Recognition of Graphs with Convex Quadratic Stability Number

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Abstract. A stable set of a graph is a set of mutually non-adjacent vertices. The determination of a maximum size stable set, which is called maximum stable set, and the determination of its size, which is called stability number, are central combinatorial optimization problems. However, given a nonnegative integer k, to determine if a graph G has a stable set of size k is NP-complete. In this paper we deal with graphs for which the stability number can be determined by solving a convex quadratic programming problem. Such graphs were introduced in [13] and are called graphs with convex-QP stability number. A few algorithmic techniques for the recognition of this type of graphs in particular families are presented.

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INTRODUCTION

Graphs with convex-QP stability number are graphs for which the stability number is equal to the optimal value of a convex quadratic program. It is known that there is an infinite number of graphs with convex-QP stability number. The purpose of this paper is to present combinatorial characterizations of graphs with convex-QP stability number and some families of graphs in which it is possible to recognize, in polynomial-time, if a graph has (or not) convex-QP stability number. Throughout this paper, G will be an undirected simple graph with n vertices for which $V = V(G)$ denotes the nonempty set of the vertices and $E = E(G)$ the set of the edges. We say that vertex u is adjacent to vertex v if they are endpoints of an edge (that will be denoted by uv) and we call neighborhood of u the set $N_G(u) = \{v \in V : uv \in E\}$. Given a vertex $u, d_G(u) = |N_G(u)|$ is the degree of u. A graph where all vertices have degree p is called a p-regular graph. A graph of order n in which all pairs of vertices are adjacent is a complete graph and will be denoted by K_n. Given a graph G and a set of vertices $U \subseteq V$, the subgraph of G induced by U is graph $G[U]$ such that $V(G[U]) = U$ and $E(G[U]) = \{uv : u, v \in U \text{ and } uv \in E(G)\}$. Graph G is strongly regular with parameters (n, p, a, c) if it is a not complete p-regular graph, with $p > 0$, such that each pair of adjacent vertices has a common neighbors and each pair of non-adjacent vertices has c common neighbors. A graph is bipartite if $V(G)$ is the disjoint union of two sets V_1 and V_2 such that $\forall uv \in E(G), |V_1 \cap \{u, v\}| = |\{u, v\} \cap V_2| = 1$. A set of mutually non adjacent vertices in a graph is called a stable set and a set of mutually adjacent vertices is called a clique. A stable set (clique) S is called maximum stable set (clique) if there is no other stable set (clique) with greater number of vertices. The number of vertices in a maximum stable set (clique) of graph G, is called the stability (clique) number of G and is denoted by $\alpha(G)(\omega(G))$. Given graph G, the line graph of G, which is denoted by $L(G)$, is constructed by taking the edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$ by an edge whenever the corresponding edges in G have a common vertex. A matching in G is a subset of edges, $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is a maximum matching. It is well known that the problem of determining a maximum matching in G is equivalent to the problem of determining a maximum stable set in $L(G)$. Given a subset of vertices S of graph G, the vector $x \in \mathbb{R}^n$ such that $x_v = 1$ if $v \in S$ and $x_v = 0$ if $v \notin S$ is the characteristic vector of S. We say that $S \subset V(G)$ is (κ, τ)-regular if it induces a κ-regular subgraph and $\forall u \notin S, |N_G(u) \cap S| = \tau$.Throughout this paper, A_G (or A) will denote the adjacency matrix of G, that is $A_G = (a_{ij})$ will be such that

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{if } ij \notin E(G) \end{cases}$$

and $\lambda_{max}(A) = \lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A) = \lambda_{min}(A)$ will denote the eigenvalues of A_G. It is well known that if G has at least one edge, then $\lambda_{min}(A_G) \leq -1$. Actually, $\lambda_{min}(A_G) = -1$ if and only if G has at least one edge and every component of G is complete.

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GRAPHS WITH CONVEX-*QP* STABILITY NUMBER

Let G be a graph of order n with at least one edge and A_G its adjacency matrix. Let $\upsilon(G)$ denote the optimal value of the quadratic program

$$\upsilon(G) = max\left\{2e^T x - x^T\left(\frac{A_G}{-\lambda_{min}(A_G)} + I_n\right)x, x \geq 0\right\} \tag{1}$$

where I_n is the identity matrix of order n and e is the all-one vector.

This program was introduced in [13] in order to obtain an upper bound for the stability number of G and analyzed in [15], where it was proved that its optimal value is the best upper bound for $\alpha(G)$, among a family of convex quadratic programs.

A graph G for which $\alpha(G) = \upsilon(G)$ is a graph with *convex quadratic stability number* or a graph with *convex QP-stability number* where *QP* stands for quadratic programming. The class of graphs with convex quadratic stability number is denoted by Q.

It is immediate that program (1) is convex.

In [13], a criteria for equality was given: if G has at least one edge, then $\alpha(G) = \upsilon(G)$ if and only if for a maximum stable set S of G (and then for all),

$$-\lambda_{min}(A_G) \leq \min\{|N_G(u) \cap S| : u \notin S\}. \tag{2}$$

This condition was improved in [6] where it was proved that $\upsilon(G) = \alpha(G)$ if and only if there is a stable set verifying (2).

We should mention that the upper bound υ for the stability number of a graph generalizes Hoffman [unpublished] and Lovász's [11] upper bound for the stability number. For regular graphs, we have

$$\upsilon(G) = \frac{-n\lambda_{min}(A_G)}{\lambda_{max}(A_G) - \lambda_{min}(A_G)}.$$

There are several famous graphs with convex-*QP* stability number. It is the case of the Petersen graph which is the strongly regular graph with parameters $(10,3,0,1)$ for which $\lambda_{min}(A_P) = -2$ and $\alpha(P) = \upsilon(P) = 4$. It is also the case of the Hoffman-Singleton graph HS, the strongly regular graph with parameters $(50,7,0,1)$ for which $\lambda_{min}(A_{HS}) = -3$ and $\alpha(HS) = \upsilon(HS) = 15$ and many others. It is not known if the fourth graph of Moore $M4$ (that is, the strongly regular graph with parameters $(3250,57,0,1)$) exists. However, if such graph exists then it is expected that it has convex-*QP* stability number, with $\lambda_{min}(A_{M4}) = -8$ and $\alpha(M4) = \upsilon(M4) = 400$. Additionally, taking into account [7], graphs defined by the disjoint union of complete subgraphs and complete bipartite graphs are trivial examples of graphs with convex-*QP* stability number (G is a complete bipartite graph if it is bipartite with $V = V_1 \cup V_2$ and $\forall u \in V_1, d_G(u) = |V_2| = q$ and $\forall v \in V_2, d_G(v) = |V_1| = p$).

RECOGNITION OF GRAPHS WITH CONVEX-*QP* STABILITY NUMBER

Theorems 3, 4, 5 and 6 in [2] give an algorithmic strategy for the recognition of Q-graphs unless the following condition is veryfied:

$$\forall u \in V, \upsilon(G) = \upsilon(G-u) = \upsilon(G - N_G(u)) \text{ and } \lambda_{min}(A_G) = \lambda_{min}(A_{G-u}) = \lambda_{min}(A_{G-N_G(u)}).$$

Graphs having an induced subgraph G with no isolated vertices such that $\upsilon(G)$ and $\lambda_{min}(A_G) \in \mathbb{Z}$, verifying

$$\forall u \in V, \upsilon(G) = \upsilon(G - N_G(u)) \text{ and } \forall u \in V, \lambda_{min}(A_G) = \lambda_{min}(A_{G-N_G(u)})$$

are called adverse graphs.

Based in Theorems 3, 4, 5 and 6 in [2], the next procedure recognizes if a graph is (or not) in Q or determines one of its adverse subgraphs. Note that the input of the procedure is graph G and $Iso(H)$ denotes the set of isolated vertices of H.

Procedure 1

(**Input:** Graph G with at least one edge)

1. **Set** $H := G, \tau := -\lambda_{min}(A_G)$;
2. **Set** $I = Iso(H), H := H - I$;
3. **If** $\exists v \in V(H) : \tau \neq -\lambda_{min}(A_{H-N_H(v)})$ **then**
   3.1 **If** $\upsilon(G) = \upsilon(G - N_G(v))$ **then** STOP ($G \in Q$);
4. **If** $\exists v \in V(H) : \upsilon(G - v) \neq \upsilon(G - N_G(v))$ **then**
   4.1 **If** $\upsilon(G) \notin \{\upsilon(G - v), \upsilon(G - N_G(v))\}$ **then** STOP ($G \notin Q$) **else**
      4.1.1 **If** $\upsilon(G) = \upsilon(G - v)$ **then set** $G := G - v$ **else set** $G := G - N_G(v)$;
      4.1.2 **goto**step 2.
   else
   4.2 **If** $\exists v \in V(H) : \upsilon(G) \neq \upsilon(G - v)$ **then** STOP ($G \notin Q$) **else** STOP (G contains the adverse subgraph H);

The following results are useful for the recognition of graphs with convex-QP stability number:

Theorem 1 *If G is adverse,* $G \in Q$ *if and only if* $\exists S \subseteq V(G)$ *such that S is* $(0,\tau)$*-regular with* $\tau = -\lambda_{min}(A_G)$.

Theorem 2 *If G is p-regular,* $G \in Q$ *if and only if* $\exists S \subset V(G)$ *such that S is* $(0,\tau)$*-regular with* $\tau = -\lambda_{min}(A_G)$.

As a consequence of Theorem 1, the recognition of Q-graphs can be done applying Procedure 1 or recognizing a $(0,\tau)$-regular set, with $\tau = \lambda_{min}(A_H)$, in an adverse subgraph H determined aplying the procedure.

ANALYSIS OF PARTICULAR FAMILIES OF GRAPHS

Bipartite graphs

According to the Perron-Frobenius Theorem ([10], Theorem 6.1), every connected graph has a simple maximum eigenvalue. On the other hand, when the graph is bipartite, its eigenvalues are symmetric in relation with the origin [9]. Therefore, the minimum eigenvalue of a connected bipartite graph G is simple and then

$$\exists u \in V(G) : \lambda_{min}(A_G) < \lambda_{min}(A_{G-u}).$$

Hence, since $G \in Q$ if and only if each component is in Q, applying Procedure 1 we can recognize (in polynomial-time) if a bipartite graph is (or not) a Q-graph.

Dismantable graphs

Dismantable graphs have the following recursive definition: the one-vertex graph is dismantable and a graph G with at least two vertices is dismantable if $\exists u, v \in V(G) : N_G[u] \subseteq N_G[v]$ (where $N_G[x] = N_G(x) \cup \{x\}$) and $G - u$ is dismantable. The next Theorem states a neighborhood inclusion condition which does not hold when the graph is adverse and then we may conclude that there are no dismantlable adverse graphs.

Theorem 3 *[3] If G has at least one edge and* $\exists u, v \in V(G) : N_G(u) \subseteq N_G(v)$*, then* $\upsilon(G) > \upsilon(G - N_G(u))$.

Graphs with low Dilworth number

Given $u, v \in V(G) : N_G(u) \subseteq N_G(v)$, we say that vertices u and v are comparable. This binary relation is a preorder (that is, it is reflexive and transitive) and is called vicinal preorder. Therefore, graph $D(G)$ such that $V(D(G)) = V(G)$ and $E(V(G)) = \{u, v \in V(G) : N_G(u) \subseteq N_G[v]$ or $N_G(v) \subseteq N_G[u]\}$ is the comparability graph of the vicinal preorder of G. Considering the Dilworth number of a graph (its largest number of pairwise incomparable vertices), we have $dilw(G) = \alpha(D(G))$.

Theorem 4 *[3] Let G be a not complete graph. If* $dilw(G) < \omega(G)$*, then G is not adverse.*

As a consequence of Theorem 4, we can apply Procedure 1 to recognize in polynomial-time if a graph with Dilworth number equal to 1 (which is a threshold graph) is a Q-graph.

Claw-free graphs

Given a graph H, graph G is said to be H-free if G has no copy of H as an induced subgraph. A claw-free graph is a $K_{1,3}$-free graph. Also, an α-redundant subset of vertices is a subset $U \subseteq V(G) : \alpha(G) = \alpha(G-U)$.

The next Theorem allows the recognition of α-redundant subsets of vertices in claw-free graphs with comparable non-adjacent vertices, in the sense of the vicinal pre-order relation already defined.

Theorem 5 *Let G be a claw-free graph and* $u,v \in V(G) : uv \notin E(G)$*. If* $N_G(u) \subseteq N_G(v)$*, then* $N_G(u)$ *is an* α*-redundant subset of vertices.*

CONCLUSIONS

It is expected that, additionally to the referred cases, there are many other classes of graphs in which we may recognize in polynomial-time if a graph has or not convex-QP stability number. The determination of such classes remains open.

Some other future challenges are the recognition of $(0, -\lambda_{min}(A_G))$- regular sets in adverse graphs and the recognition in polynomial time of adverse graphs, so that the recognition of graphs with convex-QP stability number can be made based upon the results here presented in the following way:

1. Aplying Procedure 1 to determine if graph G belongs to the class Q or to determine an adverse subgraph of G;
2. Recognizing if the adverse subgraph determined with the procedure has a $(0, -\lambda_{min}(A_G))$-regular set.

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