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Research Article

Complex Dynamics of an Adnascent-Type Game Model

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The paper presents a nonlinear discrete game model for two oligopolistic firms whose products are adnascent. (In biology, the term adnascent has only one sense, "growing to or on something else," e.g., "moss is an adnascent plant." See Webster's Revised Unabridged Dictionary published in 1913 by C. & G. Merriam Co., edited by Noah Porter.) The bifurcation of its Nash equilibrium is analyzed with Schwarzian derivative and normal form theory. Its complex dynamics is demonstrated by means of the largest Lyapunov exponents, fractal dimensions, bifurcation diagrams, and phase portraits. At last, bifurcation and chaos anticontrol of this system are studied.

1. Introduction

Economic thought has had some significant influence on the development of ecological theory [1]. (Worster claimed that Darwin was influenced in his development of the theory of evolution of species by the views of Malthus.) In the opposite direction, many scientists such as Marshall [2] and Lotka [3], have stated that biology can be a source of inspiration for economics. (Marshall [2] suggested that "The Mecca of the economist lies in economic biology rather than in economic dynamics;" Lotka [3] said that "Man's industrial activities are merely a highly specialized and greatly form of the general biological struggle for existence, . . . , the analysis of the biophysical foundations of economics, is one of the problems coming within the program of physical biology.") Thus further analogies between biology and economics can be discovered as both disciplines adopt concepts such as competition, mutualism and adnascent relation. Such ideas have greatly influenced a good many researchers in economics, for example, Barnett and Glenn [4] investigated competition and mutualism among early telephone companies; Hens and Schenk-Hoppé [5] studied evolutionary stability of portfolio rules in incomplete markets; Levine [6] Compared products and production in ecological and industrial systems.

In addition, there are a lot of phenomena with adnascent relation in economics, for example, a car key ring is adnascent to a car. In this paper, the definition of adnascent will be applied into economics to investigate a novel game model with two oligopolistic firms X and Y, where product B of the firm Y is adnascent to product A of the firm X, and the output of product B is determined by the output of product A, but not vice versa.

In 1838, Cournot proposed the classical oligopoly game model. In 1883, Bertrand reworked Cournot's duopoly game model using prices rather than quantities as the strategic variables. In 1991, Puu [7] introduced chaos and bifurcation theory into duopoly game models. Over the past decade, many researchers, such as Tramontana et al. [8], Ahmed and Agiza [9] and Ahmed et al. [10], Agiza and Elsadany [11], Bischi et al. [12], Kopel [13] and Den Haan [14], have paid a great attention to the dynamics of games.

As mentioned above, if one draws an analogy between species in biology and products in economics, it is easy to find that some of relationships among different products are substitutable or parasitic, and others are supportive or adnascent. But all the models cited above are based on the assumption that all players(firms) produce goods which are perfect substitutes in an oligopoly market. In this paper, we assume that the relationship of two players' products are not substitutable but adnascent.

This paper is organized as follows. In Section 2, a nonlinear discrete adnascent-type game model is presented. In Section 3, local stability of the Nash equilibrium of this system is studied. In Section 4, the bifurcation is studied with Schwarzian derivative and normal form theory. In Section 5, bifurcation and chaos anticontrol of the model is considered with nonlinear feedback anticontrol technology. In Section 6, the model's complex dynamics is numerically simulated by the largest Lyapunov exponents, fractal dimensions, bifurcation diagrams and phase portraits.

2. An adnascent-type dynamical game model

2.1. Assumptions

This model is based on these following assumptions.

Assumption 2.1. There are two heterogeneous firms X and Y producing adnascent products. The production decision of firm Y must depend on firm X, but not vice versa.

Assumption 2.2. Each firm is a monopoly of its products market.

Assumption 2.3. Firms have respective nonlinear variable cost functions [15] and nonlinear inverse demand functions [16]. (The linear cost function $C(x) = x$ or $C(x) = a + bx$ is usually adopted in the classical economics. Indeed, quadratic cost functions are often met in many applications (see [17–19]).)

Assumption 2.4. Firm X can compete solely on price and then make its output decision, which can have effect on firm Y.

Assumption 2.5. Firms always make the optimal output decision for the maximal margin profit in every period.

2.2. Nomenclature

The following is a list of notations that will be used throughout the paper.

(i) x_t, y_t are outputs of firms X and Y in period t, respectively, and they must be positive for any $t > 0$.

(ii) $P_{x_t} = a_1 - b_1 x_t^2$, $P_{y_t} = a_2 - b_2 y_t^2$ are nonlinear inverse demand functions [16] for firms X and Y in period t, respectively, where $a_1, b_1, a_2, b_2 > 0$.

(iii) $C_{x_t} = c_1 x_t^2$, $C_{y_t} = c_2 y_t^2$ are nonlinear variable cost functions [15] for firms X and Y in period t, respectively, where $c_1, c_2 > 0$. (The nonlinear variable cost function $C(x) = cx^2$ can be derived from a Cobb-Douglas-type production function (see [19–21]).)

(iv) $\Pi_{x_t} = P_{x_t} x_t - C_{x_t} = x_t(a_1 - b_1 x_t^2) - c_1 x_t^2$, $\Pi_{y_t} = P_{y_t} y_t - C_{y_t} = y_t(a_2 - b_2 y_t^2) - c_2 y_t^2$ are single profits of firms X and Y in period t, respectively.

(v) $\alpha_1, \alpha_2 > 0$ are respective output adjustment parameters of firms X and Y, which represent the fluctuation of two firms' output decisions. Generally speaking, the two parameters should be very small.

2.3. Model

With Assumptions (2.5), the margin profits of firms X and Y in period t are give, respectively, by

$$\begin{aligned} \frac{\partial \Pi_{x_t}}{\partial x_t} &= a_1 - 3b_1 x_t^2 - 2c_1 x_t, \\ \frac{\partial \Pi_{y_t}}{\partial y_t} &= a_2 - 3b_2 y_t^2 - 2c_2 y_t. \end{aligned} \tag{2.1}$$

One of the methods to find out the Nash equilibrium is to let (2.1) be equal to 0. Thus one can get firms' reaction functions, that is, the optimal outputs x_t^* and y_t^*. Under Assumptions (2.1) and (2.4), the dynamic adjustment of the adnascent-type game can be written as follows:

$$\begin{aligned} x_{t+1} &= x_t + \alpha_1 x_t \frac{\partial \Pi_{x_t}}{\partial x_t}, \\ y_{t+1} &= y_t + \alpha_2 x_t \frac{\partial \Pi_{y_t}}{\partial y_t}. \end{aligned} \tag{2.2}$$

The game model with bounded rational players has the following nonlinear form:

$$\begin{aligned} x_{t+1} &= x_t + \alpha_1 x_t \left(a_1 - 3b_1 x_t^2 - 2c_1 x_t\right), \\ y_{t+1} &= y_t + \alpha_2 x_t \left(a_2 - 3b_2 y_t^2 - 2c_2 y_t\right). \end{aligned} \tag{2.3}$$

Note that the model has a particular form, it is a so-called triangular map which is the class of maps in which one dynamic variable is independent on the other, that is of the type $x' = f(x)$,

$y' = g(x, y)$, while the other, y, strongly depends on the first. A peculiarity of this class of maps is that the eigenvalues in any point of the phase plane are always real, and that many bifurcations are explained via the one-dimensional map $x' = f(x)$.

3. Nash equilibrium and its local stability of system (2.3)

A Nash equilibrium, named after John Nash, is a solution concept of a game involving two or more players, such that no player has incentive to unilaterally change his or her action. In other words, players are in equilibrium if a change in strategies by any one of them would lead that he (she) to earn less than if he (she) remained with his (her) current strategy.

System (2.3) is a two-dimensional non-invertible that depends on eight parameters. The Nash equilibrium point of system (2.3) is the solution of the following algebraic system:

$$\begin{aligned} \alpha_1 x(a_1 - 3b_1 x^2 - 2c_1 x) &= 0, \\ \alpha_2 x(a_2 - 3b_2 y^2 - 2c_2 y) &= 0. \end{aligned} \tag{3.1}$$

Note that system (3.1) does not depend on the parameters α_1 and α_2. By simple computation of the above algebraic system it was found that there exists one interesting positive Nash equilibrium as follows:

$$E^*(x^*, y^*) = \left(\frac{A - c_1}{3b_1}, \frac{B - c_2}{3b_2}\right), \tag{3.2}$$

where $A = \sqrt{c_1^2 + 3a_1 b_1}$, $B = \sqrt{c_2^2 + 3a_2 b_2}$.

The Jacobian matrix of system (2.3) at the Nash equilibrium $E^*(x^*, y^*)$ has the following form:

$$J(E^*) = \begin{bmatrix} 1 - 2\alpha_1(a_1 - c_1 x^*) & 0 \\ 0 & 1 - 2\alpha_2 B x^* \end{bmatrix}. \tag{3.3}$$

Thus its eigenvalues can be expressed as $\lambda_1 = 1 - 2\alpha_1 A x^*$ and $\lambda_2 = 1 - 2\alpha_2 B x^*$. Then the condition $\lambda_1 < 1$ is always satisfied while $\lambda_1 > -1$ holds if

$$\alpha_1 < \frac{1}{Ax^*} = \frac{3b_1}{A(A - c_1)} = C, \tag{3.4}$$

and the condition $\lambda_2 < 1$ is always satisfied while $\lambda_2 > -1$ holds if

$$\alpha_2 < \frac{1}{Bx^*} = \frac{3b_1}{B(A - c_1)} = D. \tag{3.5}$$

As a result, the following proposition holds.

Proposition 3.1. *The Nash equilibrium $E^*(x^*, y^*)$ is called*

(i) *a sink if* $\alpha_1 < C$ *and* $\alpha_2 < D$*, so the sink is locally asymptotically stable;*

(ii) *a source if* $\alpha_1 > C$ *and* $\alpha_2 > D$*, so the sink is locally unstable;*

(iii) *a saddle if* $\alpha_1 < C$ *and* $\alpha_2 > D$ *or* $\alpha_1 > C$ *and* $\alpha_2 < D$*;*

(iv) *non-hyperbolic if either* $\alpha_1 = C$ *or* $\alpha_2 = D$*.*

4. Bifurcation analysis

Due to the fact that the map is triangular, the stability of the variable x is independent on the other, thus the bifurcation analysis for this variable can be easily performed with the one-dimensional map $x' = f(x)$, which is a cubic, and the interest is only in the positive part.

The best known and most popular projective differential invariant is the Schwarzian derivative. The map's Schwarzian derivative [22] is

$$\begin{aligned} S_f(x) &= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \\ &= -\frac{6\alpha_1\left(3b_1 + 3\alpha_1 a_1 b_1 + 54\alpha_1 b_1^2 x^2 + 24\alpha_1 b_1 c_1 x + 4\alpha_1 c_1^2\right)}{\left(1 + \alpha_1 a_1 - 9\alpha_1 b_1 x^2 - 4\alpha_1 c_1 x\right)^2}. \end{aligned} \tag{4.1}$$

Obviously $S_f(x) < 0$ for $x > 0$, so that all the flip bifurcations are supercritical [23].

An example of supercritical flip bifurcation will be presented with normal form theory as follows.

Generally speaking, for given firms X and Y, their parameters $a1$, $b1$, $a2$, $b2$, $c1$, and $c2$ are invariable, and their output adjustment parameters α_1 and α_2 are changeable. In what follows, for convenience of studying the bifurcation parameter α_1 and α_2, we let $a1 = 10$, $b1 = 0.5$, $a2 = 9.75$, $b2 = 0.182$, $c1 = 5$, and $c2 = 4$. Then we can get the following system:

$$\begin{aligned} x_{t+1} &= x_t + \alpha_1 x_t\left(10 - 1.5x_t^2 - 10x_t\right), \\ y_{t+1} &= y_t + \alpha_2 x_t\left(9.75 - 0.546y_t^2 - 8y_t\right). \end{aligned} \tag{4.2}$$

Howevere, (4.2) exists a Nash equilibrium point $E^*(0.883, 1.132)$ which is independent of the parameters α_1 and α_2. The Jacobian matrix at $E^*(0.883, 1.132)$ is

$$A = \begin{pmatrix} 1 - 11.17\alpha_1 & 0 \\ 0 & 1 - 8.1557\alpha_2 \end{pmatrix}. \tag{4.3}$$

Obviously, its eigenvalues satisfy (i) $\lambda_1 = -1$ if $\alpha_1 = 0.179$; (ii) $\lambda_2 = -1$ if $\alpha_2 = 0.245$. Thus system (4.2) may undergo flip bifurcation at $\alpha_1 = 0.179$ or $\alpha_2 = 0.245$.

Lemma 4.1 (Topological norm form for the flip bifurcation [24]). *Any generic, scalar, one-parameter system* $x \mapsto f(x, \alpha)$*, having at* $\alpha = 0$ *the fixed point* $x_0 = 0$ *with* $\mu = f_x(0,0) = -1$*, is locally topologically equivalent near the origin to one of the following normal forms:* $\eta \mapsto -(1 + \beta)\eta \pm \eta^3$.

The following system can be obtained with $\alpha_2 = 0.2$,

$$\begin{aligned} x_{t+1} &= x_t + \alpha_1 x_t\left(10 - 1.5x_t^2 - 10x_t\right), \\ y_{t+1} &= y_t + 0.2x_t\left(9.75 - 0.546y_t^2 - 8y_t\right). \end{aligned} \tag{4.4}$$

Proposition 4.2 (Critical norm form for flip bifurcation). *System* (4.4) *can be written as following critical normal form for flip bifurcation:*

$$\xi_{t+1} = -\xi_t + c\xi_t^3, \tag{4.5}$$

where $c = 12.23$.

Proof. To compute coefficients of normal form, we translate the origin of the coordinates to this Nash equilibrium $E^* = (0.883, 1.132)$ by the change of variables as by the change of variables

$$x = 0.883 + u, \qquad y = 1.132 + v. \tag{4.6}$$

This transforms system (4.2) with parameters $\alpha_1 = 0.179$ into

$$\begin{aligned} u_{t+1} &= 0.883 - u_t - 2.5u_t^2 - 0.269u_t^3, \\ v_{t+1} &= 1.131 - 0.001u_t - 0.63v_t - 1.85u_tv_t - 0.1v_t^2 - 0.11u_tv_t^2. \end{aligned} \tag{4.7}$$

This system can be written as

$$X_{n+1} = AX_n + \frac{1}{2}B\left(X_n, X_n\right) + \frac{1}{6}C\left(X_n, X_n, X_n\right) + O\left(X_n^4\right), \tag{4.8}$$

where

$$A_0 = A\left(E^*\right) = \begin{pmatrix} -1 & 0 \\ 0 & -0.63 \end{pmatrix}. \tag{4.9}$$

and the multilinear functions $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and $C : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ are also defined, respectively, by

$$\begin{aligned} B_i(x, y) &= \sum_{j,k=1}^{2} \left.\frac{\partial^2 X_i(\xi, 0)}{\partial \xi_j \partial \xi_k}\right|_{\xi=0} x_j y_k, \\ C_i(x, y, z) &= \sum_{j,k,l=1}^{2} \left.\frac{\partial^3 X_i(\xi, 0)}{\partial \xi_j \partial \xi_k \partial \xi_l}\right|_{\xi=0} x_j y_k z_l. \end{aligned} \tag{4.10}$$

For system (4.7),

$$B(\xi,\eta)=\begin{pmatrix}-5\xi_1\eta_1\\ -1.85\xi_1\eta_2-0.2\xi_2\eta_2\end{pmatrix},$$
$$C(\xi,\eta,\zeta)=\begin{pmatrix}-1.61\xi_1\eta_1\zeta_1\\ -0.22\xi_1\eta_2\zeta_2\end{pmatrix}. \tag{4.11}$$

The eigenvalues of the matrix J are $\lambda_1=-1$ and $\lambda_2=-0.63$.

Let $q,p\in\mathbb{R}^2$ be eigenvectors corresponding to $\lambda_1,\ \lambda_1$, respectively:

$$q=\begin{pmatrix}1\\0\end{pmatrix},\qquad p=\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.12}$$

satisfy $A_0q=-q,\ A_0^Tp=-p$ and $\langle p,q\rangle=1$.

So the coefficient of the normal form of system (4.7) can be computed by the following invariant formula:

$$c=\frac{1}{6}\langle p,C(q,q,q)\rangle-\frac{1}{2}\langle p,B\big(q,(A-I_n)^{-1}B(q,q)\big)\rangle=12.23. \tag{4.13}$$

The proposition is proved. □

The bifurcation type is determined by the stability of the Nash equilibrium as at the critical parameter value. According to the above Proposition 4.2, for system (4.7), the critical parameter $c=12.23>0$, so the flip bifurcation at the Nash equilibrium $E^*(0.883,\ 1.132)$ is supercritical.

5. Bifurcation and chaos anticontrol

A government may pay attention to chaos anticontrol on the game system. Its motivations are as follows. Chaos exhibits high sensitivity to initial conditions, which manifests itself as an exponential growth of perturbations in the initial conditions. As a result, two firms' decision behaviors of the anticontrolled chaotic game systems appear to be random. So it can weaken the negative effect of excessive monopoly at least. In addition, Huang [25] has proved that, in some sense, chaos is beneficial not only to all oligopolistic firms but also to the economy as a whole.

There are various methods can be used to control or anticontrol bifurcations and chaos, for example, impulsive control [26], adaptive feedback control [27], linear and nonlinear feedback control [28–30]. In this section, the nonlinear feedback technique will be employed to anticontrol system (4.4). As mentioned above, system (4.4) is a adnascent-type game model, that is, firm Y must depend on firm X, but not vice versa. In other words, the production decision of firm X is independent. Since firm X of system (4.4) undergoes bifurcation and chaos, one may merely anticontrol firm Y. Considering the principle of simplification and maneuverability, one may choose a generalized nonlinear feedback anticontroller (e.g., production tax rebate) on firm Y as follows:

$$u = \sum_{i=1}^{n} k_i y^i, \tag{5.1}$$

where the linear terms in the anticontroller are used to shift the location of the equilibrium and bifurcation because only the linear part affects the Jacobian matrix of the linearized system, the nonlinear terms are used to change the property of the bifurcation and chaos. But it is not necessary to take too much components unless one wants to preserve all equilibria of the original system. In this paper, since it is unnecessary to preserve all equilibria of system, the anticontroller can be greatly simplified as

$$u = ky^2. \tag{5.2}$$

Then the anticontrolled system can be represented as

$$\begin{aligned} x_{t+1} &= x_t + \alpha_1 x_t \left(10 - 1.5x_t^2 - 10x_t\right), \\ y_{t+1} &= y_t + 0.2x_t \left(9.75 - 0.546y_t^2 - 8y_t\right) + ky_t^2, \end{aligned} \tag{5.3}$$

for system (5.3), it is easy to get its Nash equilibria

$$E_1\left(0.88, \frac{106 - \sqrt{14964 - 38743k}}{150k - 14.47}\right), \qquad E_2\left(0.88, \frac{106 + \sqrt{14964 - 38743k}}{150k - 14.47}\right) \tag{5.4}$$

and Jacobian matrix

$$J(E^*) = \begin{bmatrix} 1 + \alpha_1\left(10 - 20x - 4.5x^2\right) & 0 \\ 1.95 - 0.11y^2 - 1.6y & 1 - 0.22xy - 0.6x + 2ky \end{bmatrix}. \tag{5.5}$$

As mentioned above, system (4.4) undergoes a flip bifurcation at $\alpha_1 = 0.179$ and $x = 0.88$. Like system (5.3), after a anticontroller $u = ky^2$ is put on firm Y of system (4.4), firm X is uninfluenced. As a result, in system (5.3), when $x = 0.88$ and $\alpha_1 = 0.179$, the two conditions of flip bifurcation at Nash equilibria can be expressed as follows:

$$1 + \alpha_1\left(10 - 20x - 4.5x^2\right) = -1, \qquad |1 - 0.22xy - 0.6x + 2ky| < 1 \quad \text{hold with } 0.1 < k < 0.39. \tag{5.6}$$

6. Numerical simulations

In this section, some numerical simulations are presented to confirm the above analytic results and to demonstrate added complex dynamical behaviors. To do this, one will use the

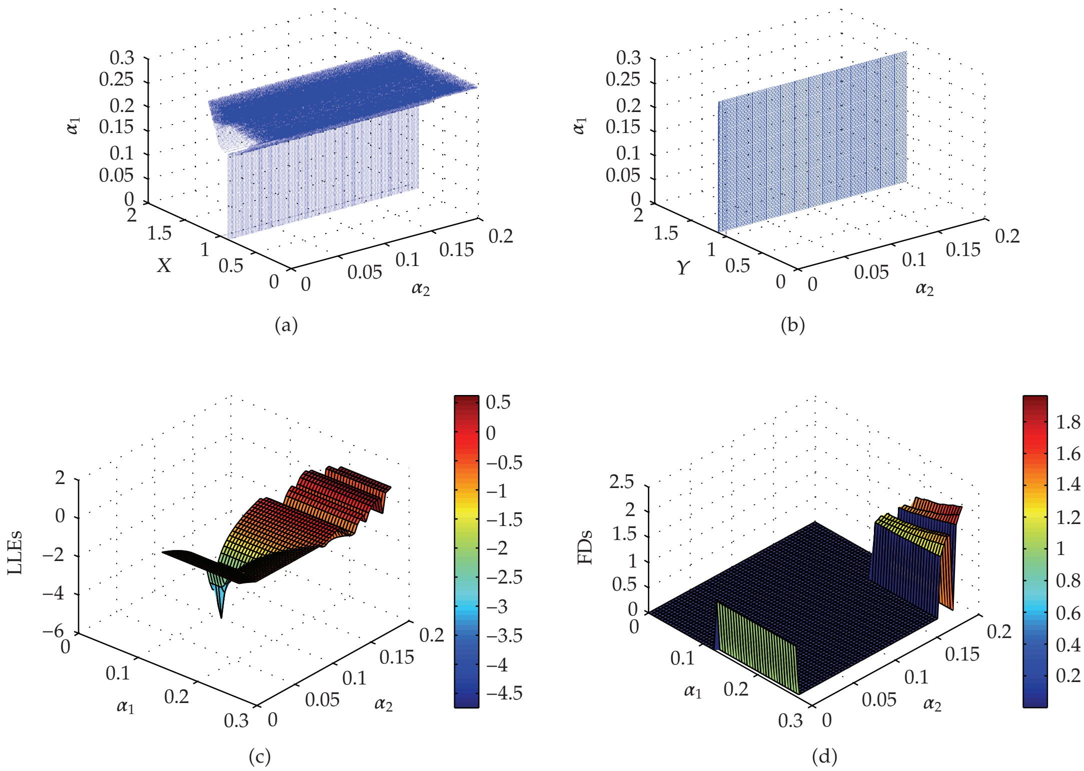

Figure 1: For system (4.2) with $\alpha_1 \in [0, 0.27]$ and $\alpha_2 \in [0, 0.2]$, (a) bifurcation diagram of firms X; (b) bifurcation diagram of firms Y; (c) largest Lyapunov exponents (LLEs); (d) fractal dimensions (FDs).

largest Lyapunov exponents, fractal dimensions, bifurcation diagrams and phase portraits to show interesting complex dynamical behaviors.

In system (4.2), the largest Lyapunov exponents, fractal dimensions and bifurcation diagrams with two parameters α_1 and α_2 are shown in Figure 1.

Figure 1(a) is the outputs bifurcation diagram of firm X with the parameters $\alpha_1 \in [0, 0.27]$ and $\alpha_2 \in [0, 0.2]$. When the output adjustment parameter α_1 increases, the outputs of firm X present complex dynamics as follows. Its outputs change from Nash equilibrium to bifurcation till chaos. Obviously the output adjustment parameter α_2 of firm Y has no effect on firm X, which just verifies the adnascent relationship between firms X and Y.

Figure 1(b) is the outputs bifurcation diagram of firm Y with the parameters $\alpha_1 \in [0, 0.27]$ and $\alpha_2 \in [0, 0.2]$. It is obviously that there is no bifurcation and chaos in Figure 1(b).

Figure 1(c) is the largest Lyapunov exponents diagram of system (4.2) with the parameters $\alpha_1 \in [0, 0.27]$ and $\alpha_2 \in [0, 0.2]$. The Lyapunov exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. A positive Lyapunov exponent is usually taken as an indication that the system is chaotic [31].

Figure 1(d) is a fractal dimensions diagram of system (4.2) with the parameters $\alpha_1 \in [0, 0.27]$ and $\alpha_2 \in [0, 0.2]$. A fractal dimension is taken as a criterion to judge whether the system is chaotic. There are many specific definitions of fractal dimension and none of them

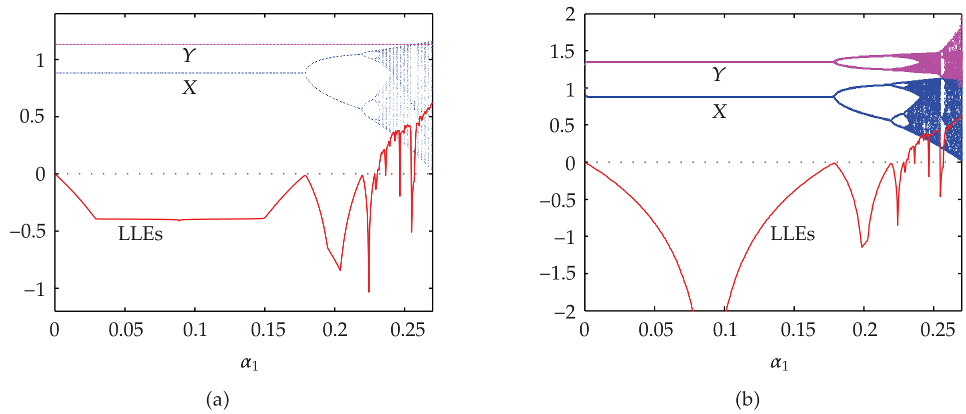

Figure 2: (a) The largest Lyapunov exponents and bifurcation diagram of system (4.4) versus $\alpha_1 \in [0, 0.27]$; (b) the largest Lyapunov exponents and bifurcation diagram of system (5.3) versus $k = 0.2$ and $\alpha_1 \in [0, 0.27]$.

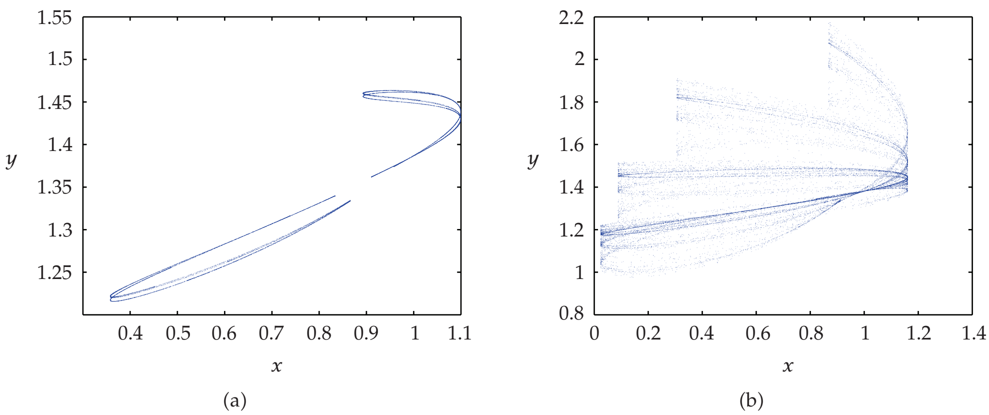

Figure 3: For system (5.3) with $k = 0.2$. (a) Chaotic attractor versus $\alpha_1 = 0.24$; (b) chaotic attractor versus $\alpha_1 = 0.27$.

should be treated as the universal one. This paper adopts the following definition of fractal dimension [32].

$$d_L = k - \frac{1}{\lambda_{k+1}} \sum_{i=1}^{k} \lambda_i \tag{6.1}$$

where $\lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_n$ are the Lyapunov exponents and k is the largest integer for which $\sum_{i=1}^{k} \lambda_i \geq 0$ and $\sum_{i=1}^{k+1} \lambda_i < 0$. If $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, n$ then $d_L = n$. If $\lambda_i < 0$ for all $i = 1, 2, \ldots, n$ then $d_L = 0$.

In system (4.4), firm X has supercritical flip bifurcation at $\alpha_1 = 0.179$ shown in Figure 2(a), while firm Y undergoes neither bifurcation nor chaos.

In system (5.3), when one fixes $k = 0.2$, he can get the largest Lyapunov exponents and bifurcations diagram shown in Figure 2(b) and chaotic attractor portrait shown in Figure 3. Obviously firms X and Y undergo synchronously bifurcations and chaos with $k = 0.2$.

The goverment can anticontrol the synchronization of bifurcation and chaos by varying the anticontrol parameter k.

7. Conclusion

In this paper, we have presented a nonlinear adnascent-type game dynamical model with two oligopolistic firms, and emphatically reported its some complex dynamics, such as Nash equilibrium, bifurcations, chaos and their anticontrol. By means of the largest Lyapunov exponents, fractal dimensions, bifurcation diagrams and phase portraits, we have demonstrated numerically its complex dynamics. For the system, other complexity anticontrol theory and methodology will be considered in future work.

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References

[1] D. Worster, *Nature's Economy: A History of Ecological Ideas*, Cambridge University Press, Cambridge, UK, 1985.
[2] A. Marshall, *Principles of Economics*, Macmillan, London, UK, 4th edition, 1898.
[3] A. J. Lotka, *Elements of Mathematical Biology*, Dover, New York, NY, USA, 1996.
[4] W. P. Barnett and R. C. Glenn, “Competition and mutualism among early telephone companies,” *Administrative Science Quarterly*, vol. 32, no. 3, pp. 400–421, 1987.
[5] T. Hens and K. R. Schenk-Hoppé, “Evolutionary stability of portfolio rules in incomplete markets,” *Journal of Mathematical Economics*, vol. 41, no. 1-2, pp. 43–66, 2005.
[6] S. H. Levine, “Comparing products and production in ecological and industrial systems,” *Journal of Industrial Ecology*, vol. 7, no. 2, pp. 33–42, 2008.
[7] T. Puu, “Chaos in duopoly pricing,” *Chaos, Solitons & Fractals*, vol. 1, no. 6, pp. 573–581, 1991.
[8] F. Tramontana, L. Gardini, and T. Puu, “Cournot duopoly when the competitors operate multiple production plants,” *Journal of Economic Dynamics and Control*, vol. 33, no. 1, pp. 250–265, 2009.
[9] E. Ahmed and H. N. Agiza, “Dynamics of a Cournot game with n-competitors,” *Chaos, Solitons & Fractals*, vol. 9, no. 9, pp. 1513–1517, 1998.
[10] E. Ahmed, H. N. Agiza, and S. Z. Hassan, “On modifications of Puu's dynamical duopoly,” *Chaos, Solitons & Fractals*, vol. 11, no. 7, pp. 1025–1028, 2000.
[11] H. N. Agiza and A. A. Elsadany, “Nonlinear dynamics in the Cournot duopoly game with heterogeneous players,” *Physica A*, vol. 320, pp. 512–524, 2003.
[12] G. I. Bischi, C. Mammana, and L. Gardini, “Multistability and cyclic attractors in duopoly games,” *Chaos, Solitons & Fractals*, vol. 11, no. 4, pp. 543–564, 2000.
[13] M. Kopel, “Simple and complex adjustment dynamics in Cournot duopoly models,” *Chaos, Solitons & Fractals*, vol. 7, no. 12, pp. 2031–2048, 1996.
[14] W. J. Den Haan, “The importance of the number of different agents in a heterogeneous asset-pricing model,” *Journal of Economic Dynamics and Control*, vol. 25, no. 5, pp. 721–746, 2001.
[15] J. Friedman, “Oligopoly theory,” in *Handbook of Mathematical Economics*, K. J. Arrow and M. D. Intriligator, Eds., vol. 2, pp. 491–534, North-Holland, Amsterdam, The Netherlands, 1982.
[16] Y. Zhang, X. Luo, and J. Xue, “Adaptive dynamic Cournot model in oligopoly market,” *Journal of Industrial Engineering and Engineering Management*, vol. 18, no. 3, pp. 78–81, 2004.
[17] H.-X. Yao and F. Xu, “Complex dynamics analysis for a duopoly advertising model with nonlinear cost,” *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 134–145, 2006.
[18] A. K. Naimzada and L. Sbragia, “Oligopoly games with nonlinear demand and cost functions: two boundedly rational adjustment processes,” *Chaos, Solitons & Fractals*, vol. 29, no. 3, pp. 707–722, 2006.

[19] G. I. Bischi, F. Lamantia, and L. Sbragia, “Competition and cooperation in natural resources exploitation: an evolutionary game approach,” in *Game Practice and the Environment*, C. Carraro and V. Fragnelli, Eds., pp. 187–211, Edward Elgar, Cheltenham, UK, 2004.

[20] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 2nd edition, 1990.

[21] F. Szidarovszky and K. Okuguchi, “An oligopoly model of commercial fishing,” *Seoul Journal of Economics*, vol. 11, no. 1, pp. 321–330, 1998.

[22] V. Ovsienko and S. Tabachnikov, *Projective Differential Geometry Old and New: From the Schwarzian Derivative to the Cohomology of Diffeomorphism Group*, vol. 165 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2005.

[23] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, vol. 42 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[24] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, vol. 112 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1998.

[25] W. Huang, “The long-run benefits of chaos to oligopolistic firms,” *Journal of Economic Dynamics and Control*, vol. 32, no. 4, pp. 1332–1355, 2008.

[26] L. Qian and Q. Lu, “Bifurcation of the competitive system with impulsive control,” *Dynamics of Continuous Discrete and Impulsive Systems. Series B*, vol. 14, supplement 5, pp. 24–28, 2007.

[27] G. Qi, Z. Chen, and Z. Yuan, “Adaptive high order differential feedback control for affine nonlinear system,” *Chaos, Solitons & Fractals*, vol. 37, no. 1, pp. 308–315, 2008.

[28] Z. Chen and P. Yu, “Controlling and anti-controlling Hopf bifurcations in discrete maps using polynomial functions,” *Chaos, Solitons & Fractals*, vol. 26, no. 4, pp. 1231–1248, 2005.

[29] X.-Y. Wang and M.-J. Wang, “Chaotic control of the coupled Logistic map,” *Acta Physica Sinica*, vol. 57, no. 2, pp. 731–736, 2008.

[30] X. F. Wang and G. Chen, “Chaotification via arbitrarily small feedback controls: theory, method, and applications,” *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 10, no. 3, pp. 549–570, 2000.

[31] Q. Lu, *Bifurcation and Singularity*, Shanghai Scientific and Technological Education, Shanghai, China, 1995.

[32] J. L. Kaplan and J. A. Yorke, “Preturbulence: a regime observed in a fluid flow model of Lorenz,” *Communications in Mathematical Physics*, vol. 67, no. 2, pp. 93–108, 1979.


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