2^2\\&\quad+\left\|\left(E-\widehat{E}\right)(t)\right\|_2^2\\&\le C\left(\Phi_0+\delta\right)e^{-\alpha t},\end{aligned}\tag{10}$$

for some constant $\alpha>0$.

Remark 2. It is more important and interesting that we should discuss the existence and large time behavior of global smooth solution for the bipolar nonisentropic Euler-Poisson system with the general ambient device temperature functions, instead of the constant ambient device temperature, as in [20]. Moreover, we also should consider the similar problem for the corresponding multi-dimensional bipolar non-isentropic Euler-Poisson systems. These are left for the forthcoming future.

The rest of this paper is arranged as follows. In Section 2, we make some necessary preliminaries. That is, we first give some well-known results on the diffusion waves and one key inequality will be used later; then we trickly construct the correction functions to delete the gaps between the solutions and the diffusion waves at the far field. We reformulate the original problem in terms of a perturbed variable and state local-in-time existence of classical solutions in Section 3. Section 4 is used to establish the uniformly a priori estimate and to show the global existence of smooth solutions, while we prove the algebraic convergence rate of smooth solutions in Section 5.

2. Some Preliminaries

In this section, we state the nonlinear diffusive wave and then construct the correction functions. First of all, we list some known results concerning the self-similar solution of the nonlinear parabolic equation (4). Let us recall that the nonlinear parabolic equation

$$\begin{gathered}\bar{n}_t-\left(\bar{n}T^*\right)_{xx}=0,\\\bar{n}\longrightarrow n_\pm,\quad\text{as }x\longrightarrow\pm\infty,\end{gathered}\tag{11}$$

possesses a unique self-similar solution $\bar{n}(x,t)\triangleq\bar{n}(\xi)$, $\xi=x/\sqrt{1+t}$, which is increasing if $n_-<n_+$ and decreasing if $n_->n_+$. The corresponding Darcy law is $\bar{j}:=-(\bar{n}T^*)_x$ satisfying $\bar{j}\to 0$ as $x\to\pm\infty$.

Lemma 3 (see [4, 15, 21]). *For the self-similar solution of* (11), *it holds*

$$\begin{aligned}&\left|\bar{n}(\xi)-n_+\right|_{\xi>0}+\left|\bar{n}(\xi)-n_-\right|_{\xi<0},\\&\quad\le C\left|n_+-n_-\right|e^{-\nu_0\xi^2},\\&\left|\partial_x^k\partial_t^l\bar{n}(x,t)\right|\\&\quad\le C\left|n_+-n_-\right|(1+t)^{-(k+2l)/2}e^{-\nu_0\xi^2}\quad k+l\ge1,\ k,l\ge0,\\&\int_{-\infty}^0\left|\bar{n}(x,t)-n_-\right|^2dx+\int_0^{+\infty}\left|\bar{n}(x,t)-n_+\right|^2dx\\&\quad\le C\left|n_+-n_-\right|^2(1+t)^{1/2},\\&\int_{\mathbb{R}}\left|\partial_x^k\partial_t^l\bar{n}\right|^2dx\\&\quad\le C\left|n_+-n_-\right|^2(1+t)^{(1/2)-2l-k},\quad k+l\ge1,\end{aligned}\tag{12}$$

where $\nu_0>0$ *is a constant.*

Next, we construct the gap function, which will be used in Sections 3 and 4. First of all, motivated by [6, 22], let us look into the behaviors of the solutions to (1)–(3) at the far fields $x=\pm\infty$. Then we may understand how big the gaps are between the solution and the diffusion waves at the far fields. Let $(n_i^\pm(t),j_i^\pm(t),T_i^\pm(t)):=(n_i(\pm\infty,t),j_i(\pm\infty,t),T_i(\pm\infty,t)),i=1,2$ and $E^\pm(t):=E(\pm\infty,t)$. From $(1)_1$ and $(1)_4$, since $\partial_x j_i|_{x=\pm\infty}=0$ for $i=1,2$, it can be easily seen that

$$n_i^\pm(t)=n_i(\pm\infty,t)\equiv n_\pm.\tag{13}$$

Differentiating $(1)_7$ with respect to t and applying $(1)_1$ and $(1)_4$, we have $E_{tx}=(n_1-n_2)_t=-(j_1-j_2)_x$, which implies

$$\begin{aligned}&\frac{d}{dt}E^+(t)-\frac{d}{dt}E^-(t)\\&\quad=-\left[j_1^+(t)-j_2^+(t)\right]+\left[j_1^-(t)-j_2^-(t)\right].\end{aligned}\tag{14}$$

Taking $x=\pm\infty$ to $(1)_{2,3}$ and $(1)_{5,6}$, we also formally have

$$\frac{d}{dt}j_1^\pm(t)=n_\pm E^\pm(t)-j_1^\pm(t),\tag{15}$$

$$\frac{d}{dt}j_2^\pm(t)=-n_\pm E^\pm(t)-j_2^\pm(t),\tag{16}$$

$$\frac{d}{dt}T_i^\pm(t)=\frac{1}{3}\left(\frac{j_i^\pm}{n_i^\pm}\right)^2-\left(T_i^\pm-T^*\right),\quad i=1,2.\tag{17}$$

It can be easily seen that (14)–(17) can uniquely determine the unknown state functions $j_i^\pm(t),T_i^\pm(t)(i=1,2)$, and $E^+(t)$ since we have known $E^-(t)=E(-\infty,t)=0$. Solving these

O.D.E and noticing (13), there exists some constant $0 < \beta_0 < 1/2$ such that

$$\begin{gathered} n_i(\pm\infty, t) = n_{\pm}, \quad i = 1, 2, \\ \left|j_i(+\infty, t)\right| = O(1)e^{-\beta_0 t}, \quad i = 1, 2, \\ j_i(-\infty, t) = O(1)e^{-t}, \quad i = 1, 2, \\ T_i(\pm\infty, t) = T^* + \left(T_{i\pm} - T^*\right)e^{-t} + O(1)e^{-\beta_0 t}, \quad i = 1, 2, \\ |E(+\infty, t)| = O(1)e^{-\beta_0 t}, \\ E(-\infty, t) = 0. \end{gathered} \tag{18}$$

Obviously, there are some gaps between $j_i(\pm\infty, t)$ and $\bar{j}(\pm\infty, t)$, $T_i(\pm\infty, t)$ and T^*, and $E(+\infty, t)$ and $\overline{E} \equiv 0$, which lead to $j_i(x,t) - \bar{j}(x,t), T_i(x,t) - T^*, E(x,t) \notin L^2(\mathbb{R})$. To delete these gaps, we need to introduce the correction functions $(\widehat{n}_1, \widehat{n}_2, \widehat{j}_1, \widehat{j}_2, \widehat{T}_1, \widehat{T}_2, \widehat{E})(x,t)$. As those done in [6, 22], we can construct these gap functions. That is, we can choose $(\widehat{n}_1, \widehat{n}_2, \widehat{j}_1, \widehat{j}_2, \widehat{E})(x,t)$, which solve the system

$$\begin{aligned} &\widehat{n}_{1t} + \widehat{j}_{1x} = 0, \\ &\widehat{j}_{1t} = \check{n}\widehat{E} - \widehat{j}_1, \\ &\widehat{n}_{2t} + \widehat{j}_{2x} = 0, \\ &\widehat{j}_{2t} = -\check{n}\widehat{E} - \widehat{j}_2, \\ &\widehat{E}_x = \widehat{n}_1 - \widehat{n}_2, \end{aligned} \tag{19}$$

with $\widehat{j}_i(x,t) \rightarrow j_i^{\pm}(t)$ as $x \rightarrow \pm\infty$, $\widehat{E}(x,t) \rightarrow 0$ as $x \rightarrow -\infty$, and $\widehat{E}(x,t) \rightarrow E^+(t)$ as $x \rightarrow +\infty$. Here, $\check{n}(x) = n_- + (n_+ - n_-)\int_{-\infty}^{x+2L_0} m_0(y)dy$ with $m_0(x) \geq 0$, $m_0 \in C_0^{\infty}(\mathbb{R})$, $\operatorname{supp} m_0 \subseteq [-L_0, L_0]$, and $\int_{-\infty}^{+\infty} m_0(y)dy = 1$. Moreover, we take $\widehat{T}_i(x,t) = \widehat{T}_i^-(t)(1 - g(x)) + \widehat{T}_i^+(t)g(x)(i = 1, 2)$ with $g(x) = \int_{-\infty}^{x} m_0(y)\, dy$, which together with (17) implies

$$\begin{aligned} \frac{\partial}{\partial t}\widehat{T}_i(x,t) &= -\widehat{T}_i(x,t) + \frac{1}{3}\left(\frac{j_i^-(t)}{n_-}\right)^2(1 - g(x)) \\ &\quad + \frac{1}{3}\left(\frac{j_i^+(t)}{n_+}\right)^2 g(x) \\ &=: -\widehat{T}_i(x,t) + S_i(x,t), \quad i = 1, 2. \end{aligned} \tag{20}$$

In conclusion, we have constructed the required correction functions $(\widehat{n}_1, \widehat{n}_2, \widehat{j}_1, \widehat{j}_2, \widehat{T}_1, \widehat{T}_2, \widehat{E})$ which satisfy

$$\begin{gathered} \widehat{n}_{1t} + \widehat{j}_{1x} = 0, \\ \widehat{j}_{1t} = \check{n}\widehat{E} - \widehat{j}_1, \\ \widehat{n}_{2t} + \widehat{j}_{2x} = 0, \\ \widehat{j}_{2t} = -\check{n}\widehat{E} - \widehat{j}_2, \\ \widehat{T}_{it} = -\widehat{T}_i + S_i(x,t), \\ \widehat{E}_x = \widehat{n}_1 - \widehat{n}_2, \\ \text{with } \begin{cases} \widehat{j}_i(x,t) \rightarrow j_i^{\pm}(t), & \text{as } x \rightarrow \pm\infty, \\ \widehat{T}_i(x,t) \rightarrow T_i^{\pm}(t) - T^*, & \text{as } x \rightarrow \pm\infty, \\ \widehat{E}(x,t) \rightarrow 0, & \text{as } x \rightarrow -\infty, \\ \widehat{E}(x,t) \rightarrow E^+(t), & \text{as } x \rightarrow +\infty. \end{cases} \end{gathered} \tag{21}$$

Since these details can be found in [6, 22], we only give the following decay time-exponentially of $(\widehat{n}_1, \widehat{n}_2, \widehat{j}_1, \widehat{j}_2, \widehat{T}_1, \widehat{T}_2, \widehat{E})(x,t)$.

Lemma 4. *There exist positive constants C and $\nu < 1/2$ independent of t, such that*

$$\left\|(\widehat{n}_i, \widehat{j}_i, \widehat{T}_i, \widehat{E})(t)\right\|_{L^{\infty}(\mathbb{R})} \leq C\delta e^{-\nu t}, \quad i = 1, 2, \tag{22}$$

and $\operatorname{supp}\widehat{n}_i = \operatorname{supp} m_0 \subseteq [-L_0, L_0]$, $i = 1, 2$.

3. Reformulation of Original Problem

In this section, we first reformulate the original problem in terms of the perturbed variables. Setting for $i = 1, 2$,

$$\begin{aligned} &(\varphi_i, \psi_i, \theta_i, \mathscr{H})(x,t) \\ &\quad := \bigg(\int_{-\infty}^{x}\left[n_i(\xi,t) - \widehat{n}_i(\xi,t) - \bar{n}(\xi + x_0, t)\right] d\xi, \\ &\qquad j_i(x,t) - \widehat{j}_i(x,t) - \bar{j}(x + x_0, t), T_i(x,t) \\ &\qquad - \widehat{T}_i(x,t) - T^*, E(x,t) - \widehat{E}(x,t)\bigg), \end{aligned} \tag{23}$$

then from (1), (11), and (21), we have for $i = 1, 2$,

$$\varphi_{it} + \psi_i = 0,$$

$$\psi_{it} + \left(\frac{\left(-\varphi_{it} + \hat{j}_i + \bar{j}\right)^2}{\varphi_{ix} + \hat{n}_i + \bar{n}} + \left(\varphi_{ix} + \hat{n}_i + \bar{n}\right) \times \left(\theta_i + \hat{T}_i + T^*\right) - \bar{n}T^* \right)_x$$
$$= (-1)^{i-1} \left(\varphi_{ix} + \hat{n}_i + \bar{n}\right) \mathscr{H} + (-1)^{i-1} \left(\varphi_{ix} + \hat{n}_i + \bar{n} - \check{n}\right) \hat{E} - \psi_i + \left(\bar{n}T^*\right)_{tx},$$
$$\theta_{it} + \frac{-\varphi_{it} + \hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \left(\theta_i + \hat{T}_i + T^*\right)_x + \frac{2}{3} \left(\frac{-\varphi_{it} + \hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \right)_x \left(\theta_i + \hat{T}_i + T^*\right) - \frac{2}{3\left(\varphi_{ix} + \hat{n}_i + \bar{n}\right)} \left(\theta_i + \hat{T}_i + T^*\right)_{xx}$$
$$= \left[\frac{1}{3} \left(\frac{-\varphi_{it} + \hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \right)^2 - S_i(x,t) \right] - \theta_i,$$
$$\mathscr{H} = \varphi_1 - \varphi_2, \tag{24}$$

with the initial data $(\varphi_i, \psi_i, \theta_i)(x,0) = (\varphi_{10}, \psi_{i0}, \theta_{i0})(x)$, $i = 1, 2$. Further, we have

$$\begin{aligned}
&\varphi_{1tt} + \varphi_{1t} - \left(\left(\theta_1 + \hat{T}_1 + T^*\right) \varphi_{1x} + \bar{n}\theta_1\right)_x + \left(\varphi_{1x} + \hat{n}_1 + \bar{n}\right) \mathscr{H} \\
&\quad = -f_1 + g_{1x} - \left(\bar{n}T^*\right)_{tx}, \\
&\varphi_{2tt} + \varphi_{2t} - \left(\left(\theta_2 + \hat{T}_2 + T^*\right) \varphi_{2x} + \bar{n}\theta_1\right)_x - \left(\varphi_{2x} + \hat{n}_2 + \bar{n}\right) \mathscr{H} \\
&\quad = f_2 + g_{2x} - \left(\bar{n}T^*\right)_{tx}, \\
&\theta_{1t} + \theta_1 - \frac{2}{3\left(\varphi_{1x} + \hat{n}_1 + \bar{n}\right)} \theta_{1xx} - \frac{2}{3} \left(\theta_1 + \hat{T}_1 + T^*\right) \left(\frac{\varphi_{1t}}{\varphi_{1x} + \hat{n}_1 + \bar{n}} \right)_x = G_1, \\
&\theta_{2t} + \theta_2 - \frac{2}{3\left(\varphi_{2x} + \hat{n}_2 + \bar{n}\right)} \theta_{2xx} - \frac{2}{3} \left(\theta_2 + \hat{T}_2 + T^*\right) \left(\frac{\varphi_{2t}}{\varphi_{2x} + \hat{n}_2 + \bar{n}} \right)_x = G_2,
\end{aligned} \tag{25}$$

with the initial data

$$\begin{aligned}
\varphi_i(x,0) &= \varphi_{i0}(x), \\
\varphi_{it}(x,0) &= -\psi_{i0}(x), \\
\theta_i(x,0) &= \theta_{i0}(x), \\
i &= 1, 2.
\end{aligned} \tag{26}$$

Here

$$f_i = \left(\varphi_{ix} + \hat{n}_i + \bar{n} - \check{n}\right) \hat{E} - \left(\left(\theta_i + \hat{T}_i + T^*\right) \hat{n}_i + \bar{n}\hat{T}_i\right)_x,$$
$$g_i = \frac{\left(-\varphi_{it} + \hat{j}_i + \bar{j}\right)^2}{\varphi_{ix} + \hat{n}_i + \bar{n}},$$
$$G_i = -\frac{-\varphi_{it} + \hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \left(\theta_i + \hat{T}_i + T^*\right)_x - \frac{2}{3} \left(\frac{\hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \right)_x \left(\theta_i + \hat{T}_i + T^*\right) - \frac{2}{3} \left(\frac{\hat{j}_i + \bar{j}}{\varphi_{ix} + \hat{n}_i + \bar{n}} \right)_x \left(\theta_i + \hat{T}_i + T^*\right) - S_i(x,t). \tag{27}$$

By the standard iteration methods (see [23]), we can prove the local existence of classical solutions of the IVP (25) and (26). Here for the sake of clarity, we only state result and omit the proof.

Lemma 5. *Suppose that $(\varphi_{i0}, -\psi_{i0}, \theta_{i0}) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}) \times H^3(\mathbb{R})$ for $i = 1, 2$. Then there is a $C_1 > 0$ such that if*

$$\left\|\left(\varphi_{10}, \theta_{10}, \varphi_{20}, \theta_{20}\right)\right\|_3^2 + \left\|\left(\psi_{10}, \psi_{20}\right)\right\|_2^2 \le C_1, \tag{28}$$

then there is a positive number T_0 such that the initial value problems (25) and (26) have a unique solution $(\varphi_1, \theta_1, \varphi_2, \theta_2)$ satisfying $\varphi_i \in C([0, T_0]; H^3(\mathbb{R})) \cap$

$C^1([0,T_0];H^2(\mathbb{R}))\cap C^2([0,T_0];\ H^1(\mathbb{R})),\theta_i\in C([0,T_0];H^3(\mathbb{R}))\cap C^1([0,T_0];H^1(\mathbb{R}))(i=1,2)$, *and*

$$\begin{aligned}&\left\|(\varphi_1,\theta_1,\varphi_2,\theta_2)(\cdot,t)\right\|_3^2+\left\|(\varphi_{1t},\varphi_{2t})(\cdot,t)\right\|_2^2\\&\quad+\left\|(\theta_{1t},\theta_{2t})(\cdot,t)\right\|_1^2\\&\le C,\end{aligned}\tag{29}$$

for some positive constant C.

To end this section, we also derive

$$\begin{aligned}&\mathscr{H}_{tt}+\mathscr{H}_t+2\bar{n}\mathscr{H}-\left(\bar{n}(\theta_1-\theta_2)+\left(\theta_1+\widehat{T}_1+T^*\right)\mathscr{H}_x\right)_x\\&\quad=h_{1x}-h_2-h_3+h_{4x},\end{aligned}\tag{30}$$

$$\begin{aligned}&(\theta_1-\theta_2)_t+(\theta_1-\theta_2)-\frac{2}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})}(\theta_1-\theta_2)_{xx}\\&\quad-\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})}\mathscr{H}_{tx}\\&=G_3,\end{aligned}\tag{31}$$

where

$$\begin{aligned}h_1&:=\left(\theta_1+\widehat{T}_1+T^*\right)(\widehat{n}_1-\widehat{n}_2)+(\varphi_{2x}+\widehat{n}_2)(\theta_1-\theta_2)\\&\quad+(\varphi_{2x}+\widehat{n}_2+\bar{n})\left(\widehat{T}_1-\widehat{T}_2\right),\\h_2&:=(\varphi_{1x}+\varphi_{2x}+\widehat{n}_1+\widehat{n}_2)\mathscr{H},\\h_3&:=[\varphi_{1x}+\varphi_{2x}+\widehat{n}_1+\widehat{n}_2+2(\bar{n}-\check{n})]\widehat{E},\\h_4&:=\frac{\left(-\varphi_{1t}+\widehat{j}_1+\bar{j}\right)^2}{\varphi_{1x}+\widehat{n}_1+\bar{n}}-\frac{\left(-\varphi_{2t}+\widehat{j}_2+\bar{j}\right)^2}{\varphi_{2x}+\widehat{n}_2+\bar{n}},\end{aligned}$$

$$\begin{aligned}G_3&:=G_1-G_2+\left[\frac{2}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})}-\frac{2}{3(\varphi_{2x}+\widehat{n}_2+\bar{n})}\right]\theta_{2xx}\\&\quad+\left[\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})}-\frac{2\left(\theta_2+\widehat{T}_2+T^*\right)}{3(\varphi_{2x}+\widehat{n}_2+\bar{n})}\right]\varphi_{2tx}\\&\quad-\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})^2}\varphi_{1t}\mathscr{H}_{xx}-\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})^2}\widehat{n}_{1x}\mathscr{H}_t\\&\quad+\left[\frac{2\left(\theta_2+\widehat{T}_2+T^*\right)}{3(\varphi_{2x}+\widehat{n}_2+\bar{n})^2}\varphi_{2t}-\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})^2}\varphi_{1t}\right]\\&\quad\times(\varphi_{2x}+\bar{n})_x\\&\quad-\left[\frac{2\left(\theta_2+\widehat{T}_2+T^*\right)}{3(\varphi_{2x}+\widehat{n}_2+\bar{n})^2}\widehat{n}_{1x}-\frac{2\left(\theta_1+\widehat{T}_1+T^*\right)}{3(\varphi_{1x}+\widehat{n}_1+\bar{n})^2}\widehat{n}_{2x}\right]\varphi_{2t}.\end{aligned}\tag{32}$$

4. Global Existence of Smooth Solutions

In this section we mainly prove global existence of smooth solutions for the initial value problems (25) and (26). To begin with, we focus on the a priori estimates of $(\varphi_1,\theta_1,\varphi_2,\theta_2)$. For this purpose, letting $T\in(0,+\infty)$, we define

$$\begin{aligned}X(T)&=\left\{(\varphi_i,\varphi_{it},\theta_i,\theta_{it}):\partial_t^j\varphi_i\in C\left([0,T];H^{3-j}(\mathbb{R})\right),\right.\\&\qquad\theta_i\in C\left([0,T];H^3(\mathbb{R})\right),\theta_{it}\in C\left([0,T];H^1(\mathbb{R})\right),\\&\qquad\left.i=1,2,\ j=0,1\right\},\end{aligned}\tag{33}$$

with the norm

$$\begin{aligned}N(T)^2=\max_{0\le t\le T}&\left\{\left\|(\varphi_1,\varphi_2,\theta_1,\theta_2)(t)\right\|_3^2+\left\|(\varphi_{1t},\varphi_{2t})(t)\right\|_2^2\right.\\&\left.+\left\|(\theta_{1t},\theta_{2t})(t)\right\|_1^2\right\}.\end{aligned}\tag{34}$$

Let $N(T)^2\le\varepsilon^2$, where ε is sufficiently small and will be determined later. Then, by Sobolev inequality, we have for $i=1,2$,

$$\left\|(\varphi_i,\varphi_{ix},\varphi_{ixx},\theta_i,\theta_{ix},\theta_{it},\theta_{ixx},\varphi_{it},\varphi_{itx})(t)\right\|_{L^\infty}\le C\varepsilon.\tag{35}$$

Clearly, there exists a positive constant c_1,c_2 such that

$$\begin{aligned}&0<\frac{1}{c_1}\le n_i=\varphi_{ix}+\widehat{n}_i+\bar{n}\le c_1,\\&0<\frac{1}{c_2}\le T_i=\theta_i+\widehat{T}_i+T^*\le c_2,\\&\qquad i=1,2.\end{aligned}\tag{36}$$

Further, from $(24)_7$, we also have $\partial_t^j\mathscr{H}\in C(0,T;H^{2-j}(\mathbb{R}))$ and

$$\left\|(\mathscr{H},\mathscr{H}_x,\mathscr{H}_t)(t)\right\|_{L^\infty}\le C\varepsilon.\tag{37}$$

Now we first establish the following basic energy estimate.

Lemma 6. *Let $(\varphi_1,\theta_1,\varphi_2,\theta_2)(x,t)\in X(T)$ be the solution of the initial value problem (25) and (26). If $\delta+\varepsilon\ll 1$, then it holds that for $0<t<T$,*

$$\sum_{i=0}^{2}\left\|(\varphi_i,\varphi_{ix},\varphi_{it},\theta_i)(\cdot,t)\right\|^2+\left\|\mathscr{H}(\cdot,t)\right\|^2+\int_0^t\left\|\mathscr{H}(\cdot,\tau)\right\|^2d\tau+\sum_{i=1}^{2}\int_0^t\left\|(\varphi_{ix},\varphi_{it},\theta_i,\theta_{ix})(\cdot,\tau)\right\|^2d\tau\le C(\Phi_0+\delta). \tag{38}$$

Proof. Multiplying $(25)_1$ and $(25)_2$ by φ_1 and φ_2, respectively, and integrating them over $\mathbb{R}$ by parts, we have for $i=1,2$,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\varphi_i\varphi_{it}+\frac{1}{2}\varphi_i^2\right)dx+\int_{\mathbb{R}}\left(\theta_i+\widehat{T}_i+T^*\right)\varphi_{ix}^2\,dx\\&\quad+(-1)^{i-1}\int_{\mathbb{R}}\left(\varphi_{ix}+\widehat{n}_i+\overline{n}\right)\mathscr{H}\varphi_i dx-\int_{\mathbb{R}}\varphi_{it}^2\,dx\\&=-\int_{\mathbb{R}}\overline{n}\theta_i\varphi_{ix}\,dx+\int_{\mathbb{R}}\left(\overline{n}T^*\right)_t\varphi_{ix}\,dx\\&\quad+(-1)^i\int_{\mathbb{R}}f_i\varphi_i dx-\int_{\mathbb{R}}g_i\varphi_{ix}dx.\end{aligned} \tag{39}$$

Using Cauchy-Schwartz's inequality, and Lemmas 3 and 4, we have

$$\begin{aligned}&-\int_{\mathbb{R}}\overline{n}\theta_i\varphi_{ix}\,dx+\int_{\mathbb{R}}\left(\overline{n}T^*\right)_t\varphi_{ix}\,dx\\&\quad\le\kappa\int_{\mathbb{R}}\varphi_{ix}^2\,dx+C\int_{\mathbb{R}}\left(\theta_i^2+\overline{n}_t^2\right)dx,\end{aligned} \tag{40}$$

where and in the subsequent $\kappa>0$ is some proper small constant, and

$$(-1)^i\int_{\mathbb{R}}f_i\varphi_i\,dx\le C\varepsilon\int_{\mathbb{R}}\varphi_{ix}^2\,dx+C\delta^2(1+t)^{1/4}e^{-\nu t}, \tag{41}$$

where we also used the facts

$$\int_{\mathbb{R}}\left(\overline{n}-\check{n}\right)^2\,dx\le C\delta^2(1+t)^{1/2} \tag{42}$$

which can be proved from the construction of $\check{n}(x)\to n_\pm$, as $x\to\pm\infty$, and the property of the diffusion wave $\overline{n}((x+x_0)/\sqrt{(1+t)})$. Similarly, we can show

$$\begin{aligned}&-\int_{\mathbb{R}}g_i\varphi_{ix}dx\\&\quad=-\int_{\mathbb{R}}\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{\varphi_{ix}+\widehat{n}_i+\overline{n}}\varphi_{ix}dx\\&\quad\le C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{ix}^2+\varphi_{it}^2\right)dx+C\delta\int_{\mathbb{R}}\overline{n}_x^2\,dx+C\delta^2e^{-\nu t},\end{aligned} \tag{43}$$

which together with (39)–(41) implies,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\varphi_i\varphi_{it}+\frac{1}{2}\varphi_i^2\right)dx+\int_{\mathbb{R}}\left[\left(\theta_i+\widehat{T}_i+T^*\right)-\kappa\right]\varphi_{ix}^2\,dx\\&\quad-\int_{\mathbb{R}}\varphi_{it}^2\,dx+(-1)^{i-1}\int_{\mathbb{R}}\left(\varphi_{ix}+\widehat{n}_i+\overline{n}\right)\mathscr{H}\varphi_i\,dx\\&\le C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{ix}^2+\varphi_{it}^2\right)dx\\&\quad+C\int_{\mathbb{R}}\left(\theta_i^2+\overline{n}_t^2+\overline{n}_x^2\right)dx+C\delta^2e^{-\nu_1 t},\end{aligned} \tag{44}$$

where $0<\nu_1<\nu$. Moreover, for the coupled term with the electric field, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(\left(\varphi_{1x}+\widehat{n}_1+\overline{n}\right)\mathscr{H}\varphi_1-\left(\varphi_{2x}+\widehat{n}_2+\overline{n}\right)\mathscr{H}\varphi_2\right)dx\\&\quad\ge\int_{\mathbb{R}}\overline{n}\mathscr{H}^2\,dx-C\varepsilon\int_{\mathbb{R}}\left(\mathscr{H}^2+\varphi_{1x}^2+\varphi_{2x}^2\right)dx-C\delta^2e^{-\nu t}.\end{aligned} \tag{45}$$

Next, multiplying $(25)_1$ and $(25)_2$ by φ_{1t} and φ_{2t}, respectively, and integrating their sum over $\mathbb{R}$ by parts, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}\varphi_{it}^2+\frac{1}{2}\left(\theta_i+\widehat{T}_i+T^*\right)\varphi_{ix}^2\right)dx\\&\quad+\int_{\mathbb{R}}\left(\varphi_{it}^2+\overline{n}\theta_i\varphi_{itx}+(-1)^{i-1}\left(\varphi_{ix}+\widehat{n}_i+\overline{n}\right)\mathscr{H}\varphi_{it}\right)dx\\&=(-1)^i\int_{\mathbb{R}}f_i\varphi_{it}dx+\int_{\mathbb{R}}g_{ix}\varphi_{it}dx\\&\quad-\int_{\mathbb{R}}\left[\left(\overline{n}T^*\right)_{tx}\varphi_{it}-\frac{1}{2}\left(\theta_i+\widehat{T}_i+T^*\right)_t\varphi_{ix}^2\right]dx.\end{aligned} \tag{46}$$

Using Schwartz's inequality, (42), and Lemmas 3 and 4, we have

$$\begin{aligned}&-\int_{\mathbb{R}}\left[\left(\overline{n}T^*\right)_{tx}\varphi_{it}-\frac{1}{2}\left(\theta_i+\widehat{T}_i+T^*\right)_t\varphi_{ix}^2\right]dx\\&\quad+(-1)^{i-1}\int_{\mathbb{R}}f_i\varphi_{it}\,dx\\&\quad\le\kappa\int_{\mathbb{R}}\varphi_{it}^2\,dx+C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{it}^2+\varphi_{ix}^2\right)dx\\&\quad+C\int_{\mathbb{R}}\overline{n}_{tx}^2\,dx+C\delta^2(1+t)^{1/4}e^{-\nu t}.\end{aligned} \tag{47}$$

Since

$$\begin{aligned}g_{ix}=&-\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{\left(\varphi_{ix}+\widehat{n}_i+\overline{n}\right)^2}\varphi_{ixx}-\frac{2\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)}{\left(\varphi_{ix}+\widehat{n}_i+\overline{n}\right)}\varphi_{ixt}\\&+O(1)\left[\left(\varphi_{it}+\widehat{j}_i+\overline{j}\right)\left(\widehat{n}_i+\overline{n}\right)_t+\left(\widehat{j}_i+\overline{j}\right)^2\left(\widehat{n}_i+\overline{n}\right)_x\right.\\&\left.+\left(\widehat{n}_i+\overline{n}\right)_x\varphi_{it}^2\right],\end{aligned} \tag{48}$$

we obtain, after integration by parts, that

$$
\begin{aligned}
&\int_{\mathbb{R}} g_{ix}\varphi_{it}\,dx \\
&\quad\le \frac{d}{dt}\int_{\mathbb{R}} \frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{2(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}\varphi_{ix}^2\,dx \\
&\qquad + C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{it}^2+\varphi_{ix}^2\right)dx \\
&\qquad + C\delta\int_{\mathbb{R}}\left(\overline{n}_t^2+\overline{n}_x^4\right)dx + C\delta^2 e^{-\nu t},
\end{aligned}
\tag{49}
$$

where we have used

$$
\begin{aligned}
&\left\|\left(\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}\right)_x\right\|_{L^\infty},\ \left\|\left(\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}\right)_t\right\|_{L^\infty} \\
&\quad\le C(\delta+\varepsilon),
\end{aligned}
\tag{50}
$$

with the aid of $|\varphi_{itt}| < C|\varphi_{ixx}+\varphi_{ixt}+\varphi_{ix}+\varphi_{it}+\varphi_i+\theta_i+\theta_{ix}+\overline{n}_{xt}| + C\delta^2 e^{-\nu t}$. Putting the above inequality into (46), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{1}{2}\varphi_{it}^2+\frac{1}{2}\left(\theta_i+\widehat{T}_i+T^*\right)\varphi_{ix}^2-\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{2(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}\varphi_{ix}^2\right]dx \\
&\qquad + (1-\kappa)\int_{\mathbb{R}}\varphi_{it}^2\,dx + (-1)^{i-1}\int_{\mathbb{R}}(\varphi_{ix}+\widehat{n}_i+\overline{n})\mathscr{H}\varphi_{it}\,dx \\
&\qquad + \int_{\mathbb{R}}\overline{n}\theta_i\varphi_{itx}\,dx \\
&\quad\le C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{it}^2+\varphi_{ix}^2\right)dx \\
&\qquad + C\int_{\mathbb{R}}\left(\overline{n}_{tx}^2+\overline{n}_t^2+\overline{n}_x^4\right)dx + C\delta^2 e^{-\nu_1 t}.
\end{aligned}
\tag{51}
$$

On the other hand, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\left((\varphi_{1x}+\widehat{n}_1+\overline{n})\mathscr{H}\varphi_{1t}-(\varphi_{2x}+\widehat{n}_2+\overline{n})\mathscr{H}\varphi_{2t}\right)dx \\
&\quad\ge \frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}\overline{n}\mathscr{H}^2\,dx - \int_{\mathbb{R}}\frac{1}{2}\overline{n}_t\mathscr{H}^2 dx \\
&\qquad - C\varepsilon\int_{\mathbb{R}}\left(\varphi_{1x}^2+\varphi_{1t}^2+\varphi_{2x}^2+\varphi_{2t}^2\right)dx - C\delta e^{-\nu t}.
\end{aligned}
\tag{52}
$$

Finally, multiplying $(25)_l$ $(l=3,4)$ by $(3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})/2(\theta_i+\widehat{T}_i+T^*))\theta_i$ $(i=1,2)$ and integrating the resultant equation by parts over $\mathbb{R}$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{4\left(\theta_i+\widehat{T}_i+T^*\right)}\theta_i^2 dx \\
&\qquad + \int_{\mathbb{R}}\left[\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\theta_i^2+\frac{\overline{n}}{\theta_i+\widehat{T}_i+T^*}\theta_{ix}^2\right]dx, \\
&- \int_{\mathbb{R}}\overline{n}\theta_i\varphi_{itx}dx = \int_{\mathbb{R}}\left(\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{4\left(\theta_i+\widehat{T}_i+T^*\right)}\right)_t\theta_i^2\,dx \\
&\qquad - \int_{\mathbb{R}}\left(\frac{\overline{n}}{\theta_i+\widehat{T}_i+T^*}\right)_x\theta_i\theta_{ix}\,dx \\
&\qquad - \int_{\mathbb{R}}\frac{\overline{n}\theta_i}{\varphi_{ix}+\widehat{n}_i+\overline{n}}\varphi_{it}(\varphi_{ix}+\widehat{n}_i+\overline{n})_x\,dx \\
&\qquad + \int_{\mathbb{R}} G_i\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\theta_i\,dx.
\end{aligned}
\tag{53}
$$

Now we estimate the term of the right hand side of (53), using Cauchy-Schwartz's inequality and Lemmas 3 and 4. First, with the help of the following equality $\widehat{j}_i=(1-g(x))j_i^-(t)+g(x)j_i^+(t)$ (see [6, 22]), we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\left(\frac{\widehat{j}_i^2}{3(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}-S_i(x,t)\right)\theta_i\,dx \\
&\quad = \left\{\int_{-\infty}^{-L_0}+\int_{L_0}^{+\infty}+\int_{-L_0}^{L_0}\right\}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)} \\
&\qquad \times\left(\frac{\widehat{j}_i^2}{3(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}-S_i(x,t)\right)\theta_i\,dx \\
&\quad = \int_{-\infty}^{-L_0}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\theta_i\widehat{j}_i^2\left(\frac{1}{3n_i^2}-\frac{1}{3n_-^2}\right)dx \\
&\qquad + \int_{L_0}^{+\infty}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\times\theta_i\widehat{j}_i^2\left(\frac{1}{3n_i^2}-\frac{1}{3n_+^2}\right)dx \\
&\qquad + \int_{-L_0}^{L_0}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\theta_i \\
&\qquad \times\left[\frac{\widehat{j}_i^2}{3n_i^2}-\frac{(1-g(x))}{3n_-^2}\left(j^-(t)\right)^2-\frac{g(x)}{3n_+^2}\left(j^+(t)\right)^2\right]dx \\
&\quad \le C\delta\int_{\mathbb{R}}\left(\varphi_{ix}^2+\theta_i^2\right)\,dx + C\delta^2 e^{-\nu t}\int_{-\infty}^{-L_0}(\overline{n}-n_-)^2\,dx \\
&\qquad + C\delta^2 e^{-\nu t}\int_{L_0}^{+\infty}(\overline{n}-n_+)^2\,dx + C\delta^2 e^{-\nu t} \\
&\quad \le C\delta\int_{\mathbb{R}}\left(\varphi_{ix}^2+\theta_i^2\right)\,dx + C\delta^2(1+t)^{1/2}e^{-\nu t},
\end{aligned}
\tag{54}
$$

which implies

$$
\begin{aligned}
&\int_{\mathbb{R}}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\left(\frac{\left(-\varphi_{it}+\widehat{j}_i+\overline{j}\right)^2}{3(\varphi_{ix}+\widehat{n}_i+\overline{n})^2}-S_i(x,t)\right)\theta_i\,dx \\
&\quad = \int_{\mathbb{R}}\frac{3\overline{n}(\varphi_{ix}+\widehat{n}_i+\overline{n})}{2\left(\theta_i+\widehat{T}_i+T^*\right)}\left(\frac{\varphi_{it}^2}{3n_i^2}-\frac{2\varphi_{it}\left(\widehat{j}_i+\overline{j}\right)}{3n_i^2}\right)\theta_i\,dx
\end{aligned}
$$

$$+\int_{\mathbb{R}} \frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{2(\theta_i+\widehat{T}_i+T^*)} \frac{\bar{j}^2+2\hat{j}_i\bar{j}}{3n_i^2}\theta_i\,dx$$
$$+\int_{\mathbb{R}} \frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{2(\theta_i+\widehat{T}_i+T^*)}\theta_i\left(\frac{\hat{j}_i^2}{3n_i^2}-S_i(x,t)\right)dx$$
$$\leq C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\varphi_{ix}^2+\varphi_{it}^2+\theta_i^2\right)dx$$
$$+C\delta\int_{\mathbb{R}}\bar{n}_x^2\,dx+C\delta^2(1+t)^{1/2}e^{-\nu t}. \tag{55}$$

From the definition of G_i ($i = 1,2$), and using Schwartz's inequality, we have

$$\int_{\mathbb{R}} G_i\frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{2(\theta_i+\widehat{T}_i+T^*)}\theta_i\,dx$$
$$\leq \kappa\int_{\mathbb{R}}\theta_i^2\,dx+C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\theta_i^2+\theta_{ix}^2+\varphi_{ix}^2+\varphi_{it}^2\right)dx$$
$$+C\int_{\mathbb{R}}\left(\bar{n}_x^2+\bar{n}_{xx}^2\right)dx+C\delta^2e^{-\nu_2 t}, \tag{56}$$

with $\nu_1 < \nu_2 < \nu$. And using Schwartz's inequality and Lemma 3 yields

$$\int_{\mathbb{R}}\left(\frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{4(\theta_i+\widehat{T}_i+T^*)}\right)_t\theta_i^2\,dx$$
$$-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_i+\widehat{T}_i+T^*}\right)_x\theta_i\theta_{ix}\,dx$$
$$\leq C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\theta_i^2+\theta_{ix}^2\right)dx, \tag{57}$$
$$-\int_{\mathbb{R}}\frac{\bar{n}\theta_i}{\varphi_{ix}+\hat{n}_i+\bar{n}}\varphi_{it}(\varphi_{ix}+\hat{n}_i+\bar{n})_x dx$$
$$\leq \frac{d}{dt}\int_{\mathbb{R}}\frac{\bar{n}\theta_i}{2(\varphi_{ix}+\hat{n}_i+\bar{n})}\varphi_{ix}^2 dx$$
$$+C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\theta_i^2+\varphi_{ix}^2+\varphi_{it}^2\right)dx.$$

Putting the above inequalities into (53) yields

$$\frac{d}{dt}\int_{\mathbb{R}}\left[\frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{4(\theta_1+\widehat{T}_i+T^*)}\theta_i^2-\frac{\bar{n}\theta_i}{2(\varphi_{ix}+\hat{n}_i+\bar{n})}\varphi_{ix}^2\right]dx$$
$$+\int_{\mathbb{R}}\left[\left(\frac{3\bar{n}(\varphi_{ix}+\hat{n}_i+\bar{n})}{2(\theta_i+\widehat{T}_i+T^*)}-\kappa\right)\theta_i^2+\frac{\bar{n}}{\theta_i+\widehat{T}_i+T^*}\theta_{ix}^2\right]dx$$
$$-\int_{\mathbb{R}}\bar{n}\theta_i\varphi_{itx}\,dx$$
$$\leq C(\delta+\varepsilon)\int_{\mathbb{R}}\left(\theta_i^2+\theta_{ix}^2+\varphi_{it}^2+\varphi_{ix}^2\right)dx$$
$$+C\int_{\mathbb{R}}\left(\bar{n}_x^2+\bar{n}_{xx}^2\right)dx+C\delta^2e^{-\nu_2 t}. \tag{58}$$

Combining (44), (45), (51), (52), and (58), we can obtain (38); this completes the proof. □

Further, in the completely similar way, we can show the following.

Lemma 7. *Let $(\varphi_1,\theta_1,\varphi_2,\theta_2)(x,t) \in X(T)$ be the solution of the initial value problems* (25) *and* (26)*; then it holds that for $0<t<T$,*

$$\sum_{i=1}^{2}\left\|\left(\varphi_{ix},\varphi_{ixx},\varphi_{itx},\varphi_{itt},\theta_{ix},\theta_{it},\theta_{ixx}\right)(\cdot,t)\right\|^2+\left\|\mathscr{H}_x(\cdot,t)\right\|^2$$
$$+\int_0^t\left(\sum_{i=1}^{2}\left\|\left(\varphi_{ixx},\varphi_{itx},\varphi_{itt},\theta_{ix},\theta_{it},\theta_{ixx},\theta_{itx},\theta_{ixxx}\right)(\cdot,\tau)\right\|^2+\left\|\mathscr{H}_x(\cdot,\tau)\right\|^2\right)d\tau$$
$$\leq C(\Phi_0+\delta),$$
$$\sum_{i=1}^{2}\left\|\left(\varphi_{ixx},\varphi_{ixxx},\varphi_{itxx},\varphi_{ittx},\theta_{ixx},\theta_{itx},\theta_{ixxx}\right)(\cdot,t)\right\|^2$$
$$+\left\|\mathscr{H}_{xx}(\cdot,t)\right\|^2$$
$$+\int_0^t\left(\sum_{i=1}^{2}\left\|\left(\varphi_{ixx},\varphi_{ix},\varphi_{itx},\theta_{ixx},\theta_{itx},\theta_{ixxx}\right)(\cdot,\tau)\right\|^2+\left\|\mathscr{H}_{xx}(\cdot,\tau)\right\|^2\right)d\tau$$
$$\leq C(\Phi_0+\delta), \tag{59}$$

provided that $\varepsilon+\delta \ll 1$.

Based on the local existence given in Lemma 5 and the a priori estimates given in Lemmas 6 and 7, by the standard continuity argument, we can prove the global existence of the unique solutions of the IVP (25) and (26).

Theorem 8. *Under the assumption of Theorem 1, the classical solution $(\varphi_1,\theta_1,\varphi_2,\theta_2,\mathscr{H})(x,t)$ of the solutions of the IVP* (25) *and* (26) *exist globally in time if $\Phi_0+\delta$ is small enough. Moreover, one has*

$$\left\|(\varphi_1,\theta_1,\varphi_2,\theta_2)(\cdot,t)\right\|_3^2+\left\|(\varphi_{1t},\varphi_{2t},\mathscr{H}(\cdot,t))\right\|_2^2$$
$$+\left\|(\theta_{1t},\theta_{2t})(\cdot,t)\right\|_1^2$$
$$+\int_0^t\left(\left\|(\varphi_{1x},\varphi_{1t},\varphi_{2x},\varphi_{2t},\mathscr{H})(\cdot,\tau)\right\|_2^2\right.$$

$$\begin{aligned}&+\left\|(\theta_1,\theta_2)(\cdot,\tau)\right\|_4^2+\left\|(\theta_{1t},\theta_{2t})(\cdot,\tau)\right\|_2^2\Big)\,d\tau\\&\le C\left(\left\|(\varphi_{10},\theta_{10},\varphi_{20},\theta_{20})\right\|_3^2+\left\|(\psi_{10},\psi_{20})\right\|_2^2+\delta\right),\quad t>0,\\&\left\|(\varphi_1,\theta_1,\varphi_2,\theta_2)(\cdot,t)\right\|_3^2+\left\|(\varphi_{1t},\varphi_{2t},\mathscr{H}(\cdot,t))\right\|_2^2\\&\quad+\left\|(\theta_{1t},\theta_{2t})(\cdot,t)\right\|_1^2\longrightarrow 0,\quad t\longrightarrow\infty.\end{aligned}\tag{60}$$

5. The Algebraic Decay Rates

In this section, we prove the time-decay rate of smooth solutions $(\varphi_1,\theta_1,\varphi_2,\theta_2)$ of (25) with the initial data $(\varphi_{10},-\psi_{10},\theta_{10},\varphi_{20},-\psi_{20},\theta_{20})$. For this aim, using the idea of [4, 15, 24], we first prove the exponential decay of $\mathscr{H}$ and $\theta_1-\theta_2$ to zero then obtain the algebraic convergence of $(\varphi_1,\theta_1,\varphi_2,\theta_2)$. Due to Theorem 8, we know that the global smooth solutions $(\varphi_1,\theta_1,\varphi_2,\theta_2)$ satisfy

$$\begin{aligned}&\left\|(\varphi_1,\theta_1,\varphi_2,\theta_2)\right\|_3^2+\left\|(\varphi_{1t},\varphi_{2t},\mathscr{H})\right\|_2^2+\left\|(\theta_{1t},\theta_{2t},\mathscr{H}_t)\right\|_1^2\\&\quad\le C\left(\Phi_0+\delta\right),\end{aligned}\tag{61}$$

which leads to, in terms of Sobolev embedding theorem, that

$$\begin{aligned}&\|(\varphi_1,\varphi_2,\varphi_{1x},\varphi_{2x},\varphi_{1xx},\varphi_{2xx},\varphi_{1t},\varphi_{1tx},\varphi_{2t},\varphi_{2tx},\\&\quad\theta_1,\theta_{1x},\theta_{1xx},\theta_2,\theta_{2x},\theta_{2xx},\mathscr{H},\mathscr{H}_x,\mathscr{H}_t)\|_{L^\infty(\mathbb{R})}\\&\quad\le C\left(\Phi_0+\delta\right).\end{aligned}\tag{62}$$

Further, by (25), we also have

$$\left\|(\varphi_{1tt},\varphi_{2tt},\theta_{1t},\theta_{2t})\right\|_{L^\infty(\mathbb{R})}\le C\left(\Phi_0+\delta\right).\tag{63}$$

Lemma 9. *Let $(\varphi_1,\theta_1,\varphi_2,\theta_2)$ be the global classical solutions of IVP (25) and (26) satisfying $\Phi_0+\delta\ll 1$. Then it holds for $\mathscr{H}$ and $\theta_1-\theta_2$ that for $t>0$,*

$$\begin{aligned}&\|(\mathscr{H},\mathscr{H}_x,\mathscr{H}_t,\mathscr{H}_{xx},\mathscr{H}_{tx},\theta_1-\theta_2,\\&\quad(\theta_1-\theta_2)_x,(\theta_1-\theta_2)_{xx})(\cdot,t)\|\\&\quad\le C\left(\Phi_0+\delta\right)e^{-\gamma_0 t}.\end{aligned}\tag{64}$$

Proof. Multiplying (30) by $\mathscr{H}$ and integrating it by parts over $\mathbb{R}$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\mathscr{H}\mathscr{H}_t+\frac{1}{2}\mathscr{H}^2\right)dx-\int_{\mathbb{R}}\mathscr{H}_t^2\,dx+\int_{\mathbb{R}}2\bar{n}\mathscr{H}^2\,dx\\&\quad+\int_{\mathbb{R}}\left(\theta_1+\widehat{T}_1+T^*\right)\mathscr{H}_x^2\,dx\\&=-\int_{\mathbb{R}}\bar{n}\left(\theta_1-\theta_2\right)\mathscr{H}_x\,dx\\&\quad+\int_{\mathbb{R}}\left(h_{1x}-h_2-h_3+h_{4x}\right)\mathscr{H}\,dx.\end{aligned}\tag{65}$$

Using Cauchy-Schwartz's inequality, Lemmas 3 and 4, (62), and (63), we have

$$\begin{aligned}&-\int_{\mathbb{R}}\bar{n}\left(\theta_1-\theta_2\right)\mathscr{H}_x\,dx+\int_{\mathbb{R}}h_{1x}\mathscr{H}\,dx\\&\quad\le\kappa\int_{\mathbb{R}}\mathscr{H}_x^2\,dx+C\int_{\mathbb{R}}\left(\theta_1-\theta_2\right)^2dx+C\delta e^{-\nu t},\\&-\int_{\mathbb{R}}\left(h_2+h_3\right)\mathscr{H}\,dx\le C\left(\Phi_0+\delta\right)\int_{\mathscr{R}}\mathscr{H}^2\,dx+C\delta e^{-\nu_1 t}.\end{aligned}\tag{66}$$

Moreover, noticing that

$$\begin{aligned}h_{4x}=&-\frac{j_1^2}{n_1^2}\mathscr{H}_{xx}-\frac{2j_1}{n_1}\mathscr{H}_{tx}+O(1)\left(\widehat{n}_{1x}+\widehat{n}_{2x}+\widehat{j}_{1x}+\widehat{j}_{2x}\right)\\&+O(1)\left(\varphi_{2xx}+\varphi_{2tx}+\bar{n}_x+\bar{j}_x+\widehat{n}_{2x}+\widehat{j}_{2x}\right)\\&\times\left(\mathscr{H}_x+\mathscr{H}_t+\widehat{n}_1+\widehat{n}_2+\widehat{j}_1+\widehat{j}_2\right),\end{aligned}\tag{67}$$

then

$$\begin{aligned}&\int_{\mathbb{R}}h_{4x}\mathscr{H}\,dx\\&\quad\le C\left(\Phi_0+\delta\right)\int_{\mathbb{R}}\left(\mathscr{H}^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)dx+C\delta e^{-\nu t}.\end{aligned}\tag{68}$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\mathscr{H}\mathscr{H}_t+\frac{1}{2}\mathscr{H}^2\right)dx-\int_{\mathbb{R}}\mathscr{H}_t^2\,dx+2\int_{\mathbb{R}}\bar{n}\mathscr{H}^2\,dx\\&\quad+\int_{\mathbb{R}}\left[\left(\theta_1+\widehat{T}_1+T^*\right)-\kappa\right]\mathscr{H}_x^2\,dx\\&\quad\le C\left(\Phi_0+\delta\right)\int_{\mathbb{R}}\left(\mathscr{H}^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)dx\\&\qquad+C\int_{\mathbb{R}}\left(\theta_1-\theta_2\right)^2dx+C\delta e^{-\nu_1 t}.\end{aligned}\tag{69}$$

While multiplying (30) by $\mathscr{H}_t$ and integrating the resultant equation by parts over $\mathbb{R}$, similarly, we can show

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\left(\frac{1}{2}\mathscr{H}_t^2+\bar{n}\mathscr{H}^2+\left(\frac{1}{2}\left(\theta_1+\widehat{T}_1+T^*\right)-\frac{j_1^2}{n_1^2}\right)\mathscr{H}_x^2\right)dx\\&\quad+\int_{\mathbb{R}}\mathscr{H}_t^2\,dx+\int_{\mathbb{R}}\bar{n}\left(\theta_1-\theta_2\right)\mathscr{H}_{tx}\,dx\\&\le C\left(\Phi_0+\delta\right)\\&\quad\times\int_{\mathbb{R}}\left(\left(\theta_1-\theta_2\right)^2+\left(\theta_1-\theta_2\right)_x^2+\mathscr{H}^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)dx\\&\quad+C\delta e^{-\nu_1 t}.\end{aligned}\tag{70}$$

Next, multiplying (31) by $(3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})/2(\theta_1+\widehat{T}_1+T^*))(\theta_1-\theta_2)$ and integrating the resultant equation by parts over $\mathbb{R}$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{4(\theta_1+\widehat{T}_1+T^*)}(\theta_1-\theta_2)^2\,dx\\
&\quad+\int_{\mathbb{R}}\left(\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{2(\theta_1+\widehat{T}_1+T^*)}(\theta_1-\theta_2)^2\right.\\
&\qquad\left.+\frac{\bar{n}}{\theta_1+\widehat{T}_1+T^*}(\theta_1-\theta_2)_x^2\right)dx\\
&\quad-\int_{\mathbb{R}}\bar{n}(\theta_1-\theta_2)\mathscr{H}_{tx}\,dx\\
&=\int_{\mathbb{R}}\left(\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{4(\theta_1+\widehat{T}_1+T^*)}\right)_t(\theta_1-\theta_2)^2\,dx\\
&\quad-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_1+\widehat{T}_1+T^*}\right)_x(\theta_1-\theta_2)(\theta_1-\theta_2)_x\,dx\\
&\quad+\int_{\mathbb{R}}G_3\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{2(\theta_1+\widehat{T}_1+T^*)}(\theta_1-\theta_2)\,dx\\
&\le C(\Phi_0+\delta)\\
&\quad\times\int_{\mathbb{R}}\left((\theta_1-\theta_2)^2+(\theta_1-\theta_2)_x^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)dx\\
&\quad+C(\Phi_0+\delta)e^{-\nu_2 t},
\end{aligned}\tag{71}$$

where in the last inequality, we have used

$$\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{4(\theta_1+\widehat{T}_1+T^*)}\right)_t(\theta_1-\theta_2)^2dx\\
&\quad-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_1+\widehat{T}_1+T^*}\right)_x(\theta_1-\theta_2)(\theta_1-\theta_2)_x dx\\
&\le C(\Phi_0+\delta)\int_{\mathbb{R}}\left((\theta_1-\theta_2)^2+(\theta_1-\theta_2)_x^2\right)dx,
\end{aligned}\tag{72}$$

$$\begin{aligned}
&\int_{\mathbb{R}}G_3\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{2(\theta_1+\widehat{T}_1+T^*)}(\theta_1-\theta_2)\,dx\\
&\le C(\Phi_0+\delta)\int_{\mathbb{R}}\left((\theta_1-\theta_2)^2+(\theta_1-\theta_2)_x^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)\\
&\quad+C(\Phi_0+\delta)e^{-\nu_2 t},
\end{aligned}\tag{73}$$

with the aid of

$$\begin{aligned}
&\sum_{i=1}^{2}(-1)^{i-1}\int_{\mathbb{R}}\left(\left(\frac{-\varphi_{it}+\hat{j}_i+\bar{j}}{\varphi_{ix}+\hat{n}_i+\bar{n}}\right)^2-S_i(x,t)\right)\\
&\quad\times\frac{3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})}{2(\theta_1+\widehat{T}_1+T^*)}(\theta_1-\theta_2)\,dx\\
&\le C(\Phi_0+\delta)\int_{\mathbb{R}}\left((\theta_1-\theta_2)^2+\mathscr{H}_x^2+\mathscr{H}_t^2\right)dx\\
&\quad+C(\Phi_0+\delta)e^{-\nu_2 t}.
\end{aligned}\tag{74}$$

Combine (69), (70), and (71), and choose proper positive constant λ_1 and Λ_1 such that

$$\begin{aligned}
&\lambda_1\times(70)+\Lambda_1\times((71)+(72))\\
&\quad\sim\mathscr{H}_t^2+\mathscr{H}^2+\mathscr{H}_x^2+(\theta_1-\theta_2)^2.
\end{aligned}\tag{75}$$

Then, we have

$$\begin{aligned}
&\frac{d}{dt}\left\|\left(\mathscr{H}_t,\mathscr{H},\mathscr{H}_x,(\theta_1-\theta_2)\right)(\cdot,\tau)\right\|^2\\
&\quad+C\left\|\left(\mathscr{H}_t,\mathscr{H},\mathscr{H}_x,(\theta_1-\theta_2),(\theta_1-\theta_2)_x\right)(\cdot,t)\right\|^2\\
&\le C(\Phi_0+\delta)e^{-\nu_2 t},
\end{aligned}\tag{76}$$

which, together with Gronwall's inequality, yields

$$\left\|\left(\mathscr{H},\mathscr{H}_x,\mathscr{H}_t,(\theta_1-\theta_2)\right)(\cdot,t)\right\|^2\le C(\Phi_0+\delta)e^{-\gamma_1 t},\tag{77}$$

for some positive constants γ_1 and C. In the completely same way, treating $\int_{\mathbb{R}}\lambda_2(30)_x\mathscr{H}_x+\Lambda_2((30)_x\mathscr{H}_{tx}+(31)_x(3\bar{n}(\varphi_{1x}+\hat{n}_1+\bar{n})/2(\theta_1+\widehat{T}_1+T^*))(\theta_1-\theta_2)_x)dx$ for proper positive constants λ_2 and Λ_2, we can show

$$\left\|\left(\mathscr{H}_x,\mathscr{H}_{xx},\mathscr{H}_{tx},(\theta_1-\theta_2)_x(\cdot,t)\right)\right\|^2\le C(\Phi_0+\delta)e^{-\gamma_2 t},\tag{78}$$

for some constant γ_2.

Moreover, from (30), (77), and (78), we obtain

$$\left\|\mathscr{H}_{tt}\right\|^2\le C(\Phi_0+\delta)e^{-\gamma_3 t},\tag{79}$$

for $\gamma_3=\min\{\gamma_1,\gamma_2\}$. Finally, by $\int_{\mathbb{R}}(31)_t(\theta_1-\theta_2)_t\,dx$ and using (77)–(79), there is a positive constant γ_4 such that

$$\left\|(\theta_1-\theta_2)_t\right\|^2\le C(\Phi_0+\delta)e^{-\gamma_4 t},\tag{80}$$

while from (31) and (77)–(80), we have

$$\left\|(\theta_1-\theta_2)_{xx}\right\|^2\le C(\Phi_0+\delta)e^{-\gamma_5 t},\tag{81}$$

with $\gamma_5=\min\{\gamma_3,\gamma_4\}$. Combination of (77)–(80) and (81) yields (64). This completes the proof. □

In the following, using the idea of [4, 15], we turn to derive the time-decay rate of $(\varphi_1,\theta_1,\varphi_2,\theta_2)$ by which we are able to obtain the algebraical decay rate of $(\varphi_1,\theta_1,\varphi_2,\theta_2)$ in large time.

Lemma 10. *Let $(\varphi_1, \theta_1, \varphi_2, \theta_2)$ be the global classical solution of the IVP* (25) *and* (26) *with initial data satisfying $\Phi_0 + \delta \ll 1$. If it holds for $(\varphi_1, \varphi_2, \theta_1, \theta_2)$ $(t > 0)$ that*

$$\begin{aligned}
&\sum_{k=0}^{3} (1+t)^k \left\| \partial_x^k (\varphi_1, \varphi_2) \right\|^2 + \sum_{k=0}^{2} (1+t)^{k+2} \left\| \partial_x^k (\varphi_{1t}, \varphi_{2t}) \right\|^2 \\
&\quad + \sum_{k=0}^{1} (1+t)^{k+2} \left\| \partial_x^k (\theta_{1t}, \theta_{2t}) \right\|^2 \\
&\quad + \sum_{k=0}^{2} (1+t)^{k+1} \left\| \partial_x^k (\theta_1, \theta_2) \right\|^2 \\
&\quad + (1+t)^3 \left\| \partial_x^3 (\theta_1, \theta_{2*}) \right\|^2 \ll 1,
\end{aligned} \tag{82}$$

then one has

$$\begin{aligned}
&\sum_{k=0}^{3} (1+t)^k \left\| \partial_x^k (\varphi_1, \varphi_2) \right\|^2 + \sum_{k=1}^{3} \int_0^t (1+\tau)^k \left\| \partial_x^k (\varphi_1, \varphi_2) \right\|^2 d\tau \\
&\quad + \sum_{k=0}^{2} (1+t)^{k+1} \left\| \partial_x^k (\theta_1, \theta_2) \right\|^2 \\
&\quad + \sum_{k=0}^{2} \int_0^t \Big((1+\tau)^{k+1} \left\| \partial_x^k (\theta_1, \theta_2) \right\|^2 \\
&\qquad + (1+\tau)^3 \left\| \partial_x^3 (\theta_{1t}, \theta_{2t}) \right\|^2 \Big) d\tau \\
&\quad \le C (\Phi_0 + \delta), \\
&\sum_{k=0}^{2} (1+t)^{k+2} \left\| \partial_x^k (\varphi_{1t}, \varphi_{2t}) \right\|^2 \\
&\quad + \sum_{k=0}^{2} \int_0^t (1+\tau)^{k+2} \left\| \partial_x^k (\varphi_{1\tau}, \varphi_{2\tau}) \right\|^2 d\tau \\
&\quad + \sum_{k=0}^{1} (1+t)^{k+2} \left\| \partial_x^k (\theta_{1t}, \theta_{2t}) \right\|^2 + (1+t)^3 \left\| \partial_x^3 (\theta_{1t}, \theta_{2t}) \right\|^2 \\
&\quad + \int_0^t \Big(\sum_{k=0}^{1} (1+\tau)^{k+2} \left\| \partial_x^k (\theta_1, \theta_2) \right\|^2 \\
&\qquad + (1+\tau)^3 \left\| \partial_x^3 (\theta_{1\tau}, \theta_{2\tau}) \right\|^2 \Big) d\tau \\
&\quad \le C (\Phi_0 + \delta).
\end{aligned} \tag{83}$$

Since the proof is similar as that in [15], we can omit the details.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research is partially supported by the National Science Foundation of China (Grant no. 11171223), the Ph.D. Program Foundation of Ministry of Education of China (Grant no. 20133127110007), and the Innovation Program of Shanghai Municipal Education Commission (Grant no. 13ZZ109).

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