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Research Article

Generalized $(d\text{-}\rho\text{-}\eta\text{-}\theta)$-Type I Univex Functions in Multiobjective Optimization

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A new class of generalized functions $(d\text{-}\rho\text{-}\eta\text{-}\theta)$-type I univex is introduced for a nonsmooth multiobjective programming problem. Based upon these generalized functions, sufficient optimality conditions are established. Weak, strong, converse, and strict converse duality theorems are also derived for Mond-Weir-type multiobjective dual program.

1. Introduction

Generalizations of convexity related to optimality conditions and duality for nonlinear single objective or multiobjective optimization problems have been of much interest in the recent past and thus explored the extent of optimality conditions and duality applicability in mathematical programming problems. Invexity theory was originated by Hanson [1]. Many authors have then contributed in this direction.

For a nondifferentiable multiobjective programming problem, there exists a generalisation of invexity to locally Lipschitz functions with gradients replaced by the Clarke generalized gradient. Zhao [2] extended optimality conditions and duality in nonsmooth scalar programming assuming Clarke generalized subgradients under type I functions. However, Antczak [3] used directional derivative in association with a hypothesis of an invex kind following Ye [4]. On the other hand, Bector et al. [5] generalized the notion of convexity to univex functions. Rueda et al. [6] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions. Mishra [7] obtained optimality results and saddle point results for multiobjective programs under generalized type I univex functions which were further generalized to univex type I-vector-valued functions by Mishra et al. [8]. Jayswal [9] introduced new classes of generalized α-univex type I vector valued functions and established sufficient optimality conditions and various duality results for Mond-Weir type dual program. Generalizing the work of Antczak [3], recently Nahak and Mohapatra [10] obtained duality results for multiobjective programming problem under (d-ρ-η-θ) invexity assumptions.

In this paper, by combining the concepts of Mishra et al. [8] and Nahak and Mohapatra [10], we introduce a new generalized class of (d-ρ-η-θ)-type I univex functions and establish weak, strong, converse, and strict converse duality results for Mond-Weir type dual.

2. Preliminaries and Definitions

The following convention of vectors in R^n will be followed throughout this paper: $x \geqq y \Leftrightarrow x_i \geqq y_i$, $i = 1,2,\dots,n$; $x \geq y \Leftrightarrow x \geqq y$, $x \neq y$; $x > y \Leftrightarrow x_i > y_i$, $i = 1,2,\dots,n$. Let D be a nonempty subset of R^n, $\eta : D \times D \to R^n$, x_o be an arbitrary point of D and $h : D \to R$, $\phi : R \to R$, $b : D \times D \to R_+$. Also, we denote $R^p_{\geq} = \{y : y \in R^p \text{ and } y \geq 0\}$ and $R^k_{\geqq} = \{y : y \in R^k \text{ and } y \geqq 0\}$.

Definition 2.1 (Ben-Israel and Mond [11]). Let $D \subseteq R^n$ be an invex set. A function h is called preinvex on D with respect to η, if for all $x, x_o \in D$,

$$\lambda h(x) + (1-\lambda) h(x_o) \geqq h\big(x_o + \lambda \eta(x, x_o)\big), \quad \forall \lambda \in [0,1]. \tag{2.1}$$

Definition 2.2 (Clarke [12]). The function h is said to be locally Lipschitz at $x_o \in D$, if there exists a neighbourhood $v(x_o)$ of x_o and a constant $k > 0$ such that

$$\left|h(y) - h(x)\right| \leqq k\left\|y - x\right\| \quad \forall x, y \in v(x_o), \tag{2.2}$$

where $\|\cdot\|$ denotes the euclidean norm. Also, we say that h is locally Lipschitz on D if it is locally Lipschitz at every point of D.

Definition 2.3 (Bector et al. [5]). A differentiable function h is said to be univex at x_o if for all $x \in D$, we have

$$b(x, x_o)\phi(h(x) - h(x_o)) \geqq [\nabla h(x_o)]^T \eta(x, x_o). \tag{2.3}$$

We consider the following nonlinear multiobjective programming problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) = \big(f_1(x), f_2(x), \dots, f_p(x)\big) \\ &\text{subject to} \quad g(x) \leqq 0, \end{aligned} \tag{MP}$$

where $x \in D$ and the functions $f : D \to R^p$, $g : D \to R^k$. Let $X = \{x \in D : g(x) \leqq 0\}$ be a set of feasible solutions of (MP) . For $x_o \in D$, if we denote by

$$J(x_o) = \{j \in \{1,2,\dots,k\} : g_j(x_o) = 0\},$$
$$\widetilde{J}(x_o) = \{j \in \{1,2,\dots,k\} : g_j(x_o) < 0\},$$
$$\overline{J}(x_o) = \{j \in \{1,2,\dots,k\} : g_j(x_o) > 0\}, \tag{2.4}$$

then

$$J(x_o) \cup \widetilde{J}(x_o) \cup \overline{J}(x_o) = \{1,2,\dots,k\}. \tag{2.5}$$

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of efficient or weak efficient solution in the following sense by Miettinen [13].

Definition 2.4. A point $x_o \in X$ is said to be an efficient solution of (MP) , if there exists no $x \in X$ such that

$$f(x) \leq f(x_o). \tag{2.6}$$

Definition 2.5. A point $x_o \in X$ is said to be a weak efficient solution of (MP) , if there exists no $x \in X$ such that

$$f(x) < f(x_o). \tag{2.7}$$

Now we define a new class of (d-ρ-η-θ)-type I univex functions which generalize the work of Mishra et al. [8] and Nahak and Mohapatra [10]. Let functions $f = (f_1,\dots,f_p) : D \to R^p$ and $g = (g_1,\dots,g_k) : D \to R^k$ are directionally differentiable at $x_o \in X$, $\eta : X \times D \to R^n$, b_o and b_1 are nonnegative functions defined on $X \times D$, $\phi_o : R^p \to R^p$ and $\phi_1 : R^k \to R^k$, while $\rho \in R^{p+k}$ and $\theta(\cdot,\cdot) : X \times D \to R^n$ be vector-valued functions.

Definition 2.6. (f, g) is said to be (d-ρ-η-θ)-type I univex at $x_o \in D$ if for all $x \in X$

$$\begin{gathered} b_o(x,x_o)\phi_o(f(x) - f(x_o)) \geqq f'(x_o;\eta(x,x_o)) + \rho^1\|\theta(x,x_o)\|^2, \\ -b_1(x,x_o)\phi_1(g(x_o)) \geqq g'(x_o;\eta(x,x_o)) + \rho^2\|\theta(x,x_o)\|^2. \end{gathered} \tag{2.8}$$

If the inequalities in f are strict (whenever $x \neq x_o$), then (f, g) is said to be semistrictly (d-ρ-η-θ)-type I univex at x_o.

Remark 2.7. (i) If $\rho^1, \rho^2 = 0$, $b_o(x,x_o) = b_1(x,x_o) = 1$, $\phi_o(t) = t, \phi_1(t) = t$, then above definition becomes that of d-type I function [14].

(ii) If in the above definition, the functions f and g are differentiable functions such that $f'(x_o;\eta(x,x_o)) = [\triangledown f(x_o)]^T\eta(x,x_o)$, $g'(x_o;\eta(x,x_o)) = [\triangledown g(x_o)]^T\eta(x,x_o)$; ρ^1, $\rho^2 = 0$; $b_o(x,x_o) = b_1(x,x_o) = 1$, $\phi_o(t) = t$, $\phi_1(t) = t$, then we obtain the definition of type I function [15].

Definition 2.8. (f,g) is said to be a weak strictly pseudo-quasi (d-ρ-η-θ)-type I univex at $x_o \in D$ if for all $x \in X$

$$\begin{gathered} b_o(x,x_o)\phi_o(f(x)-f(x_o)) \leq 0 \Longrightarrow f'(x_o;\eta(x,x_o)) < -\rho^1\|\theta(x,x_o)\|^2, \\ b_1(x,x_o)\phi_1(g(x_o)) \geqq 0 \Longrightarrow g'(x_o;\eta(x,x_o)) \leqq -\rho^2\|\theta(x,x_o)\|^2. \end{gathered} \tag{2.9}$$

Definition 2.9. (f,g) is said to be strong pseudo-quasi (d-ρ-η-θ)-type I univex at $x_o \in D$ if for all $x \in X$

$$\begin{gathered} b_o(x,x_o)\phi_o(f(x)-f(x_o)) \leq 0 \Longrightarrow f'(x_o;\eta(x,x_o)) \leq -\rho^1\|\theta(x,x_o)\|^2, \\ b_1(x,x_o)\phi_1(g(x_o)) \geqq 0 \Longrightarrow g'(x_o;\eta(x,x_o)) \leqq -\rho^2\|\theta(x,x_o)\|^2. \end{gathered} \tag{2.10}$$

Definition 2.10. (f,g) is said to be weak strictly-pseudo (d-ρ-η-θ)-type I univex at $x_o \in D$ if for all $x \in X$

$$\begin{gathered} b_o(x,x_o)\phi_o(f(x)-f(x_o)) \leq 0 \Longrightarrow f'(x_o;\eta(x,x_o)) < -\rho^1\|\theta(x,x_o)\|^2, \\ b_1(x,x_o)\phi_1(g(x_o)) \geqq 0 \Longrightarrow g'(x_o;\eta(x,x_o)) < -\rho^2\|\theta(x,x_o)\|^2. \end{gathered} \tag{2.11}$$

Definition 2.11. (f,g) is said to be a weak quasistrictly-pseudo (d-ρ-η-θ)-type I univex at $x_o \in D$ if for all $x \in X$

$$\begin{gathered} b_o(x,x_o)\phi_o(f(x)-f(x_o)) \leq 0 \Longrightarrow f'(x_o;\eta(x,x_o)) \leqq -\rho^1\|\theta(x,x_o)\|^2, \\ b_1(x,x_o)\phi_1(g(x_o)) \geqq 0 \Longrightarrow g'(x_o;\eta(x,x_o)) \leq -\rho^2\|\theta(x,x_o)\|^2. \end{gathered} \tag{2.12}$$

Remark 2.12. In the above definitions, if f and g are differentiable functions such that $f'(x_o;\eta(x,x_o)) = [\triangledown f(x_o)]^T\eta(x,x_o)$; $g'(x_o;\eta(x,x_o)) = [\triangledown g(x_o)]^T\eta(x,x_o)$; $\rho^1 = \rho^2 = 0$, then we obtain the functions given in Mishra et al. [8].

3. Sufficient Optimality Conditions

In this section, we discuss sufficient optimality conditions for a point to be an efficient solution of (MP) under generalized (d-ρ-η-θ)-type I univex assumptions. In the following theorems, $\mu = (\mu_1,\ldots,\mu_p) \in R^p$ and $\lambda = (\lambda_1,\ldots,\lambda_{J(x_o)}) \in J(x_o)$.

Theorem 3.1. *Suppose there exists a feasible solution x_o for* (MP)*, vector functions $\eta : X \times D \to R^n$ and vectors $\mu > \mathbf{0}$ and $\lambda \geqq \mathbf{0}$, such that*

(i) $\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) \geqq 0$,

(ii) $(f, g_{J(x_o)})$ *is a strong pseudo-quasi $(d$-ρ-η-$\theta)$-type I univex at x_o,*

(iii) *for any $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ and $v \in R^{J(x_o)}$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x, x_o) > 0$, $b_1(x, x_o) \geqq 0$,*

(iv) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \geqq 0$,

then x_o is an efficient solution of (MP).

Proof. Suppose x_o is not an efficient solution of (MP), then there exists $x \in X$ such that $f(x) \leq f(x_o)$.

Since $g_j(x_o) = 0$, $j \in J(x_o)$, therefore by hypothesis (iii), we get

$$\begin{gathered} b_o(x, x_o)\phi_o(f(x) - f(x_o)) \leq 0, \\ b_1(x, x_o)\phi_1(g_{J(x_o)}(x_o)) \geqq 0. \end{gathered} \tag{3.1}$$

which using hypothesis (ii) yields

$$\begin{gathered} f'(x_o; \eta(x, x_o)) \leq -\rho^1 \|\theta(x, x_o)\|^2, \\ g'_{J(x_o)}(x_o; \eta(x, x_o)) \leqq -\rho^2_{J(x_o)} \|\theta(x, x_o)\|^2. \end{gathered} \tag{3.2}$$

Also $\mu > 0$ and $\lambda \geqq 0$, so, we get

$$\begin{gathered} \sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) < -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x, x_o)\|^2, \\ \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) \leqq -\sum_{j \in J(x_o)} \lambda_j \rho_j^2 \|\theta(x, x_o)\|^2. \end{gathered} \tag{3.3}$$

Adding the above inequalities, we obtain

$$\begin{aligned} \sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) &< -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \right) \|\theta(x, x_o)\|^2 \\ &\leqq 0 \quad (\text{By hypothesis (iv)}), \end{aligned} \tag{3.4}$$

which contradicts hypothesis (i). Hence the proof. □

Theorem 3.2. *Suppose there exists a feasible solution x_o for* (MP)*, vector functions $\eta : X \times D \to R^n$ and vectors $\mu \geq \mathbf{0}$ and $\lambda \geqq \mathbf{0}$, such that*

(i) $\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) \geqq 0$,

(ii) $(f, g_{J(x_o)})$ *is a weak strictly-pseudo-quasi (d-ρ-η-θ)-type I univex at* x_o,

(iii) *for any* $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^{J(x_o)}$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x, x_o) > 0$, $b_1(x, x_o) \geqq 0$,

(iv) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \geqq 0$,

then x_o is an efficient solution of (MP).

Proof. Suppose x_o is not an efficient solution of (MP), then there exists $x \in X$ such that $f(x) \leq f(x_o)$.

As $g_j(x_o) = 0$, $j \in J(x_o)$, so, hypothesis (iii) yields

$$b_o(x, x_o)\phi_o(f(x) - f(x_o)) \leq 0,$$
$$b_1(x, x_o)\phi_1(g_{J(x_o)}(x_o)) \geqq 0. \tag{3.5}$$

By hypothesis (ii), the above inequalities imply

$$f'(x_o; \eta(x, x_o)) < -\rho^1 \|\theta(x, x_o)\|^2,$$
$$g'_{J(x_o)}(x_o; \eta(x, x_o)) \leqq -\rho^2_{J(x_o)} \|\theta(x, x_o)\|^2. \tag{3.6}$$

Since $\mu \geq 0$ and $\lambda \geqq 0$, we get

$$\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) < -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x, x_o)\|^2,$$
$$\sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) \leqq -\sum_{j \in J(x_o)} \lambda_j \rho_j^2 \|\theta(x, x_o)\|^2. \tag{3.7}$$

Adding the above inequalities, we obtain

$$\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) < -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \right) \|\theta(x, x_o)\|^2$$
$$\leqq 0 \quad (\text{using hypothesis (iv)}), \tag{3.8}$$

which contradicts hypothesis (i). Hence the proof. □

Theorem 3.3. *Suppose there exists a feasible solution x_o for* (MP)*, vector functions $\eta : X \times D \to R^n$ and vectors $\mu \geq \mathbf{0}$ and $\lambda \geqq \mathbf{0}$, such that*

(i) $\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) \geqq 0$,

(ii) $(f, g_{J(x_o)})$ *is a weak strictly-pseudo* $(d\text{-}\rho\text{-}\eta\text{-}\theta)$*-type I univex at* x_o,

(iii) *for any* $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^{J(x_o)}$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x, x_o) > 0$, $b_1(x, x_o) \geqq 0$,

(iv) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \geqq 0$,

then x_o is an efficient solution of (MP).

Proof. Suppose x_o is not an efficient solution of (MP), then there exists $x \in X$ such that $f(x) \leq f(x_o)$.

As $g_j(x_o) = 0$, $j \in J(x_o)$, so hypothesis (iii) implies

$$\begin{aligned} b_o(x, x_o)\phi_o\big(f(x) - f(x_o)\big) &\leq 0, \\ b_1(x, x_o)\phi_1\big(g_{J(x_o)}(x_o)\big) &\geqq 0. \end{aligned} \tag{3.9}$$

Since hypothesis (ii) holds, above inequalities imply

$$\begin{aligned} f'(x_o; \eta(x, x_o)) &< -\rho^1 \left\| \theta(x, x_o)^2 \right\|, \\ g'_{J(x_o)}(x_o; \eta(x, x_o)) &< -\rho^2_{J(x_o)} \|\theta(x, x_o)\|^2. \end{aligned} \tag{3.10}$$

Also $\mu \geq 0$ and $\lambda \geqq 0$, so we obtain

$$\begin{aligned} \sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) &< -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x, x_o)\|^2, \\ \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) &\leqq -\sum_{j \in J(x_o)} \lambda_j \rho_j^2 \|\theta(x, x_o)\|^2. \end{aligned} \tag{3.11}$$

On adding and using hypothesis (iv), above inequalities yield

$$\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j \in J(x_o)} \lambda_j g_j'(x_o; \eta(x, x_o)) < -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j \in J(x_o)} \lambda_j \rho_j^2 \right) \|\theta(x, x_o)\|^2 \leqq 0 \tag{3.12}$$

which contradicts hypothesis (i). Hence the proof. □

Theorem 3.4. *Suppose there exists a feasible solution x_o for* (MP), *vector functions* $\eta : X \times D \to R^n$ *and vectors* $\mu \geqq \mathbf{0}$ *and* $\lambda > \mathbf{0}$, *such that*

(i) $\sum_{i=1}^{p} \mu_i f_i'(x_o;\eta(x,x_o)) + \sum_{j\in J(x_o)} \lambda_j g_j'(x_o;\eta(x,x_o)) \geqq 0$,

(ii) $(f, g_{J(x_o)})$ *is weak quasi-strictly-pseudo* $(d$-ρ-η-$\theta)$*-type I univex at* x_o,

(iii) *for any* $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^{J(x_o)}$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x,x_o) > 0$, $b_1(x,x_o) \geqq 0$,

(iv) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j\in J(x_o)} \lambda_j \rho_j^2 \geqq 0$,

then x_o *is an efficient solution of* (MP).

Proof. Suppose x_o is not an efficient solution of (MP), then there exists $x \in X$ such that $f(x) \leq f(x_o)$.

Since $g_j(x_o) = 0$, $j \in J(x_o)$, therefore hypothesis (iii) yields

$$\begin{aligned} b_o(x,x_o)\phi_o\big(f(x) - f(x_o)\big) &\leq 0, \\ b_1(x,x_o)\phi_1\big(g_{J(x_o)}(x_o)\big) &\geqq 0. \end{aligned} \tag{3.13}$$

By hypothesis (ii), we get

$$\begin{aligned} f'(x_o;\eta(x,x_o)) &\leqq -\rho^1 \|\theta(x,x_o)\|^2, \\ g_{J(x_o)}'(x_o;\eta(x,x_o)) &\leq -\rho_{J(x_o)}^2 \|\theta(x,x_o)\|^2. \end{aligned} \tag{3.14}$$

Also $\mu \geqq 0$ and $\lambda > 0$, so, we obtain

$$\sum_{i=1}^{p} \mu_i f_i'(x_o;\eta(x,x_o)) \leqq -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x,x_o)\|^2, \tag{3.15}$$

$$\sum_{j\in J(x_o)} \lambda_j g_j'(x_o;\eta(x,x_o)) < -\sum_{j\in J(x_o)} \lambda_j \rho_j^2 \|\theta(x,x_o)\|^2. \tag{3.16}$$

On adding and using hypothesis (iv), above inequalities yield

$$\begin{aligned} \sum_{i=1}^{p} \mu_i f_i'(x_o;\eta(x,x_o)) + \sum_{j\in J(x_o)} \lambda_j g_j'(x_o;\eta(x,x_o)) &< -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j\in J(x_o)} \lambda_j \rho_j^2\right) \|\theta(x,x_o)\|^2 \\ &\leqq 0, \end{aligned} \tag{3.17}$$

which contradicts hypothesis (i). Hence the proof. □

Now, following Antczak [3], we state following necessary optimality conditions.

Theorem 3.5 (Karush-Kuhn-Tucker type necessary optimality conditions). *If*

(i) x_o *is a weakly efficient solution of* (MP),

(ii) g_j *is continuous at* x_o *for* $j \in \widetilde{J}(x_o)$,

(iii) *there exists a vector functions* $\eta : X \times D \to R^n$,

(iv) *for all* $i = \overline{1,p}$ *and* $j \in J(x_o)$, f_i *and* g_j *are directionally differentiable at* x_o *and the functions* $f_i'(x_o; \eta(x, x_o))$, $i = \overline{1,p}$ *and* $g_j'(x_o; \eta(x, x_o))$, $j \in J(x_o)$ *are preinvex functions of* x *on* X,

(v) *the function* g *satisfies the generalized Slater's constraint qualification at* x_o,

then there exists $\mu \in R^p_{\geqslant}$ *and* $\lambda \in R^k_{\geqq}$ *such that*

$$\sum_{i=1}^{p} \mu_i f_i'(x_o; \eta(x, x_o)) + \sum_{j=1}^{k} \lambda_j g_j'(x_o; \eta(x, x_o)) \geqq 0 \quad \forall x \in X,$$
$$\lambda_j g_j(x_o) = 0, \quad j = \overline{1,k}. \tag{3.18}$$

4. Mond-Weir Type Duality

In this section, we consider Mond-Weir type dual of (MP) and establish weak, strong, converse, and strict converse duality theorems. In this section, we denote $g_\lambda = (\lambda_1 g_1, \ldots, \lambda_k g_k)$.

$$\begin{aligned} &\text{Max} && f(y) \\ &\text{subject to} && \sum_{i=1}^{p} \mu_i f_i'(y, \eta(x, y)) + \sum_{j=1}^{k} \lambda_j g_j'(y, \eta(x, y)) \geqq 0 \quad \forall x \in X, \\ & && \lambda_j g_j(y) \geqq 0, \quad j = \overline{1,k}, \end{aligned} \tag{MWD}$$

where $y \in D$, $\mu \in R^p_{\geqslant}$, $\lambda \in R^k_{\geqq}$, $\eta : X \times D \to R^n$. Let W be the set of feasible points of (MWD).

Theorem 4.1 (Weak Duality). *Let* x *and* (y, μ, λ, η) *be the feasible solutions for* (MP) *and* (MWD) *respectively. If*

(i) (f, g_λ) *is a weak strictly-pseudo-quasi* $(d\text{-}\rho\text{-}\eta\text{-}\theta)$*-type I univex at* y,

(ii) *for any* $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^k$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x, y) > 0$, $b_1(x, y) \geqq 0$,

(iii) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \geqq 0$,

then

$$f(x) \nleq f(y). \tag{4.1}$$

Proof. Suppose to the contrary that

$$f(x) \leq f(y). \tag{4.2}$$

Since $\lambda_j g_j(y) \geqq 0$, $j = \overline{1,k}$, hypothesis (ii) yields

$$b_o(x,y)\phi_o(f(x) - f(y)) \leq 0,$$
$$b_1(x,y)\phi_1(g_\lambda(y)) \geqq 0. \tag{4.3}$$

As hypothesis (i) holds, therefore the above inequalities imply

$$f'(y;\eta(x,y)) < -\rho^1 \|\theta(x,y)\|^2,$$
$$g'_\lambda(y;\eta(x,y)) \leqq -\rho^2 \|\theta(x,y)\|^2. \tag{4.4}$$

Also $\mu \in R^p_{\geqslant}$, so, we obtain

$$\sum_{i=1}^{p} \mu_i f'_i(y;\eta(x,y)) < -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x,y)\|^2,$$
$$\sum_{j=1}^{k} \lambda_j g'_j(y;\eta(x,y)) \leqq -\sum_{j=1}^{k} \rho_j^2 \|\theta(x,y)\|^2. \tag{4.5}$$

On adding above inequalities and using hypothesis (iii), we get

$$\sum_{i=1}^{p} \mu_i f'_i(y;\eta(x,y)) + \sum_{j=1}^{k} \lambda_j g'_j(y;\eta(x,y)) < -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2\right) \|\theta(x,y)\|^2 \leqq 0, \tag{4.6}$$

which is a contradiction to the dual constraint. Hence the proof. □

The proofs of the following weak duality theorems are similar to Theorem 4.1 and hence are omitted.

Theorem 4.2 (Weak Duality). *Let x and (y,μ,λ,η) be the feasible solutions for (MP) and (MWD), respectively, with $\mu_i > 0$, $i = \overline{1,p}$. If*

(i) *(f, g_λ) is a strong pseudo-quasi $(d$-ρ-η-$\theta)$-type I univex at y,*

(ii) *for any $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ and $v \in R^k$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x,y) > 0$, $b_1(x,y) \geqq 0$,*

(iii) *$\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \geqq 0$,*

then $f(x) \nleq f(y)$.

Theorem 4.3 (Weak Duality). *Let x and (y,μ,λ,η) be the feasible solutions for* (MP) *and* (MWD), *respectively. If*

(i) *(f, g_λ) is weak strictly-pseudo $(d$-ρ-η-$\theta)$-type I univex at y,*

(ii) *for any $u \in R^p$, $u \leq 0 \Rightarrow \phi_o(u) \leq 0$ and $v \in R^k$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x,y) > 0$, $b_1(x,y) \geqq 0$,*

(iii) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \geqq 0$,

then $f(x) \nleq f(y)$.

Corollary 4.4. *Let* x_o *and* $(y_o, \mu, \lambda, \eta)$ *be the feasible solutions for* (MP) *and* (MWD), *respectively, such that* $f(x_o) = f(y_o)$. *If the weak duality holds between* (MP) *and* (MWD) *for all feasible solutions of two problems, then* x_o *is efficient for* (MP) *and* $(y_o, \mu, \lambda, \eta)$ *is efficient for* (MWD).

Proof. Suppose that x_o is not efficient for (MP), then for some $x \in X$

$$f(x) \leq f(x_o) = f(y_o). \tag{4.7}$$

which contradicts weak duality theorems as $(y_o, \mu, \lambda, \eta)$ is feasible for (MWD) and x is feasible for (MP). So, x_o is efficient for (MP). Similarly $(y_o, \mu, \lambda, \eta)$ is efficient for (MWD). □

Theorem 4.5 (Strong Duality). *Let* x_o *be a weakly efficient solution of* (MP), g_j *is continuous at* x_o *for* $j \in \widetilde{J}(x_o)$, f, g *are directionally differentiable at* x_o *with* $f_i'(x_o, \eta(x, x_o))$, *and* $g_j'(x_o, \eta(x, x_o))$ *as preinvex functions on* X. *Also if* g *satisfies the generalized Slater's constraint qualification at* x_o, *then* $\exists\ \mu \in R_{\geqslant}^{p},\ \lambda \in R_{\geqq}^{k}$ *such that* $(x_o, \mu, \lambda, \eta)$ *is feasible for* (MWD) *and the objective function values of* (MP) *and* (MWD) *are equal. Moreover, if any of weak duality theorem holds, then* $(x_o, \mu, \lambda, \eta)$ *is an efficient solution of* (MWD).

Proof. Since x_o is a weakly efficient solution of (MP), therefore by Theorem 3.5, there exists $\mu \in R_{\geqslant}^{p},\ \lambda \in R_{\geqq}^{k}$ such that

$$\begin{gathered} \sum_{i=1}^{p} \mu_i f_i'(x_o, \eta(x, x_o)) + \sum_{j=1}^{k} \lambda_j g_j'(x_o, \eta(x, x_o)) \geqq 0 \quad \forall x \in X, \\ \lambda_j g_j(x_o) = 0, \quad j = \overline{1, k}. \end{gathered} \tag{4.8}$$

It follows that $(x_o, \mu, \lambda, \eta) \in W$ and therefore feasible for (MWD). Clearly objective function values of (MP) and (MWD) are equal at optimal points.

Suppose $(x_o, \mu, \lambda, \eta)$ is not an efficient solution for (MWD). Then $\exists\ (\tilde{y}, \tilde{\mu}, \tilde{\lambda}, \tilde{\eta}) \in W$ such that $f(x_o) \leq f(\tilde{y})$, which contradicts weak duality theorems. Therefore $(x_o, \mu, \lambda, \eta)$ is an efficient solution of (MWD). Hence the proof. □

Theorem 4.6 (Converse Duality). *Let* $(y_o, \mu, \lambda, \eta)$ *be a feasible solution of* (MWD). *If*

(i) (f, g_λ) *is a weak strictly-pseudo-quasi* $(d$-ρ-η-$\theta)$*-type I univex at* y_o,

(ii) *for any* $u \in R^p,\ u \leq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^k,\ v \geqq 0 \Rightarrow \phi_1(v) \geqq 0;\ b_o(x_o, y_o) > 0, b_1(x_o, y_o) \geqq 0$,

(iii) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \geqq 0$,

then y_o *is an efficient solution of* (MP).

Proof. Suppose that y_o is not an efficient solution of (MP). Then $\exists\ x_o \in X$ such that

$$f(x_o) \leq f(y_0). \tag{4.9}$$

Now proceeding as in Theorem 4.1 (Weak Duality), we obtain a contradiction. Hence y_o is an efficient solution of (MP). □

Theorem 4.7 (Strict Converse Duality). *Let x_o and $(y_o, \mu, \lambda, \eta)$ be the feasible solutions of* (MP) *and* (MWD), *respectively. If*

(i) $f(x_o) \leqq f(y_o)$,

(ii) (f, g_λ) *is a weak quasi-strictly-pseudo* $(d\text{-}\rho\text{-}\eta\text{-}\theta)$*-type I univex at* y_o,

(iii) *for any* $u \in R^p$, $u \leqq 0 \Rightarrow \phi_o(u) \leq 0$ *and* $v \in R^k$, $v \geqq 0 \Rightarrow \phi_1(v) \geqq 0$; $b_o(x_o, y_o) > 0, b_1(x_o, y_o) \geqq 0$,

(iv) $\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \geqq 0$,

then $x_o = y_o$.

Proof. Suppose $x_o \neq y_o$.

Since y_o is a feasible solution of (MWD), therefore by hypothesis (i) and hypothesis (iii), we get

$$\begin{gathered} b_o(x_o, y_o)\phi_o(f(x_o) - f(y_o)) \leq 0, \\ b_1(x_o, y_o)\phi_1(g_\lambda(y_o)) \geqq 0. \end{gathered} \tag{4.10}$$

By hypothesis (ii), we obtain

$$\begin{gathered} f'(y_o; \eta(x_o, y_o)) \leqq -\rho^1 \|\theta(x_o, y_o)\|^2, \\ g'_\lambda(y_o; \eta(x_o, y_o)) \leq -\rho^2 \|\theta(x_o, y_o)\|^2. \end{gathered} \tag{4.11}$$

Since $\mu \in R^p_{\geqslant}$, therefore the above inequalities yield

$$\sum_{i=1}^{p} \mu_i f'_i(y_o; \eta(x_o, y_o)) \leqq -\sum_{i=1}^{p} \mu_i \rho_i^1 \|\theta(x_o, y_o)\|^2, \tag{4.12}$$

$$\sum_{j=1}^{k} \lambda_j g'_j(y_o; \eta(x_o, y_o)) < -\sum_{j=1}^{k} \rho_j^2 \|\theta(x_o, y_o)\|^2, \tag{4.13}$$

which on adding gives

$$\begin{aligned} \sum_{i=1}^{p} \mu_i f'_i(y_o; \eta(x_o, y_o)) + \sum_{j=1}^{k} \lambda_j g'_j(y_o; \eta(x_o, y_o)) &< -\left(\sum_{i=1}^{p} \mu_i \rho_i^1 + \sum_{j=1}^{k} \rho_j^2 \right) \|\theta(x_o, y_o)\|^2 \\ &\leqq 0 \quad \text{(using hypothesis (iv))}, \end{aligned} \tag{4.14}$$

which is a contradiction to feasibility of y_o. Hence $x_o = y_o$. □

References

[1] M. A. Hanson, "On sufficiency of the Kuhn-Tucker conditions," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 2, pp. 545–550, 1981.

[2] F. A. Zhao, "On sufficiency of the Kuhn-Tucker conditions in nondifferentiable programming," *Bulletin of the Australian Mathematical Society*, vol. 46, no. 3, pp. 385–389, 1992.

[3] T. Antczak, "Multiobjective programming under d-invexity," *European Journal of Operational Research*, vol. 137, no. 1, pp. 28–36, 2002.

[4] Y. L. Ye, "d-invexity and optimality conditions," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 1, pp. 242–249, 1991.

[5] C. R. Bector, S. K. Suneja, and S. Gupta, "Univex functions and univex nonlinear programming," in *Proceeding of the Administrative Sciences Association of Canada*, pp. 115–124, 1992.

[6] N. G. Rueda, M. A. Hanson, and C. Singh, "Optimality and duality with generalized convexity," *Journal of Optimization Theory and Applications*, vol. 86, no. 2, pp. 491–500, 1995.

[7] S. K. Mishra, "On multiple-objective optimization with generalized univexity," *Journal of Mathematical Analysis and Applications*, vol. 224, no. 1, pp. 131–148, 1998.

[8] S. K. Mishra, S.-Y. Wang, and K. K. Lai, "Optimality and duality for multiple-objective optimization under generalized type I univexity," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 1, pp. 315–326, 2005.

[9] A. Jayswal, "On sufficiency and duality in multiobjective programming problem under generalized α-type I univexity," *Journal of Global Optimization*, vol. 46, no. 2, pp. 207–216, 2010.

[10] C. Nahak and R. N. Mohapatra, "$d - \rho - \eta - \theta$-invexity in multiobjective optimization," *Nonlinear Analysis*, vol. 70, no. 6, pp. 2288–2296, 2009.

[11] A. Ben-Israel and B. Mond, "What is invexity?" *Journal of the Australian Mathematical Society B*, vol. 28, no. 1, pp. 1–9, 1986.

[12] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1983.

[13] K. Miettinen, *Nonlinear Multiobjective Optimization*, vol. 12 of *International Series in Operations Research & Management Science*, Kluwer Academic, Boston, Mass, USA, 1999.

[14] S. K. Mishra, S. Y. Wang, and K. K. Lai, "Optimality and duality in nondifferentiable and multiobjective programming under generalized d-invexity," *Journal of Global Optimization*, vol. 29, no. 4, pp. 425–438, 2004.

[15] M. A. Hanson and B. Mond, "Necessary and sufficient conditions in constrained optimization," *Mathematical Programming*, vol. 37, no. 1, pp. 51–58, 1987.