Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 326434, 11 pages
http://dx.doi.org/10.1155/2014/326434

Research Article

Global Regularity for the $\overline{\partial}_b$-Equation on *CR* Manifolds of Arbitrary Codimension

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Received 9 April 2014; Accepted 12 May 2014; Published 12 June 2014

Academic Editor: Dumitru Baleanu

Let M be a $\mathscr{C}^\infty$ compact *CR* manifold of *CR*-codimension $\ell \geq 1$ and *CR*-dimension $n-\ell$ in a complex manifold X of complex dimension $n \geq 3$. In this paper, assuming that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$, we prove an L^2-existence theorem and global regularity for the solutions of the tangential Cauchy-Riemann equation for $(0,s)$-forms on M.

1. Introduction and Basic Notations

The tangential Cauchy-Riemann complex (or $\overline{\partial}_b$-complex) was first introduced by Kohn and Rossi [1] for studying the holomorphic extension of *CR* functions from the boundary of a complex manifold. The closed range property is related to existence and regularity theorems for $\overline{\partial}_b$ and for *CR* manifolds to a reason of embedding. It is worth then to mention that the $\overline{\partial}_b$-operator has closed range in the L^2-sense on boundaries of smooth bounded pseudoconvex domains in $\mathbb{C}^n$ due to Shaw [2] for all $1 \leq s < n-2$ and Boas and Shaw [3] for $s = n-2$. Later, Kohn [4] obtained results analogue to those of [2, 3] on boundaries of smooth bounded pseudoconvex domains in a complex manifold. Nicoara [5] extended the results of Kohn [4] to compact, orientable, pseudoconvex *CR* manifold of real dimension $2n-1$, at least five, embedded in $\mathbb{C}^N$, $N \geq n$, leading to global regularity for the $\overline{\partial}_b$-equation on such *CR* manifolds. The main tool in his proof is that of microlocalizations using a new type of weight functions called strongly *CR* plurisubharmonic functions (see also [6]).

In addition, Harrington and Raich [7] adapted the microlocal analysis used by Nicoara [5] to establish the closed range property for the $\overline{\partial}_b$-operator on *CR* manifold of hypersurface type satisfying weak $Y(s)$ condition. More precisely, by using the weighted $\overline{\partial}$-theory, they showed that the complex Green's operator is continuous in the L^2-Sobolev spaces W^k, $k \in \mathbb{N}$, and they further obtained a global solution with $\mathscr{C}^\infty$-regularity for solutions of the $\overline{\partial}_b$-equation for $(0,s)$-forms.

This paper is concerned with proving an L^2-existence theorem for the $\overline{\partial}_b$-Neumann problem on a $\mathscr{C}^\infty$ *CR* compact manifold M of real dimension $2n-\ell$ ($\ell \geq 1$) that satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$ in an n-dimensional complex manifold X and with establishing the global regularity properties of the $\overline{\partial}_b$-equation. In particular, our $\overline{\partial}_b$-problem is set up in the usual L^2-setting with no weights using our arguments in [8, 9]. Namely, via a partition of unity, we globalize first the local maximal L^2-Sobolev estimates obtained by [10] for $\Box_b$ and patching them together to obtain global ones on M. Further, we explore an L^2-existence theorem for the $\overline{\partial}_b$-equation on M. These L^2 results allow us to prove that the complex Green operator G_b and the Szegö projection operators S_s are continuous in the Sobolev spaces $W^k_{0,s}(M)$ for some s such that $1 \leq s \leq n-\ell-1$ and $k \geq 0$. Furthermore, we obtain a global smooth solution for the $\bar{\partial}_b$-equation given smooth data on M. Before we proceed, we recall first some basic definitions and notations on CR manifolds.

Definition 1. Let M be a $\mathscr{C}^\infty$-manifold of real dimension $2n-\ell$. Then a CR structure on M is given by a complex subbundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}T(M) = T(M) \otimes \mathbb{C}$ such that the following conditions are satisfied.

(1) $\dim_{\mathbb{C}} T_z^{1,0}(M) = n-\ell$, where $T_z^{1,0}(M)$ is the fiber at each $z \in M$.
(2) If we define $T^{0,1}(M) = \overline{T^{1,0}(M)}$, then $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$.
(3) $T^{1,0}(M)$ is involutive (or formally integrable); that is, if L_1 and L_2 are two smooth sections of $T^{1,0}(M)$, defined on an open subset $\mathbb{U}$ of M, then so is their Lie bracket $[L_1, L_2] = L_1L_2 - L_2L_1$, for every open subset $\mathbb{U}$ of M.

A $\mathscr{C}^\infty$ manifold M endowed with this CR structure is called a CR manifold of CR-dimension $n-\ell$ and CR codimension ℓ.

Let M be a generic CR manifold of real dimension $2n-\ell$ embedded in an n-dimensional complex manifold X. Such a manifold M can be represented locally in the following form: for each $z \in M$ there exists an open neighborhood U of z in X such that

$$M \cap U = \{\zeta \in U \mid \rho_1(\zeta) = \cdots = \rho_\ell(\zeta) = 0\}, \quad (1)$$

where $\{\rho_\nu\}_{\nu=1,\dots,\ell}$ are $\mathscr{C}^\infty$ real-valued functions on U such that

$$\bar{\partial}\rho_1(\zeta) \wedge \cdots \wedge \bar{\partial}\rho_\ell(\zeta) \neq 0 \quad \text{on } M \cap U. \quad (2)$$

The complex subbundle which defines the induced CR structure on M is given by $T^{1,0}(M) = T^{1,0}(X) \cap \mathbb{C}T(M)$. Denote by $\mathscr{C}^\infty_{0,s}(M)$ the space of $(0,s)$-forms with $\mathscr{C}^\infty$-coefficients on M. The involution condition (3) of Definition 1 implies that there is a restriction of the de Rham exterior derivative d to $\mathscr{C}^\infty_{0,s}(M)$, which is defined by $\bar{\partial}_b : \mathscr{C}^\infty_{0,s}(M) \to \mathscr{C}^\infty_{0,s+1}(M)$.

Let us equip X with a Hermitian metric such that $T^{1,0}(X) \perp T^{0,1}(X)$ and consider on M the induced metric, then $T^{1,0}(M) \perp T^{0,1}(M)$. Let $\mathscr{D}_{0,s}(M)$ be the space of $(0,s)$-forms whose coefficients are $\mathscr{C}^\infty$ with compact support in M. We then can define a Hermitian inner product on $\mathscr{D}_{0,s}(M)$ by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle_z dv, \quad (3)$$

where dv is the volume element associated with the induced metric on M and $\langle \varphi, \psi \rangle_z$ is the pointwise inner product induced on $\mathscr{C}^\infty_{0,s}(M)$ by the metric on $\mathbb{C}T(M)$ at each $z \in M$. Let $\|\varphi\|^2 = (\varphi, \varphi)$ be the corresponding norm and $L^2_{0,s}(M)$ the L^2-completion of $\mathscr{D}_{0,s}(M)$ with respect to this norm. Let $\bar{\partial}_b : L^2_{0,s}(M) \to L^2_{0,s+1}(M)$ be the maximal closed extension of the original $\bar{\partial}_b$ on $\mathscr{C}^\infty_{0,s}(M)$. A form $u \in L^2_{0,s}(M)$ is in the domain of $\bar{\partial}_b$ if $\bar{\partial}_b u$, defined in the sense of distributions, belongs to $L^2_{0,s+1}(M)$. In this way, $\bar{\partial}_b$ defines a linear, closed, densely defined operator. Let $\bar{\partial}_b^* : L^2_{0,s+1}(M) \to L^2_{0,s}(M)$ be the L^2-Hilbert space adjoint of $\bar{\partial}_b$ such that $(\varphi, \bar{\partial}_b\psi) = (\bar{\partial}_b^*\varphi, \psi)$ for all ψ in $\mathrm{Dom}(\bar{\partial}_b)$ and φ in $\mathrm{Dom}(\bar{\partial}_b^*)$. The Kohn-Laplacian $\Box_b$ is defined by

$$\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b : \mathrm{Dom}(\Box_b) \longrightarrow L^2_{0,s}(M), \quad (4)$$

where

$$\begin{aligned} &\mathrm{Dom}(\Box_b) \\ &= \Big\{\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*) \\ &\quad \subset L^2_{0,s}(M) \mid \bar{\partial}_b\varphi \in \mathrm{Dom}(\bar{\partial}_b^*), \bar{\partial}_b^*\varphi \in \mathrm{Dom}(\bar{\partial}_b)\Big\}. \end{aligned} \quad (5)$$

We recall that the Kohn-Laplacian $\Box_b$ is not elliptic, so it has a characteristic set of dimension ℓ. Let $N(M)$ be the ℓ-dimensional bundle such that

$$\mathbb{C}T(M) = T^{1,0}(M) \oplus T^{0,1}(M) \oplus N(M). \quad (6)$$

Let $N^*(M)$ be the dual bundle of $N(M)$. Let $\gamma \in N^*(M)$, then γ annihillates $T^{1,0}(M) \oplus T^{0,1}(M)$. Thus $N^*(M)$ is called the characteristic bundle. The Levi form of M at a point $z \in M$ is defined as the Hermitian form on $T^{1,0}(M)$ with values in $N(M)$ such that

$$\mathscr{L}_z(L_1, L_2) = i\pi_z\left(\left[L_1, \bar{L}_2\right]_z\right), \quad L_1, L_2 \in T^{1,0}(M), \quad (7)$$

where π_z is the projection of $\mathbb{C}T_z(M)$ onto $N_z(M)$.

The Levi form of M at a point $z \in M$ in the direction $\gamma \in N^*(M)$ is the scalar Hermitian form denoted $\mathscr{L}_z(\gamma)$ and is given by

$$\begin{aligned} \mathscr{L}_z(\gamma) &= \langle \mathscr{L}_z(L_1, L_2), \gamma \rangle \\ &= i\left\langle \left[L_1, \bar{L}_2\right], \gamma \right\rangle_z, \quad L_1, L_2 \in T^{1,0}(M). \end{aligned} \quad (8)$$

Definition 2 (see [10, Definition 1.2]). A CR manifold M of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in a complex manifold of complex dimension n is said to satisfy condition $Z(s)$, $1 \leq s \leq n-\ell-1$, at a point $z \in M$ in the direction $\gamma \in N^*(M)$ if the Levi form $\mathscr{L}_z(\gamma)$ has at least $n-\ell-s+1$ positive eigenvalues or at least $s+1$ negative eigenvalues. M is said to satisfy condition $Y(s)$ at $z \in M$ if it satisfies condition $Z(s)$ for all directions $\gamma \in N_z^*(M)$.

Note that in the hypersurface case, that is, $\ell = 1$, the condition $Y(s)$ defined above is equivalent to the classical $Y(s)$ condition of Kohn for hypersurfaces (see, e.g., [11] for more details). In particular, if the CR structure is strictly pseudoconvex; that is, the Levi form of M is positive or negative definite, condition $Y(s)$ holds for all $1 \leq s \leq n-2$.

2. L^2-Existence Theory for $\bar{\partial}_b$

Let M be a $\mathscr{C}^\infty$ generic CR manifold of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in a complex manifold X

of complex dimension n. For each point $p_0 \in M$, there is then a neighborhood U of p_0 in X and a local orthonormal basis consisting of smooth vector fields $L_1, \ldots, L_{n-\ell}$ for $T^{1,0}(U)$ (see, e.g., [12, Section 7.2; Theorem 3]). The collection of vector fields $\{\overline{L}_1, \ldots, \overline{L}_{n-\ell}\}$ forms a local orthonormal basis for $T^{0,1}(U)$. Let $T_1, \ldots, T_\ell$ be real vector fields on U such that the set $\{L_1, \ldots, L_{n-\ell}, \overline{L}_1, \ldots, \overline{L}_{n-\ell}, T_1, \ldots, T_\ell\}$ forms a local orthonormal basis for $\mathbb{C}T(U)$. Denote by $\{\omega^1, \ldots, \omega^{n-\ell}, \overline{\omega}^1, \ldots, \overline{\omega}^{n-\ell}, \gamma_1, \ldots, \gamma_\ell\}$ the basis for $\mathbb{C}T^*(U)$ dual to $\{L_1, \ldots, \overline{L}_{n-\ell}, T_1, \ldots, T_\ell\}$. In terms of this basis, an element φ in $\mathscr{C}^\infty_{0,s}(U)$ can be uniquely expressed as a sum:

$$\varphi = \sum_{|I|=s} \varphi_I \overline{\omega}^I, \tag{9}$$

where $I = (i_1, i_2, \ldots, i_s)$ is an s-tuple of integers with $1 \le i_1 < \cdots < i_s \le n-\ell$ and $\overline{\omega}^I = \overline{\omega}^{i_1} \wedge \cdots \wedge \overline{\omega}^{i_s}$.

We then have

$$\begin{aligned} \bar{\partial}_b \varphi &= \sum_{|I|=s} \sum_{j=1}^{n-\ell} \overline{L}_j (\varphi_I) \overline{\omega}^j \wedge \overline{\omega}^I + \cdots \\ &= \sum_{|J|=s+1} \left(\sum_{j,I} \varepsilon^{jI}_J \overline{L}_j (\varphi_I) \right) \overline{\omega}^J + \cdots, \end{aligned} \tag{10}$$

where ε^{jI}_J is zero if $j \cup \{I\} \ne J$ as sets and is the sign of the permutation that reorders jI as J if $j \cup \{I\} = J$, and the $\cdots$ stands for terms of order zero. Using integration by parts, we obtain

$$\begin{aligned} \bar{\partial}^*_b \varphi &= -\sum_{|I|=s} \sum_{j=1}^{n-\ell} L_j (\varphi_{jI}) \overline{\omega}^I + \cdots \\ &= -\sum_{|K|=s-1} \left(\sum_{j,I} \varepsilon^{jI}_K L_j (\varphi_I) \right) \overline{\omega}^K + \cdots. \end{aligned} \tag{11}$$

For φ in $\mathscr{C}^\infty_{0,s}(\overline{U})$, the subspace of smooth $(0,s)$-forms on U that can be extended smoothly up to and including the boundary, we set

$$\begin{aligned} \|\varphi\|^2_{\mathscr{L}(U)} &= \sum_{j=1}^{n-\ell} \left\| L_j(\varphi) \right\|^2 + \|\varphi\|^2, \\ \|\varphi\|^2_{\overline{\mathscr{L}}(U)} &= \sum_{j=1}^{n-\ell} \left\| \overline{L}_j(\varphi) \right\|^2 + \|\varphi\|^2. \end{aligned} \tag{12}$$

If we further assume that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$, for each $p_0 \in M$, we can find a constant $C = C(p_0) > 0$ such that

$$\|\varphi\|^2_{\mathscr{L}(U)} + \|\varphi\|^2_{\overline{\mathscr{L}}(U)} \le C \left(\left\| \bar{\partial}_b \varphi \right\|^2 + \left\| \bar{\partial}^*_b \varphi \right\|^2 + \|\varphi\|^2 \right) \tag{13}$$

uniformly for all $\varphi \in \mathscr{D}_{0,s}(U)$ (see, e.g., [10]).

Set $L_j = X_{2j-1} + iX_{2j}$; $j = 1, \ldots, n-\ell$. The condition $Y(s)$ implies that the real vector $X_1, \ldots, X_{2n-2\ell}$ and their commutators of length at most two span the tangent space at each point in U. Thus $X_1, \ldots, X_{2n-2\ell}$ satisfy Hörmander's finite rank condition of order two. It follows then from [13, Theorem A] (see also [14]) that there is a positive constant $C = C(U)$ satisfying the following $1/2$-subelliptic estimate:

$$\|\varphi\|^2_{1/2(U)} \le C \left(\sum_{i=1}^{2n-2\ell} \left\| X_i \varphi \right\|^2 + \|\varphi\|^2 \right), \quad \varphi \in \mathscr{D}_{0,s}(U). \tag{14}$$

Here and always $\|\cdot\|_{k(U)}$ denotes the L^2 Sobolev space k-norm, $\|\cdot\|_{-k}$ is the norm of its dual space, and $\|\cdot\|$ is the usual L^2-norm. We may omit the subscript U from the norm notation when there is no danger of confusion.

Combining the above $1/2$-subelliptic estimate with (13), as in [10], we get the following theorem.

Theorem 3. *Let M be a $\mathscr{C}^\infty$ CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in a complex manifold X of complex dimension n. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. For each point $p_0 \in M$, there is then an open neighborhood U on which the Kohn Laplacian $\Box_b$ satisfies the $1/2$-subelliptic estimate*

$$\|\varphi\|_{1/2(U)} \le C \left(\left\| \bar{\partial}_b \varphi \right\|^2 + \left\| \bar{\partial}^*_b \varphi \right\|^2 + \|\varphi\|^2 \right) \tag{15}$$

uniformly for all φ in $\mathscr{D}_{0,s}(U)$.

In addition, if M is compact, the estimate (15) *holds uniformly on M for all φ in $\mathscr{C}^\infty_{0,s}(M)$.*

Theorem 4 (see [10]). *Let M be given as in Theorem 3 and ϕ the unique solution of the equation $(\Box_b + Id)\phi = f$ for $f \in L^2_{0,s}(M)$, where Id is the identity operator. Let $U \subset\subset M$ be a relatively compact subset of M. If the restriction of f to U is in $\mathscr{C}^\infty_{0,s}(U)$, the restriction of ϕ to U is then in $\mathscr{C}^\infty_{0,s}(U)$. In addition, suppose that η and η_1 are two cut-off functions supported in U such that $\eta = 1$ on the support of η_1; then if the restriction of f to U is in the L^2-Sobolev space $W^k_{0,s}(U)$ for some nonnegative integer k, the restriction of $\eta_1\phi$ to U is in $W^{k+1}_{0,s}(U)$ and there is a constant $C_k > 0$ (independent of f) such that*

$$\|\eta_1 \phi\|_{k+1(U)} \le C_k \left(\|\eta f\|_{k(U)} + \|f\| \right). \tag{16}$$

Patching the above local estimates, we obtain the following global one.

Theorem 5. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. Let $\phi \in \mathrm{Dom}(\Box_b)$ such that $(\Box_b + Id)\phi = f$ for f in $W^k_{0,s}(M)$, $k \ge 0$, then ϕ is in $W^{k+1}_{0,s}(M)$ and there exists a constant $C_k > 0$ (independent of f) such that*

$$\|\phi\|_{k+1(M)} \le C_k \|f\|_{k(M)}. \tag{17}$$

Using Theorem 5 and following an induction argument on k, we get the following result.

Proposition 6. *Let M be given as in Theorem 5. Then the Kohn Laplacian $\Box_b$ is hypoelliptic. Moreover, if $\Box_b\phi = f$ for f*

in $W^k_{0,s}(M)$, $k \ge 0$, *then* ϕ *is in* $W^{k+1}_{0,s}(M)$ *and there is a constant* $C_k > 0$ *(independent of* f*) such that*

$$\|\phi\|^2_{k+1(M)} \le C_k \left(\|f\|^2_{k(M)} + \|\phi\|^2\right). \tag{18}$$

Let

$$\mathscr{H}^b_{0,s}(M) = \left\{\alpha \in \mathrm{Dom}\left(\bar\partial_b\right) \cap \mathrm{Dom}\left(\bar\partial^\star_b\right) \subset L^2_{0,s}(M) \mid \bar\partial_b\alpha = \bar\partial^\star_b\alpha = 0\right\} \tag{19}$$

be the closed subspace of $L^2_{0,s}(M)$ consisting of harmonic forms and

$$^\perp\mathscr{H}^b_{0,s}(M) = \left\{\alpha \in L^2_{0,s}(M) \mid (\alpha,\phi) = 0 \ \forall\phi \in \mathscr{H}^b_{0,s}(M)\right\}. \tag{20}$$

The main L^2-result is the following theorem.

Theorem 7. *Let* M *be a* $\mathscr{C}^\infty$ *compact CR manifold of real dimension* $2n-\ell$ *and codimension* $\ell \ge 1$ *in an* n*-dimensional complex manifold* X*. Suppose that* M *satisfies condition* $Y(s)$ *for some* s *such that* $1 \le s \le n-\ell-1$*. Then the following holds.*

(1) *The space of harmonic* $(0,s)$*-forms* $\mathscr{H}^b_{0,s}(M)$ *is of finite dimensional.*

(2) *The operators* $\bar\partial_b : L^2_{0,s}(M) \to L^2_{0,s+1}(M)$, $\bar\partial^\star_b : L^2_{0,s+1}(M) \to L^2_{0,s}(M)$, *and* $\Box_b = \bar\partial_b\bar\partial^*_b + \bar\partial^*_b\bar\partial_b : \mathrm{Dom}(\Box_b) \to L^2_{0,s}(M)$ *have closed ranges.*

(3) *The complex Green operator* $G_b : L^2_{0,s}(M) \to \mathrm{Dom}(\Box_b)$ *exists and is a compact operator in* $L^2_{0,s}(M)$.

(4) *For any* f *in* $L^2_{0,s}(M)$, *we have*

$$f = \bar\partial_b\bar\partial^\star_b G_b f + \bar\partial^\star_b\bar\partial_b G_b f + H^b_{0,s} f, \tag{21}$$

where $H^b_{0,s}$ *is the orthogonal projection of* $L^2_{0,s}(M)$ *onto* $\mathscr{H}^b_{0,s}(M)$.

(5) $G_b H^b_{0,s} = H^b_{0,s} G_b = 0$. $G_b\Box_b = \Box_b G_b = Id - H^b_{0,s}$ *on* $\mathrm{Dom}(\Box_b)$.

(6) *If* G_b *is defined on* $L^2_{0,s+1}(M)$ *(resp.,* $L^2_{0,s-1}(M)$*),* $\bar\partial_b G_b = G_b\bar\partial_b$ *on* $\mathrm{Dom}(\bar\partial_b)$ *(resp.,* $\bar\partial^\star_b G_b = G_b\bar\partial^\star_b$ *on* $\mathrm{Dom}(\bar\partial^\star_b)$*).*

(7) *If* f *is in* $L^2_{0,s}(M)$ *such that* $\bar\partial_b f = 0$ *and* $f \perp \mathscr{H}^b_{0,s}(M)$, *then* $f = \bar\partial_b\bar\partial^\star_b G_b f$ *and* $u = \bar\partial^\star_b G_b f$ *is the unique solution to the equation* $\bar\partial_b u = f$ *which is orthogonal to* $\mathrm{Ker}(\bar\partial_b)$ *and satisfies* $\|u\|^2 \le C\|f\|^2$.

(8) $G_b(\mathscr{C}^\infty_{0,s}(M)) \subseteq \mathscr{C}^\infty_{0,s}(M)$, *and for each* $k \in \mathbb{R}$ *there is a positive constant* C_s *such that the estimate* $\|G_b f\|_{k+1} \le C_s\|f\|_k$ *holds uniformly for all* f *in* $\mathscr{C}^\infty_{0,s}(M)$.

Proof. Since M is compact, via a partition of unity, the estimate (15) holds globally on M. Suppose that f_k is a sequence in $\mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b) \cap L^2_{0,s}(M)$ such that $\|f_k\|$ is bounded, $\bar\partial_b f_k \to 0$ in the $L^2_{0,s+1}(M)$-norm and $\bar\partial^*_b f_k \to 0$ in the $L^2_{0,s-1}(M)$-norm as $k \to \infty$. Thus, we have $\|f_k\|_{1/2(M)} \le c$ for some constant c. By Rellich's Lemma, the inclusion map $i_M : W^{1/2}_{0,s}(M) \to L^2_{0,s}(M)$ is compact; we can then extract a subsequence of f_k which converges in $L^2_{0,s}(M)$. Then the hypotheses of Theorem 1.1.3 in Hörmander [15] are satisfied which implies that $\mathscr{H}^b_{0,s}(M)$ is finite dimensional and the estimate

$$\|f\|^2 \le C\left(\left\|\bar\partial_b f\right\|^2 + \left\|\bar\partial^*_b f\right\|^2\right) \tag{22}$$

holds for every f in $\mathrm{Dom}(\bar\partial_b) \cap \mathrm{Dom}(\bar\partial^*_b)$ with $f \perp \mathscr{H}^b_{0,s}(M)$.

By Theorem 1.1.2 in [15], we then conclude that the operators $\bar\partial_b : L^2_{0,s}(M) \to L^2_{0,s+1}(M)$ and $\bar\partial^\star_b : L^2_{0,s}(M) \to L^2_{0,s-1}(M)$ have closed ranges. We obtain also from (22) that

$$\|f\| \le C\|\Box_b f\|, \quad f \in \mathrm{Dom}(\Box_b),\ f \perp \mathscr{H}^b_{0,s}(M). \tag{23}$$

This estimate implies that $\Box_b$ is one-to-one and in view of Theorem 1.1.1 in [15] that the range of $\Box_b$ is closed. It forces, since $\Box_b$ is self-adjoint, the strong Hodge decomposition:

$$\begin{aligned} L^2_{0,s}(M) &= \mathrm{Range}(\Box_b) \oplus \mathscr{H}^b_{0,s}(M) \\ &= \bar\partial_b\bar\partial^*_b \,\mathrm{Dom}(\Box_b) \oplus \bar\partial^*_b\bar\partial_b\,\mathrm{Dom}(\Box_b) \oplus \mathscr{H}^b_{0,s}(M). \end{aligned} \tag{24}$$

Thus $\Box_b : \mathrm{Dom}(\Box_b) \to {}^\perp\mathscr{H}^b_{0,s}(M)$ is one-to-one and onto. This implies the existence of the complex Green operator $G_b : L^2_{0,s}(M) \to \mathrm{Dom}(\Box_b)$ as a unique operator that inverts $\Box_b$ on ${}^\perp\mathscr{H}^b_{0,s}(M)$. The operator G_b is defined as follows: if f is in $\mathrm{Range}(\Box_b)$, we define $G_b f = \phi$, where ϕ is the unique solution of $\Box_b\phi = f$ with $f \perp \mathscr{H}^b_{0,s}(M)$. G_b is extended to the whole $L^2_{0,s}(M)$ space by setting $G_b = 0$ on $\mathscr{H}^b_{0,s}(M)$. The boundedness of G_b in $L^2_{0,s}(M)$ follows from (23).

To show that G_b is compact in $L^2_{0,s}(M)$, it suffices to show compactness on ${}^\perp\mathscr{H}^b_{0,s}(M)$ (since $G_b \equiv 0$ on $\mathscr{H}^b_{0,s}(M)$). When $f \perp \mathscr{H}^b_{0,s}(M)$ (and hence $G_b f \perp \mathscr{H}^b_{0,s}(M)$), the integration by parts, Cauchy-Schwarz inequality ($|(u,v)| \le \|u\|\|v\|$), and (23) imply

$$\begin{aligned} \left\|\bar\partial_b G_b f\right\|^2 + \left\|\bar\partial^*_b G_b f\right\|^2 &= \left(\bar\partial_b G_b f, \bar\partial_b G_b f\right) + \left(\bar\partial^*_b G_b f, \bar\partial^*_b G_b f\right) \\ &= \left(\bar\partial^*_b\bar\partial_b G_b f, G_b f\right) + \left(\bar\partial_b\bar\partial^*_b G_b f, G_b f\right) \\ &= (f, G_b f) \le \|f\|\,\|G_b f\| \le C\|f\|^2. \end{aligned} \tag{25}$$

By applying (15) to $G_b f$ and using (23), we get

$$\begin{aligned} \|G_b f\|^2_{1/2(M)} &\le C\left(\left\|\bar\partial_b G_b f\right\|^2 + \left\|\bar\partial^*_b G_b f\right\|^2 + \|G_b f\|^2\right) \\ &\le K\|f\|^2, \end{aligned} \tag{26}$$

where K is a positive constant. Thus the compactness of G_b in $L^2_{0,s}(M)$ follows from Rellich's Lemma.

The assertions in (5) follow immediately from the definition of G_b. For assertion (6), if $f \in \mathrm{Dom}(\bar{\partial}_b)$ and G_b is also defined on $L^2_{0,s+1}(M)$, by (21) and the first assertion of (5), we have

$$\begin{aligned} G_b\bar{\partial}_b f &= G_b\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f \\ &= G_b\left(\bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b\right)\bar{\partial}_b G_b f \\ &= G_b\Box_b\bar{\partial}_b G_b f = \bar{\partial}_b G_b f. \end{aligned} \tag{27}$$

A similar equation holds for $\bar{\partial}_b^*$. Assertions (1)–(6) have been established.

To show assertion (7), if $f \perp \mathscr{H}^b_{0,s}(M)$ and $\bar{\partial}_b f = 0$, then $\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f = 0$ as well (from (21)). Consequently, $\|\bar{\partial}_b^*\bar{\partial}_b G_b f\|^2 = (\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f, \bar{\partial}_b G_b f) = 0$, since $\bar{\partial}_b G_b f \in \mathrm{Dom}(\bar{\partial}_b^*)$, and hence $\bar{\partial}_b^*\bar{\partial}_b G_b f = 0$. Thus $f = \bar{\partial}_b(\bar{\partial}_b^* G_b f)$ and $u = \bar{\partial}_b^* G_b f$ is orthogonal to $\mathrm{Ker}(\bar{\partial}_b)$. Following assertion (3) and the fact that G_b is bounded, u satisfies the following L^2-estimate:

$$\begin{aligned} \|u\|^2 &= \left\|\bar{\partial}_b^* G_b f\right\|^2 = \left(\bar{\partial}_b^* G_b f, \bar{\partial}_b^* G_b f\right) \\ &= \left(\bar{\partial}_b\bar{\partial}_b^* G_b f, G_b f\right) = \left(\left(\bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b\right) G_b f, G_b f\right) \\ &= \left(f, G_b f\right) \le \|f\| \, \|G_b f\| \le C\|f\|^2. \end{aligned} \tag{28}$$

Finally, we show assertion (8); if $f \in \mathscr{C}^\infty_{0,s}(M)$, then $f - H^b_{0,s} f \in \mathscr{C}^\infty_{0,s}(M)$ and, since M is compact, $f \in \mathrm{Dom}(\Box_b)$. On other hand, from assertion (5), $\Box_b G_b f = f - H^b_{0,s} f$. Since $\Box_b$ is hypoelliptic, by Proposition 6, $G_b f \in \mathscr{C}^\infty_{0,s}(M)$.

Again Proposition 6 implies

$$\begin{aligned} \left\|G_b f\right\|_{k+1(M)} &\le C_k\left(\left\|\Box_b G_b f\right\|_{k(M)} + \left\|G_b f\right\|\right) \\ &\le C_k\left(\|f\|_{k(M)} + \left\|H^b_{0,s} f\right\|_{k(M)} + (\text{const.})\,\|f\|\right) \\ &\le C\|f\|_{k(M)}. \end{aligned} \tag{29}$$

Here we have used the fact that $\mathscr{H}^b_{0,s}(M)$ is of finite dimension to conclude the estimate

$$\left\|H^b_{0,s} f\right\|_{k(M)} \le C_k\left\|H^b_{0,s} f\right\| \le C_k\|f\|_{k(M)} \tag{30}$$

for some constant C_k. The theorem is proved. □

3. Sobolev Space Estimates

In this section, we prove that the complex Green operator G_b, the canonical solution operators $\bar{\partial}_b G_b$ and $\bar{\partial}_b^* G_b$, and the Szegö projection S_s operators enjoy some regularity properties in the L^2-Sobolev spaces $W^k_{0,s}(M)$, $k \ge 0$, for some s with $1 \le s \le n - \ell - 1$. Furthermore, we obtain a global regularity for the solutions of the $\bar{\partial}_b$-equation.

By the same way for bounded pseudoconvex domains, a differential operator is said to be exactly regular if it maps all L^2-Sobolev spaces $W^k_{0,s}(M)$ $(k \ge 0)$ to themselves and globally regular if it maps the space $\mathscr{C}^\infty_{0,s}(M)$ continuously to itself.

3.1. Continuity of the Complex Green Operator. We prove first the continuity of the complex Green operator G_b on $W^k_{0,s}(M)$, $k \ge 0$.

Theorem 8. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \ge 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \le s \le n-\ell-1$. Then the complex Green operator G_b is continuous on the Sobolev space $W^k_{0,s}(M)$, $k \ge 0$; that is, there is a constant $C = C(k) > 0$ such that*

$$\left\|G_b f\right\|_{k(M)} \le C\|f\|_{k(M)}, \quad f \in W^k_{0,s}(M). \tag{31}$$

Proof. We consider the special case when $k = 0, 1, 2, 3, \ldots$. Indeed the general case is then derived by means of interpolation of linear operators. Since M is compact, it is easy to show that $\mathscr{C}^\infty_{0,s}(M)$ is a dense subspace in $W^k_{0,s}(M)$. Further, by Theorem 7 (8), we have $G_b f \in \mathscr{C}^\infty_{0,s}(M)$ for $f \in \mathscr{C}^\infty_{0,s}(M)$. Thus it suffices to establish (31) for $f \in \mathscr{C}^\infty_{0,s}(M)$. For $k = 0$, (31) follows from (23).

For each $k \ge 0$, let $\Lambda^k(\xi)$ be a pseudodifferential operator of order k with symbol $(1 + |\xi|^2)^{k/2}$. Let U be an open neighborhood of ζ in M and let η and η_1 be two cutoff functions with supports in U such that $\eta = 1$ on supp η_1; then $\eta\Lambda^k\eta_1 f \in \mathscr{D}_{0,s}(U)$ whenever $f \in \mathscr{D}_{0,s}(U)$.

Recall that the compactness of G_b in $L^2_{0,s}(U)$ is equivalent to the compactness estimate: for every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that for every $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}_b^*)$

$$\|\varphi\|^2 \le \epsilon Q_b(\varphi, \varphi) + C(\epsilon)\|\varphi\|^2_{-1(U)}, \tag{32}$$

where $Q_b(\varphi, \varphi) = (\bar{\partial}_b\varphi, \bar{\partial}_b\varphi) + (\bar{\partial}_b^*\varphi, \bar{\partial}_b^*\varphi)$. For this estimate and further results on the compactness of the complex Green operator see, e.g., [16–19].

Applying (32) for $\eta\Lambda^k\eta_1 G_b f$, we obtain

$$\begin{aligned} \left\|\eta\Lambda^k\eta_1 G_b f\right\|^2 &\le \epsilon Q_b\left(\eta\Lambda^k\eta_1 G_b f, \eta\Lambda^k\eta_1 G_b f\right) \\ &\quad + C(\epsilon)\left\|\eta\Lambda^k\eta_1 G_b f\right\|^2_{-1(U)}. \end{aligned} \tag{33}$$

We sometimes use A for $\eta\Lambda^k\eta_1$ and A^* for its formal adjoint, which is also a tangential operator of order k. We estimate the first term on the right hand side in (33), it is a standard consequence of [20, Corollary 3.1] (or [11, Lemma 2.4.2]) that

$$\begin{aligned} Q_b\left(AG_b f, AG_b f\right) &= \mathrm{Re}\, Q_b\left(G_b f, A^* A G_b f\right) \\ &\quad + \mathscr{O}\left(|||DG_b f|||^2_{k-1(U)}\right) \\ &\le \mathrm{Re}\, Q_b\left(G_b f, A^* A G_b f\right) + C\|G_b f\|^2_{k(U)}. \end{aligned} \tag{34}$$

Here we have used the fact that the tangential derivative D^α of order $|\alpha| = \lambda$ satisfies the tangential Sobolev estimate $|||D^\alpha f|||_r \le \|f\|_{r+\lambda}$.

Taking $v = A^* A f$ in the form $Q_b(G_b u, v) = (u, v)$, we get

$$\begin{aligned} Q_b\left(AG_b f, AG_b f\right) &\le \operatorname{Re}\left(f, A^* AG_b f\right) + C\|G_b f\|_{k(U)}^2 \\ &\le \left|\left(f, A^* AG_b f\right)\right| + C\|G_b f\|_{k(U)}^2. \end{aligned} \tag{35}$$

The Cauchy-Schwarz inequality implies

$$Q_b\left(AG_b f, AG_b f\right) \le \|Af\| \, \|AG_b f\| + C\|G_b f\|_{k(U)}^2. \tag{36}$$

Inequality (33) becomes

$$\left\|\eta \Lambda^k \eta_1 G_b f\right\|^2 \le \epsilon \|f\|_{k(U)} \|G_b f\|_{k(U)} + C(\epsilon) \left\|\eta \Lambda^k \eta_1 G_b f\right\|_{-1(U)}^2. \tag{37}$$

Summing over a partition of unity subordinate to an open covering of M by patches $\{U_i\}_{i=1}^m$, we obtain estimate like (37) on each of these patches and using the interior regularity properties, we get

$$\|G_b f\|_{k(M)}^2 \le \epsilon \|f\|_{k(M)} \|G_b f\|_{k(M)} + C(\epsilon) \|G_b f\|_{k-1(M)}^2. \tag{38}$$

The first term in the right-hand side of (38) is estimated by $\epsilon(\text{s.c.})\|G_b f\|_{k(M)}^2 + \epsilon(\text{l.c.})\|f\|_{k(M)}^2$, where s.c. and l.c. denote a small and a large constants, respectively, in the inequality $|ab| \le (\text{s.c.})a^2 + (\text{l.c.})b^2$. The second term is estimated by interpolation of Sobolev norms ($\|G_b f\|_{k-1(M)}^2 \le \varepsilon \|G_b f\|_{k(M)}^2 + C(\varepsilon)\|G_b f\|^2$) and then by using the continuity of G_b in $L^2_{0,s}(M)$ with L^2-bounded norm.

Adding up the analogues terms and absorbing, by choosing ϵ and ε to be small enough, $\|G_b f\|_{k(M)}^2$ into the left, this gives

$$\|G_b f\|_{k(M)}^2 \le C\|f\|_{k(M)}^2 + K\|f\|^2, \tag{39}$$

where $C = C(\epsilon, k) > 0$ and $K = K(\epsilon, k) > 0$. The embedding Sobolev space implies (31) for $k = 0, 1, 2, 3, \ldots$. The general case is obtained from interpolation of linear operators. As mentioned above, the density of $\mathscr{C}^\infty_{0,s}(M)$ in $W^k_{0,s}(M)$ passes (31) to forms f in $W^k_{0,s}(M)$. This proves the continuity of G_b in $W^k_{0,s}(M)$. □

Corollary 9. *Let M be given as in Theorem 8, then the canonical solution operators $\overline{\partial}_b G_b$ and $\overline{\partial}_b^* G_b$ are continuous on $W^k_{0,s}(M)$ for all $k \ge 0$.*

Proof. We argue by induction on k. The case when $k = 0$ follows from (25). Suppose that the assertions hold for positive integers less than k and assume that ζ, U, η, and η_1 are given as in the proof of Theorem 8. By the interior elliptic regularity properties, we prove first a priori estimate for $\overline{\partial}_b G_b f$ and $\overline{\partial}_b^* G_b f$ with $f \in \mathscr{D}_{0,s}(U)$ as follows:

$$\begin{aligned} &\left\|\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f\right\|^2 + \left\|\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f\right\|^2 \\ &= \left(\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f, \overline{\partial}_b \eta \Lambda^k \eta_1 G_b f\right) \\ &\quad + \left(\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f, \overline{\partial}_b^* \eta \Lambda^k \eta_1 G_b f\right) \\ &\quad + \mathscr{O}\left(\left(\left\|\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f\right\| + \left\|\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f\right\|\right) \|G_b f\|_{k(U)}\right) \\ &= \left(\eta \Lambda^k \eta_1 \overline{\partial}_b^* \overline{\partial}_b G_b f, \eta \Lambda^k \eta_1 G_b f\right) \\ &\quad + \left(\eta \Lambda^k \eta_1 \overline{\partial}_b \overline{\partial}_b^* G_b f, \eta \Lambda^k \eta_1 G_b f\right) \\ &\quad + \mathscr{O}\left(\left(\left\|\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f\right\| + \left\|\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f\right\|\right) \|G_b f\|_{k(U)} \right. \\ &\qquad \left. + \|G_b f\|_{k(U)}^2\right) \\ &= \left(\eta \Lambda^k \eta_1 \Box_b G_b f, \eta \Lambda^k \eta_1 G_b f\right) \\ &\quad + \mathscr{O}\left(\left(\left\|\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f\right\| + \left\|\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f\right\|\right) \|G_b f\|_{k(U)} \right. \\ &\qquad \left. + \|G_b f\|_{k(U)}^2\right) \\ &\le C_1 \|f\|_{k(U)} \|G_b f\|_{k(U)} \\ &\quad + C_2 \left(\left(\left\|\eta \Lambda^k \eta_1 \overline{\partial}_b G_b f\right\| + \left\|\eta \Lambda^k \eta_1 \overline{\partial}_b^* G_b f\right\|\right) \|G_b f\|_{k(U)} \right. \\ &\qquad \left. + \|G_b f\|_{k(U)}^2\right). \end{aligned} \tag{40}$$

Summing over a partition of unity, using the small and large constants for the resulting terms $\|f\|_k \|G_b f\|_k$, $\|\overline{\partial}_b G_b f\|_k \|G_b f\|_k$, and $\|\overline{\partial}_b^* G_b f\|_k \|G_b f\|_k$, using (31) and adding up the analogues terms, we see that the terms on the right-hand side containing $\|\overline{\partial}_b G_b f\|_k^2$ and $\|\overline{\partial}_b^* G_b f\|_k^2$ can be absorbed into the left hand side. We therefore obtain

$$\left\|\overline{\partial}_b G_b f\right\|_{k(M)}^2 + \left\|\overline{\partial}_b^* G_b f\right\|_{k(M)}^2 \le C\|f\|_{k(M)}^2, \quad f \in \mathscr{D}_{0,s}(M). \tag{41}$$

This completes the induction on k for the norms of $\overline{\partial}_b G_b$ and $\overline{\partial}_b^* G_b$. By the density of $\mathscr{C}^\infty_{0,s}(M)$ in $W^k_{0,s}(M)$, the estimates extend to forms in $W^k_{0,s}(M)$. As before, the general case is obtained from interpolation of linear operators. Then $\overline{\partial}_b G_b$ and $\overline{\partial}_b^* G_b$ are continuous on $W^k_{0,s}(M)$. □

3.2. Exact and Global Regularity Theorems. We now show the expression of the complex Green operator by Szegö projections.

Theorem 10. *The Szegö projections* $S_s : L^2_{0,s}(M) \to \operatorname{Ker}(\overline{\partial}_b)$ *are given by the following relations:*

$$S_s = Id - \overline{\partial}_b^* \overline{\partial}_b G_b = Id - G_b \overline{\partial}_b^* \overline{\partial}_b, \quad s \geq 0, \tag{42}$$

$$S_{s-1} = Id - \overline{\partial}_b^* G_b \overline{\partial}_b, \quad s \geq 1. \tag{43}$$

Proof. We first show that $\overline{\partial}_b^* \overline{\partial}_b G_b = G_b \overline{\partial}_b^* \overline{\partial}_b$. For $\alpha, \beta \in^{\perp} \mathscr{H}^b_{0,s}(M)$, we observe that

$$\overline{\partial}_b \alpha = 0 \Longrightarrow \overline{\partial}_b^* \overline{\partial}_b G_b \alpha = 0 \Longrightarrow \alpha = \overline{\partial}_b \overline{\partial}_b^* G_b \alpha = G_b \overline{\partial}_b \overline{\partial}_b^* \alpha, \tag{44}$$

$$\overline{\partial}_b^* \beta = 0 \Longrightarrow \overline{\partial}_b \overline{\partial}_b^* G_b \beta = 0 \Longrightarrow \beta = \overline{\partial}_b^* \overline{\partial}_b G_b \beta = G_b \overline{\partial}_b^* \overline{\partial}_b \beta. \tag{45}$$

As Range $(\overline{\partial}_b) \perp \operatorname{Ker}(\overline{\partial}_b^*)$ and Range $(\overline{\partial}_b^*) \perp \operatorname{Ker}(\overline{\partial}_b)$, one has

$$\overline{\partial}_b \alpha = 0 \Longrightarrow \overline{\partial}_b G_b \alpha = 0, \tag{46}$$

$$\overline{\partial}_b^* \beta = 0 \Longrightarrow \overline{\partial}_b^* G_b \beta = 0. \tag{47}$$

Any $f \perp \mathscr{H}^b_{0,s}(M)$ can then be written as $f = \alpha + \beta$ so that $\overline{\partial}_b \alpha = 0$ and $\overline{\partial}_b^* \beta = 0$. By (45) and (46), we then have

$$\begin{aligned} \overline{\partial}_b^* \overline{\partial}_b G_b f &= \overline{\partial}_b^* \overline{\partial}_b G_b (\alpha + \beta) = \overline{\partial}_b^* \overline{\partial}_b G_b \beta \\ &= G_b \overline{\partial}_b^* \overline{\partial}_b \beta = G_b \overline{\partial}_b^* \overline{\partial}_b f. \end{aligned} \tag{48}$$

This implies the second equality in (42). Now, If $f \in \operatorname{Ker}(\overline{\partial}_b)$, then $(Id - G_b \overline{\partial}_b^* \overline{\partial}_b) f = f$, so the expression for S_s holds. Next, if $f \perp \operatorname{Ker}(\overline{\partial}_b)$ and hence $f \perp \mathscr{H}^b_{0,s}(M)$, so $f = \overline{\partial}_b \overline{\partial}_b^* G_b f + \overline{\partial}_b^* \overline{\partial}_b G_b f$ and $u = \overline{\partial}_b^* \overline{\partial}_b G_b f$ is the canonical solution to the equation $\overline{\partial}_b u = \overline{\partial}_b f$. Thus $\overline{\partial}_b (f - u) = 0$, that is, $f - u \in \operatorname{Ker}(\overline{\partial}_b)$. We claim that $u \perp \operatorname{Ker}(\overline{\partial}_b)$. Indeed, for all $g \in \operatorname{Ker}(\overline{\partial}_b)$ one has $(u, g) = (\overline{\partial}_b^* \overline{\partial}_b G_b f, g) = (\overline{\partial}_b G_b f, \overline{\partial}_b g) = 0$. Since $f \perp \operatorname{Ker}(\overline{\partial}_b)$, it turns out that $f - u \perp \operatorname{Ker}(\overline{\partial}_b)$ so $f - u = 0$ and then $0 = f - u = (Id - \overline{\partial}_b^* \overline{\partial}_b G_b f)$. This proves (42). Similarly, we get (43). □

Theorem 11. *Let M be given as in Theorem 8. Then the Szegö projections operators S_{s-1} and S_s are continuous in the Sobolev spaces $W^k_{0,s-1}(M)$ and $W^k_{0,s}(M)$ for all $k \geq 0$, respectively.*

Proof. We investigate first the continuity of S_{s-1}. For the case $k = 0$, when $f \in L^2_{0,s}(M)$, we have

$$\begin{aligned} \left\| \overline{\partial}_b^* G_b \overline{\partial}_b f \right\|^2 &= \left(\overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* G_b \overline{\partial}_b f \right) \\ &= \left(G_b \overline{\partial}_b f, \overline{\partial}_b \overline{\partial}_b^* G_b \overline{\partial}_b f \right) \\ &= \left(G_b \overline{\partial}_b f, \overline{\partial}_b f \right) = \left(\overline{\partial}_b^* G_b \overline{\partial}_b f, f \right) \\ &\leq \left\| \overline{\partial}_b^* G_b \overline{\partial}_b f \right\| \|f\|. \end{aligned} \tag{49}$$

Here we have used the fact that $\overline{\partial}_b \overline{\partial}_b^* G_b \overline{\partial}_b f = \overline{\partial}_b f$, because $\overline{\partial}_b^2 = 0$. The relation (43) thus implies that $\|S_{s-1} f\| \leq C \|f\|$. This proves the continuity in $L^2_{0,s-1}(M)$.

The case $k \geq 1$. Applying (32) for $\varphi = \eta \Lambda^k \eta_1 G_s \overline{\partial}_b f$ on U, we obtain

$$\begin{aligned} \left\| \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right\|^2 \leq{}& \epsilon Q_b \left(\eta \Lambda^k \eta_1 G_b \overline{\partial}_b f, \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right) \\ &+ C(\epsilon) \left\| \eta \Lambda^k \eta_1 G_b \overline{\partial}_b f \right\|^2_{-1(U)}. \end{aligned} \tag{50}$$

The first term on the right-hand side of (50) is estimated as

$$\begin{aligned} Q_b \left(A G_b \overline{\partial}_b f, A G_b \overline{\partial}_b f \right) &= \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\|^2 + \left\| \overline{\partial}^* A G_b \overline{\partial}_b f \right\|^2 \\ &= \left(\overline{\partial}_b A G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\overline{\partial}_b^* A G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &= \left(A \overline{\partial}_b G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(A \overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\left[\overline{\partial}_b, A \right] G_b \overline{\partial}_b f, \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &\quad + \left(\left[\overline{\partial}_b^*, A \right] G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right). \end{aligned} \tag{51}$$

The sum of the last two terms on the right-hand side of the preceding equality is estimated by

$$\begin{aligned} & \left\| \left[\overline{\partial}_b, A \right] G_b \overline{\partial}_b f \right\| \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| \\ &\quad + \left\| \left[\overline{\partial}_b^*, A \right] G_b \overline{\partial}_b f \right\| \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \\ &\leq \left\| D G_b \overline{\partial}_b f \right\|_{k-1(U)} \left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| \\ &\quad + \left\| D G_b \overline{\partial}_b f \right\|_{k-1(U)} \left\| \overline{\partial}_b^* A G_s \overline{\partial}_b f \right\| \\ &\leq \left\| G_b \overline{\partial}_b f \right\|_{k(U)} \left(\left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| + \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \right) \\ &= \mathscr{O} \left((\text{l.c.}) \left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right. \\ &\qquad \left. + (\text{s.c.}) \left(\left\| \overline{\partial}_b A G_b \overline{\partial}_b f \right\| + \left\| \overline{\partial}_b^* A G_b \overline{\partial}_b f \right\| \right)^2 \right) \\ &= \mathscr{O} \left(\left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right). \end{aligned} \tag{52}$$

We then have

$$\begin{aligned} Q_b \left(A G_b \overline{\partial}_b f, A G_b \overline{\partial}_b f \right) \leq{}& \left(\overline{\partial}_b G_b \overline{\partial}_b f, A^* \overline{\partial}_b A G_b \overline{\partial}_b f \right) \\ &+ \left(A \overline{\partial}_b^* G_b \overline{\partial}_b f, \overline{\partial}_b^* A G_b \overline{\partial}_b f \right) \\ &+ \mathscr{O} \left(\left\| G_b \overline{\partial}_b f \right\|^2_{k(U)} \right). \end{aligned} \tag{53}$$

The first term on the right-hand side of (53) equals zero due to the fact that $\bar{\partial}_b G_b \bar{\partial}_b f = \bar{\partial}_b^2 G_b f = 0$.

We now analyze the second term as follows:

$$\begin{aligned}
&\left(A\bar{\partial}_b^* G_b\bar{\partial}_b f, \bar{\partial}_b^* A G_b\bar{\partial}_b f\right)\\
&= \left(\bar{\partial}_b A\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right)\\
&= \left(A\bar{\partial}_b\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right)\\
&= \left(A\bar{\partial}_b f, AG_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right)\\
&= \left(\bar{\partial}_b A f, AG_b\bar{\partial}_b f\right) + \left(\left[A, \bar{\partial}_b\right] f, AG_b\bar{\partial}_b f\right)\\
&\quad + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right)\\
&= \left(Af, \bar{\partial}_b^* A G_b\bar{\partial}_b f\right) + \cdots\\
&= \left(Af, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right) + \left(Af, \left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right)\\
&\quad + \left(\left[A, \bar{\partial}_b\right] f, AG_b\bar{\partial}_b f\right) + \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right).
\end{aligned} \tag{54}$$

Thus

$$\begin{aligned}
&Q_b\left(AG_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right)\\
&\quad\le \left(Af, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right) + E + \mathcal{O}\left(\left\|G_b\bar{\partial} f\right\|_{k(U)}^2\right)\\
&\quad\le \left|\left(Af, A\bar{\partial}_b^* G_b\bar{\partial}_b f\right)\right| + |E| + \mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\right),
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
E = &\left(Af, \left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right) + \left(\left[A, \bar{\partial}_b\right] f, AG_b\bar{\partial}_b f\right)\\
&+ \left(\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f, AG_b\bar{\partial}_b f\right).
\end{aligned} \tag{56}$$

As above, the three terms on the right-hand side of (56) are estimated, respectively, by

$$\begin{aligned}
&\|Af\|\left\|\left[\bar{\partial}_b^*, A\right] G_b\bar{\partial}_b f\right\|\\
&\quad\le \|f\|_{k(U)}\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\\
&\quad\le (\text{s.c.})\,\|f\|_{k(U)}^2 + (\text{l.c.})\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2,\\
&\|f\|_{k(U)}\left\|AG_b\bar{\partial}_b f\right\|\\
&\quad\le (\text{s.c.})\,\|f\|_{k(U)}^2 + (\text{l.c.})\left\|AG_b\bar{\partial}_b f\right\|^2\\
&\quad= \mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\right),\\
&\left\|\left[\bar{\partial}_b, A\right]\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|\left\|AG_b\bar{\partial}_b f\right\|\\
&\quad\le (\text{s.c.})\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)}^2 + (\text{l.c.})\left\|AG_b\bar{\partial}_b f\right\|^2.
\end{aligned} \tag{57}$$

Now we are left with the first term in the right-hand side of (55) which, by applying the Cauchy-Schwarz inequality, is estimated by $\|f\|_{k(U)}\|\bar{\partial}_b^* G_b\bar{\partial}_b f\|_{k(U)}$. By choosing the s.c. small enough we can absorb the first term in the right-hand side of the last inequality into $\|f\|_{k(U)}\|\bar{\partial}_b^* G_b\bar{\partial}_b f\|_{k(U)}$. This completes the estimation of the first term on the right-hand side of (50). Therefore (50) becomes

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f\right\|^2\\
&\quad\le \epsilon\|f\|_{k(U)}\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)}\\
&\qquad + \epsilon\,(\text{s.c.})\,\|f\|_{k(U)}^2 + \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\\
&\qquad + C(\epsilon)\left\|\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f\right\|_{-1(U)}^2\\
&\quad\le \epsilon\|f\|_{k(U)}\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(U)}\\
&\qquad + \epsilon\,(\text{s.c.})\,\|f\|_{k(U)}^2 + \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(U)}^2\\
&\qquad + C'(\epsilon)\left\|G_b\bar{\partial}_b f\right\|_{k-1(U)}^2.
\end{aligned} \tag{58}$$

By summing over a partition of unity subordinate to an open covering of M by patches $\{U_i\}_{i=1}^m$ so that on each of these patches an estimate like (58) is satisfied, using the interior regularity properties, we get

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon\|f\|_{k(M)}\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)} + \epsilon\,\text{s.c.}\|f\|_{k(M)}^2\\
&+ \epsilon C\left\|G_b\bar{\partial}_b f\right\|_{k(M)} + C'(\epsilon)\left\|G_b\bar{\partial}_b f\right\|_{k-1(M)}^2.
\end{aligned} \tag{59}$$

By using the small and large constants, the first term on the right-hand side in (59) is estimated as

$$\epsilon\left((\text{s.c.})\,\|f\|_{k(M)}^2 + (\text{l.c.})\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2\right). \tag{60}$$

Then adding and choosing ϵ and the s.c. small enough we can absorb the third term on the right-hand side of (59) into the left-hand side; we obtain

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon C\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2\\
&+ C'(\epsilon)\left(\|f\|_{k(M)}^2 + \left\|G_b\bar{\partial}_b f\right\|_{k-1(M)}^2\right).
\end{aligned} \tag{61}$$

Applying this inequality with k replaced by $k-1$ to the last term on the right-hand side and repeating, we obtain

$$\begin{aligned}
\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2 \le{}& \epsilon C\left\|\bar{\partial}_b^* G_b\bar{\partial}_b f\right\|_{k(M)}^2\\
&+ C'(\epsilon)\left(\|f\|_{k(M)}^2 + \left\|G_b\bar{\partial}_b f\right\|^2\right).
\end{aligned} \tag{62}$$

We have

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|^2 \\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\quad=\left(\bar{\partial}_b^*\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b\bar{\partial}_b^*G_b\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f,\bar{\partial}_b\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad=\left(\bar{\partial}_b^*\eta\Lambda^k\eta_1 G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^* G_b\bar{\partial}_b f,\eta\Lambda^k\eta_1 f\right)\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right)\\
&\quad\le\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\left\|\eta\Lambda^k\eta_1 f\right\|\\
&\qquad+\mathcal{O}\left(\left\|G_b\bar{\partial}_b f\right\|_{k(U)}\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|\right)\right).
\end{aligned}\tag{63}$$

Again summing over a partition of unity, using the interior regularity properties and the small and large constants technique, we obtain

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C\left(\left\|G_b\bar{\partial}_b f\right\|_{k(M)}^2+\|f\|_{k(M)}^2\right).\tag{64}$$

Substituting (62) into (64), we obtain

$$\begin{aligned}\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2&\le K\epsilon\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\\&\quad+C'(\epsilon)\left(\|f\|_{k(M)}^2+\left\|G_b\bar{\partial}_b f\right\|^2\right).\end{aligned}\tag{65}$$

Choosing $\epsilon>0$ small enough allows us to absorb the first term on the right-hand side into the left, we then get

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C'(\epsilon)\left(\|f\|_{k(M)}^2+\left\|G_b\bar{\partial}_b f\right\|^2\right).\tag{66}$$

As the operator $\bar{\partial}_b^*$ has $L^2(M)$-closed range, it follows from Theorem 1.1.1 in Hörmander [15] that there is a positive constant C such that

$$\left\|G_b\bar{\partial}_b f\right\|\le C\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|.\tag{67}$$

Then, by (49), we obtain

$$\left\|G_b\bar{\partial}_b f\right\|\le C\|f\|.\tag{68}$$

Substituting (68) into (66), we get

$$\left\|\bar{\partial}_b^*G_b\bar{\partial}_b f\right\|_{k(M)}^2\le C\|f\|_{k(M)}^2.\tag{69}$$

By (43), the Szegö projection S_{s-1} is therefore continuous on $W^k_{0,s-1}(M)$ for each $k=0,1,2\ldots$. The general case is obtained from interpolation of linear operators.

For the continuity of the Szegö projection S_s, in view of (42), it suffices to show that

$$\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|_{k(M)}^2\le C\|f\|_{k(M)}^2,\quad k\ge 0.\tag{70}$$

For $k=0$, we have

$$\begin{aligned}\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|^2&=\left(\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f,\bar{\partial}_b G_b f\right)=\left(\bar{\partial}_b f,\bar{\partial}_b G_b f\right)\\&=\left(f,\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\le C\|f\|\left\|\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|.\end{aligned}\tag{71}$$

For $k\ge 1$, as before, an elliptic regularity argument implies

$$\begin{aligned}
&\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|^2\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\qquad+\left(\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f,\left[\eta\Lambda^k\eta_1,\bar{\partial}_b^*\right]\bar{\partial}_b G_b f\right)\\
&\qquad+\left(\left[\bar{\partial}_b,\eta\Lambda^k\eta_1\right]\bar{\partial}_b^*\bar{\partial}_b G_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\quad=\left(\eta\Lambda^k\eta_1\bar{\partial}_b f,\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right)\\
&\quad=\left(\eta\Lambda^k\eta_1 f,\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right)\\
&\qquad+\mathcal{O}\left(\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right)\\
&\qquad+\mathcal{O}\left(\|f\|_{k(U)}\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\right)\\
&\quad\le\left\|\eta\Lambda^k\eta_1 f\right\|\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\\
&\qquad+\mathcal{O}\left(\left(\|f\|_{k(U)}+\left\|\eta\Lambda^k\eta_1\bar{\partial}_b G_b f\right\|\right)\left\|\eta\Lambda^k\eta_1\bar{\partial}_b^*\bar{\partial}_b G_b f\right\|\right).
\end{aligned}\tag{72}$$

Summing over a partition of unity, using the small and large constants argument, absorbing the terms containing $\|\bar{\partial}_b^*\bar{\partial}_b G_b f\|_{k(M)}$, and finally using the fact that $\bar{\partial}_b G_b$ is continuously bounded on $W^k_{0,s}(M)$, we conclude (70) which proves the continuity of S_s on $W^k_{0,s}(M)$. □

Corollary 12. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$. Then for any f in $W^k_{0,s}(M)$ ($k \geq 0$) such that $\bar{\partial}_b f = 0$ and $f \perp \mathscr{H}^b_{0,s}(M)$, there exists u in $W^k_{0,s-1}(M)$ which solves the equation $\bar{\partial}_b u = f$.*

Theorem 13. *Let M be a $\mathscr{C}^\infty$ compact CR manifold of real dimension $2n-\ell$ and codimension $\ell \geq 1$ in an n-dimensional complex manifold X. Suppose that M satisfies condition $Y(s)$ for some s with $1 \leq s \leq n-\ell-1$. Then for any f in $\mathscr{C}^\infty_{0,s}(M)$, with $\bar{\partial}_b f = 0$ and $f \perp \mathscr{H}^b_{0,s}(M)$, there exists a global solution u in $\mathscr{C}^\infty_{0,s-1}(M)$ to the equation $\bar{\partial}_b u = f$.*

Proof. By Corollary 12, for each $k \geq 0$, there exists some $u_k \in W^k_{0,s-1}(M)$ such that $\bar{\partial}_b u_k = f$. We modify each u_k by an element of $\operatorname{Ker}(\bar{\partial}_b)$ in order to construct a telescoping series that belongs to $W^k_{0,s}(M)$ for each $k \geq 1$. To conclude the proof, we first claim that $W^k_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ is dense in $W^m_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ for any $k > m \geq 0$. Since $\mathscr{C}^\infty_{0,s}(M)$ is dense in $W^m_{0,s}(M)$, $m \geq 0$, in the W^m-norm, then for a given $\eta \in W^m_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ there is a sequence $\eta_j \in \mathscr{C}^\infty_{0,s}(M)$ converging to η in the $W^m_{0,s}(M)$-norm; that is, $\|\eta_j - \eta\|_{m(M)} \to 0$ as $j \to \infty$. $\bar{\partial}_b \eta = 0$ implies that $\eta - S_s\eta = \bar{\partial}_b^* G_b \bar{\partial}_b \eta = 0$, so $\eta = S_s u$. Let $\hat{\eta}_j = S_s \eta_j$. $\hat{\eta}_j \in W^k_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ since the Szegö projection S_s is a bounded operator on $W^k_{0,s}(M)$. By the same reason we have $\|\hat{\eta}_j - \eta\|_{m(M)} = \|S_s(\eta_j - \eta)\|_{m(M)} \leq C\|\eta_j - \eta\|_{m(M)} \to 0$ as $j \to \infty$. This implies that $\hat{\eta}_j \to \eta$ in the W^m-norm. Thus, indeed, $W^k_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ is dense in $W^m_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ for any $k > m \geq 0$.

Next, using this result and following the inductive argument due to [21, page 230], we can construct a sequence $\tilde{u}_k \in W^k_{0,s-1}(M)$, $\bar{\partial}_b \tilde{u}_k = f$, and $\|\tilde{u}_{k+1} - u_k\|_{k(M)} \leq 2^{-k}$ as follows:

$$\tilde{u}_1 = u_1, \qquad \tilde{u}_2 = u_2 + v_2, \tag{73}$$

where $v_2 \in W^2_{0,s-1}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ is such that

$$\|\tilde{u}_2 - u_1\|_{1(M)} \leq 2^{-1} \tag{74}$$

and in general

$$\tilde{u}_{k+1} = u_{k+1} + v_{k+1}, \tag{75}$$

where $v_{k+1} \in W^{k+1}_{0,s}(M) \cap \operatorname{Ker}(\bar{\partial}_b)$ is such that

$$\|\tilde{u}_{k+1} - u_k\|_{k(M)} \leq 2^{-k}. \tag{76}$$

Clearly $\bar{\partial}_b \tilde{u}_k = f$, so set

$$u = \tilde{u}_j + \sum_{k=j}^{\infty} (\tilde{u}_{k+1} - \tilde{u}_k), \quad j \in \mathbb{N}. \tag{77}$$

It follows that $u \in W^k_{0,s-1}(M)$ for each $k \in \mathbb{N}$, and hence $u \in \mathscr{C}^\infty_{0,s-1}(M)$ and $\bar{\partial}_b u = f$. The general case is obtained from interpolation of linear operators. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant no. 130-248-D1435. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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