EPJ Web of Conferences **175**, 10008 (2018) https://doi.org/10.1051/epjconf/201817510008
Lattice 2017

Non-perturbative determination of improvement b-coefficients in $N_f = 3^\star$

Giulia Maria de Divitiis[1,2,★★], *Maurizio* Firrotta[1,2], *Jochen* Heitger[3], *Carl Christian* Köster[3], and *Anastassios* Vladikas[2]

[1] *Dipartimento di Fisica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy*
[2] *INFN, Sezione di Tor Vergata, c/o Dipartimento di Fisica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy*
[3] *Institut für Theoretische Physik, Universität Münster, Wilhelm-Klemm-Str. 9, 48149 Münster, Germany*

Abstract. We present our preliminary results of the non-perturbative determination of the valence mass dependent coefficients $b_A - b_P$ and b_m as well as the ratio $Z_P Z_m / Z_A$ entering the flavour non-singlet PCAC relation in lattice QCD with $N_f = 3$ dynamical flavours. We apply the method proposed in the past for quenched approximation and $N_f = 2$ cases, employing a set of finite-volume ALPHA configurations with Schrödinger functional boundary conditions, generated with $O(a)$ improved Wilson fermions and the tree-level Symanzik-improved gauge action for a range of couplings relevant for simulations at lattice spacings of about 0.09 fm and below.

1 Introduction

Discretisation effects of lattice quantities computed with Wilson fermions are linear in the lattice spacing a, and may be a source of significant systematic errors, resulting in poor control of the continuum extrapolations of physical observables. In the Symanzik improvement programme these $O(a)$ effects can be removed by adding irrelevant operators both to the lattice action and to the local operators inserted in bare correlation functions. These so-called Symanzik counterterms have coefficients which must be tuned non-perturbatively, in order to remove all $O(a)$ contributions from physical quantities. The improvement coefficients which multiply mass dependent Symanzik counterterms are referred in the literature as b-coefficients. We will present preliminary results for the b-coefficients related to the renormalised quark masses in QCD with three dynamical sea quarks. For analogous results on the renormalisation and improvement of the vector current see Ref. [1].

2 Improvement condition

The improvement coefficients are short distance quantities. They can be determined by imposing suitable conditions in small physical volumes. We adopt the Schrödinger functional setup, with $L^3 \times T$ lattices having periodic (Dirichlet) boundary conditions in space (time). The renormalisation scale is

★ Talk given at the 35th International Symposium on Lattice Field Theory, 18-24 June 2017, Granada, Spain.
★★ Speaker, e-mail: giulia.dedivitiis@roma2.infn.it

$\mu = 1/L$. As we will exploit the freedom to keep sea- and valence-quark masses distinct, our setup is non-unitary. Sea quark masses are tuned to the chiral limit, in line with the usual ALPHA choice of a mass-independent renormalisation scheme. As the bare coupling g_0 is varied, all other bare parameters (such as the valence quark masses) are tuned so as to stay on a line of constant physics. This ensures that the b-coefficients are smooth functions of g_0.

The non-pertubative definition of the b-coefficients is not unique and depends upon the chosen improvement condition. The one we use is the standard non-singlet PCAC relation among renormalised quantities [2]:

$$\tilde{\partial}_\mu \left\langle A_{R\mu}^{ij}(x)\, O^{ji} \right\rangle = (m_{\mathrm{R},i} + m_{\mathrm{R},j}) \left\langle P_R{}^{ij}(x)\, O^{ji} \right\rangle + O(a^2)\,, \tag{1}$$

where $A_{R\mu}^{ij}$, $P_{R\mu}^{ij}$, $m_{\mathrm{R},i}$, $m_{\mathrm{R},j}$ denote the renormalised axial current, pseudoscalar density and masses with flavour indices i, j. In the following, quantities with the same flavour index, such as A_μ^{11}, m_{22} etc., are intended as defined for two distinct but degenerate valence flavours, so as to avoid Wick contractions that give rise to diagrams with disconnected quark lines. Improvement enforces this Ward identity, which holds in the continuum, to have no corrections linear in the lattice spacing, extending its validity up to $O(a^2)$ violations. Starting from the bare quantities

$$\begin{aligned} &A_\mu^{ij} \equiv \bar{\psi}_i \gamma_\mu \gamma_5 \psi_j\,, \quad P^{ij} \equiv \bar{\psi}_i \gamma_5 \psi_j\,, \\ &m_{q,ij} \equiv \tfrac{1}{2}(m_{q,i} + m_{q,j})\,, \quad m_{q,i} \equiv m_{0,i} - m_{crit} = \tfrac{1}{2a}(\tfrac{1}{\kappa_i} - \tfrac{1}{\kappa_{crit}})\,, \end{aligned} \tag{2}$$

we can write the renormalised masses and operators, in standard notation, as [3]:

$$\begin{aligned} A_{R\mu}^{ij} &= Z_{\mathrm{A}}\,(1 + b_{\mathrm{A}}\, am_{q,ij} + \bar{b}_{\mathrm{A}}\, a\,\cancel{\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}})\,\{A_\mu^{ij} + c_{\mathrm{A}} a\,\tilde{\partial}_\mu P^{ij}\}\,, \\ P_{R\mu}^{ij} &= Z_{\mathrm{P}}\,(1 + b_{\mathrm{P}}\, am_{q,ij} + \bar{b}_{\mathrm{P}}\, a\,\cancel{\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}})\, P^{ij}\,, \\ m_{\mathrm{R},i} &= Z_{\mathrm{m}}\Big\{m_{q,i}(1 + b_{\mathrm{m}}\, am_{q,i} + \bar{b}_{\mathrm{m}}\, a\,\cancel{\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}}) + \cancel{x\,\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}} + \cancel{y\, a\,\mathrm{tr}\,\hat{m}^{2(\mathrm{sea})}} + \cancel{z\, a\,(\mathrm{tr}\,\hat{m}^{(\mathrm{sea})})^2}\Big\}\,. \end{aligned} \tag{3}$$

$$x{\equiv}(1-r_{\mathrm{m}})/N_f \qquad y{\equiv}(r_{\mathrm{m}}d_{\mathrm{m}}-b_{\mathrm{m}})/N_f \qquad z{\equiv}(r_{\mathrm{m}}\bar{d}_{\mathrm{m}}-\bar{b}_{\mathrm{m}})/N_f$$

In small print we give the expressions for x, y and z in terms of the parameters $r_{\mathrm{m}}, b_{\mathrm{m}}, \bar{b}_{\mathrm{m}}, d_{\mathrm{m}}, \bar{d}_{\mathrm{m}}$ defined in Ref. [3]. It is important to keep in mind that the coefficients $b_{\mathrm{A}}, b_{\mathrm{P}}$ and b_{m}, multiplying valence quark masses, arise from the mass dependence of the valence quark propagators and contain also mass-independent contributions from the fermion loops. On the other hand $\bar{b}_{\mathrm{A}}, \bar{b}_{\mathrm{P}}, x, y, z$ arise from the mass dependence of quark fermion loops. By keeping valence and sea quark masses distinct and tuning the bare (subtracted) sea-quark mass-matrix $\hat{m}^{(\mathrm{sea})}$ to the chiral limit, the above expressions simplify as indicated.

3 Non-perturbative definitions of $b_{\mathrm{A}} - b_{\mathrm{P}}$, b_{m}, and Z

We compute Schrödinger functional correlation functions

$$\begin{aligned} f_{\mathrm{A}}^{ij}(x_0) &\equiv -a^3 \sum_{\mathbf{x}} \langle A_0^{ij}(x)\, O^{ji} \rangle\,, \\ f_{\mathrm{P}}^{ij}(x_0) &\equiv -a^3 \sum_{\mathbf{x}} \langle P^{ij}(x)\, O^{ji} \rangle\,, \qquad\qquad O^{ji} \equiv a^6 \sum_{\mathbf{u},\mathbf{v}} \bar{\zeta}_j(\mathbf{u})\, \gamma_5 \zeta_i(\mathbf{v})\,, \end{aligned} \tag{4}$$

with the operators A, P located in the bulk ($0 < x_0 < T$) and the source operator O^{ji} located on the boundary ($x_0 = 0$). We also compute the correlation functions $g_{\mathrm{A,P}}^{ij}(T - x_0)$ with the same operator

insertions in the bulk and sources $\mathcal{O}'^{ji}$ at ($x_0 = T$). Due to the symmetric boundary conditions on the gauge fields, we can symmetrise $f^{ij}_{\mathrm{A,P}}$ and $g^{ij}_{\mathrm{A,P}}$, thus reducing statistical fluctuations. The renormalisation pattern and improvement constraint imply that the current (PCAC) mass m_{ij}, defined by

$$m_{ij}(x_0) \equiv \frac{\tilde{\partial}_0 f^{ij}_{\mathrm{A}}(x_0) + a c_{\mathrm{A}} \partial^*_0 \partial_0 f^{ij}_{\mathrm{P}}(x_0)}{2\, f^{ij}_{\mathrm{P}}(x_0)}\,, \tag{5}$$

can be parametrised as

$$\begin{aligned} m_{ij}(x_0) = Z\Big(& \cancel{x\,\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}} + \cancel{[z + x(\bar{b}_{\mathrm{A}} - \bar{b}_{\mathrm{P}})]\, a\,(\mathrm{tr}\,\hat{m}^{(\mathrm{sea})})^2} + \cancel{y\, a\,\mathrm{tr}\,\hat{m}^{2(\mathrm{sea})}} \\ & + m_{q,ij}\,(1 + \cancel{[x(b_{\mathrm{A}} - b_{\mathrm{P}}) + \bar{b}_{\mathrm{m}} - (\bar{b}_{\mathrm{A}} - \bar{b}_{\mathrm{P}})]\, a\,\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}}) + a m^2_{q,ij}\,(b_{\mathrm{P}} - b_{\mathrm{A}}) + \tfrac{1}{2} a (m^2_{q,i} + m^2_{q,j})\, b_{\mathrm{m}}\Big), \end{aligned} \tag{6}$$

where the slashed terms nearly vanish at $\hat{m}^{(\mathrm{sea})} \approx 0$ and Z indicates the ratio of renormalisation constants $Z(g_0^2) \equiv Z_{\mathrm{m}}(g_0^2, a/L) Z_{\mathrm{P}}(g_0^2, a/L)/Z_{\mathrm{A}}(g_0^2)$. For the various lattice derivatives standard notation is used: symmetric $\tilde{\partial}$, forward ∂, backward ∂^*. Nearest-neighbour derivatives $\tilde{\partial}$ and $\partial^*\partial$ suffer from $O(a^2)$ discretisation errors; we label results produced with them with "standard derivative". In Refs. [4, 5], next-to-nearest-neighbour definitions have been proposed, with $O(a^4)$ errors. Results obtained with these definitions are labelled with "improved derivative".

We determine the improvement coefficients adopting the same strategy introduced for quenched QCD in [4–7] and applied later for the two flavour case [8]. We consider three different valence flavours $i, j = 1, 2, 3$ and compute the four different PCAC masses m_{11}, m_{22}, m_{33}, m_{12}. Up to renormalisation, these are physical quantities. We keep m_{11} and m_{22} fixed along our line of constant physics. The hopping parameter κ_1 of the first valence flavour is set equal to the value of the dynamical quarks, in order to have nearly vanishig m_{11}. For the second valence flavour, κ_2 is chosen so that m_{22} is approximately equal to four arbitrary reference values:

$$\begin{aligned} & Lm_{11} \approx 0.0\,, \\ & Lm_{22} \approx 0.25,\ 0.5,\ 0.75,\ 1.0\,. \end{aligned} \tag{7}$$

The third flavour is such that the corresponding bare mass is halfway the two others:

$$m_{0,3} = \tfrac{1}{2}(m_{0,1} + m_{0,2})\,, \quad \text{equivalently} \quad m_{q,3} = \tfrac{1}{2}(m_{q,1} + m_{q,2})\,. \tag{8}$$

The renormalisation and improvement structure of PCAC mass differences is as follows:

$$\begin{cases} \Delta_{22,11} \equiv \frac{1}{2}(m_{22} - m_{11}) & = Z\delta\,\Big(1 + aA^{(\mathrm{sea})} + 2\,a\bar{m}\,b_{\mathrm{mAP}}\Big) + \ldots \\ \Delta_{22,33} \equiv (m_{22} - m_{33}) & = Z\delta\,\Big(1 + aA^{(\mathrm{sea})} + (2\,a\bar{m} + a\delta)\,b_{\mathrm{mAP}}\Big) + \ldots \\ \Delta_{33,11} \equiv (m_{33} - m_{11}) & = Z\delta\,\Big(1 + aA^{(\mathrm{sea})} + (2\,a\bar{m} - a\delta)\,b_{\mathrm{mAP}}\Big) + \ldots \\ \Delta_{22,12} \equiv (m_{22} - m_{12}) & = Z\delta\,\Big(1 + aA^{(\mathrm{sea})} + 2\,a\bar{m}\,b_{\mathrm{mAP}} - a\delta\,b_{\mathrm{AP}}\Big) + \ldots \\ \Delta_{12,11} \equiv (m_{12} - m_{11}) & = Z\delta\,\Big(1 + aA^{(\mathrm{sea})} + 2\,a\bar{m}\,b_{\mathrm{mAP}} + a\delta\,b_{\mathrm{AP}}\Big) + \ldots\,, \end{cases} \tag{9}$$

$$\begin{aligned} & a\bar{m} \equiv \big(am_{q,2} + am_{q,1}\big)/2\,, \qquad a\delta \equiv \big(am_{q,2} - am_{q,1}\big)/2\,, \\ & aA^{(\mathrm{sea})} \equiv (x\,b_{\mathrm{AP}} + \bar{b}_{\mathrm{mAP}})\,a\,\mathrm{tr}\,\hat{m}^{(\mathrm{sea})}\,, \qquad b_{\mathrm{mAP}} \equiv b_{\mathrm{m}} - (b_{\mathrm{A}} - b_{\mathrm{P}}), \qquad b_{\mathrm{AP}} \equiv b_{\mathrm{A}} - b_{\mathrm{P}}\,. \end{aligned}$$

Both $aA^{(\mathrm{sea})}$ and Z cancel in the ratio of mass differences, enabling us to single out $b_\mathrm{A} - b_\mathrm{P}$, b_m, as well as Z:

$$\begin{aligned} R_\mathrm{AP} &\equiv \frac{(2m_{12} - m_{11} - m_{22})}{\Delta\left(am_{q,2} - am_{q,1}\right)} = b_\mathrm{A} - b_\mathrm{P} + O(am_{q,1} + am_{q,2})\,, \\ R_\mathrm{m} &\equiv \frac{2\,(m_{12} - m_{33})}{\Delta\left(am_{q,2} - am_{q,1}\right)} = b_\mathrm{m} + O(am_{q,1} + am_{q,2})\,, \\ R_Z &\equiv \frac{m_{11} - m_{22}}{m_{q,1} - m_{q,2}} + (R_\mathrm{AP} - R_\mathrm{m})\,(am_{11} + am_{22}) = Z + O(a\,\mathrm{tr}\,\hat{m}^{(\mathrm{sea})})\,. \end{aligned} \tag{10}$$

In the above expressions, Δ without subscripts indicates any of the five Δ's in Eqs. (9), leading to five possible determinations of the b's, which differ by $O(a)$ terms. This ambiguity becomes $O(a^2)$ when the b's are inserted in the definition of renormalised, improved quark-masses. With exactly massless sea quarks the ambiguity in Z is $O(a^2)$. These formulae generalise the ones in previous works [4, 5, 7, 8].

4 Simulation details

As already mentioned, our simulations are performed on a constant-physics trajectory in the space of bare parameters, with all physical scales held fixed, as illustrated in Fig. 1. We use the gauge configurations generated by the ALPHA collaboration, with the coupling constant $\beta = 6/g_0^2$ tuned so that the physical lattice extent is fixed to $L \approx 1.2$ fm. The tuning is based on the 2-loop perturbative expression for the lattice spacing. Subsequently, the value of κ (corresponding to the mass of the degenerate sea quarks) is fixed for each lattice, so as to obtain a vanishing PCAC mass. The parameters of the available configurations are shown in Tab. 1. The values of β span a range which is suitable for large-volume simulations. They correspond to the interval of lattice spacings $0.045\,\mathrm{fm} \lesssim a \lesssim 0.090\,\mathrm{fm}$. All lattices (except E1k1 and E1k2 where $T = 3L/2$) have temporal size $T = 3L/2 - a$. For details, see Ref. [9, 10].

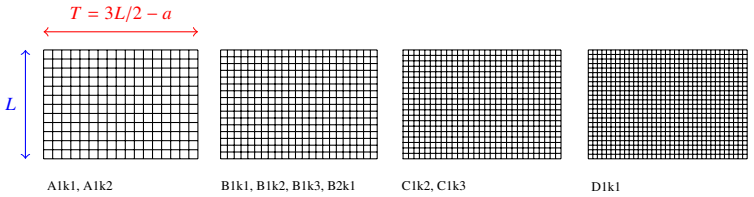


Figure 1. Lattices with varying lattice spacing but identical physical size $L \approx 1.2$ fm.

The SF simulations have been performed using the `openQCD` code [11], with improved Lüscher–Weisz gauge action [12], $N_f = 3$ massless Wilson-clover fermions, vanishing boundary gauge fields $C = C' = 0$ and boundary fermion parameter $\theta = 0$. The value of the improvement coefficient c_SW is taken from Ref. [13]. The RHMC algorithm [14–16] is used for the third dynamical quark.

5 Results

The preliminary results presented in the this work are obtained from the analysis of the B1k3 ensemble, marked in red in Tab. 1.

Table 1. Simulation parameters L, T, β, κ, number of replicas #REP (i.e. number of statistically independent sets of configurations from Monte Carlo runs at identical parameters) and number of molecular dynamics units #MDU for each ensemble ID.

$L^3 \times T/a^4$	β	κ	#REP	#MDU	ID
$12^3 \times 17$	3.3	0.13652	10	10240	A1k1
		0.13660	10	12620	A1k2
$14^3 \times 21$	3.414	0.13690	32	10360	E1k1
		0.13695	48	13984	E1k2
$16^3 \times 23$	3.512	0.13700	2	20480	B1k1
		0.13703	1	8192	B1k2
		0.13710	3	24560	B1k3
$16^3 \times 23$	3.47	0.13700	3	29584	B2k1
$20^3 \times 29$	3.676	0.13700	4	15232	C1k2
		0.13719	4	15472	C1k3
$24^3 \times 35$	3.810	0.13712	5	10240	D1k1

The time dependence of the PCAC masses $m_{11}, m_{22}, m_{33}, m_{12}$ is shown in Fig. 2. These results are obtained with improved derivatives; those obtained with standard derivatives do not show appreciable differences. All masses show wide plateaux, and the statistical errors are smaller than the symbols. The red points correspond to the chiral flavour $m_{11} \approx 0$, while the blue data represent m_{22}. As can be seen on the right vertical axis, m_{22} is tuned with good precision to the chosen reference values $Lm_{22} \approx 0.25,\ 0.5,\ 0.75,\ 1.0$ of Eq. (7).

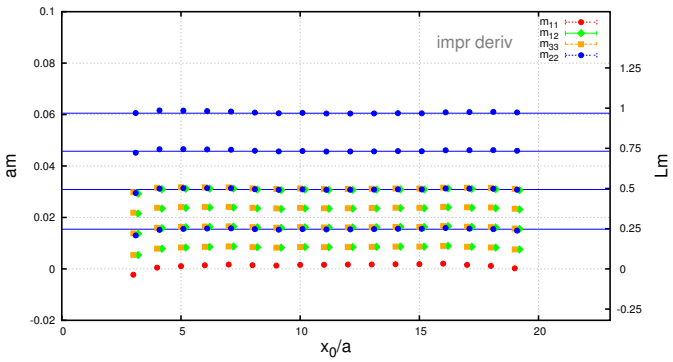


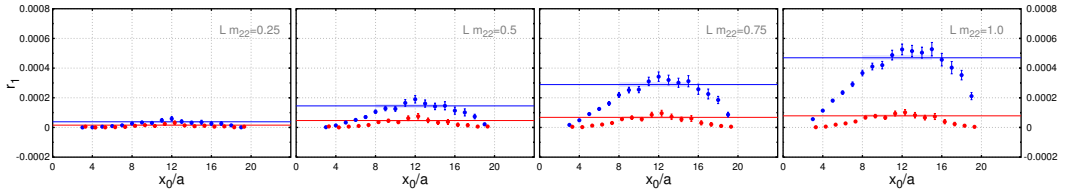
Figure 2. Time dependence of PCAC masses for the ensemble B1k3.

To check the consistency of our data with the parametrisation of the cutoff effects given in Eqs. (9), we verify that the quantities

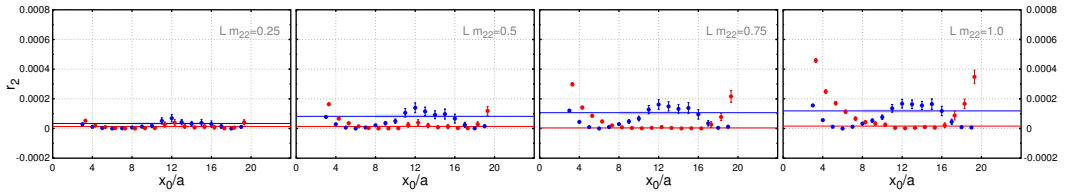
$$r_1 \equiv \frac{1}{4}\,\frac{(m_{22}-m_{11})\,(m_{22}-m_{11})}{(m_{22}-m_{33})\,(m_{33}-m_{11})} - 1 = O(a^2)\,,$$
$$r_2 \equiv \frac{1}{4}\,\frac{(m_{22}-m_{11})\,(m_{22}-m_{11})}{(m_{22}-m_{12})\,(m_{12}-m_{11})} - 1 = O(a^2)\,, \qquad (11)$$

are close to zero. As can be seen in Fig. 3, these ratios are of order 10^{-4} and less, significantly smaller than the values $am_{q,2} \approx 0.015,\ 0.06$, with improved-derivative data having the smaller values. Moreover, they tend to increase with the mass m_{22} and time x_0, as expected. The smallness of r_1 and

r_2 demonstrates that results for the b's are insensitive to the choice of Δ in the denominator. In what follows we set $\Delta = \Delta_{22,11}$, which is the one kept fixed on the line of constant physics.



(a) r_1.



(b) r_2.

Figure 3. Time dependence of the ratios r_1 and r_2 for the ensemble B1k3. Blue points refer to PCAC masses computed with standard derivatives, red points to those computed with improved ones. The four plots correspond to the four reference values $Lm_{22} = 0.25,\ 0.5,\ 0.75,\ 1.0$.

The main results of our preliminary analysis are presented in Fig. 4. The plots (a),(b) and (c) show the time dependence of estimators for $b_{\rm AP}$, $b_{\rm m}$ and Z, respectively, with blue points corresponding to the standard derivative and red points to the improved one. The horizontal lines in the plots indicate the averages over the time window $x_0/a = [8; 15]$, corresponding to the middle third of the time extent T. Averaging over time slices is part of our operative definition of the parameters $b_{\rm AP}$, $b_{\rm m}$, Z. Note that $R_{\rm AP}$ data show a significant ambiguity with respect to the choice of the lattice derivative, as previously observed in the quenched and $N_f = 2$ studies [4, 5, 8].

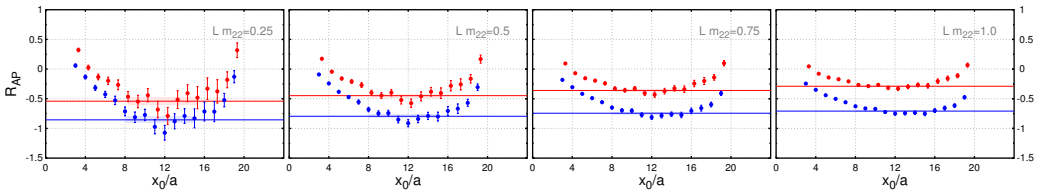
In general all signals show better plateaux and smaller statistical errors at larger values of m_{22}, where however discretisation effects are expected to be larger.

5.1 Topological sectors

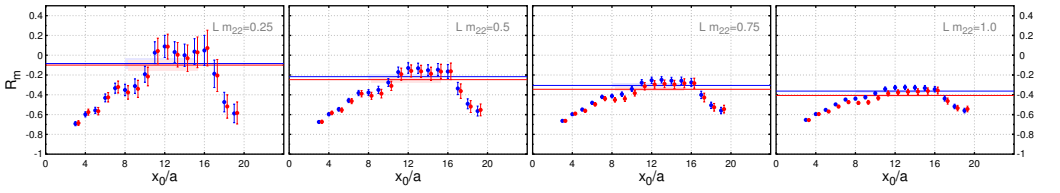
Since Ward identities hold in any topological sector and the improvement coefficients are short distance quantities, our results should be insensitive to the topological charge Q. Following Ref. [9], we repeated our data analysis only considering configurations belonging to the trivial (i.e. $Q = 0$) topological sector, using a topological charge defined through gradient-flow fields [17, 18]

$$Q(t) \equiv -\frac{a^4}{32\pi^2} \sum_x \epsilon_{\mu\nu\alpha\beta}\, \mathrm{tr}\{G_{\mu\nu}(x,t)\, G_{\alpha\beta}(x,t)\}\,, \qquad G_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,, \tag{12}$$

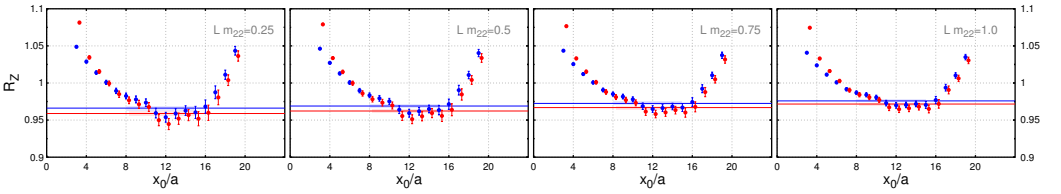
where t is the flow time, kept fixed in units of physical volume, and B_μ is the gluon field. The results were in agreement with the full statistics (i.e. including all topological charges), while only reflecting fluctuations consistent with the reduction of statistics. This confirms the aforementioned expectation of the results' insensitivity to topology.



(a) R_{AP}.



(b) R_{m}.



(c) R_Z.

Figure 4. Time dependence of R_{AP}, R_{m} and R_Z for the ensemble B1k3. Blue points refer to PCAC masses computed with standard derivatives, red points to those computed with improved ones. The four plots correspond to the four reference values $Lm_{22} = 0.25, 0.5, 0.75, 1.0$.

6 Conclusion

To complete our work we will compute the correlation functions for full statistics and on all available lattices at different lattice spacings (see Tab. 1). Combining the known analytic perturbative expressions for these quantities, valid towards vanishing g_0^2, with our data points, we aim at obtaining suitable interpolation functions for $b_{\mathrm{AP}}(g_0^2)$, $b_{\mathrm{m}}(g_0^2)$, and $Z(g_0^2)$. These non-perturbative formulae are needed for reaching $O(a)$ improved results in simulations of lattice QCD with $N_f = 3$ Wilson quarks in large volume. It will be interesting to compare our results to those recently obtained by Korcyl and Bali [19], using a different non-perturbative renormalisation method.

Acknowledgements

Computer resources were provided by the INFN (GALILEO cluster at CINECA) and the ZIV of the University of Münster (PALMA HPC cluster). This work was supported by the grant HE 4517/3-1 (J. H.) of the Deutsche Forschungsgemeinschaft. C. C. K., scholar of the German Academic Scholarship Foundation (Studienstiftung des deutschen Volkes), gratefully acknowledges their financial and academic support. *The speaker wishes to thank Isabel Campos, Elvira Gámiz and all the organizers of the 2017 lattice conference for their wonderful hospitality in Granada.*

References

[1] J. Heitger, F. Joswig, A. Vladikas, C. Wittemeier (ALPHA), *Non-perturbative determination of c_V, Z_V and Z_S/Z_P in $N_f = 3$ lattice QCD*, in *Proceedings, 35th International Symposium on Lattice Field Theory (Lattice2017): Granada, Spain*, to appear in EPJ Web Conf.

[2] M. Lüscher, S. Sint, R. Sommer, P. Weisz, U. Wolff, Nucl. Phys. **B491**, 323 (1997), hep-lat/9609035

[3] T. Bhattacharya, R. Gupta, W. Lee, S.R. Sharpe, J.M. Wu, Phys.Rev. **D73**, 034504 (2006), hep-lat/0511014

[4] G.M. de Divitiis, R. Petronzio, Phys. Lett. **B419**, 311 (1998), hep-lat/9710071

[5] M. Guagnelli, R. Petronzio, J. Rolf, S. Sint, R. Sommer, U. Wolff (ALPHA), Nucl. Phys. **B595**, 44 (2001), hep-lat/0009021

[6] J. Heitger, J. Wennekers (ALPHA), JHEP **02**, 064 (2004), hep-lat/0312016

[7] T. Bhattacharya, R. Gupta, W.J. Lee, S.R. Sharpe, Phys. Rev. **D63**, 074505 (2001), hep-lat/0009038

[8] P. Fritzsch, J. Heitger, N. Tantalo (ALPHA), JHEP **08**, 074 (2010), 1004.3978

[9] J. Bulava, M. Della Morte, J. Heitger, C. Wittemeier (ALPHA), Nucl. Phys. **B896**, 555 (2015), 1502.04999

[10] J. Bulava, M. Della Morte, J. Heitger, C. Wittemeier (ALPHA), Phys. Rev. **D93**, 114513 (2016), 1604.05827

[11] *openQCD, Simulation program for lattice QCD*, http://luscher.web.cern.ch/luscher/openQCD/

[12] M. Lüscher, P. Weisz, Commun. Math. Phys. **97**, 59 (1985), [Erratum: Commun. Math. Phys.98,433 (1985)]

[13] J. Bulava, S. Schaefer, Nucl. Phys. **B874**, 188 (2013), 1304.7093

[14] A.D. Kennedy, I. Horvath, S. Sint, Nucl. Phys. Proc. Suppl. **73**, 834 (1999), hep-lat/9809092

[15] M.A. Clark, A.D. Kennedy, Phys. Rev. Lett. **98**, 051601 (2007), hep-lat/0608015

[16] M. Lüscher, F. Palombi, PoS **LATTICE2008**, 049 (2008), 0810.0946

[17] M. Lüscher, JHEP **08**, 071 (2010), [Erratum: JHEP03,092 (2014)], 1006.4518

[18] M. Lüscher, P. Weisz, JHEP **02**, 051 (2011), 1101.0963

[19] P. Korcyl, G.S. Bali, Phys. Rev. **D95**, 014505 (2017), 1607.07090