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An $O(n \log n)$ algorithm for finding edge span of cacti

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Abstract Let $G = (V, E)$ be a nonempty graph and $\xi : E \to \mathbb{N}$ be a function. In the paper we study the computational complexity of the problem of finding vertex colorings c of G such that:

(1) $|c(u) - c(v)| \geq \xi(uv)$ for each edge $uv \in E$;
(2) the edge span of c, i.e. $\max\{|c(u) - c(v)| : uv \in E\}$, is minimal.

We show that the problem is NP-hard for subcubic outerplanar graphs of a very simple structure (similar to cycles) and polynomially solvable for cycles and bipartite graphs. Next, we use the last two results to construct an algorithm that solves the problem for a given cactus G in $O(n \log n)$ time, where n is the number of vertices of G.

Keywords Cacti · Edge span · Vertex coloring

Mathematics Subject Classification 05C15

1 Introduction

In the literature one can find some variants of vertex coloring which model the frequency assigment problem (Hale 1980), e.g. the backbone coloring (Broersma 2003) or the $L(p, q)$-labeling (Griggs and Yeh 1992). These variants impose similar requirements on the colors assigned to adjacent or close enough vertices: their distance have

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to be greater than or equal to a given number. For instance, in the backbone coloring problem we are given a graph G, its spanning subgraph H (backbone) and the following condition: the distance of the colors assigned to vertices u, v have to be at least 2 if they are adjacent in H or at least 1 if they are adjacent in G. In this paper we introduce a similar concept which we call ξ-colorings.

Definition 1 Let $G = (V, E)$ be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function. A function $c \colon V \to \mathbb{Z}$ is a *ξ-coloring* of G if and only if $|c(u) - c(v)| \geq \xi(uv)$ for each edge $uv \in E$.

Clearly, ξ-colorings generalize backbone colorings. They are also related to $L(p, q)$-colorings since these are just ξ-colorings of G^2 for function ξ defined as follows:

$$\xi(uv) = \begin{cases} p, & \text{if } u, v \text{ are adjacent in } G, \\ q, & \text{if } u, v \text{ are of distance } 2 \text{ in } G. \end{cases}$$

In most works related to the $L(p, q)$-labeling or the backbone coloring (see e.g. Calamoneri (2006); Yeh (2006) for a survey of results) the authors study the problem of finding colorings with minimal *span*, i.e. the difference between the largest and the smallest color used. For instance, it was shown that the $L(p, q)$-labeling problem is NP-complete even for bipartite planar graphs with degree $\Delta \leq 4$ (Janczewski et al. 2009) and polynomially solvable for trees in the case $q = 1$ (Yeh 2006) and for many simple graph classes: paths, cycles etc. It was also shown that the $L(2, 1)$-labeling problem is polynomially solvable for cacti (Jonas 1993) and p-almost trees (Fiala and Kratochvil 2001) for every fixed p.

Herein we focus on similar minimization criterion: the edge span.

Definition 2 Let $c \colon V \to \mathbb{Z}$ be a ξ-coloring of a nonempty graph $G = (V, E)$. The number $\operatorname{esp}(c) := \max\{|c(u) - c(v)| \colon uv \in E\}$ is the *edge span* of c.

Definition 3 Let $G = (V, E)$ be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function. The number $\operatorname{esp}(G, \xi) := \min\{\operatorname{esp}(c) \colon c \text{ is a } \xi\text{-coloring of } G\}$ is the *edge ξ-span* of G.

The edge span was studied in the context of the $L(2, 1)$-labeling (Yeh 2000) and other coloring variants (Chang et al. 1999). It is important since it is used as a local optimization criterion in the frequency assigment problem.

The remainder of the paper is organized as follows. We begin by showing that it is NP-hard to compute the edge ξ-span of subcubic outerplanar graphs. Next, we show that the edge ξ-span and optimal ξ-colorings, i.e. ξ-colorings with minimal possible edge span, of a bipartite graph G may be found in $O(n + m)$ time, where n is the number of vertices and m is the number of edges of G. Section 4 deals with odd cycles. We present a formula for the edge ξ-span and an $O(n \log n)$ algorithm that produces optimal ξ-colorings of C_{2n+1}. The last section shows how these results can be used to obtain an $O(n \log n)$ algorithm that produces optimal ξ-colorings of cacti.

2 Subcubic outerplanar graphs

It appears that computing the edge ξ-span is NP-hard even for relatively simple graphs.

Lemma 1 *Let $a < b$ be positive integers. If d_1, d_2 and d_3 are integers such that $|d_1| = b$, $a \leq |d_2| \leq b$, $b - a \leq |d_3| \leq b$ and $d_1 + d_2 + d_3 = 0$ then $|d_2| = a$ and $|d_3| = b - a$.*

Proof Observe that $|d_2 + d_3| = |-d_1| = b$. There are two cases to consider.

(a) $|d_2+d_3| = |d_2|-|d_3|$ or $|d_2+d_3| = |d_3|-|d_2|$. Then $|d_2+d_3| < \max\{|d_2|, |d_3|\} \leq b$—a contradiction.
(b) $|d_2 + d_3| = |d_2| + |d_3|$. Then $b = a + b - a \leq |d_2| + |d_3| = |d_2 + d_3| = b$, which yields $|d_2| = a$ and $|d_3| = b - a$.

□

Theorem 1 *The following problem is NP-complete:*

Instance: *A nonempty subcubic outerplanar graph $G = (V, E)$, a function $\xi : E \to \mathbb{N}$ and an integer k.*
Question: *Does* $\mathrm{esp}(G, \xi) \leq k$*?*

Proof The problem is clearly in NP. To complete the proof we will show that there is a polynomial-time reduction from the well-known partition problem to our problem. Recall that the partition problem is NP-complete (Karp 1972) even in the following version:

Instance: A sequence of positive integers $a_1, a_2, \ldots, a_{2s}$.
Question: Is there a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2s}$ such that $|\varepsilon_i| = 1$ for all i and $\sum_{i=1}^{2s} \varepsilon_i a_i = 0$?

Given a sequence $a_1, a_2, \ldots, a_{2s}$, we construct an instance of our problem in the following way. Let H_i $(1 \leq i \leq 2s)$ be a complete graph with vertex set $\{u_i, v_i, w_i\}$ and H be a graph whose connected components are $H_1, H_2, \ldots, H_{2s}$. G arises from H by adding edges v_1u_2, v_2u_3, ..., $v_{2s-1}u_{2s}$ and $v_{2s}u_1$ (see Fig. 1 for an example). We set $k = 1 + \sum_{i=1}^{2s} a_i$ and

$$\xi(e) = \begin{cases} a_i, & \text{if } e = u_i v_i, \\ k - a_i, & \text{if } e = u_i w_i, \\ k, & \text{otherwise.} \end{cases}$$

It is easy to see that G is subcubic, outerplanar and the above construction takes $O(s)$ time. To complete the proof it suffices to show that $\mathrm{esp}(G, \xi) \leq k$ if and only if there is a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2s}$ such that $|\varepsilon_i| = 1$ for all i and $\sum_{i=1}^{2s} \varepsilon_i a_i = 0$.

($\Rightarrow$) Let c be an optimal ξ-coloring of G. Since $\mathrm{esp}(G, \xi) \leq k$, we have $|c(u) - c(v)| = k$ for all edges satisfying $\xi(uv) = k$. Combining this with Lemma 1, we get

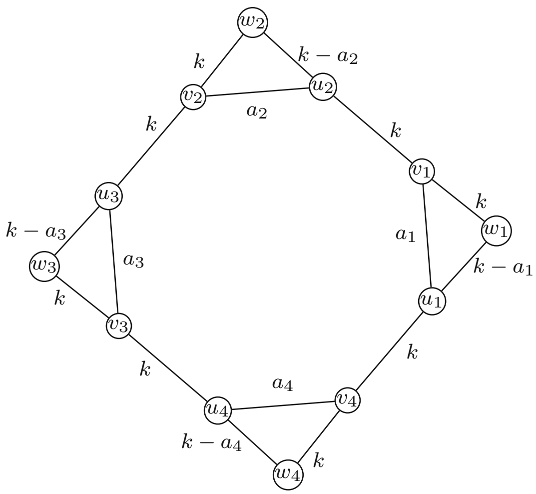

Fig. 1 An example of the reduction for $s = 2$

$\xi(uv) = |c(u) - c(v)|$ for all edges of G since $|c(v_i) - c(w_i)| = k$, $k \geq |c(u_i) - c(v_i)| \geq \xi(u_i v_i) = a_i$ and $k \geq |c(u_i) - c(w_i)| \geq \xi(u_i w_i) = k - a_i$ for all i. Obviously

$$\sum_{i=1}^{2s} \Big((c(v_i) - c(u_i)) + (c(u_{1+(i \bmod 2s)}) - c(v_i)) \Big) = 0.$$

This sum is of form $\sum_{i=1}^{2s} \varepsilon_i a_i + Ak$. Since $k > \sum_{i=1}^{2s} a_i$ and the sum is 0, this gives $A = 0$ and $\sum_{i=1}^{2s} \varepsilon_i a_i = 0$.

($\Leftarrow$) Let $c \colon V \to \mathbb{Z}$ be function given by

$$c(u) = \begin{cases} \sum_{i=1}^{j-1} \varepsilon_i a_i, & \text{if } u = u_j \text{ and } j \text{ is odd}, \\ k + \sum_{i=1}^{j-1} \varepsilon_i a_i, & \text{if } u = u_j \text{ and } j \text{ is even}, \\ \sum_{i=1}^{j} \varepsilon_i a_i, & \text{if } u = v_j \text{ and } j \text{ is odd}, \\ k + \sum_{i=1}^{j} \varepsilon_i a_i, & \text{if } u = v_j \text{ and } j \text{ is even}, \\ c(u_i) + k - a_i, & \text{if } u = w_i. \end{cases}$$

It is easy to verify that c is a ξ-coloring of G and $\operatorname{esp}(c) = k$. □

3 Bipartite graphs

The following proposition implies that to obtain an optimal ξ-coloring of a bipartite graph G it suffices to find its partitions and color the vertices of the first one with 0 and the second with $\max \xi(E)$. All these steps can be done in $O(n + m)$ time.

Proposition 1 *Let G be a nonempty graph and $\xi \colon E \to \mathbb{N}$ be a function. Then*

$$\max \xi(E) \leq \operatorname{esp}(G, \xi) \leq \max \xi(E)(\chi - 1),$$

where χ is the chromatic number of G.

Proof The left-hand side inequality follows directly from the definition of the edge span and the ξ-coloring. The right-hand side is an easy consequence of the fact that if c is a coloring of G that uses colors 0, 1, …, $\chi - 1$ then $\max \xi(E) \cdot c$ is a ξ-coloring of G with edge span that equals $\max \xi(E)(\chi - 1)$. □

4 Odd cycles

In this section, we consider an odd cycle C_{2n+1} and a function $\xi : E(C_{2n+1}) \to \mathbb{N}$. Let $v_1, v_2, \ldots, v_{2n+1}$ be the vertices of C_{2n+1}, numbered in such a way that v_i is adjacent to $v_{1+i \bmod (2n+1)}$ for $i = 1, 2, \ldots, 2n+1$. Let π be a permutation of $\{1, 2, \ldots, 2n+1\}$ such that the sequence

$$\xi_i := \xi(v_{\pi(i)} v_{1+\pi(i) \bmod (2n+1)})$$

is nondecreasing.

Definition 4 A sequence $r = (r_1, r_2, \ldots, r_{2n+1})$ is a sequence of *cyclic differences* if and only if

(1) $|r_i| \geq \xi_i$ for $1 \leq i \leq 2n+1$;
(2) $\sum_{i=1}^{2n+1} r_i = 0$.

The number $\|r\| := \max_{1 \leq i \leq 2n+1} |r_i|$ will be called the *norm* of r.

Theorem 2 $\operatorname{esp}(C_{2n+1}, \xi) = \min\{\|r\| : r$ *is a sequence of cyclic differences*$\}$.

Proof It suffices to show that for every positive integer s the existence of a sequence of cyclic differences of norm s is equivalent to the existence of a ξ-coloring of C_{2n+1} with edge span s.

($\Rightarrow$) Let r be a sequence of cyclic differences with norm s. The formula $c(v_i) = \sum_{j : \pi(j) < i} r_j$ defines the required ξ-coloring. Indeed,

$$\left|c(v_{1+i \bmod (2n+1)}) - c(v_i)\right| = \left|r_{\pi^{-1}(i)}\right|$$

which gives $|c(v_{1+i \bmod (2n+1)}) - c(v_i)| \geq \xi_{\pi^{-1}(i)} = \xi(v_i v_{1+i \bmod (2n+1)})$ and $\operatorname{esp}(c) = s$.

($\Leftarrow$) Let c be a ξ-coloring of C_{2n+1} with edge span s. It is easy to verify that the formula $r_i = c(v_{1+\pi(i) \bmod (2n+1)}) - c(v_{\pi(i)})$ defines the required sequence of cyclic differences. □

The proof of the above theorem shows that finding an optimal ξ-coloring of C_{2n+1} reduces to finding of a sequence of cyclic differences with minimal possible norm. The transition from one problem to the other requires two steps: we must sort the multiset $\xi(E)$, compute the permutation π and its inverse π^{-1}. All these steps can be done in $O(n \log n)$ time.

Lemma 2 *Let r be a sequence of cyclic differences.*

(1) *$-r$ is a sequence of cyclic differences.*
(2) *The sequence r' that arises as a result of sorting r in order of nondecreasing absolute values is a sequence of cyclic differences.*

Proof (1) Obvious.
(2) It suffices to show that swapping r_i with r_j, where $i < j$ and $|r_i| > |r_j|$, results in a sequence of cyclic differences. It holds since swapping does not change the sum of the sequence, $|r_i| > |r_j| \geq \xi_j$ and $|r_j| \geq \xi_j \geq \xi_i$.

□

Lemma 3 *If there exists an integer k such that $n + 1 \leq k \leq 2n$ and $\sum_{i=1}^{k} \xi_i \leq (2n + 1 - k)\xi_{2n+1}$ then there exist an integer s such that $0 \leq s \leq k - 2$ and the sequence r given by*

$$r_i = \begin{cases} -\xi_{2n+1}, & \text{if } 1 \leq i \leq s, \\ \sum_{j=s+2}^{k} \xi_j - (2n+1-k-s)\xi_{2n+1}, & \text{if } i = s+1, \\ -\xi_i, & \text{if } s+2 \leq i \leq k, \\ \xi_{2n+1}, & \text{if } k < i \leq 2n+1 \end{cases}$$

is a sequence of cyclic differences with norm ξ_{2n+1}.

Proof Let us notice that for all possible values of s we have $\xi_i \leq |r_i| \leq \xi_{2n+1}$ for $i \neq s+1$ and $\sum_{i=1}^{2n+1} r_i = -s\xi_{2n+1} + \sum_{i=s+2}^{k} \xi_i - (2n+1-k-s)\xi_{2n+1} - \sum_{i=s+2}^{k} \xi_i + (2n + 1 - k)\xi_{2n+1} = 0$. To complete the proof it suffices to show that there is s such that $\xi_{s+1} \leq |r_{s+1}| \leq \xi_{2n+1}$.

Let $\delta\colon \{0, 1, \ldots, k-1\} \to \mathbb{Z}$ be the function given by $\delta(j) = (2n + 1 - k - j)\xi_{2n+1} - \sum_{i=j+1}^{k} \xi_i$. We know that $\delta(0) = \xi_{2n+1}(2n + 1 - k) - \sum_{i=1}^{k} \xi_i \geq 0$. Moreover $\delta(k-1) = 2(n + 1 - k)\xi_{2n+1} - \xi_k \leq -\xi_k < 0$ and $\delta(j+1) - \delta(j) = \xi_{j+1} - \xi_{2n+1} \leq 0$, so there must be s such that $0 \leq s \leq k-2$, $\delta(s) \geq 0$ and $\delta(s+1) < 0$. To complete the proof it suffices to note that $|r_{s+1}| = \xi_{s+1} + \delta(s)$ and $\xi_{s+1} \leq \xi_{s+1} + \delta(s) < \xi_{s+1} + \delta(s) - \delta(s+1) = \xi_{2n+1}$. □

Theorem 3 *The following conditions are equivalent:*

(1) *there exists an integer k such that $n+1 \leq k \leq 2n$ and $\sum_{i=1}^{k} \xi_i \leq (2n+1-k)\xi_{2n+1}$;*
(2) *there exists a sequence of cyclic differences with norm ξ_{2n+1}.*

Proof ($\Rightarrow$) Follows immediately from Lemma 3.

($\Leftarrow$) For every sequence of cyclic differences r with norm ξ_{2n+1} we define a parameter $\zeta(r) = \sum_{i\colon r_i>0} |\xi_{2n+1} - r_i|$. It is easy to see that $\zeta(r) \geq 0$ and $\zeta(r) = 0$ if and only if $r_i = \xi_{2n+1}$ for each $r_i > 0$.

Let r be a sequence of cyclic differences with norm ξ_{2n+1} such that $\zeta(r)$ is minimal. Suppose that $\zeta(r) > 0$. Then there is j such that $0 < r_j < \xi_{2n+1}$. By Lemma 2, $-r$ is a sequence of cyclic differences. Since $\|-r\| = \|r\|$, we have $\zeta(-r) > 0$ and there must be l such that $0 < -r_l < \xi_{2n+1}$. But then the sequence r' given by

$$r'_i = \begin{cases} r_j + 1, & \text{if } i = j, \\ r_l - 1, & \text{if } i = l, \\ r_i, & \text{otherwise,} \end{cases}$$

would be a sequence of cyclic differences with $\|r'\| = \xi_{2n+1}$ and $\zeta(r') = \zeta(r) - 1$—a contradiction. Hence $\zeta(r) = 0$ and $r_i = \xi_{2n+1}$ for all $r_i > 0$. Let $k = |\{i : r_i < 0\}|$. By Lemma 2, the sequence r'' resulting from r by sorting it in order of nondecreasing absolute values is a sequence of cyclic differences. Since only negative elements of r may have absolute value less than ξ_{2n+1}, we may assume that $r''_i < 0$ if and only if $i \le k$. Hence

$$(2n+1-k)\xi_{2n+1} = \sum_{i=k+1}^{2n+1} r''_i = -\sum_{i=1}^{k} r''_i = \sum_{i=1}^{k} |r''_i|,$$

which, combined with $\sum_{i=1}^{k} \xi_i \le \sum_{i=1}^{k} |r''_i| \le k\xi_{2n+1}$, gives immediately $k \ge n+1$ and $\sum_{i=1}^{k} \xi_i \le (2n+1-k)\xi_{2n+1}$. This completes the proof since $k \le 2n$ is obvious. □

The above theorem combined with Lemma 3 leads to a linear algorithm that can verify if there is a sequence of cyclic differences of norm ξ_{2n+1}. The algorithm, if the answer is yes, will give us a formula describing the required sequence. Indeed, it suffices to verify for $k = n+1, n+2, \ldots, 2n$ whether $\sum_{i=1}^{k} \xi_i \le (2n+1-k)\xi_{2n+1}$, and, if such k was found, use the formula of Lemma 3.

Lemma 4 *If there exists an integer k such that* $1 \le k \le 2n$ *and* $\sum_{i=1}^{k} \xi_i > (2n+1-k)\xi_{2n+1}$ *then the sequence r given by*

$$r_i = \begin{cases} -\xi_i, & \text{if } i \le k, \\ q, & \text{if } k+1 \le i \le 2n+1-s, \\ q+1, & \text{if } i > 2n+1-s, \end{cases}$$

where q is the quotient and s is the remainder from the division of $\sum_{i=1}^{k} \xi_i$ *by* $2n+1-k$*, is a sequence of cyclic differences with norm* $\lceil \sum_{i=1}^{k} \xi_i/(2n+1-k) \rceil$.

Proof Since $\sum_{i=1}^{k} \xi_i > (2n+1-k)\xi_{2n+1}$, we have $q \ge \xi_{2n+1}$. To complete the proof, it suffices to observe that $\sum_{i=1}^{2n+1} r_i = -\sum_{i=1}^{k} \xi_i + q(2n+1-s-k) + (q+1)s = -\sum_{i=1}^{k} \xi_i + q(2n+1-k) + s = 0$. □

Theorem 4 *Suppose that every sequence of cyclic differences is of norm greater than* ξ_{2n+1}*. If r is a sequence of cyclic differences with the minimal possible norm then there is an integer k such that* $1 \le k \le 2n$, $\sum_{i=1}^{k} \xi_i > (2n+1-k)\xi_{2n+1}$ *and*

$$\|r\| \ge \left\lceil \sum_{i=1}^{k} \frac{\xi_i}{2n+1-k} \right\rceil.$$

Proof For a given sequence of cyclic differences s we define a parameter

$$\zeta_1(s) = \min\left\{ \sum_{i\,:\,s_i<0} |s_i + \xi_i|, \sum_{i\,:\,s_i>0} |s_i - \xi_i| \right\}.$$

It is easy to see that $\zeta_1(s) \geq 0$ and $\zeta_1(s) = 0$ if and only if holds at least one of the following conditions:

(a) for all i such that $s_i < 0$ we have $s_i = -\xi_i$;
(b) for all i such that $s_i > 0$ we have $s_i = \xi_i$.

Let s be a sequence of cyclic differences such that $\|s\| = \|r\|$ and the value of $\zeta_1(s)$ is minimal. If $\zeta_1(s) > 0$ then there would exist l and j such that $s_l < -\xi_l$, $s_j > \xi_j$ and the function given by

$$s'_i = \begin{cases} s_i, & \text{if } i \neq j \text{ and } i \neq l, \\ s_l + 1, & \text{if } i = l, \\ s_j - 1, & \text{if } i = j, \end{cases}$$

would be a sequence of cyclic differences satisfying $\|s'\| = \|r\|$ and $\zeta_1(s') = \zeta_1(s) - 1$—a contradiction. Hence $\zeta_1(s) = 0$. Since $\zeta_1(-s) = \zeta_1(s)$ and $\|s\| = \|-s\|$, we may assume without loss of generality that s satisfies condition (a).

For all sequences of cyclic differences s that satisfy (a) and the equality $\|s\| = \|r\|$ we define a parameter

$$\zeta_2(s) = \min\left\{ \sum_{i\,:\,s_i>0} \max\{s_i - \xi_{2n+1}, 0\}, \sum_{i\,:\,s_i>0} \max\{\xi_{2n+1} - s_i, 0\} \right\}.$$

Obviously, $\zeta_2(s) \geq 0$ and $\zeta_2(s) = 0$ if and only if at least one of the following conditions hold

(c) for all i such that $s_i > 0$ we have $s_i \leq \xi_{2n+1}$;
(d) for all i such that $s_i > 0$ we have $s_i \geq \xi_{2n+1}$.

Let s be a sequence of cyclic differences that satisfies (a), the equality $\|s\| = \|r\|$ and has the minimal value of $\zeta_2(s)$. If $\zeta_2(s) > 0$ then there would exist l and j such that $0 < s_l < \xi_{2n+1}$, $s_j > \xi_{2n+1}$ and the function given by

$$s'_i = \begin{cases} s_i, & \text{if } i \neq j \text{ and } i \neq l, \\ s_l + 1, & \text{if } i = l, \\ s_j - 1, & \text{if } i = j, \end{cases}$$

would be a sequence of cyclic differences satisfying (a), the equality $\|s'\| = \|r\|$ and $\zeta_2(s') = \zeta_2(s) - 1$—a contradiction. Hence $\zeta_2(s) = 0$, which means that s satisfies (d), otherwise (c) would be satisfied and $\|s\| = \xi_{2n+1} < \|r\|$. Let $k = |\{i : s_i < 0\}|$. By Lemma 2 the sequence s'' resulting from sorting s in order of nondecreasing absolute values is a sequence of cyclic differences. Since only the negative elements of s may have absolute value less than ξ_{2n+1}, we may assume that $s''_i < 0$ if and only if $i \leq k$. Hence

$$\sum_{i=1}^{k} \xi_i = -\sum_{i=1}^{k} s''_i = \sum_{i=k+1}^{2n+1} s''_i,$$

which along with $\sum_{i=k+1}^{2n+1} s_i'' > (2n+1-k)\xi_{2n+1}$ (at least one is greater than ξ_{2n+1}) and $\|r\| = \|s''\| \geq \max\{s_i'': s_i'' > 0\} \geq \lceil \sum_{i=k+1}^{2n+1} s_i''/(2n+1-k) \rceil = \lceil \sum_{i=1}^{k} \xi_i/(2n+1-k) \rceil$) completes the proof. □

Observe that from Lemma 4 it follows that the inequality $\|r\| \geq \lceil \sum_{i=1}^{k} \xi_i/(2n+1-k) \rceil$ must be equality for some k. This means that this theorem along with Lemma 4 leads to another linear algorithm which this time finds a sequence of cyclic differences with minimal norm under the assumption that there is no sequence of cyclic differences with norm ξ_{2n+1}. Combining these algorithms we obtain a linear algorithm that finds a sequence of cyclic differences with minimal norm. We also obtain the following formula:

(1) if there is k such that $n+1 \leq k \leq 2n$ and $\sum_{i=1}^{k} \xi_i \leq (2n+1-k)\xi_{2n+1}$ then $\operatorname{esp}(G, \xi) = \xi_{2n+1}$;
(2) otherwise $\operatorname{esp}(G, \xi) = \min\{\lceil \sum_{i=1}^{k} \frac{\xi_i}{2n+1-k} \rceil : 1 \leq k \leq 2n \wedge \sum_{i=1}^{k} \xi_i > (2n+1-k)\xi_{2n+1}\}$.

5 Cacti

A connected graph G is a cactus if and only if every edge belongs to at most one cycle. It is known that G is a cactus if and only if there exists a sequence of graphs G_1, G_2, ..., G_k, called a *decomposition* of G, such that:

(1) G_i is a cycle or a path P_2 for $i = 1, 2, \ldots, k$;
(2) G_i has exactly one vertex in common with $G_1 \cup G_2 \cup \ldots \cup G_{i-1}$ for $i = 2, 3, \ldots, k$;
(3) $G = G_1 \cup G_2 \cup \ldots \cup G_k$.

Verification whether G is a cactus and, if the answer is yes, obtaining a decomposition is executable in a linear time. Now we are ready to formulate and prove our main result.

Theorem 5 *Let G be a nonempty cactus, $G_1, G_2, \ldots, G_k$ be its decomposition and $\xi : E \to \mathbb{N}$ be a function. Then*

$$\operatorname{esp}(G, \xi) = \max_{1 \leq i \leq k} \operatorname{esp}(G_i, \xi_i),$$

where $\xi_i := \xi|_{E(G_i)}$.

Proof The inequality $\operatorname{esp}(G, \xi) \geq \max_{1 \leq i \leq k} \operatorname{esp}(G_i, \xi_i)$ follows from the fact that G_i is a subgraph of G and $\xi_i = \xi|_{E(G_i)}$. To complete the proof we use induction on j to show that $\operatorname{esp}(G'_j, \xi'_j) \leq \max_{1 \leq i \leq j} \operatorname{esp}(G_i, \xi_i)$, where $G'_j := G_1 \cup G_2 \cup \ldots \cup G_j$ and $\xi'_j := \xi|_{E(G'_j)}$.

This is obvious for $j = 1$ since $G'_1 = G_1$ and $\xi'_1 = \xi_1$. Assume that the inequality holds for $j-1$, i.e. $\operatorname{esp}(G'_{j-1}, \xi'_{j-1}) \leq \max_{1 \leq i \leq j-1} \operatorname{esp}(G_i, \xi_i)$. Let c' be an optimal ξ'_{j-1}-coloring of G'_{j-1} and c_j be an optimal ξ_j-coloring of G_j. Let v be the only common vertex of G'_{j-1} and G_j. It is easy to see that the function $c\colon V(G'_j) \to \mathbb{Z}$ given by

$$c(u) = \begin{cases} c'(u), & \text{if } u \in V(G'_{j-1}), \\ c_j(u) + c'(v) - c_j(v), & \text{if } u \in V(G_j), \end{cases}$$

is a well-defined ξ'_j-coloring of G'_j. Moreover $\text{esp}(c) = \max\{\text{esp}(c'), \text{esp}(c_j)\}$, so $\text{esp}(G'_j, \xi'_j) \le \text{esp}(c) = \max\{\text{esp}(G'_{j-1}, \xi'_{j-1}), \text{esp}(G_j, \xi_j)\} \le \max_{1 \le i \le j} \text{esp}(G'_i, \xi'_i)$. □

The above proof shows how to construct an optimal ξ-coloring of a cactus G provided that we have its decomposition $G_1, G_2, \ldots, G_k$ and optimal ξ_i-colorings. The former can be done in $O(n)$, the latter was proved to be executable in $O(n \log n)$ time since the elements of the decomposition are paths and cycles. The construction presented in the proof is clearly linear, thus the overall complexity of this construction is $O(n \log n)$.

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References

Broersma HJ (2003) A general framework for coloring problems: old results, new results, and open problems. In: Combinatorial geometry and graph theory. Springer, Berlin, p 65–79

Calamoneri T (2006) The $L(h, k)$-labelling problem: a survey and annotated bibliography. Comput J 49:585–608

Chang GJ, Juan S-T, Hu S-J (1999) T-colorings and T-edge spans of graphs. Graphs Combin 15:295–301

Fiala J, Kratochvil J (2001) Complexity of partial covers of graphs., Lectures notes in computer science-Springer, Berlin, pp 537–549

Griggs JR, Yeh RK (1992) Labeling graphs with a condition at distance two. SIAM J Discret Math 5:586–595

Hale WK (1980) Frequency assigment: theory and applications. Proc IEEE 68:1497–1514

Janczewski R, Kosowski A, Małafiejski M (2009) The complexity of the $L(p, q)$-labeling problem for bipartite planar graphs of small degree. Discret Math 309:3270–3279

Jonas K (1993) Graph coloring analogues with a condition at distance two: $L(2, 1)$-labelings and list λ-labelings. Ph.D. Thesis, University of South Carolina, Columbia

Karp RM (1972) Reducibility among combinatorial problems. In: Miller RE, Thatcher JW (eds) Complex Comput Comput. Plenum, New York, pp 85–103

Yeh RK (2000) The edge span of distance two labellings of graphs. Taiwan J Math 4:675–683

Yeh RK (2006) A survey on labeling graphs with a condition at distance two. Discret Math 306:1217–1231