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MILNOR-WOOD TYPE INEQUALITIES

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Introduction

The Gauss-Bonnet Theorem, which was generalized by Shiing-Shen Chern in 1944 [12] to all closed oriented even-dimensional smooth manifolds, correlates the curvature of the Levi-Civita connection of a Riemannian manifold with its Euler characteristic. This result provides a strong restriction on the kind of geometry such a manifold can support. For instance, let us consider Euclidean manifolds, i.e. manifolds which admit an atlas whose coordinate change functions are isometries of $\mathbb{R}^n$. Since the curvature of the Levi-Civita connection they inherit from $\mathbb{R}^n$ vanishes, the Euler characteristic represents an obstruction to the existence of Euclidean structures: indeed if $\chi(M) \neq 0$ then M cannot support a flat metric.

On the other hand, let us consider affine manifolds, i.e. manifolds which admit an atlas whose coordinate change functions are affine isomorphisms of $\mathbb{R}^n$. These can be characterized as those manifolds whose tangent bundle supports flat and symmetric connections. Although they may seem to be a mild generalization of Euclidean manifolds, the attempt to generalize the above result to affine manifolds resulted in the formulation of a long standing open conjecture:

Conjecture 1. The Euler characteristic of a closed oriented affine manifold vanishes.

The key point is that the Euler characteristic of a manifold cannot be computed from the curvature of an arbitrary linear connection ∇, because it is essential for ∇ to be compatible with a Riemannian metric. Now, although Conjecture 1 was shown to hold true for complete affine manifolds by Bertram Kostant and Dennis Sullivan [22], the non-complete case is much more difficult. There are known examples, due to John Smillie [28], of manifods with non-zero Euler characteristic and flat tangent bundle in every even dimension greater than 2. However, as William Goldman points out in [15], since the torsion of their connections seems hard to control they do not disprove Conjecture 1. None of Smillie's manifolds is aspherical, and indeed another open conjecture is:

Conjecture 2. The Euler characteristic of a closed oriented aspherical manifold whose tangent bundle is flat vanishes.

The first important breakthrough was made by John Milnor in 1958 [24], when he proved both conjectures for closed oriented surfaces. He exploited the fact that the existence of a flat connection on a rank-m vector bundle $\pi : E \to M$ is equivalent to the existence of a holonomy representation $\rho : \pi_1(M, x_0) \to \mathrm{GL}^+(m, \mathbb{R})$ which induces the bundle. The study of all possible holonomy representations for closed oriented surfaces enabled him to establish a much more detailed result, which is very interesting on its own: he managed to characterize all flat oriented plane bundles over closed oriented surfaces by means of their Euler class, that is a cohomology class in the cohomology ring of the base space which generalizes the Euler characteristic. What happens is that the Euler class of flat bundles over a fixed surface Σ is bounded, that is, just a finite number (up to isomorphism) of oriented plane bundles over Σ can support flat connections. In particular, none of these is the tangent bundle if Σ is not the torus. This remarkable result is now known as Milnor-Wood inequality (the name celebrates John Wood's generalization to S^1-bundles [31]).

While it has been proven that the boundedness of the Euler class of flat bundles generalizes to all dimensions, Conjectures 1 and 2 remain elusive. Indeed one needs explicit inequalities in order to determine whether the tangent bundle can be ruled out from the flat ones or not. Many attemps have been made to generalize Milnor-Wood inequality to other dimensions, but very little progress has been made until very recently. In 2011 Michelle Bucher and Tsachik Gelander proved that the Euler class of flat bundles over closed oriented manifolds whose universal cover is isometric to $(\mathbb{H}^2)^n$ satisfies an inequality of Milnor-Wood type [8], thus confirming both conjectures for all manifolds which are locally isometric to a product of surfaces of constant curvature. Their work, which takes up the largest part of our exposition, uses the theory of bounded cohomology developed by Mikhaïl Gromov in 1982 [17] and some deep results about the super-rigidity of lattices in semisimple Lie groups due to Gregori Margulis [23].

Chapter 1

Affine manifolds, flat bundles and the Euler class

This chapter is devoted to the presentation of the objects of our studies and their properties. Throughout the exposition the term vector bundle will stand for real vector bundle except where explicitly noted.

1.1 Affine manifolds

A Euclidean manifold is a smooth manifold which admits an atlas whose coordinate change functions are isometries of $\mathbb{R}^n$. Affine manifolds are a generalization obtained by allowing coordinate change functions which are affine isomorphisms of $\mathbb{R}^n$.

Geometric structures

A Hausdorff and paracompact space M is a (*topological*) *n-manifold* if it admits an open covering formed by subsets of M which are homeomorphic to open subsets of $\mathbb{R}^n$. An *n-atlas* for M is a collection of *n-charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ where:

(*i*) $\{U_\alpha\}_{\alpha\in I}$ is an open covering of M;

(*ii*) φ_α is an open map from U_α to $\mathbb{R}^n$ which is a homeomorphism onto its image.

If $U_\alpha \cap U_\beta \neq \varnothing$ then the restriction of $\varphi_\beta \circ \varphi_\alpha^{-1}$ to $\varphi_\alpha(U_\alpha \cap U_\beta)$ is a *coordinate change function.* If all the coordinate change functions of an n-atlas are smooth functions between open subsets of $\mathbb{R}^n$ then the n-atlas is *smooth.* Analogously, if they are analytic we get an *analytic n-atlas.* Two smooth n-atlases for M are *compatible* if their union is a smooth n-atlas. A smooth n-atlas is *maximal* if it contains all the smooth n-atlases compatible with it. A *smooth structure* on a topological n-manifold M is a maximal smooth n-atlas. A topological manifold equipped with a smooth structure is a *smooth manifold.* The analogous definition of compatibility for analytic atlases yields *analytic structures* and *analytic manifolds*

A continuous map $f : M \to M'$ between smooth manifolds is *smooth* if for every $x \in M$ there exist charts $(U_\alpha, \varphi_\alpha)$ in the smooth structure of M and $(U'_\beta, \varphi'_\beta)$ in the smooth structure of M' such that $x \in U_\alpha$, $f(U_\alpha) \subset U'_\beta$ and $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ is a smooth map. A smooth homeomorphism with smooth inverse is a *diffeomorphism*, and we dentoe by $\mathrm{Diff}(M)$ the group of self-diffeomorphisms of a smooth manifold M. Analogously we can define an *analytic map* between analytic manifolds to be a map which is analytic in charts and an *analytic diffeomorphism* to be an analytic homeomorphism with analytic inverse. The group of analytic self-diffeomorphisms of an analytic manifold M will be denoted by $\mathrm{Diff}^\omega(M)$.

Remark 1.1.1. An analytic map $f : M \to M'$ is uniquely determined by its behaviour on any open subset of M.

Let X be an n-manifold and let $G < \mathrm{Homeo}(X)$ be a group of self-homeomorphisms of X. An (X, G)-*atlas* for M is a collection of (X, G)-*charts* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ where:

(*i*) $\{U_\alpha\}_{\alpha \in I}$ is an open covering of M;

(*ii*) φ_α is an open map from U_α to X which is a homeomorphism onto its image;

(*iii*) if $U_\alpha \cap U_\beta \neq \varnothing$ then the restriction of $\varphi_\beta \circ \varphi_\alpha^{-1}$ to each connected component of $\varphi_\alpha(U_\alpha \cap U_\beta)$ is the restriction of an element of G.

Two (X, G)-atlases for M are *compatible* if their union is an (X, G)-atlas. An (X, G)-atlas is *maximal* if it contains all the (X, G)-atlases compatible with it. An (X, G)-*structure* on a topological n-manifold M is a maximal (X, G)-atlas. A topological manifold equipped with an (X, G)-structure is an (X, G)-*manifold*.

A continuous map $f : M \to M'$ between (X, G)-manifolds is a *local* (X, G)-*isomorphism* if it is a local homeomorphism and if for every $x \in M$ there exist (X, G)-charts $(U_\alpha, \varphi_\alpha)$ in the (X, G)-structure of M and $(U'_\beta, \varphi'_\beta)$ in the (X, G)-structure of M' such that $x \in U_\alpha$, $f(U_\alpha) \subset U'_\beta$ and $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ is given, on each connected component of $\varphi_\alpha(U_\alpha)$, by the restriction of an element of G. An (X, G)-*isomorphism* is a local (X, G)-isomorphism which is a global homeomorphism.

Remark 1.1.2. If the manifold X is smooth and $G < \mathrm{Diff}(X)$ then an (X, G)-structure for M automatically equips M with a smooth structure. In this case we will say that M admits a *smooth* (X, G)-*structure*. Analogously, if X is analytic and $G < \mathrm{Diff}^\omega(X)$ then every (X, G)-structure for M yields an analytic structure, and we'll speak of *analytic* (X, G)-*structures*. An extremely important feature of the latter is that each coordinate change function $\varphi_\beta \circ \varphi_\alpha^{-1}$ in an analytic (X, G)-structure having a connected domain uniquely determines an element of G.

Example 1.1.3. A *Euclidean structure* on a topological n-manifold M is given by an analytic $(\mathbb{R}^n, \mathrm{Isom}(\mathbb{R}^n))$-structure. An *affine structure* is given by an analytic $(\mathbb{R}^n, \mathrm{Aff}(n, \mathbb{R}))$-structure. A (local) $(\mathbb{R}^n, \mathrm{Aff}(n, \mathbb{R}))$-isomorphism will be called a (local) affine isomorphism.

Developing maps and holonomy representations

Let M be a connected n-manifold and let $\pi : \tilde{M} \to M$ denote its universal cover. With each analytic (X, G)-structure over M we can associate local diffeomorphisms $D : \tilde{M} \to X$ called *developing maps*. To construct a developing map let us fix a base point $x_0 \in M$ and let $\tilde{x}_0 \in \tilde{M}$ be a basepoint in the fiber of x_0. Define $D(\tilde{x}_0)$ to be $\varphi_{\alpha_0}(x_0)$ for any (X, G)-chart $(U_{\alpha_0}, \varphi_{\alpha_0})$ which trivializes $\tilde{M}$ and contains x_0. To extend D to any point $\tilde{y}$ in $\tilde{M}$ consider a smoothly embedded curve $\gamma : I \to \tilde{M}$ from $\tilde{x}_0$ to $\tilde{y}$ such that, if $\tilde{U}_0$ denotes the unique lifting of U_{α_0} containing $\tilde{x}_0$, then $\gamma^{-1}(\tilde{U}_0) = I_0$ is an interval. Now we can find open subsets $\tilde{U}_i$ for $i = 0, \ldots, k$ such that:

(*i*) $\pi(\tilde{U}_i)$ is the domain of an (X, G)-chart $(U_{\alpha_i}, \varphi_{\alpha_i})$ for all $i = 0, \ldots, k$;

(*ii*) $\gamma^{-1}(\tilde{U}_i) = I_i$ is an interval for all $i = 0, \ldots, k$;

(*iii*) $\gamma(I) \subset \bigcup_{i=0}^k \tilde{U}_i$;

(*iv*) $I_i \cap I_j = \varnothing$ if $|i - j| > 1$;

(*v*) $\tilde{U}_i \cap \tilde{U}_{i+1}$ is connected for all $i = 0, \ldots, k - 1$;

Now if $g_i(\gamma)$ is the unique element of G extending $\varphi_{\alpha_{i-1}} \circ \varphi_{\alpha_i}^{-1}$ we can define

$$D(\tilde{y}) = (g_1(\gamma) \circ \ldots \circ g_k(\gamma))(\varphi_{\alpha_k}(\pi(\tilde{y})))$$

Proposition 1.1.4. *For every choice of base points $x_0 \in M$, $\tilde{x}_0 \in \tilde{M}$ and of an initial (X,G)-chart $(U_{\alpha_0}, \varphi_{\alpha_0})$ the developing map D is well-defined. In particular its value in $\tilde{y}$ does not depend on the choice of the curve γ joining $\tilde{x}_0$ and $\tilde{y}$.*

In a neighborhood of $\tilde{y}$ the developing map D can be written as a composition of analytic diffeomorphisms, and thus it is indeed a local diffeomorphism. If α is a loop in M based at x_0 representing an element of $\pi_1(M,x_0)$ let us consider its unique lifting $\tilde{\alpha}$ starting from $\tilde{x}_0$. Then, since $\tilde{\alpha}$ determines $\tilde{U}_0,\dots,\tilde{U}_h \subset \tilde{M}$ and a well-defined element $g_1(\tilde{\alpha}) \circ \dots \circ g_h(\tilde{\alpha}) \in G$, we can construct the *holonomy representation*

$$\begin{array}{rccc} \rho: & \pi_1(M,x_0) & \to & G \\ & [\alpha] & \mapsto & g_1(\tilde{\alpha}) \circ \dots \circ g_h(\tilde{\alpha}) \end{array}$$

Now ρ makes D into a $\pi_1(M,x_0)$-equivariant map, i.e. we have $D(\tilde{y}\cdot\alpha^{-1}) = \rho(\alpha)(D(\tilde{y}))$ for all $\tilde{y} \in \tilde{M}$.

Proposition 1.1.5. *If we choose another pair of base points $x_0 \in M$, $\tilde{x}_0 \in \tilde{M}$ and another initial (X,G)-chart $(U_{\alpha_0}, \varphi_{\alpha_0})$ the developing map changes by composition with an element of G, while the holonomy representation changes by composition with the respective inner automorphism of* $\mathrm{Diff}^\omega(X)$.

Completeness

A manifold M endowed with an analytic (X,G)-structure is *complete* if the developing maps are covering maps. Since covering maps of simply connected smooth manifolds are diffeomorphisms, affine structures are complete if and only if their developing maps are diffeomorphisms.

Remark 1.1.6. If an analytic (X,G)-structure over M is complete and X is simply connected, the holonomy representation gives a free and properly discontinuous action of $\pi_1(M,x_0)$ on the left of X.

Proposition 1.1.7. *If X is simply connected and M is endowed with a complete analytic (X,G)-structure, then $M \simeq \pi_1(M,x_0)\backslash X$.*

Corollary 1.1.8. *An affine n-manifold M is complete if and only if it is affinely isomorphic to $\Gamma\backslash\mathbb{R}^n$ for some discrete group $\Gamma < \mathrm{Aff}(n,\mathbb{R})$ acting freely and properly discontinuously on $\mathbb{R}^n$.*

1.2 Flat bundles

A vector bundle or a principal bundle is flat if it supports a flat connection. Affine manifolds can be seen as special cases of flat bundles, as they can be characterized as those manifolds whose tangent bundle supports flat and symmetric connections.

Flat connections

Let G be a Lie group and M be a smooth manifold. The projection onto the first factor $\pi_1 : M \times G \to M$ defines a trivial principal G-bundle on M. The connection defined by $\Gamma_0 : (x,a) \mapsto T_{(x,a)}(M \times \{a\})$ is called the *standard flat connection* on $M \times G$.

Remark 1.2.1. A connection on $M \times G$ is the standard flat connection if and only if it is reducible to a connection on the sub-bundle $M \times \{e\}$.

Remark 1.2.2. Let ϑ be the $\mathfrak{g}$-valued left-invariant 1-form on G determined by $\vartheta_e(A) = A$ for all $A \in \mathfrak{g}$, which is called the *Maurer-Cartan form on G*. Then the connection form of Γ_0 coincides with $\omega := \pi_2^*\vartheta$.

Proposition 1.2.3 (Maurer-Cartan equation)**.** *Let G be a Lie group and let V be a finite-dimensional real vector space. Let η be a V-valued left-invariant 1-form on G. Then:*

$$\mathrm{d}\eta = -[\eta, \eta]$$

Since the exterior derivative commutes with pull-backs, Proposition 1.2.3 implies:

$$\mathrm{d}\omega = \mathrm{d}(\pi_2^*\vartheta) = \pi_2^*(\mathrm{d}\vartheta) = -\pi_2^*[\vartheta, \vartheta] = -[\pi_2^*\vartheta, \pi_2^*\vartheta] = -[\omega, \omega]$$

Therefore the curvature form of Γ_0 vanishes (compare with the structure equation for the curvature, Theorem 6.3.2).

A connection Γ on a principal G-bundle $\pi : P \to M$ is *flat* if for all $u \in P$ there exists a trivializing neighborhood U such that $\iota : P\big|_U \hookrightarrow P$ induces the standard flat connection on $P\big|_U$, i.e. $\iota^*\Gamma = \Gamma_0$. If $\pi : P \to M$ supports a flat connection then it is a *flat principal bundle.*

Proposition 1.2.4. *Let $\pi : P \to M$ be a principal G-bundle and let Γ be a connection on P whose curvature form Ω vanishes. If M is connected and simply connected then $\pi : P \to M$ is trivial and Γ is isomorphic to the standard flat connection.*

Proof. Since M is simply connected then for all $u \in P$ the holonomy group $\Phi(u)$ equals the reduced holonomy group $\Phi^0(u)$. Then, thanks to Theorem 6.3.9 and to the Holonomy Theorem 6.3.14, the holonomy group in $u \in P$ is $\{e\}$ for all $u \in P$. Thus Theorem 6.3.13 implies that the total space $P(u)$ of the holonomy bundle through u is isomorphic to $M \times \{e\}$, and therefore there exists a global section $\sigma : M \to P$ given by $x \mapsto P(u)_x$. Then $\pi : P \to M$ is isomorphic to the trivial bundle $\pi_1 : M \times G \to M$ and Γ is isomorphic to the standard flat connection. $\square$

As a corollary we get the following:

Theorem 1.2.5. *Let $\pi : P \to M$ be a principal G-bundle and let Γ be a connection on P. Then Γ is flat if and only if its curvature form Ω vanishes.*

Analogously, for a rank-m vector bundle $\pi : E \to M$ we say that a connection ∇ on E is *flat* if its curvature tensor R vanishes. If $\pi : E \to M$ supports a flat connection then it is a *flat vector bundle.*

Remark 1.2.6. Since direct sums and pull-backs of flat connections are flat, Whitney sums and pull-backs of flat bundles are flat.

Characterization of affine manifolds

Let $\pi : E \to M$ be a rank-m vector bundle and let U be an open subset of M which trivializes E. A *parallel frame on U* is a local frame $V_1, \dots, V_m$ defined on U such that all sections V_i are parallel, i.e. the Christoffel symbols on U with respect to $V_1, \dots, V_m$ vanish. Then, as a corollary to Proposition 1.2.4, we get:

Proposition 1.2.7. *There exists a parallel frame on $U \subset M$ if and only if $R\big|_U \equiv 0$.*

A linear connection ∇ on the tangent bundle $\pi : TM \to M$ is *symmetric* if its torsion tensor T vanishes. If U is an open subset of M a local frame $X_1, \dots, X_n$ for TU is *holonomic* if $[X_i, X_j] = 0$, while it is a *coordinate frame* if for all $x \in U$ there exists an open coordinate neighborhood $x \in W \subset U$ with coordinates $(x_1, \dots, x_n)$ such that $X_i = \frac{\partial}{\partial x_i}$ on W.

Proposition 1.2.8. *A local frame $X_1, \dots, X_n$ is holonomic if and only if it is a coordinate frame.*

Remark 1.2.9. If a linear connection ∇ on TM admits parallel frames on U then these are all holonomic if $T\big|_U \equiv 0$, while they are all non-holonomic otherwise. Indeed let $X_1, \dots, X_n$ be a parallel frame on U. Then $\Gamma^k_{ij}\Big|_U \equiv 0$, and thus

$$T(X_i, X_j) = \Gamma^k_{ij} X_k - \Gamma^k_{ji} X_k - [X_i, X_j] = -[X_i, X_j]$$

Let (U, φ) and (V, ψ) be charts on M inducing coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ respectively. If

$$\frac{\partial^2 y_i}{\partial x_j \partial x_k} \equiv 0 \tag{1.1}$$

then the Jacobian of the coordinate change function $\psi \circ \varphi^{-1}$ is locally constant. An atlas for M whose coordinate change functions all satisfy condition 1.1 is called an *almost-affine atlas*.

Remark 1.2.10. If $U \cap V$ is connected, then a sufficient condition for the coordinate change function $\psi \circ \varphi^{-1}$ to be the restriction of an affine isomorphism is condition 1.1.

Proposition 1.2.11. *If M is a closed manifold then for all almost-affine atlas there exists a countable refinement which determines an affine structure on M.*

Thus we have that M admits an affine structure if and only if it admits an almost-affine atlas.

Theorem 1.2.12. *M admits an affine structure if and only if it admits a flat and symmetric linear connection ∇.*

Proof. Let's consider an affine structure on M. On each affine chart $(U_\alpha, \varphi_\alpha)$ inducing coordinates $(x^\alpha_1, \dots, x^\alpha_n)$ we can define a flat and symmetric linear connection by setting $\nabla^\alpha_{\partial^\alpha_i} \partial^\alpha_j = 0$, where $\partial^\alpha_i := \frac{\partial}{\partial x^\alpha_i}$. Then these local connections induce a well-defined global connection ∇, since on any overlapping of affine charts we have:

$$\nabla^\beta_{\partial^\alpha_i} \partial^\alpha_j = \partial^\alpha_i \left(\frac{\partial x^\beta_k}{\partial x^\alpha_j} \right) \partial^\beta_k + \frac{\partial x^\beta_k}{\partial x^\alpha_j} \nabla^\beta_{\partial^\alpha_i} \partial^\beta_k = \frac{\partial^2 x^\beta_k}{\partial x^\alpha_i \partial x^\alpha_j} \partial^\beta_k + \frac{\partial x^\beta_k}{\partial x^\alpha_j} \frac{\partial x^\beta_h}{\partial x^\alpha_i} \nabla^\beta_{\partial^\beta_h} \partial^\beta_k = 0 = \nabla^\alpha_{\partial^\alpha_i} \partial^\alpha_j$$

Conversely, let ∇ be a flat and symmetric linear connection on TM. Then each connected and simply connected open subset of M admits parallel frames, and these are all holonomic. Thus we get an almost-affine atlas for M: indeed if U and V are two such subsets inducing coordinates $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ respectively, set $X_i := \frac{\partial}{\partial x_i}$ and $Y_i := \frac{\partial}{\partial y_i}$. Then we have:

$$0 = \nabla_{X_i} X_j = X_i \left(\frac{\partial y_k}{\partial x_j} \right) Y_k + \frac{\partial y_k}{\partial x_j} \nabla_{X_i} Y_k = \frac{\partial^2 y_k}{\partial x_i \partial x_j} Y_k + \frac{\partial y_k}{\partial x_j} \frac{\partial y_h}{\partial x_i} \nabla_{Y_h} Y_k = \frac{\partial^2 y_k}{\partial x_i \partial x_j} Y_k$$

□

Characterization of flat bundles

Lemma 1.2.13. *A principal G-bundle $\pi_P : P \to M$ is flat if and only if it is associated with the universal cover $\pi_{\tilde{M}} : \tilde{M} \to M$ via a homomorphism $h : \pi_1(M, x_0) \to G$.*

Proof. Let Γ be a flat connection on P. Then for all curves $\gamma : [0,1] \to M$ the parallel transport $\tau_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ remains unchanged under homotopies fixing the endpoints of γ. Consider any G-equivariant diffeomorphism $g : G \to P_{x_0}$ (i.e. $g(ab) = g(a)b$) and define the map:

$$\begin{array}{rccc} h: & \pi_1(M, x_0) & \to & G \\ & \alpha & \mapsto & g^{-1}(\tau_\alpha^{-1}(g(e))) \end{array}$$

Then, since g, g^{-1} and τ_α are all G-equivariant, we have $g^{-1}(\tau_\alpha^{-1}(g(a))) = g^{-1}(\tau_\alpha^{-1}(g(e)))a$. Therefore:

$$h(\alpha\beta) = g^{-1}(\tau_{\alpha\beta}^{-1}(g(e))) = g^{-1}(\tau_\alpha^{-1}(\tau_\beta^{-1}(g(e)))) = g^{-1}(\tau_\alpha^{-1}(g(h(\beta)))) = h(\alpha)h(\beta)$$

Now fix a base point $\tilde{x}_0$ for $\tilde{M}$ lifting x_0. Consider, for all $\tilde{y} \in \tilde{M}$, a curve in $\tilde{M}$ from $\tilde{y}$ to $\tilde{x}_0$ which projects onto a curve $\gamma_{\tilde{y}}$ in M going from $\pi_{\tilde{M}}(\tilde{y})$ to x_0. Then the parallel transport $\tau_{\gamma_{\tilde{y}}}$ depends only on $\tilde{y}$, since every two choices for a curve going from $\tilde{y}$ to $\tilde{x}_0$ project onto homotopic curves. Therefore we can write $\tau_{\tilde{y}}$ for $\tau_{\gamma_{\tilde{y}}}$. Moreover, since a curve from $\tilde{y} \cdot \alpha$ to $\tilde{x}_0$ is given by the lifting of $\gamma_{\tilde{y}} * \alpha$ starting from $\tilde{y} \cdot \alpha$, we have $\tau_{\tilde{y}\cdot\alpha} = \tau_\alpha \circ \tau_{\tilde{y}}$. Thus we can define the map

$$\begin{array}{rccc} \eta : & \tilde{M} \times G & \to & P \\ & (\tilde{y}, a) & \mapsto & \tau_{\tilde{y}}^{-1}(g(a)) \end{array}$$

which satisfies $\eta(\tilde{y} \cdot \alpha, a) = \tau_{\tilde{y}}^{-1}(\tau_\alpha^{-1}(g(a))) = \tau_{\tilde{y}}^{-1}(g(h(\alpha)a)) = \eta(\tilde{y}, h(\alpha)a)$. Hence $\pi_P : P \to M$ is indeed the principal G-bundle associated with $\pi_{\tilde{M}} : \tilde{M} \to M$ via h.

Conversely, given a homomorphism $h : \pi_1(M, x_0) \to G$, let $\pi_P : P \to M$ be the associated principal G-bundle and let $p : \tilde{M} \times G \to P$ be the projection. The standard flat connection Γ_0 on $\tilde{M} \times G \to \tilde{M}$ induces a connection on P as follows: the right action of $\pi_1(M, x_0)$ onto $\tilde{M} \times G$ preserves Γ_0 so that we can define a horizontal distribution on P by setting $\Gamma := \mathrm{d}p \circ \Gamma_0$. Then, since p is G-equivariant (where G acts by right translation on itself and on the second factor of $\tilde{M} \times G$) and since Γ_0 is G-invariant, Γ is indeed G-invariant. Therefore we actually defined a connection on P. To show the flatness of Γ it suffices to note that if U is connected and trivializes $\tilde{M}$ via $\chi_{\tilde{M}} : \tilde{M}\big|_U \to U \times \pi_1(M, x_0)$ then the second coordinate of $\chi_{\tilde{M}}$ is locally constant. Thus if $\chi_P : P\big|_U \to U \times G$ is the corresponding trivialization for P then every horizontal curve γ for Γ whose image is contained in $P\big|_U$ satisfies $\chi_P(\gamma(t)) = (\pi_P(\gamma(t)), a)$ for some $a \in G$. Therefore Γ pulls back to the standard flat connection on each connected local trivialization. □

Lemma 1.2.14. *Let $\rho_0, \rho_1 : \pi_1(M, x_0) \to G$ be two representations inducing flat G-bundles $\pi_i : P_i \to M$ for $i = 0, 1$. If ρ_0 and ρ_1 lie in the same path-connected component of the space of representations $\mathrm{Rep}(\pi_1(M, x_0), G)$ then the induced bundles are isomorphic.*

Proof. If we consider a path $\gamma : \mathbb{R} \to \mathrm{Rep}(\pi_1(M, x_0), G)$ with $\gamma(0) = \rho_0$ and $\gamma(1) = \rho_1$ we can construct a rank-m vector bundle over $M \times \mathbb{R}$ using the equivalence relation defined on $\tilde{M} \times \mathbb{R} \times \mathbb{R}^m$ by $(\tilde{x}, t, \xi) \sim (\tilde{x} \cdot \alpha, t, \rho_t(\alpha) \cdot \xi)$ for all $\alpha \in \pi_1(M, x_0)$. Then clearly the inclusions $\iota_i : M \hookrightarrow M \times \{i\} \subset M \times \mathbb{R}$ for $i = 0, 1$ are homotopic, and hence they induce isomorphic pull-back bundles, which coincide with $\pi_i : P_i \to M$ for $i = 0, 1$. □

1.3 Euler class

The Euler class of an oriented rank-m vector bundle $\pi : E \to M$ is an element in the m-th cohomology group of M which generalizes the Euler characteristic. It measures the non-triviality of the bundle and coincides with the obstruction to defining a nowhere-zero section on the m-th skeleton of M.

Definition via Thom's isomorphism

For each vector space W let W_0 denote $W \smallsetminus \{0\}$. Analogously for each vector bundle $\pi : E \to M$ let E_0 denote $E \smallsetminus s_0(M)$, where s_0 is the zero section of $\pi : E \to M$.

Remark 1.3.1. For a real m-dimensional vector space W the exact sequence of the pair (W, W_0) gives an isomorphism $H^m(W, W_0; \mathbb{Z}) \simeq \mathbb{Z}$. Then the choice of an orientation for W is equivalent to the choice of a generator for $H^m(W, W_0; \mathbb{Z})$. Indeed, a linear embedding $\sigma : \triangle^m \hookrightarrow W$ such that the baricenter of $\triangle^m$ is mapped to 0 determines a generator $\alpha = [\sigma]$ for $H_m(W, W_0)$. But, since $\mathrm{Ext}(H_{m-1}(W, W_0), \mathbb{Z}) = 0$, we have $H^m(W, W_0; \mathbb{Z}) \simeq \mathrm{Hom}(H_m(W, W_0), \mathbb{Z})$ and therefore we can specify the unique generator $a \in H^m(W, W_0; \mathbb{Z})$ such that $a(\alpha) = 1$. Thus we can associate a with the orientation of W that makes σ into an orientation-preserving embedding and $-a$ with the other one.

Remark 1.3.2. For a disc neighborhood U of $x \in M$ we have that, for all $y \in U$, the inclusions $\iota_y : (E_y, (E_y)_0) \hookrightarrow (E\big|_U, (E\big|_U)_0)$ yield homotopy equivalences. Then we have isomorphisms $(\iota_y)_* : H_m(E_y, (E_y)_0) \to H_m(E\big|_U, (E\big|_U)_0)$ for all $y \in U$. Since, once again, we have an isomorphism $H^m(E\big|_U, (E\big|_U)_0; \mathbb{Z}) \simeq \mathrm{Hom}(H_m(E\big|_U, (E\big|_U)_0), \mathbb{Z})$, each cohomology class $a \in H^m(E\big|_U, (E\big|_U)_0; \mathbb{Z})$ is uniquely determined by $a((\iota_y)_*(\alpha_y)) = (\iota_y)^* a(\alpha_y)$, where α_y is a generator for $H_m(E_y, (E_y)_0)$ and y is any point in U.

Therefore an orientation of a vector bundle can be realized as a choice of a generator $a_x \in H^m(E_x, (E_x)_0; \mathbb{Z})$ for all $x \in M$ satisfying the following local compatibility condition: for each $x \in M$ there exists an open neighborhood U and a generator $a_U \in H^m(E\big|_U, (E\big|_U)_0; \mathbb{Z})$ such that, for all $y \in U$, the cohomology class $(\iota_y)^*(a_U)$ equals the preferred generator a_y for $H^m(E_y, (E_y)_0; \mathbb{Z})$.

Theorem 1.3.3 (Thom's isomorphism). *For every oriented rank-m vector bundle $\pi : E \to M$ there exists a unique orientation class $u_E \in H^m(E, E_0; \mathbb{Z})$ whose restriction under the homomorphism $(\iota_x)^* : H^m(E, E_0; \mathbb{Z}) \to H^m(E_x, (E_x)_0; \mathbb{Z})$ equals the preferred generator a_x for every $x \in M$. Furthermore the map:*

$$\begin{array}{ccc} H^k(E; \mathbb{Z}) & \to & H^{k+m}(E, E_0; \mathbb{Z}) \\ a & \mapsto & a \smile u_E \end{array}$$

is an isomorphism.

Let $\pi : E \to M$ be an oriented rank-m vector bundle. The projection π induces an isomorphism $\pi^* : H^k(M; \mathbb{Z}) \to H^k(E; \mathbb{Z})$ and the inclusion $\iota : (E, \varnothing) \to (E, E_0)$ yields a restriction homomorphism $\iota^* : H^k(E, E_0; \mathbb{Z}) \to H^k(E; \mathbb{Z})$ for every k. Then the *Euler class* of $\pi : E \to M$ is defined as the cohomology class $e(E) \in H^m(M; \mathbb{Z})$ given by $(\pi^*)^{-1}(\iota^*(u_E))$.

Properties of the Euler class

Let $\mathfrak{C}$ be a category of fiber bundles and define for each manifold M the set $b_{\mathfrak{C}}(M)$ of isomorphism classes of bundles in $\mathfrak{C}$ with base M, which coincides with the set $[M, X]$ of homotopy classes of maps from M to some classifying space X. Then $b_{\mathfrak{C}}$ can be seen as a contravariant functor from the category of smooth manifolds to the category of sets which associates with each smooth map $f : N \to M$ the function $f^* : b_{\mathfrak{C}}(M) \to b_{\mathfrak{C}}(N)$ mapping each isomorphism class to its pull-back by f. Note that for every coefficient group Λ the cohomology functor $H^*(\,\cdot\,; \Lambda)$ which assigns to each manifold M its cohomology ring $\bigoplus_{k \geqslant 0} H^k(M; \Lambda)$ can be seen as a contravariant functor from manifolds to sets (up to composing it with the forgetful functor from rings to sets). A *characteristic class* is a natural tranformation between $b_{\mathfrak{C}}$ and $H^*(\,\cdot\,; \Lambda)$, i.e. it associates each fiber bundle $\pi : E \to M$ in $\mathfrak{C}$ with a cohomology class $c(E) \in \bigoplus_{k \geqslant 0} H(M; \Lambda)$ which satisfies

$$c(f^* E) = f^*(c(E))$$

for all $f : N \to M$.

Example 1.3.4. The *Stiefel-Whitney classes* are characteristic classes with $\mathbb{Z}_2$-coefficients defined on the class of vector bundles. They can be characterized as the only characteristic classes satisfying the following list of axioms: if $\pi : E \to M$ and $\pi' : E' \to M$ are vector bundles then (*i*) $w_k(E)$ is an element of $H^k(M;\mathbb{Z}_2)$ and $w_0(E)$ is the unit element of the cohomology ring $\bigoplus_{k\geqslant 0} H^k(M;\mathbb{Z}_2)$;

(*ii*) $w_k(E\oplus E') = \sum_{i=0}^{k} w_i(E)\smile w_{k-i}(E')$;

(*iii*) if $\pi_\gamma : \gamma_1(\mathbb{R}^2)\to \mathbb{P}^1(\mathbb{R})$ is the tautological line bundle over $\mathbb{P}^1(\mathbb{R})$ then $w_1(\gamma_1(\mathbb{R}^2))\neq 0$.

The Euler class is a characteristic class with integer coefficients defined on the class of oriented vector bundles. Indeed for every pull-back

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_{f^*E}"'] & E \arrow[d, "\pi_E"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

an orientation on $\pi_E : E\to M$ induces an orientation on $\pi_{f^*E} : f^*E\to N$ obtained by specifying for every $y\in N$ the generator $a_y := \tilde{f}^*(a_{f(y)})\in H^m((f^*E)_y,((f^*E)_y)_0;\mathbb{Z})$. Then, since we have $\tilde{f}\circ\iota_{f^*E} = \iota_E\circ\tilde{f}$ and $f\circ\pi_{f^*E} = \pi_E\circ\tilde{f}$, we get $e(f^*E) = f^*(e(E))$. In particular, since every trivial bundle can be realized as the pull-back of a bundle over a point, the Euler class of trivial bundles vanishes.

Remark 1.3.5. If $\pi : \bar{E}\to M$ is $\pi : E\to M$ endowed with the opposite orientation, the orientation class $u_{\bar{E}}\in H^m(E,E_0;\mathbb{Z})$ equals $-u_E$, and therefore $e(\bar{E}) = -e(E)$.

Remark 1.3.6. If the rank m of $\pi : E\to M$ is odd the Euler class is a torsion element of order two. Indeed in this case there exist orientation preserving isomorphisms between $\pi : \bar{E}\to M$ and $\pi : E\to M$, and thus we have $e(E) = -e(E)$.

Proposition 1.3.7. *The Euler class satisfies:*

(*i*) $e(E\oplus E') = e(E)\smile e(E')$;

(*ii*) $e(E\times E') = e(E)\times e(E')$;

(*iii*) *if M is an oriented closed manifold then* $\langle e(TM),[M]\rangle = \chi(M)$.

Remark 1.3.8. Every homomorphism $\varphi : \Lambda\to\Lambda'$ of abelian groups yields a natural *change-of-coefficients homomorphism* between cohomology groups. In other words, for every space X and every $k\in\mathbb{N}$ we have a map $\varphi_* : H^k(X;\Lambda)\to H^k(X;\Lambda')$ which sends the class represented by the Λ-valued cocycle ψ to the class represented by the Λ'-valued cocycle $\varphi\circ\psi$ and which satisfies

$$\begin{tikzcd} H^k(Y;\Lambda) \arrow[r, "\varphi_*"] \arrow[d, "f^*"'] & H^k(Y;\Lambda') \arrow[d, "f^*"] \\ H^k(X;\Lambda) \arrow[r, "\varphi_*"'] & H^k(X;\Lambda') \end{tikzcd}$$

for every continuous map $f : X\to Y$.

The *real Euler class* is the characteristic class with real coefficients obtained by assigning to each oriented vector bundle $\pi : E\to M$ the image $e_{\mathbb{R}}(E)$ of $e(E)$ under the natural homomorphism $\iota_* : H^k(X;\mathbb{Z})\to H^k(X;\mathbb{R})$ induced by the inclusion of coefficients $\iota : \mathbb{Z}\hookrightarrow\mathbb{R}$. This real version of the Euler class will prove to be useful when dealing with related objects which are naturally defined with real coefficients.

Proposition 1.3.9. *If $p : \mathbb{Z}\to\mathbb{Z}_2$ denotes the projection homomorphism then for every oriented rank-m vector bundle $\pi : E\to M$ we have* $p_*(e(E)) = w_m(E)\in H^m(M;\mathbb{Z}_2)$.

The classification of oriented plane bundles

As we pointed out earlier the Euler class measures the non-triviality of an oriented real vector bundle. In general it is not a very sensitive instrument, but for oriented plane bundles over a fixed base it yields a complete classification. Indeed, oriented plane bundles over a fixed manifold M are classified by homotopy classes of maps from M to some classifying space X. Now, since oriented rank-2 real vector bundles are equivalent to rank-1 complex vector bundles, a classifying space is given by $\mathbb{P}^\infty(\mathbb{C})$ with universal bundle given by the tautological bundle $\pi_\gamma : \gamma_1(\mathbb{C}^\infty) \to \mathbb{P}^\infty(\mathbb{C})$. Incidentally $\mathbb{P}^\infty(\mathbb{C})$ is also a $K(\mathbb{Z}, 2)$-space, and therefore homotopy classes of maps from M to $\mathbb{P}^\infty(\mathbb{C})$ correspond also to cohomology classes in $H^2(M; \mathbb{Z})$.

Theorem 1.3.10. *Let $f : M \to \mathbb{P}^\infty(\mathbb{C})$ be a map. The element in $H^2(M; \mathbb{Z})$ determined by the homotopy class of f is $e(f^*\gamma_1(\mathbb{C}^\infty))$. In particular two oriented rank-2 real vector bundles over M are isomorphic if and only if their Euler classes coincide.*

Remark 1.3.11. It is useful to remark that the Euler class is essentially the unique 2-dimensional characteristic class with integer coefficients for oriented rank-2 real vector bundles. Indeed every such characterstic class can be naturally identified with an element in $H^2(\mathbb{P}^\infty(\mathbb{C}); \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$. It can be shown that the Euler class corresponds to one of the two generators of $H^2(\mathbb{P}^\infty(\mathbb{C}); \mathbb{Z})$. Therefore every characteristic class which associates each oriented plane bundle $\pi : E \to M$ with an element $c(E) \in H^2(M; \mathbb{Z})$ must satisfy $c(E) = k \cdot e(E)$ for some $k \in \mathbb{Z}$. Analogously, since $H^2(\mathbb{P}^\infty(\mathbb{C}); \mathbb{R}) \simeq \mathbb{R}$, every characteristic class with real coefficients for oriented plane bundles must be a real multiple of the real Euler class $e_\mathbb{R}$.

The Euler class as a primary obstruction

Let W be an m-dimensional $\mathbb{K}$-vector space. An *r-frame* on W is an ordered set of r linearly independent vectors of W. The *Stiefel manifold* $V_r(W)$ is the set of all r-frames on W. It can be made into a smooth manifold as follows: recall that there is a free and transitive right action of $\mathrm{GL}(m, \mathbb{K})$ onto $V_m(W)$ given by

$$\{v_1, \ldots, v_m\} \cdot a_{ij} := \{a_{j1}v_j, \ldots, a_{jm}v_j\}$$

The injective homomorphism

$$\begin{array}{rccc} \iota : & \mathrm{GL}(m-r, \mathbb{K}) & \to & \mathrm{GL}(m, \mathbb{K}) \\ & a & \mapsto & \begin{pmatrix} I_r & 0 \\ 0 & a \end{pmatrix} \end{array}$$

induces a free action of $\mathrm{GL}(m-r, \mathbb{K})$ onto $V_m(W)$, and therefore we can identify the quotient $V_m(W)/\mathrm{GL}(m-r, \mathbb{K})$ with $V_r(W)$ via the bijection $[\{v_1, \ldots, v_m\}] \mapsto \{v_1, \ldots, v_r\}$. Thus the natural identification between $V_m(W)$ and $\mathrm{GL}(m, \mathbb{K})$ endows $V_r(W)$ with the smooth structure of $\mathrm{GL}(m, \mathbb{K})/\mathrm{GL}(m-r, \mathbb{K})$.

Let $\pi_E : E \to M$ be a rank-m vector bundle. The *associated Stiefel manifold bundle* $\pi_V : V_r(E) \to M$ is the $V_r(\mathbb{R}^m)$-bundle with total space $V_r(E) := \bigsqcup_{x \in M} V_r(E_x)$ and projection given by $\pi_V(V_r(E_x)) = x$. Every local trivialization of E

$$\begin{array}{rccc} \chi^E_\alpha : & \pi_E^{-1}(U_\alpha) & \to & U_\alpha \times \mathbb{R}^m \\ & v & \mapsto & (\pi_E(v), \psi^E_\alpha(v)) \end{array}$$

induces a local trivializations of $V_r(E)$

$$\begin{array}{rccc} \chi^V_\alpha : & \pi_V^{-1}(U_\alpha) & \to & U_\alpha \times V_r(\mathbb{R}^m) \\ & u & \mapsto & (\pi_V(u), \psi^V_\alpha(u)) \end{array}$$

where ψ^V_α maps $u = \{v_1, \ldots, v_r\} \in V_r(E_x)$ to $\{\psi^E_\alpha(v_1), \ldots, \psi^E_\alpha(v_r)\} \in V_r(\mathbb{R}^m)$.

Remark 1.3.12. A section of $V_r(E)$ determines r linearly independent sections of E.

Now let us choose a base point $x_0 \in M$. If $m = 1$ the orientation of $\pi_E : E \to M$ gives a nowhere zero section which yields $E \simeq M \times \mathbb{R}$, and therefore both the Euler class $e(E)$ and the obstruction cocycle $o(V_1(E))$ vanish. If $m > 1$ then the Stiefel manifold $V_1(E_{x_0}) = (E_{x_0})_0$ is $(m-1)$-connected and m-simple. Thus we have an isomorphism $\psi : \pi_{m-1}(V_1(E_{x_0})) \to \mathbb{Z}$ given by the chain of isomophisms

$$\pi_{m-1}(V_1(E_{x_0})) \longrightarrow H_{m-1}((E_{x_0})_0) \longrightarrow H_m(E_{x_0}, (E_{x_0})_0) \longrightarrow \mathbb{Z}$$

(note that the orientation of E determines uniquely the rightmost one). Another consequence of the existence of an orientation is that the twisting homomorphism

$$\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\pi_{m-1}(V_1(E_{x_0})))$$

is trivial, and the cohomology groups with twisted coefficients $H^*(M; \pi_{m-1}(V_1(E_{x_0}))_\rho)$ are actually standard cohomology groups. Therefore ψ induces an isomorphism between cohomology groups $\psi_\# : H^*(M; \pi_{m-1}(V_1(E_{x_0}))) \to H^m(M; \mathbb{Z})$.

Theorem 1.3.13. *The Euler class $e(E) \in H^m(M; \mathbb{Z})$ is the image under $\psi_\#$ of the primary obstruction $o(V_1(E)) \in H^m(M; \pi_{m-1}(V_1(E_{x_0})))$.*

Remark 1.3.14. For an oriented plane bundle $\pi_E : E \to M$ the Euler class corresponds to the primary obstruction of the associated principal $\mathrm{GL}^+(2, \mathbb{R})$-bundle $\pi_L : L_{\mathrm{GL}^+(2,\mathbb{R})}(E) \to M$. Indeed the choice of a Riemannian metric induces a complex structure on $\pi_E : E \to M$ and we can define bundle maps

$$\begin{tikzcd} L_{\mathrm{GL}^+(2,\mathbb{R})}(E) \arrow[r, "\rho"] \arrow[d, "\pi_L"'] & V_1(E) \arrow[d, "\pi_V"] \\ M \arrow[r, "\mathrm{id}"'] & M \end{tikzcd} \qquad \begin{tikzcd} L_{\mathrm{GL}^+(2,\mathbb{R})}(E) \arrow[d, "\pi_L"'] & V_1(E) \arrow[l, "\iota"'] \arrow[d, "\pi_V"] \\ M & M \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

where ρ maps each positive frame in $L_{\mathrm{GL}^+(2,\mathbb{R})}(E)$ to its first vector, and ι maps each non-zero vector $v \in V_1(E)$ to the positive frame $\{v, i \cdot v\}$. Then the twisting homomorphism

$$\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\pi_1(L_{\mathrm{GL}^+(2,\mathbb{R})}(E)_{x_0}))$$

is trivial too and any isomorphism $\xi : \pi_1(L_{\mathrm{GL}^+(2,\mathbb{R})}(E)_{x_0}) \to \mathbb{Z}$ induces an isomorphism between cohomology groups which maps the primary obstruction of $\pi_L : L_{\mathrm{GL}^+(2,\mathbb{R})}(E) \to M$ to $\pm e(E)$ (where the sign depends on wether ξ preserves the orientation or not).

Remark 1.3.15. An obstruction theoretic interpretation can be given of Stiefel-Whitney classes too. Indeed the k-th Stiefel-Whitney class of a rank-m vector bundle $\pi : E \to M$ coincides with the modulo 2 reduction of the obstruction to defining $m - k + 1$ linearly independent sections of E over the k-th skeleton of M. More precisely, the primary obstruction for the bundle $\pi_V : V_{m-k+1}(E) \to M$ is an element of $H^k(M; \pi_{k-1}(V_{m-k+1}(E_{x_0}))_\rho)$. Now we have that $\pi_{k-1}(V_{m-k+1}(\mathbb{R}^m)) \in \{\mathbb{Z}, \mathbb{Z}_2\}$ for all $0 \leqslant k \leqslant m \in \mathbb{N}$ and we have a natural reduction homomorphism $H^k(M; \pi_{k-1}(V_{m-k+1}(E_{x_0}))_\rho) \to H^k(M; \mathbb{Z}_2)$ which sends the primary obstruction to $w_k(E)$. Moreover if $\pi_{k-1}(V_{m-k+1}(\mathbb{R}^m)) \simeq \mathbb{Z}_2$ then the reduction homomorphism is an isomorphism and $w_k(E)$ coincides with the primary obstruction. For example, since $\pi_0(V_m(\mathbb{R}^m)) \simeq \pi_0(\mathrm{GL}(m, \mathbb{R})) \simeq \mathbb{Z}_2$, a rank-$m$ vector bundle $\pi : E \to M$ admits m linearly independent sections defined on the 1-skeleton of M if and only if $w_1(E) = 0$. In particular, every vector bundle is orientable if and only if its first Stiefel-Whitney class vanishes.

Chapter 2

Chern-Gauss-Bonnet Theorem and complete affine manifolds

In 1944 Chern generalized the Gauss-Bonnet Theorem to all even dimensions. What he proved is that we can realize the Euler class of an oriented vector bundle $\pi : E \to M$ as a homogeneous polynomial in the curvature of an *orthogonal* connection on E. As we will later see, the condition that the connection should be compatible with a metric tensor is necessary, as we can actually construct flat connections on vector bundles whose Euler class is non-zero.

2.1 Pfaffian polynomial

Let $X = \{X_{ij}\}_{1 \leqslant i < j \leqslant k}$ be a set of $\frac{k(k-1)}{2}$ indeterminates. Then any polynomial $P \in \mathbb{R}[X]$ can be evaluated on all $k \times k$ skew-symmetric matrices $A \in \mathfrak{so}(k)$ in the obvious way. Such a polynomial P is $\mathrm{SO}(k)$-*invariant* if it satisfies $P(\mathrm{Ad}_b(A)) = P(bAb^t) = P(A)$ for all $A \in \mathfrak{so}(k)$ and $b \in \mathrm{SO}(k)$.

Lemma 2.1.1. *For every $k \in \mathbb{N}$ there exists one and, up to sign, only one polynomial $\mathrm{Pf} \in \mathbb{Z}[X]$ such that $\mathrm{Pf}(A)^2 = \det(A)$ for all $A \in \mathfrak{so}(k)$. It satisfies $\mathrm{Pf}(BAB^t) = \mathrm{Pf}(A)\det(B)$ for all $A \in \mathfrak{so}(k)$ and $B \in \mathfrak{gl}(k, \mathbb{R})$.*

Remark 2.1.2. Pf is $\mathrm{SO}(k)$-invariant.

Proof. If we consider a symplectic basis for $\mathbb{R}^{2n}$ with respect to the skew-symmetric bilinear form represented by A, then the coordinates of its vectors with respect to the standard basis of $\mathbb{R}^{2n}$ fit into a matrix $C(A)$ satisfying $J = C(A)AC(A)^t$ where

$$J = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

The entries of the matrix $C(A)$, which are obtained via a Gram-Schmidt-like process, are rational functions of the entries of A. Let Λ be the ring of polynomials $\mathbb{Z}[X]$ and let $Q(\Lambda)$ be its field of fractions. Let D denote the polynomial in Λ such that $D(A) = \det A$ for all $A \in \mathfrak{so}(k)$. Since $\det A(\det C(A))^2 = 1$ we have the equality $D = (\det \circ C)^{-2}$ between the polynomial $D \in \Lambda$ and the rational function $(\det \circ C)^{-2} \in Q(\Lambda)$. But Λ is a unique factorization domain, and therefore D must be a square in Λ too. Therefore there exists a polynomial $\mathrm{Pf} \in \Lambda$ such that $\mathrm{Pf}^2 = D$. Moreover, if $Y = \{Y_{ij}\}_{1 \leqslant i,j \leqslant k}$ is a set of k^2 indeterminates and if $P(X, Y) \in \Lambda[Y]$ is the polynomial satisfying $P(A, B) = \mathrm{Pf}(BAB^t)$ for all $A \in \mathfrak{so}(k)$ and $B \in \mathfrak{gl}(k, \mathbb{R})$, we have that $\det(BAB^t) = \det A(\det B)^2$ and thus $P^2(X,Y) = (\mathrm{Pf}(X)\det(Y))^2$. Therefore, since $\Lambda[Y]$ is a unique factorization domain, we must have $P(X,Y) = \pm\mathrm{Pf}(X)\det(Y)$. We actually have $P = \mathrm{Pf}\cdot\det$ as can be seen by evaluating $P(A,I)$. $\square$

Remark 2.1.3.

$$A_1 \in \mathfrak{so}(2k_1),\ A_2 \in \mathfrak{so}(2k_2),\ A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \Rightarrow \mathrm{Pf}(A) = \mathrm{Pf}(A_1)\mathrm{Pf}(A_2)$$

Indeed $\det(A) = \det(A_1)\det(A_2) = (\mathrm{Pf}(A_1)\mathrm{Pf}(A_2))^2$ and we can invoke the uniqueness of Pf. The sign can be determined by considering the case where A is the matrix J defined before.

2.2 Orthogonal connections

Let $\pi : E \to M$ be a vector bundle of rank m and let g be a metric tensor on E. A connection ∇ on E is *compatible with* g, or g*-orthogonal*, if

$$X(g(V,W)) = g(\nabla_X V, W) + g(V, \nabla_X W)$$

for all $V, W \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$.

Remark 2.2.1. A connection ∇ is g-orthogonal if and only if it is $\mathrm{O}(m)$-compatible with respect to the $\mathrm{O}(m)$-structure induced by g.

If we have a map $f : N \to M$ every metric tensor g on E can be pulled back to f^*E: indeed, since every vector in f^*E_y can be written as f^*V for $V \in E_{f(y)}$, we can define

$$(f^*g)_y(f^*V, f^*W) = g_{f(y)}(V,W)$$

Lemma 2.2.2. *If ∇ is g-orthogonal then $f^*\nabla$ is f^*g-orthogonal.*

Proof. For all $y \in N$, $Y \in T_yN$ and $V, W \in \Gamma(E)$ we have:

$$\begin{aligned}(f^*g)_y((f^*\nabla)_Y(f^*V), f^*W) + (f^*g)_y(f^*V, (f^*\nabla)_Y(f^*W)) &= g_{f(y)}(\nabla_{d_yf(Y)}V, W) + \\ &+ g_{f(y)}(V, \nabla_{d_yf(Y)}W) = d_yf(Y)(g(V,W)) = \\ &= Y(g(V,W)\circ f) = Y(f^*g(f^*V, f^*W))\end{aligned}$$

$\square$

Lemma 2.2.3. *If ∇_0, ∇_1 are g-orthogonal connections and $h \in \mathscr{C}^\infty_{\mathbb{R}}(M)$ then the connection $\nabla = (1-h)\nabla_0 + h\nabla_1$ is g-orthogonal.*

Proof.

$$\begin{aligned}g(\nabla_X V, W) + g(V, \nabla_X W) &= (1-h)\,(g((\nabla_0)_X V, W) + g(V, (\nabla_0)_X W)) + \\ &+\ h\,(g((\nabla_1)_X V, W) + g(V, (\nabla_1)_X W)) = (1-h+h)\,(X(g(V,W)))\end{aligned}$$

$\square$

Lemma 2.2.4. *∇ is g-orthogonal if and only if it the matrix of local connection forms with respect to any orthonormal local frame is skew-symmetric.*

Proof. Let $V_1, \dots, V_m$ be a local orthonormal frame over $U \subset M$ and let $\omega = (\omega_{ij})$ denote the matrix of local connection forms with respect to $V_1, \dots, V_m$. Then we have:

$$g(\nabla_X V_i, V_j) + g(V_i, \nabla_X V_j) = \omega_{ij}(X) + \omega_{ji}(X)$$

If $\omega_{ji} = -\omega_{ij}$ for all i, j, then the right-hand side is zero for all $X \in \mathfrak{X}(M)$, and therefore the left-hand side is equal to $0 = X(\delta_{ij}) = X(g(V_i, V_j))$. Conversely, if ∇ is g-orthogonal the left-hand side is zero for all $X \in \mathfrak{X}(M)$ and we conclude. $\square$

2.3 The Pfaffian of the curvature

Let $\pi : E \to M$ be an oriented rank-$2m$ vector bundle, let g be a metric tensor on E and let ∇ be a g-orthogonal connection.

Remark 2.3.1. For every commutative $\mathbb{R}$-algebra A let $\mathfrak{so}_A(k)$ denote the $\frac{k(k-1)}{2}$-dimensional vector space of $k \times k$ skew-symmetric matrices with entries in A. Then every polynomial $P \in \mathbb{R}[\{X_{ij}\}_{1\leqslant i<j\leqslant k}]$ can be evaluated on all matrices in $\mathfrak{so}_A(k)$.

Remark 2.3.2. The $\mathbb{R}$-algebra $\Omega^{2*}(U)$ of differential forms of even degree over an open subset $U \subset M$ is commutative with respect to the wedge product.

Let $\Omega^U = (\Omega^U_{ij})$ be the matrix of curvature forms of ∇ with respect to a positive local orthonormal frame $V_1, \dots, V_{2m}$ over U. Then Ω^U is skew-symmetric thanks to the structure equation $\Omega^U_{ij} = \mathrm{d}\omega^U_{ij} - \omega^U_{ir} \wedge \omega^U_{rj}$ and every polynomial $P \in \mathbb{R}[\{X_{ij}\}_{1\leqslant i<j\leqslant 2m}]$ can be evaluated on Ω^U. Moreover if P is $\mathrm{SO}(2m)$-invariant then the differential form $P(\Omega^U)$ is independent of the chosen positive local orthonormal frame. Indeed if $\tilde{V}_i = a_{ji}V_j$ is another positive local orthonormal frame with $a : U \to \mathrm{SO}(2m)$ and $X, Y \in \mathfrak{X}(M)$ we have

$$R_{X,Y}\tilde{V}_i = R_{X,Y}(a_{ki}V_k) = a_{ki}\Omega^U_{kh}(X,Y)V_h = a_{ki}\Omega^U_{kh}(X,Y)a_{hj}\tilde{V}_j$$

Therefore $\tilde{\Omega}^U_{ij} = a_{ki}\Omega^U_{kh}a_{hj}$, i.e. $\tilde{\Omega}^U = a^t\Omega^U a$. Letting U vary inside an open cover of M all these local differential forms piece together to yield a global differential form $P(\Omega) \in \Omega^{2*}(M)$.

Lemma 2.3.3. *If P is $\mathrm{SO}(2m)$-invariant then the differential form $P(\Omega)$ is closed.*

Proof. Let's consider the $2m \times 2m$ skew-symmetric matrix D whose entries D_{ij} are polynomials in $\mathbb{Z}[X_{12}, \dots, X_{2m-1\,2m}]$ given by formal partial derivatives of P in the following way:

$$D_{ij} = \begin{cases} \frac{\partial P}{\partial X_{ij}} & \text{if } i < j \\ -\frac{\partial P}{\partial X_{ji}} & \text{if } i > j \\ 0 & \text{if } i = j \end{cases}$$

Let $\mathrm{d}\Omega$ denote the matrix $(\mathrm{d}\Omega_{ij})$ of exterior derivatives of curvature forms. Then we have

$$\mathrm{d}P(\Omega) = \sum_{i<j} \frac{\partial P}{\partial X_{ij}}(\Omega) \wedge \mathrm{d}\Omega_{ij} = \frac{1}{2}\sum_{i,j} D_{ij}(\Omega) \wedge \mathrm{d}\Omega_{ij} = \frac{1}{2}\mathrm{tr}\left(D(\Omega)^t \wedge \mathrm{d}\Omega\right)$$

Applying the exterior derivative to the equality

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega$$

we get

$$\mathrm{d}\Omega = \omega \wedge \Omega - \Omega \wedge \omega$$

Moreover, for every skew-symmetric matrix A we have $AD(A)^t = D(A)^t A$. Indeed if we differentiate the equality $P((I + t(E_{ij} - E_{ji}))A(I + t(E_{ji} - E_{ij}))) = P(A)(1 + t^2)$ and evaluate it for $t = 0$ we get

$$\frac{\partial P}{\partial X_{ik}}(A)A_{jk} - \frac{\partial P}{\partial X_{jk}}(A)A_{ik} + \frac{\partial P}{\partial X_{hi}}(A)A_{hj} - \frac{\partial P}{\partial X_{hj}}(A)A_{hi} = 0$$

and thus

$$2(D_{hi}(A)A_{hj} - A_{ik}D_{jk}(A)) = 0$$

Therefore we get:

$$\begin{aligned} 2\mathrm{d}P(\Omega) &= \mathrm{tr}\left(D(\Omega)^t \wedge \mathrm{d}\Omega\right) = \mathrm{tr}\left(D(\Omega)^t \wedge \omega \wedge \Omega - D(\Omega)^t \wedge \Omega \wedge \omega\right) = \\ &= \mathrm{tr}\left((D(\Omega)^t \wedge \omega) \wedge \Omega - \Omega \wedge (D(\Omega)^t \wedge \omega)\right) = \\ &= (D(\Omega)^t \wedge \omega)_{ij} \wedge \Omega_{ji} - \Omega_{ij} \wedge (D(\Omega)^t \wedge \omega)_{ji} \end{aligned}$$

and, since Ω_{ji} commutes with all differential forms, we conclude. $\square$

Thus for any oriented rank-$2m$ vector bundle $\pi_E : E \to M$ the choice of a metric tensor g and of a g-orthogonal connection determines a well-defined de Rham cohomology class given by $[\mathrm{Pf}(\Omega)] \in H^{2m}_{\mathrm{dR}}(M)$. Every map $f : N \to M$ induces an orientation and an f^*g-orthogonal connection $f^*\nabla$ on $\pi_{f^*E} : f^*E \to N$. Every positive g-orthonormal local frame for E induces a positive f^*g-orthonormal local frame for f^*E whose associated matrix of local curvature forms is $f^*\Omega$. Therefore we have the equality $[\mathrm{Pf}(f^*\Omega)] = f^*[\mathrm{Pf}(\Omega)]$.

Proposition 2.3.4. *The cohomology class* $[\mathrm{Pf}(\Omega)]$ *is independent of the g-orthogonal connection* ∇.

Proof. Let ∇_0 and ∇_1 be g-orthogonal connections over M and let us consider the product $M \times \mathbb{R}$. Thanks to Lemma 2.2.2 the projection onto the first factor $\pi_1 : M \times \mathbb{R} \to M$ induces two π_1^*g-orthogonal connections $\pi_1^*\nabla_0$ and $\pi_1^*\nabla_1$ over π_1^*E. Thus we can define the connection $\nabla = (1 - \pi_2) \cdot (\pi_1^*\nabla_0) + \pi_2 \cdot (\pi_1^*\nabla_1)$, which is π_1^*g-orthogonal thanks to Lemma 2.2.3. Then the inclusions $\iota_\varepsilon : M \hookrightarrow M \times \mathbb{R}$ given by $x \mapsto (x, \varepsilon)$ satisfy $\iota_0^*\nabla = \nabla_0$ and $\iota_1^*\nabla = \nabla_1$. Therefore $\iota_0^*[\mathrm{Pf}(\Omega_\nabla)] = [\mathrm{Pf}(\Omega_{\nabla_0})]$ and $\iota_1^*[\mathrm{Pf}(\Omega_\nabla)] = [\mathrm{Pf}(\Omega_{\nabla_1})]$. Hence, since maps ι_0 and ι_1 are homotopic, we conclude. □

Remark 2.3.5. The above discussion is a special case of a more general construction. Indeed it can be shown that $\mathbb{R}[\{X_{ij}\}_{1\leqslant i<j\leqslant k}]$ is isomorphic to the algebra $S^*(\mathfrak{so}(k)^*) := \bigoplus_{h\geqslant 0} S^h(\mathfrak{so}(k)^*)$ where $S^h(\mathfrak{so}(k)^*)$ is the vector space of h-linear symmetric functions from the h-fold product $\mathfrak{so}(k) \times \ldots \times \mathfrak{so}(k)$ to $\mathbb{R}$. The isomorphism associates every function $f \in S^h(\mathfrak{so}(k)^*)$ with the homogeneous polynomial P of degree h satisfying $P(A) = f(A, \ldots, A)$ for all $A \in \mathfrak{so}(k)$. Under this isomorphism the vector space of $\mathrm{SO}(k)$-invariant polynomials corresponds to the subspace $I(\mathrm{SO}(k)) = \bigoplus_{h\geqslant 0} I^h(\mathrm{SO}(k))$ where every $I^h(\mathrm{SO}(k)) \subset S^h(\mathfrak{so}(k)^*)$ is the subspace given by those elements which satisfy $f(\mathrm{Ad}_a(A_1), \ldots, \mathrm{Ad}_a(A_h)) = f(A_1, \ldots, A_h)$ for all $a \in \mathrm{SO}(k)$ and all $A_1, \ldots, A_h \in \mathfrak{so}(k)$. Therefore we have that every $f \in I^h(\mathrm{SO}(k))$ defines an element in $H^{2h}_{\mathrm{dR}}(M)$ given by $[f(\Omega, \ldots, \Omega)]$. More in general for any linear Lie group $G < \mathrm{GL}(m, \mathbb{R})$ with Lie algebra $\mathfrak{g}$ let $S^h\mathfrak{g}^*$ denote the vector space of h-linear symmetric functions from the h-fold product $\mathfrak{g} \times \ldots \times \mathfrak{g}$ to $\mathbb{R}$ and let $I^h(G) \subset S^h\mathfrak{g}^*$ denote the subspace of $\mathrm{Ad}(G)$-invariant forms, i.e. the subspace whose elements satisfy $f(\mathrm{Ad}_a(A_1), \ldots, \mathrm{Ad}_a(A_h)) = f(A_1, \ldots, A_h)$ for all $a \in G$ and all $A_1, \ldots, A_h \in \mathfrak{g}$. Now if $\pi : E \to M$ is a rank-m vector bundle with structure group G let ∇ be a G-compatible connection. If Ω^U is the matrix of local curvature forms with respect to a local G-frame (that is a local frame defining a local section of the principal G-bundle associated with $\pi : E \to M$) then every element $f \in I^h(G)$ gives well defined local differential forms $f(\Omega^U, \ldots, \Omega^U)$. These local forms piece together to yield a global differential form $f(\Omega, \ldots, \Omega)$ which is closed. Therefore we can define the *Chern-Weil homomorphism*

$$\begin{array}{rccc} w_E : & I^h(G) & \to & H^{2h}_{\mathrm{dR}}(M) \\ & f & \mapsto & [f(\Omega, \ldots, \Omega)] \end{array}$$

It can be shown that w_E does not depend on the chosen G-compatible connection ∇ and that for every map $h : N \to M$ we have the equality $w_{h^*E} = h^* \circ w_E$

Now, going back to the Pfaffian polynomial, we will show that $[\mathrm{Pf}(\Omega)]$ does not depend on the choice of the metric tensor g either, and therefore every $\mathrm{SO}(2m)$-structure on $\pi : E \to M$ induces the same class in $H^{2m}_{\mathrm{dR}}(M)$.

Remark 2.3.6. If g_0 and g_1 are metric tensors over E and $h \in \mathscr{C}^\infty_{\mathbb{R}}(M)$ is $[0, 1]$-valued then $(1 - h)g_0 + hg_1$ is a metric tensor.

Proposition 2.3.7. *The cohomology class* $[\mathrm{Pf}(\Omega)]$ *is independent of the metric tensor* g.

Proof. Let g_0 and g_1 be metric tensors over E and let us consider a function $\varphi \in \mathscr{C}^\infty_{\mathbb{R}}(\mathbb{R})$ such that $\varphi(t) = 0$ for $t \leqslant 0$ and $\varphi(t) = 1$ for $t \geqslant 1$. If we pull back g_0 and g_1 by $\pi_1 : M \times \mathbb{R} \to M$

we can define the metric tensor $g = (1 - (\varphi \circ \pi_2))g_0 + (\varphi \circ \pi_2)g_1$ over $\pi_1^* E$. If we choose a g-orthogonal connection over $\pi_1^* E$ then $\iota_0^* \nabla$ will be g_0-orthogonal and $\iota_1^* \nabla$ will be g_1-orthogonal on E. Therefore we can conclude as before. □

Thus we actually defined a characteristic class with $\mathbb{R}$ coefficients for oriented vector bundles of even rank. Of course its square, which is

$$[\mathrm{Pf}(\Omega)] \smile [\mathrm{Pf}(\Omega)] = [\mathrm{Pf}(\Omega) \wedge \mathrm{Pf}(\Omega)] = [\det(\Omega)],$$

defines a characteristic class too.

Remark 2.3.8. It is easily verified that both these characteristic classes satisfy the Whitney sum formula $c(E \oplus E') = c(E) \smile c(E')$. Indeed this can be seen by choosing a Riemannian metric and an orthogonal connection on both bundles separately and then considering the direct sum of these objects.

2.4 Chern-Gauss-Bonnet Theorem

Since the Pfaffian of the curvature restricts to a characteristic class with real coefficients on the category of oriented plane bundles over closed manifolds, Remark 1.3.11 implies that it must be a real multiple of the real Euler class.

Theorem 2.4.1 (Gauss-Bonnet Theorem)**.** *Let $\pi : E \to M$ be an oriented rank-2 vector bundle over a closed manifold endowed with a metric tensor g and a g-orthogonal connection ∇. Then $[\mathrm{Pf}(\Omega)] = -2\pi e_{\mathbb{R}}(E)$.*

Proof. Since we must have $[\mathrm{Pf}(\Omega)] = \lambda \cdot e_{\mathbb{R}}(E)$, in order to evaluate λ it suffices to compute $\langle [\mathrm{Pf}(\Omega)], [\Sigma] \rangle$ and $\langle e(E), [\Sigma] \rangle$ for a particular oriented plane bundle $\pi : E \to \Sigma$ over a closed oriented surface Σ. Let's consider the tangent bundle $\pi : TS^2 \to S^2$ endowed with the Riemannian metric inherited from the standard embedding $\iota : S^2 \hookrightarrow \mathbb{R}^3$ and with the associated Levi-Civita connection ∇. Then $\mathrm{Pf}(\Omega) = \Omega_{12} = -K\nu_g$ where K is the Gaussian curvature function of S^2 and ν_g is the oriented volume form. Therefore $\langle [\mathrm{Pf}(\Omega)], [S^2] \rangle = \int_{S^2} \mathrm{Pf}(\Omega) = -\int_{S^2} K\nu_g = -4\pi$, while $\langle e(TS^2), [S^2] \rangle = \chi(S^2) = 2$. Thus we established $\lambda = -2\pi$. □

Theorem 2.4.2. *Let $\pi : E \to M$ be an oriented rank-$2m$ vector bundle. Then*

$$[\det(\Omega)] = (2\pi)^{2m}(e_{\mathbb{R}}(E) \smile e_{\mathbb{R}}(E))$$

Proof. If $m = 1$ then the Gauss-Bonnet Theorem gives

$$(2\pi)^2(e_{\mathbb{R}}(E) \smile e_{\mathbb{R}}(E)) = [\mathrm{Pf}(\Omega)] \smile [\mathrm{Pf}(\Omega)] = [\det(\Omega)]$$

Thus for a bundle of the form $E_1 \oplus \ldots \oplus E_m$ where E_j is an oriented rank-2 vector bundle we can choose a metric tensor g of the form $g_1 \oplus \ldots \oplus g_m$ and a g-orthogonal connection ∇ of the form $\nabla_1 \oplus \ldots \oplus \nabla_m$ where each ∇_j is g_j-orthogonal. Therefore a local orthonormal frame obtained as the ordered disjoint union of local orthonormal frames for the metric tensors g_j gives

$$[\det(\Omega)] = [\det(\Omega_1) \wedge \ldots \wedge \det(\Omega_m)] = (2\pi)^{2m}\left(e_{\mathbb{R}}(E_1)^2 \smile \ldots \smile e_{\mathbb{R}}(E_m)^2\right) = (2\pi)^{2m} e_{\mathbb{R}}(E)^2$$

For a general oriented rank-$2m$ vector bundle $\pi : E \to M$ with orthogonal connection ∇ we use the fact that the complexification $E_{\mathbb{C}}$ is isomorphic, as an oriented rank-$4m$ real vector bundle, to $E \oplus E$, and therefore $e(E)^2 = e(E_{\mathbb{C}})$. Thus for the splitting principle we have that, if $\pi_F : F(E_{\mathbb{C}}) \to M$ is the flag manifold bundle associated with $E_{\mathbb{C}}$, the pull-back bundle $\pi_F^* E_{\mathbb{C}} \to F(E_{\mathbb{C}})$ splits as a Whitney sum of complex line bundles. Therefore, if $\nabla_{\mathbb{C}} = \nabla \oplus \nabla$ is an orthogonal connection on $E_{\mathbb{C}}$, we have $(2\pi)^{4m} e_{\mathbb{R}}(\pi_F^* E_{\mathbb{C}})^2 = [\det(\pi_F^* \Omega_{\mathbb{C}})] = \pi_F^*([\det(\Omega_{\mathbb{C}})])$.

Hence the injectivity of $\pi_F^* : H(M;\mathbb{R}) \to H(F(E_{\mathbb{C}});\mathbb{R})$ gives $(2\pi)^{4m} e_{\mathbb{R}}(E_{\mathbb{C}})^2 = [\det(\Omega_{\mathbb{C}})]$, and thus $\pm(2\pi)^{2m} e_{\mathbb{R}}(E)^2 = [\det(\Omega)]$. To determine the sign let us consider, alongside a completely arbitrary oriented rank-$2m$ vector bundle, a second oriented rank-$2m$ vector bundle which splits as an m-fold Whitney sum of complex line bundles and has non-trivial Euler class. Since there exists a sufficiently large N such that both of them admit a characteristic map into $\tilde{G}_{2m}(\mathbb{R}^N)$ the sign must coincide. Therefore, since for split bundles it was proved to be $+1$, we conclude. □

We can finally prove:

Theorem 2.4.3 (Chern-Gauss-Bonnet Theorem)**.** *Let $\pi : E \to M$ be an oriented rank-$2m$ vector bundle over a closed manifold endowed with a metric tensor g and a g-orthogonal connection ∇. Then $[\mathrm{Pf}(\Omega)] = (-2\pi)^m e_{\mathbb{R}}(E)$.*

Proof. We are only left to check the sign, which can be done exactly as before. Indeed for complex line bundles we have the standard Gauss-Bonnet Theorem, for rank-$2m$ split bundles the sign is $(-1)^m$ and for general rank-$2m$ bundles the sign must be the same because we can simultaneoulsy realize any of them and a split bundle with non trivial Euler class as pull-backs of the same bundle. □

Chapter 3

Milnor-Wood inequality for surfaces

We present now a complete solution, due to Milnor, to the problem of determining which oriented rank-2 vector bundles over oriented closed surfaces support flat connections. As it was already announced we will see that the Euler class of an oriented vector bundle cannot be derived from arbitrary connections.

Remark 3.0.4. We shall suppose the genus g of the surface to be greater than zero. Indeed a principal bundle on a sphere admits a flat connection if and only if it is trivial because S^2 is simply connected.

3.1 Primary obstructions of flat principal bundles

Let Σ_g be an oriented closed surface of genus g. Let's consider a CW-complex structure for Σ_g featuring:

(i) a unique 0-cell x_0 which serves as a base point for $\pi_1(\Sigma_g, x_0)$;

(ii) $2g$ 1-cells parametrized by curves $\alpha_j : I \to \Sigma_g$ which determine a set of generators for $\pi_1(\Sigma_g, x_0)$ whose only relation is given by the curve

$$\alpha = \alpha_1 * \alpha_2 * \alpha_1^{-1} * \alpha_2^{-1} * \ldots * \alpha_{2g-1} * \alpha_{2g} * \alpha_{2g-1}^{-1} * \alpha_{2g}^{-1};$$

(iii) a unique 2-cell ε whose attaching map is

$$\begin{array}{rccc} \varphi : & \partial D^2 \subset \mathbb{C} & \to & \Sigma_g \\ & \cos(2\pi t) + i \cdot \sin(2\pi t) & \mapsto & \alpha(t) \end{array}$$

The universal cover $\tilde{\Sigma}_g$ inherits from Σ_g a CW-complex structure and if we specify a base point $\tilde{x}_0 \in \tilde{\Sigma}_g$ in the fiber of x_0 then we induce a free right action of $\pi_1(\Sigma_g, x_0)$ onto $\tilde{\Sigma}_g$ which permutes the cells of the same dimension. If we denote by $\tilde{\alpha}$ the unique lifting of α starting from $\tilde{x}_0$ and if we denote by $\tilde{\varepsilon}$ the unique 2-cell of $\tilde{\Sigma}_g$ whose attaching map is given by

$$\begin{array}{rccc} \tilde{\varphi} : & \partial D^2 \subset \mathbb{C} & \to & \tilde{\Sigma}_g \\ & \cos(2\pi t) + i \cdot \sin(2\pi t) & \mapsto & \tilde{\alpha}(t) \end{array}$$

then all other 2-cells of $\tilde{\Sigma}_g$ have the form $\tilde{\varepsilon} \cdot \beta$ for some $\beta \in \pi_1(\Sigma_g, x_0)$.

Now with every principal G-bundle $\pi_P : P \to \Sigma_g$ we can associate a twisting homomorphism $\rho : \pi_1(\Sigma_g, x_0) \to \mathrm{Aut}(\pi_1(P_{x_0}))$ which induces cohomology groups with twisted coefficients $H^*(\Sigma_g; \pi_1(P_{x_0})_\rho)$. The primary obstruction for $\pi_P : P \to \Sigma_g$ is then an element $o(P) \in H^2(\Sigma_g; \pi_1(P_{x_0})_\rho)$ represented by a cellular cocycle $o(f)$ which maps the 2-cells of $\tilde{\Sigma}_g$ to $\pi_1(P_{x_0})$ and satisfies $o(f)(\tilde{\varepsilon} \cdot \beta) = \rho(\beta^{-1})(o(f)(\tilde{\varepsilon}))$ for all $\beta \in \pi_1(\Sigma_g, x_0)$

If $\pi_P : P \to \Sigma_g$ is flat then it is induced by a homomorphism $h : \pi_1(\Sigma_g, x_0) \to G$. For such a homomorphism, set $a_j := h([\alpha_j]) \in G$. If we consider the short exact sequence of groups

$$\{e\} \longrightarrow \pi_1(G) \xrightarrow{\iota_G} \tilde{G} \xrightarrow{\pi_{\tilde{G}}} G \longrightarrow \{e\}$$

we can choose elements $A_j \in \tilde{G}$ such that $\pi_{\tilde{G}}(A_j) = a_j$. Then, since

$$\pi_{\tilde{G}}([A_1, A_2] \cdots [A_{2g-1}, A_{2g}]) = e,$$

where $[\,\cdot\,,\,\cdot\,]$ denotes the commutator, we get an element $\iota_G^{-1}([A_1, A_2] \cdots [A_{2g-1}, A_{2g}]) \in \pi_1(G)$. This element is represented by the projection under $\pi_{\tilde{G}}$ of any curve in $\tilde{G}$ starting from $[A_1, A_2] \cdots [A_{2g-1}, A_{2g}]$ and ending in $\tilde{e}$.

Denote by $\varphi_{\tilde{x}_0}$ the diffeomorphism between G and P_{x_0} given by

$$\begin{tikzcd} G \arrow[r, hook, "\iota_{\tilde{x}_0}"'] \arrow[rr, bend left, "\varphi_{\tilde{x}_0}"] & \tilde{\Sigma}_g \times G \arrow[r, "p"'] & P \end{tikzcd}$$

where $\iota_{\tilde{x}_0}(a) = (\tilde{x}_0, a)$ for all $a \in G$ and $p : \tilde{\Sigma}_g \times G \to P$ is the standard projection.

Lemma 3.1.1. *The primary obstruction for the flat principal G-bundle $\pi_P : P \to \Sigma_g$ is represented by the cocycle*

$$o(f) : \tilde{\varepsilon} \mapsto (\varphi_{\tilde{x}_0})_*(-\iota_G^{-1}([A_1, A_2] \cdots [A_{2g-1}, A_{2g}])) \in \pi_1(P_{x_0})$$

Proof. Let $\tilde{\alpha} : I \to \tilde{M}$ be the closed loop obtained by lifting the curve α starting from $\tilde{x}_0$. Let $\tilde{e}$ be the identity in $\tilde{G}$ and consider curves in $\tilde{G}$ from $\tilde{e}$ to A_j which project onto curves $\lambda_j : I \to G$ from e to a_j. If we denote by $\tilde{\alpha}_j$ the unique lifting of α_j starting from $\tilde{x}_0$ we can define a section f on the 1-skeleton of Σ_g given by:

$$f : \alpha_j(t) \mapsto p(\tilde{\alpha}_j(t), \lambda_j(t)) \in P$$

Therefore, if we define the curve

$$\lambda := (\lambda_1 * (L_{a_1} \circ \lambda_2) * (L_{a_1 a_2} \circ \lambda_1^{-1}) * (L_{a_1 a_2 a_1^{-1}} \circ \lambda_2^{-1}) * \ldots * (L_{a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1}} \circ \lambda_{2g}^{-1}))$$

where L_a denotes the left translation by $a \in G$, the section f induces a map of the form:

$$\begin{array}{cccc} f^{\#} : & \partial D^2 & \to & P \\ & \cos(2\pi t) + i \cdot \sin(2\pi t) & \mapsto & p(\tilde{\alpha}(t), \lambda(t)) \end{array}$$

If $\Phi : D^2 \to \Sigma_g$ and $\tilde{\Phi} : D^2 \to \tilde{\Sigma}_g$ denote the characteristic maps of the 2-cells ε and $\tilde{\varepsilon}$ respectively then the pull-back bundle $\Phi^* P$ can be trivialized via the isomorphism

$$\begin{array}{cccc} \Psi : & \Phi^* P \subset D^2 \times P & \to & D^2 \times P_{x_0} \\ & (y, p(\tilde{\Phi}(y), a)) & \mapsto & (y, \varphi_{\tilde{x}_0}(a)) \end{array}$$

Therefore the primary obstruction cocycle $o(f)$ assigns to the 2-cell $\tilde{\varepsilon}$ the homotopy class of the map

$$\begin{array}{cccc} f_{\tilde{\varepsilon}} : & \partial D^2 & \to & P_{x_0} \\ & \cos(2\pi t) + i \cdot \sin(2\pi t)) & \mapsto & \varphi_{\tilde{x}_0}(\lambda(t)) \end{array}$$

In other words $o(f)(\tilde{\varepsilon})$ is the image under $(\varphi_{\tilde{x}_0})_* : \pi_1(G) \to \pi_1(P_{x_0})$ of $[\lambda] \in \pi_1(G)$. Now since

$$\iota_G([\lambda]^{-1}) = [A_1, A_2] \cdots [A_{2g-1}, A_{2g}]$$

we conclude. $\square$

3.2 Real polar matrix decomposition

Every $a \in \mathrm{GL}(n,\mathbb{R})$ can be written uniquely in the form $a = r(a)s(a)$ where $r(a) \in \mathrm{O}(n)$ and $s(a)$ is symmetric and positive definite. Indeed $a^t a$ is symmetric and positive-definite, and therefore the spectral theorem gives $a^t a = bdb^t$ where $b \in \mathrm{O}(n)$ and d is diagonal with strictly positive eigenvalues. Thus if we define $r(a) := ab\sqrt{d}^{-1}b^t$ and $s(a) := b\sqrt{d}b^t$ we have that clearly $s(a)$ is symmetric and positive definite and

$$r(a)r(a)^t = ab\sqrt{d}^{-1}b^t b\sqrt{d}^{-1}b^t a^t = abd^{-1}b^t a^t = a(a^t a)^{-1}a^t = I.$$

The decomposition is unique since, if $a = \rho\sigma$, then $s(a)^2 = a^t a = \sigma^t\rho^t\rho\sigma = \sigma^2$ and since both $s(a)$ and σ are positive definite we have $\sigma = s(a)$. The map r defines a retraction of $\mathrm{GL}^+(n,\mathbb{R})$ onto $\mathrm{SO}(n)$ which is covered by a retraction $\tilde{r} : \tilde{\mathrm{GL}}^+(n,\mathbb{R}) \to \tilde{\mathrm{SO}}(n)$ such that $\tilde{r}(\tilde{I}) = \tilde{I}$. These maps fit into the commutative diagram

$$\begin{array}{ccc} \tilde{\mathrm{GL}}^+(n,\mathbb{R}) & \xrightarrow{\tilde{r}} & \tilde{\mathrm{SO}}(n) \\ {\scriptstyle \pi_{\tilde{\mathrm{GL}}^+}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\tilde{\mathrm{SO}}}} \\ \mathrm{GL}^+(n,\mathbb{R}) & \xrightarrow[r]{} & \mathrm{SO}(n) \end{array}$$

Lemma 3.2.1. *The retractions r and $\tilde{r}$ satisfy:*

(i) $r(a^{-1}) = r(a)^{-1}$ *for all* $a \in \mathrm{GL}^+(n,\mathbb{R})$*;*

(ii) $a \in \mathrm{SO}(n) \Rightarrow r(ab) = ar(b),\ r(ba) = r(b)a$*;*

(iii) $\tilde{r}(A^{-1}) = \tilde{r}(A)^{-1}$ *for all* $A \in \tilde{\mathrm{GL}}^+(n,\mathbb{R})$*;*

(iv) $A \in \tilde{\mathrm{SO}}(n) \Rightarrow \tilde{r}(AB) = A\tilde{r}(B),\ \tilde{r}(BA) = \tilde{r}(B)A.$

Proof. *(i)* $a^{-1} = s(a)^{-1}r(a)^t = r(a)^t(r(a)s(a)^{-1}r(a)^t)$ and $r(a)s(a)^{-1}r(a)^t$ is clearly symmetric and positive-definite.

(ii) The first case is obvious, while for the second case we have $r(b)s(b)a = r(b)a(a^t s(b)a)$ and again $(a^t s(b)a)$ is symmetric and positive-definite.

(iii) Consider a curve $\tilde{\gamma}$ in $\tilde{\mathrm{GL}}^+(n,\mathbb{R})$ from $\tilde{I}$ to A which projects onto a curve γ in $\mathrm{GL}^+(n,\mathbb{R})$. Then $\tilde{r}\circ\tilde{\gamma}^{-1}$ and $(\tilde{r}\circ\tilde{\gamma})^{-1}$ project onto the curves $r\circ\gamma^{-1}$ and $(r\circ\gamma)^{-1}$ in $\mathrm{SO}(n)$ (where the inversion denotes the group operation). These curves coincide thanks to (i). But since both $\tilde{r}\circ\tilde{\gamma}^{-1}$ and $(\tilde{r}\circ\tilde{\gamma})^{-1}$ start from $\tilde{I}$ they must be the same lifting.

(iv) Consider a curve $\tilde{\gamma}$ in $\tilde{\mathrm{GL}}^+(n,\mathbb{R})$ from $\tilde{I}$ to B which projects onto a curve γ in $\mathrm{GL}^+(n,\mathbb{R})$. If $A \in \tilde{\mathrm{SO}}(n)$ and $a = \pi_{\tilde{\mathrm{SO}}}(A)$ then the curves $\tilde{r}\circ L_A\circ\tilde{\gamma}$ and $L_A\circ\tilde{r}\circ\tilde{\gamma}$ project onto the curves $r\circ L_a\circ\gamma$ and $L_a\circ r\circ\gamma$ (where the maps L_* denote left translations). These curves coincide thanks to (ii). But since both $\tilde{r}\circ L_A\circ\tilde{\gamma}$ and $L_A\circ\tilde{r}\circ\tilde{\gamma}$ start from A they must be the same lifting. The second assertion is completely analogous.

□

Lemma 3.2.2. *If $n = 2$ then:*

(i) $\mathrm{tr}(a) > 0 \Rightarrow \mathrm{tr}(r(a)) > 0$ *for all* $a \in \mathrm{GL}^+(2,\mathbb{R})$*;*

(ii) a, b *symmetric and positive-definte* $\Rightarrow \mathrm{tr}(r(ab)) > 0$*;*

(iii) $\mathrm{tr}(r(ab)r(a)^t r(b)^t) > 0$ *for all* $a, b \in \mathrm{GL}^+(2,\mathbb{R})$*.*

Proof. (*i*) By direct computation we have

$$r\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

where $x = a + d$ and $y = b - c$.

(*ii*) For every orthogonal matrix c the diagonal entries of cbc^t are positive because they equal $e_i^t(cbc^t)e_i = (ce_i)^t b(ce_i) > 0$. Therefore if c is chosen so that cac^t is a diagonal matrix then $\mathrm{tr}(ab) = \mathrm{tr}((cac^t)(cbc^t))$ is a sum of positive terms. Now we can apply (*i*).

(*iii*) $ab = (r(a)r(b))\left(r(b)^t s(a)r(b)s(b)\right)$. Since the first term is orthogonal, we have

$$r(ab) = (r(a)r(b))\, r\left(r(b)^t s(a)r(b)s(b)\right)$$

But $r(b)^t s(a)r(b)$ and $s(b)$ are both symmetric and positive-definite, and therefore (*ii*) gives $\mathrm{tr}(r(b)^t r(a)^t r(ab)) > 0$. Now the invariance of the trace under cyclic permutations allows us to conclude.

□

The isomorphism

$$\begin{aligned} \exp : \quad S^1 &\to \mathrm{SO}(2) \\ \alpha &\mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \end{aligned}$$

is covered by an isomorphism $\widetilde{\exp} : \mathbb{R} \to \widetilde{\mathrm{SO}}(2)$. The map $\vartheta := \widetilde{\exp}^{-1} \circ \tilde{r} : \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathbb{R}$, which is not a homomorphism, fits into the following commutative diagram:

$$\begin{tikzcd}
& \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \arrow[dl, "\vartheta"'] \arrow[dr, "\tilde{r}"] & \\
\mathbb{R} \arrow[rr, "\widetilde{\exp}"] \arrow[d, "\pi_{\mathbb{R}}"'] & & \widetilde{\mathrm{SO}}(2) \arrow[d, "\pi_{\widetilde{\mathrm{SO}}}"] \\
S^1 \arrow[rr, "\exp"'] & & \mathrm{SO}(2)
\end{tikzcd}$$

Lemma 3.2.3. *$|\vartheta(AB) - \vartheta(A) - \vartheta(B)| < \frac{\pi}{2}$ for all $A, B \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.*

Proof. Given elements $A, B \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ we define

$$\triangle(A,B) = \vartheta(AB) - \vartheta(A) - \vartheta(B)$$

Then $\pi_{\widetilde{\mathrm{SO}}}(\widetilde{\exp}(\triangle(A,B))) = r(ab)r(a)^{-1}r(b)^{-1}$ has positive trace thanks to (*iii*) in Lemma 3.2.2. Since

$$\pi_{\widetilde{\mathrm{SO}}} \circ \widetilde{\exp} \circ \triangle = \exp \circ \pi_{\mathbb{R}} \circ \triangle = \begin{pmatrix} \cos\circ\triangle & -\sin\circ\triangle \\ \sin\circ\triangle & \cos\circ\triangle \end{pmatrix}$$

we must have $\cos\circ\triangle > 0$. But since $\triangle$ is continuous and vanishes for $A = \tilde{I}$, we must have $-\frac{\pi}{2} < \triangle < \frac{\pi}{2}$. □

3.3 Milnor-Wood inequality

Let $\pi : E \to \Sigma_g$ be an oriented rank-2 vector bundle over an oriented closed surface of genus g.

Theorem 3.3.1. *If $|\langle e(E), [\Sigma_g]\rangle| \geqslant g$ then E does not admit flat connections.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\{e\} \arrow[r] & \pi_1(\mathrm{GL}^+(2,\mathbb{R})) \arrow[r, "\iota_{\mathrm{GL}^+}"] \arrow[d, "r_*"] & \tilde{\mathrm{GL}}^+(2,\mathbb{R}) \arrow[r, "\pi_{\tilde{\mathrm{GL}}^+}"] \arrow[d, "\tilde{r}"] & \mathrm{GL}^+(2,\mathbb{R}) \arrow[r] \arrow[d, "r"] & \{e\} \\
\{e\} \arrow[r] & \pi_1(\mathrm{SO}(2)) \arrow[r, "\iota_{\mathrm{SO}}"] & \tilde{\mathrm{SO}}(2) \arrow[r, "\pi_{\tilde{\mathrm{SO}}}"] & \mathrm{SO}(2) \arrow[r] & \{e\} \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\iota_{S^1}"] \arrow[u, "\exp_*"'] & \mathbb{R} \arrow[r, "\pi_{\mathbb{R}}"] \arrow[u, "\tilde{\exp}"'] & S^1 \arrow[r] \arrow[u, "\exp"'] & 0
\end{tikzcd}$$

Since $\pi_{\mathbb{R}} \circ \vartheta \circ \iota_{\mathrm{GL}^+} = \exp^{-1} \circ r \circ \pi_{\tilde{\mathrm{GL}}^+} \circ \iota_{\mathrm{GL}^+} = 0$ then im $(\vartheta \circ \iota_{\mathrm{GL}^+}) \subset$ im ι_{S^1}. Therefore we can define $\psi = \iota_{S^1}^{-1} \circ \vartheta \circ \iota_{\mathrm{GL}^+} : \pi_1(\mathrm{GL}^+(2,R)) \to \mathbb{Z}$, which is an isomorphism since it equals $(\exp_*)^{-1} \circ r_*$. Now suppose $\pi : E \to \Sigma_g$ admits a flat connection, let $P = L_{\mathrm{GL}^+(2,\mathbb{R})}(E)$ be the total space of the associated oriented frame bundle and consider the diffeomorphism $\varphi_{\tilde{x}_0} : \mathrm{GL}^+(2,\mathbb{R}) \to P_{x_0}$ given by

$$\begin{tikzcd}
\mathrm{GL}^+(2,\mathbb{R}) \arrow[r, hook, "\iota_{\tilde{x}_0}"'] \arrow[rr, bend left, "\varphi_{\tilde{x}_0}"] & \tilde{\Sigma}_g \times \mathrm{GL}^+(2,\mathbb{R}) \arrow[r, "p"'] & P
\end{tikzcd}$$

as in the first section of this chapter. Remark 1.3.14 gives, for every isomorphism $f : \pi_1(P_{x_0}) \to \mathbb{Z}$, the equality

$$|\langle e(E), [\Sigma_g]\rangle| = |f(\langle o(P), [\Sigma_g]\rangle)|$$

Hence, if we choose $f = \psi \circ (\varphi_{x_0})_*^{-1}$, we obtain

$$|\langle e(E), [\Sigma_g]\rangle| = \left|\psi((\varphi_{x_0})_*^{-1}(\langle o(P), [\Sigma_g]\rangle))\right| = \left|\psi(\iota_{\mathrm{GL}^+}^{-1}([A_1, A_2]\cdots[A_{2g-1}, A_{2g}]))\right|$$

Therefore, since ι_{S^1} is the multiplication by 2π, we have:

$$|\langle e(E), [\Sigma_g]\rangle| = \left|\frac{1}{2\pi}\vartheta([A_1, A_2]\cdots[A_{2g-1}, A_{2g}])\right|$$

Now if we apply Lemma 3.2.3 $4g-1$ times we get:

$$\begin{aligned}
\left|\vartheta(A_1A_2A_1^{-1}A_2^{-1}\cdots) - \vartheta(A_1) - \vartheta(A_2) - \vartheta(A_1^{-1}) - \vartheta(A_2^{-1}) - \ldots\right| &\leqslant \\
\leqslant \left|\vartheta(A_1A_2A_1^{-1}A_2^{-1}\cdots) - \vartheta(A_1) - \vartheta(A_2A_1^{-1}A_2^{-1}\cdots)\right| &+ \\
+ \left|\vartheta(A_2A_1^{-1}A_2^{-1}\cdots) - \vartheta(A_2) - \vartheta(A_1^{-1}A_2^{-1}\cdots)\right| &+ \ldots \\
\ldots + \left|\vartheta(A_{2g-1}^{-1}A_{2g}^{-1}) - \vartheta(A_{2g-1}^{-1}) - \vartheta(A_{2g}^{-1})\right| &< (4g-1)\frac{\pi}{2}
\end{aligned}$$

But $\vartheta(A_i^{-1}) = -\vartheta(A_i)$ thanks to (iii) in Lemma 3.2.1, and thus we have:

$$|\langle e(E), [\Sigma_g]\rangle| < g - \frac{1}{4} < g$$

□

In dimension 2 the previous result admits a converse, i.e. if $|\langle e(E), [\Sigma_g]\rangle| < g$ then E admits flat connections. In order to show that let us consider the matrix

$$a_0 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R})$$

Since $r(a_0) = I$ we have $\pi_{\mathbb{R}}(\vartheta(A)) = \exp^{-1}(r(a_0)) = 0$ for each $A \in \pi^{-1}_{\tilde{\mathrm{GL}}^+}(a_0)$. Therefore we can choose $A_0 \in \pi^{-1}_{\tilde{\mathrm{GL}}^+}(a_0)$ such that $\vartheta(A_0) = 0$. Consider the element $\tilde{\exp}(\pi) \in \tilde{\mathrm{SO}}(2) \subset \tilde{\mathrm{GL}}^+(2,\mathbb{R})$. Then $\tilde{\exp}(\pi) \in Z(\tilde{\mathrm{GL}}^+(2,\mathbb{R}))$: indeed, for all $A \in \tilde{\mathrm{GL}}^+(2,\mathbb{R})$ we can consider a curve $\tilde{\gamma}$ from $\tilde{I}$ to A, which projects onto a curve γ in $\mathrm{GL}^+(2,\mathbb{R})$. Then the curve

$$\sigma : t \mapsto \tilde{\exp}(\pi)\tilde{\gamma}(t)\tilde{\exp}(\pi)^{-1}$$

projects onto γ, and therefore must coincide with the unique lifting of γ starting from $\tilde{\sigma}(0) = \tilde{I}$. Now let us consider the conjugacy classes K and $\tilde{K}$ of a_0 in $\mathrm{GL}^+(2,\mathbb{R})$ and of A_0 in $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ respectively.

Lemma 3.3.2. *Every element in $\tilde{\exp}(\pi)\tilde{K}$ can be written as a product of two elements in $\tilde{K}$.*

Proof. Let's consider the equality

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{5}{2} & \frac{9}{2} \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 9 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$$

Let a_1 and a_2 denote the second and the third matrix respectively. Then, since $\det(a_1) = 1$ and $\mathrm{tr}(a_1) = \frac{5}{2}$, its eigenvalues are 2 and $\frac{1}{2}$, and therefore $a_1 \in K$. Analogously $a_2 \in -IK$. Now, since the kernel of the projection $\pi_{\tilde{\mathrm{GL}}^+}$ is central, there exists a unique element $A_1 \in \tilde{K}$ corresponding to a_1. Therefore, since $\pi^{-1}_{\tilde{\mathrm{GL}}^+}(-I) = \{\tilde{\exp}(n\pi) \mid n \text{ odd}\}$, we must have that $A_0A_1 \in \tilde{\exp}(n\pi)\tilde{K}$ for some $n \in \mathbb{Z}$. Now, we have $\cos\vartheta(A) = \frac{1}{2}\mathrm{tr}(r(\pi_{\tilde{\mathrm{GL}}^+}(A)))$ and thus, since $\mathrm{tr}(a) = \frac{5}{2}$ for all $a \in K$, assertion (i) in Lemma 3.2.2 gives $\cos\vartheta(A) > 0$ for all $A \in \tilde{K}$. Then, since $\tilde{K}$ is connected, we have:

(i) $\vartheta(A_0) = 0 \;\Rightarrow\; |\vartheta(A)| < \frac{\pi}{2} \;\; \forall\, A \in \tilde{K}$;

(ii) $\vartheta(\tilde{\exp}(n\pi)A_0) = \tilde{\exp}^{-1}(\tilde{\exp}(n\pi) + \tilde{r}(A_0)) = n\pi \;\Rightarrow\; |\vartheta(A)| > \frac{5\pi}{2} \;\; \forall\, A \in \tilde{\exp}(n\pi)\tilde{K},\;\; n \geqslant 3.$

But since $|\vartheta(A_0A_1)| < |\vartheta(A_0)| + |\vartheta(A_1)| + \frac{\pi}{2} < \frac{3\pi}{2}$ we have $A_0A_1 \in \tilde{\exp}(\pm\pi)\tilde{K}$. Then, if $A_0A_1 \in \tilde{\exp}(\pi)\tilde{K}$ we have

$$BA_0A_1B^{-1} = \left(BA_0B^{-1}\right)\left(BA_1B^{-1}\right)$$

while if $A_0A_1 \in \tilde{\exp}(-\pi)\tilde{K}$ we have

$$BA_1^{-1}A_0^{-1}B^{-1} = \left(BA_1^{-1}B^{-1}\right)\left(BA_0^{-1}B^{-1}\right)$$

□

Lemma 3.3.3. $A \in \tilde{\exp}(\pi)\tilde{K} \;\Rightarrow\; A = [B_1, B_2]$ *for some* $B_1, B_2 \in \tilde{\mathrm{GL}}(2,\mathbb{R})$.

Proof. We can write $A = B_1B_3$ with $B_1, B_3 \in \tilde{K}$. But since $B_1^{-1} \in \tilde{K}$ there must exist some $B_2 \in \tilde{\mathrm{GL}}(2,\mathbb{R})$ such that $B_2B_1^{-1}B_2^{-1} = B_3$. □

We are now ready to prove the following:

Theorem 3.3.4. *If $|\langle e(E), [\Sigma_g]\rangle| < g$ then E admits flat connections.*

Proof. Since oriented rank-2 vector bundles over a fixed manifold are classified by their Euler class, then for an oriented closed surface Σ_g the equality $\langle e(E), [\Sigma_g]\rangle = 0$ implies the triviality of E, which in turn implies its flatness. In particular for $g = 1$ the Theorem is proved. Now suppose $g \geqslant 2$ and consider $0 < k < g$. Let's consider k elements $A_1, \dots, A_k \in \tilde{K}$ such that $\tilde{\exp}((k-1)\pi)A_0 = A_1 \cdots A_k$. Then, setting $A_{k+1} := A_0^{-1}$, we get $\tilde{\exp}((k-1)\pi) = A_1 \cdots A_{k+1}$. Now let us consider $B_1, \dots, B_{2k+2}$ such that $\tilde{\exp}(\pi)A_i = [B_{2i-1}, B_{2i}]$. We have that

$$[B_1, B_2] \cdots [B_{2k+1}, B_{2k+2}] = \tilde{\exp}(\pi)A_1 \cdots \tilde{\exp}(\pi)A_{k+1} = \tilde{\exp}((k+1)\pi)\tilde{\exp}((k-1)\pi) = \tilde{\exp}(2k\pi)$$

is an element of $\ker \pi_{\tilde{\mathrm{GL}}^+}$. Therefore we can define

$$\begin{array}{cccc} h: & \pi_1(M, x_0) & \to & \mathrm{GL}^+(2, \mathbb{R}) \\ & \alpha_i & \mapsto & \begin{cases} \pi_{\tilde{\mathrm{GL}}^+}(B_i) \text{ if } i \leqslant 2k+2 \\ I \text{ if } i > 2k+2 \end{cases} \end{array}$$

The associated vector bundle E satisfies

$$|\langle e(E), [\Sigma_g]\rangle| = \frac{1}{2\pi}\vartheta(\tilde{\exp}(2k\pi)) = k$$

and the two possible orientations realize both k and $-k$. $\square$

Chapter 4

Kostant-Sullivan Theorem and Smillie's counterexamples

We present now a result which confirms Conjecture 1 for closed *complete* affine manifolds. It is interesting to remark that, although Kostant and Sullivan's proof is very short and uses only the Chern-Gauss-Bonnet Theorem, it was published in 1975, almost 30 years after Chern's work and almost 20 years after Milnor-Wood inequalities. We construct also Smillie's counterexamples, which were found the following year, and which show that Conjecture 2 cannot be extended to non-aspherical manifolds.

4.1 Invariant metric tensors

Let V be a real m-dimensional vector space and let $\rho : G \to \mathrm{GL}(V)$ be a linear representation for a Lie group G. We can make G act on the right of the space $B(V)$ of scalar products on V (where by scalar product we mean a positive definite symmetric bilinear form) by defining for each $a \in G$ and each $\Phi \in B(V)$ the scalar product

$$(\Phi \cdot a)(v, w) := \Phi(\rho(a)(v), \rho(a)(w)) \quad \forall\, v, w \in V$$

We denote by $B^G(V)$ the set of G-*invariant* scalar products, i.e. the set of scalar products whose stabilizer is G.

Remark 4.1.1. Every compact Lie group G is oriented and admits a unique right-invariant volume form ν which satisfies $\int_G \nu = 1$. Indeed any non-zero vector $\omega_e \in \bigwedge^n \mathfrak{g}^*$ (where n is the dimension of G) can be uniquely extended to a right-invariant nowhere-zero n-form ω by defining

$$\omega_a(X_1, \ldots, X_n) = \omega_e(d_a R_{a^{-1}} X_1, \ldots, d_a R_{a^{-1}} X_n) \quad \forall\, X_1, \ldots, X_n \in T_a G$$

Thus, since G is compact, the integral $\int_G \omega$ is finite, and therefore we can define $\nu := \omega / \int_G \omega$. Every other right-invariant volume form is a scalar multiple of ν because the space of right-invariant n-forms on G is isomorphic to $\bigwedge^n \mathfrak{g}^*$, which is 1-dimensional. The right-invariance of ν gives $\int_G (f \circ R_a)\nu = \int_G f\nu$ for all $f \in \mathscr{C}^\infty_{\mathbb{R}}(G)$ and all $a \in G$.

Proposition 4.1.2. *Let V be a real m-dimensional vector space and let $\rho : G \to \mathrm{GL}(V)$ be a linear representation for a compact Lie group G. Then $B^G(V) \neq \varnothing$.*

Proof. If we consider the right-invariant volume form ν defined above and any scalar product Φ on V we can define

$$\Phi_G(v, w) := \int_G \Phi(\rho(a)(v), \rho(a)(w))\nu$$

Then, since $f_{v,w} : a \mapsto \Phi(\rho(a)v, \rho(a)v)$ is a strictly positive function on G, Φ_G is indeed positive definite. Moreover the right invariance of ν gives

$$\Phi_G(\rho(b)(v), \rho(b)(w)) := \int_G \Phi(\rho(ab)(v), \rho(ab)(w))\nu = \int_G (f_{v,w} \circ R_b)\nu = \int_G f_{v,w}\nu = \Phi_G(v,w)$$

□

Now let $\pi_E : E \to M$ be a rank-m vector bundle with structure group G and let $\pi_P : P \to M$ be the associated principal G-bundle. Then the choice of a metric tensor g on E is equivalent to the choice of a G-equivariant map $f : P \to B(\mathbb{R}^m)$. A metric tensor g on a rank-m vector bundle $\pi : E \to M$ with structure group G is *G-invariant* if the associated G-equivariant map $f : P \to B(\mathbb{R}^m)$ factorizes as

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\pi_P"'] & & B(\mathbb{R}^m) \\ & M \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

for some map $\bar{f}$.

Lemma 4.1.3. *If a vector bundle $\pi : E \to M$ has compact structure group G then it admits a G-invariant metric tensor g.*

Proof. Since $B^G(\mathbb{R}^m) \neq \varnothing$ there exist maps from M to $B^G(\mathbb{R}^m)$. □

4.2 Kostant-Sullivan Theorem

Let $L : \mathrm{Aff}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the homomorphism which sends each affine isomorphism f of $\mathbb{R}^n$ to its linear part $f - f(0)$. Then the holonomy representation $\rho : \pi_1(M, x_0) \to \mathrm{Aff}(n, \mathbb{R})$ and the developing map $D : \tilde{M} \to \mathbb{R}^n$ associated with each affine structure over a manifold M allow us to reduce the structure group of the tangent bundle $\pi_T : TM \to M$ to $L(\rho(\pi_1(M, x_0)))$. Indeed consider an open covering $\{U_i\}_{i \in I}$ which trivializes the universal cover $\pi_{\tilde{M}} : \tilde{M} \to M$. If $s_i : U_i \to \tilde{M}$ is any section we can define the atlas $\{(U_{i,\alpha}, \varphi_{i,\alpha}) \mid i \in I, \alpha \in \pi_1(M, x_0)\}$ by setting $U_{i,\alpha} := U_i$ and $\varphi_{i,\alpha} := \rho(\alpha) \circ D \circ s_i$. Then each $U_{i,\alpha}$ trivializes TM as

$$\begin{array}{cccc} \chi^T_{i,\alpha} : & TU_{i,\alpha} & \to & U_{i,\alpha} \times \mathbb{R}^n \\ & u & \mapsto & (\pi_T(u), d_{\pi_T(u)}\varphi_{i,\alpha}(u)) \end{array}$$

and therefore the transition functions are

$$\begin{array}{cccc} \varphi^T_{(i,\alpha),(j,\beta)} : & U_{i,\alpha} \cap U_{j,\beta} & \to & \mathrm{GL}(n, \mathbb{R}) \\ & x & \mapsto & \mathrm{d}\rho(\beta\gamma\alpha^{-1}) = L(\rho(\beta\gamma\alpha^{-1})) \end{array}$$

where $s_j(x) = s_i(x) \cdot \gamma^{-1}$ for all $x \in U_{i,\alpha} \cap U_{j,\beta}$.

Lemma 4.2.1. *Let G be a subgroup of $\mathrm{Aff}(n, \mathbb{R})$ acting freely on $\mathbb{R}^n$ and let $\overline{G}$ be its Zariski closure. Then for every $f \in \overline{G}$ the linear part $L(f)$ has 1 as an eigenvalue.*

Proof. If $f(x) = L(f)(x) + \xi$ then equation $f(x) = x$ has no solution if and only if $-\xi$ is not in the range of the linear map $L(f) - I$. Clearly if $f \in G$ then ξ cannot be zero because G acts freely, and therefore the condition is satisfied only if $\det(L(f) - I) = 0$. The continuity of the function $f \mapsto \det(L(f) - I)$ with respect to the Zariski topology gives the result also for $\overline{G}$. □

Lemma 4.2.2. *Let G be a connected closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ such that every matrix in G has 1 as an eigenvalue. Then all matrices in $\mathfrak{g}$ are singular.*

Proof. Given any curve $a : (-\varepsilon, \varepsilon) \to G$ with $a(0) = I$ we can find a curve $x : (-\varepsilon, \varepsilon) \to S^{n-1}$ satisfying $a(t)x(t) = x(t)$. Therefore we get $a'(0)x(0) + x'(0) = x'(0)$. □

Theorem 4.2.3. *If M^n is a closed complete affine manifold then $\chi(M) = 0$*

Proof. For n odd there is nothing to prove. Therefore suppose n is even. Then M can be realized as $\pi_1(M, x_0)\backslash\mathbb{R}^n$ where $\pi_1(M, x_0)$ acts on the left of $\mathbb{R}^n$ via the holonomy representation $\rho : \pi_1(M, x_0) \to \mathrm{Aff}(n,\mathbb{R})$. Let G be the Zariski closure of $\rho(\pi_1(M, x_0))$, let G^0 be the connected component of G containing e and let H be the subgroup $\rho(\pi_1(M, x_0)) \cap G^0$. Then, since G is a Zariski-closed set, it has finitely many connected components, and thus H is a subgroup of finite index of $\rho(\pi_1(M, x_0))$. Therefore the complete affine manifold $M' := H\backslash\mathbb{R}^n$ is a finite covering of M, and in particular it is still closed. Let us enlarge the structure group of TM' from $L(H)$ to $L(G^0)$, which is a connected group since it is the image of the connected group G^0 under L. Therefore the Iwasawa decomposition of $L(G^0)$ yields a compact maximal subgroup K onto which $L(G^0)$ deformation retracts. Therefore we can reduce the structure group of TM' from $L(G^0)$ to K. Now, since $K < \mathrm{GL}(n,\mathbb{R})$ is compact, there exists a K-invariant metric tensor g over E. If we choose a g-orthogonal connection ∇ then its local curvature form matrix with respect to a g-orthonormal local frame over $U \subset M$ can be expressed as $a\Omega a^{-1}$ for some $a : U \to \mathrm{GL}(n,\mathbb{R})$ (the inversion denotes the group operation in $\mathrm{GL}(n,\mathbb{R})$) and some $\mathfrak{k}$-valued 2-form Ω over U (where $\mathfrak{k} = \mathrm{Lie}(K)$). But since every matrix $b \in K$ satisfies $\det(b - I) = 0$, every matrix $B \in \mathfrak{k}$ must satisfy $\det(B) = 0$, and therefore

$$e_{\mathbb{R}}(TM')^2 = [\det(a\Omega a^{-1})] = [\det(\Omega)] = 0$$

To conclude it suffices to use the fact that if M' is a k-sheeted covering of M then its Euler characteristic $\chi(M')$ equals $k \cdot \chi(M)$. □

4.3 Smillie's counterexamples

We will construct a 4-manifold N and a 6-manifold Q whose tangent bundles are flat, whose Euler characteristics are non-zero and whose products produce analogs in every even dimension greater than 2.

A rank-m vector bundle $\pi_E : E \to M$ is *almost trivial* if the rank-$(m+1)$ vector bundle $\pi_E \oplus \pi_1 : E \oplus (M \times \mathbb{R}) \to M$ is trivial. A manifold whose tangent bundle is almost trivial is *almost parallelizable*.

Lemma 4.3.1. *If $\pi_i : E_i \to M$ is an almost trivial vector bundle for $i = 1, 2$ then the Whitney sum $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$ is almost trivial too. If $\pi_E : E \to M$ is an almost trivial vector bundle and $f : N \to M$ is a map then the pull-back $\pi_{f^*E} : f^*E \to N$ is almost trivial too.*

Proof. If m_i denotes the rank of the vector bundle $\pi_i : E_i \to M$ for $i = 1, 2$ then we have that $E_1 \oplus E_2 \oplus (M \times \mathbb{R}) \simeq E_1 \oplus (M \times \mathbb{R}^{m_2+1}) \simeq M \times \mathbb{R}^{m_1+m_2+1}$. The second assertion is obvious. □

Remark 4.3.2. If $\pi_i : E_i \to M$ is a vector bundle for $i = 1, 2$ and $p_i : M_1 \times M_2 \to M_i$ denotes the projection onto the i-th component then $E_1 \times E_2 \simeq p_1^*E_1 \oplus p_2^*E_2$. Therefore, if two vector bundles are almost trivial, their direct product is almost trivial too.

Proposition 4.3.3. *Let $\pi_E : E \to \Sigma_g$ be an oriented rank-2 vector bundle. If $\langle e(E), [\Sigma_g]\rangle$ is even then $\pi_E : E \to \Sigma_g$ is almost trivial.*

Proof. Since $\pi_1(V_2(\mathbb{R}^3)) \simeq \pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}_2$, a rank-3 vector bundle admits 2 linearly independent sections defined over the 2-skeleton of the base space if and only if its second Stiefel-Whitney class vanishes. Therefore $\pi_E \oplus \pi_1 : E \oplus (\Sigma_g \times \mathbb{R}) \to M$ admits 2 linearly independent global sections if and only if we have $w_2(E \oplus (\Sigma_g \times \mathbb{R})) = 0 \in H^2(\Sigma_g; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. But that's precisely what happens since $w_2(E \oplus (\Sigma_g \times \mathbb{R})) = w_2(E)$ and $\langle w_2(E), [\Sigma_g]\rangle \equiv \langle e(E), [\Sigma_g]\rangle \pmod 2$. Therefore we have that $\pi_E \oplus \pi_1 : E \oplus (\Sigma_g \times \mathbb{R}) \to M$ is isomorphic to $\pi_{E'} \oplus \pi_1 : E' \oplus (\Sigma_g \times \mathbb{R}^2) \to M$ for some line bundle $\pi_{E'} : E' \to M$. But $w_1(E') = w_1(E' \oplus (\Sigma_g \times \mathbb{R}^2)) = w_1(E \oplus (\Sigma_g \times \mathbb{R})) = w_1(E) = 0$ because E is oriented, and therefore $\pi_{E'} : E' \to M$ must be trivial. □

Remark 4.3.4. A closed and orientable n-manifold M is almost parallelizable if and only if it admits an immersion f into $\mathbb{R}^{n+1}$. Indeed if $f : M \to \mathbb{R}^{n+1}$ is an immersion then the pull-back by f of the trivial bundle $T(\mathbb{R}^{n+1})$ is a trivial rank-$(n+1)$ bundle over M which is isomorphic to $TM \oplus N_f M$ where $N_f M \to M$ is the normal bundle of the immersion f. The latter is trivial because M is orientable. Conversely suppose that M is almost parallelizable and consider and immersion $g : M \to \mathbb{R}^{2n-1}$, whose existence is implied by Whitney's immersion Theorem. Then the normal bundle $N_g M \to M$ is trivial thanks to the almost triviality of TM. Therefore, thanks to Theorem 6.5 in [19], we have that M admits an immersion into $\mathbb{R}^{n+1}$.

Proposition 4.3.5. *If M^n and N^n are almost parallelizable then $M\#N$ is almost parallelizable too.*

Proof. Immersions of M and N into $\mathbb{R}^{n+1}$ induce an immersion of $M\#N$. □

Proposition 4.3.6. *If $\pi_E : E \to M$ is an almost trivial rank-m vector bundle then it is isomorphic to $\pi_{g^*TS^m} : g^*TS^m \to M$ for some map $g : M \to S^m$.*

Proof. There exists a bundle map

$$\begin{array}{ccc} E \oplus (M \times \mathbb{R}) & \xrightarrow{f} & \mathbb{R}^{m+1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & \{*\} \end{array}$$

If we consider a nowhere-vanishing section $\sigma : M \to (M \times \mathbb{R}) \hookrightarrow E \oplus (M \times \mathbb{R})$ we get a map $f \circ \sigma : M \to \mathbb{R}^{m+1}$. Using a Gram-Schmidt procedure we may assume that $f(\sigma(M)) \subset S^m$ and that for all $x \in M$ we have $f(E_x) \perp f(\sigma(x))$. Therefore $\pi_E : E \to M$ is isomorphic to the pull-back by $f \circ \sigma$ of the tangent bundle TS^m. □

Proposition 4.3.7. *Two almost trivial oriented rank-$2m$ vector bundles $\pi_i : E_i \to M$ over an oriented closed manifold M are isomorphic if and only if $e(E_1) = e(E_2)$.*

Proof. We use Hopf's degree Theorem, which states that for an oriented closed n-manifold M two maps $f_1, f_2 : M \to S^n$ are homotopic if and only if they have the same degree, i.e. if and only if $(f_1)_*[M] = (f_2)_*[M]$ in $H_n(S^n, \mathbb{Z})$. This happens if and only if the induced maps $f_i^* : H^n(S^n; \mathbb{Z}) \to H^n(M; \mathbb{Z})$ coincide, thanks to the Universal Coefficient Theorem. □

Now if we consider 4-manifolds $P := S^1 \times S^3$ and $M := \Sigma_3 \times \Sigma_3$ we have that P is parallelizable (because it is a product of parallelizable manifolds) and M is almost parallelizable (because it is a product of almost parallelizable manifolds). Therefore, thanks to Proposition 4.3.5, the 4-manifold

$$N := M\#(\underbrace{P\#\ldots\#P}_{6 \text{ times}})$$

must be almost parallelizable too. Moreover, since every pair of n-manifolds M_1, M_2 satisfies

$$\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - (1 + (-1)^n),$$

we have that $\chi(N) = 4$. Now let $\pi_E : E \to \Sigma_3$ denote the unique oriented rank-2 vector bundle satisfying $\langle e(E), [\Sigma_3]\rangle = 2$, which is flat thanks to Milnor-Wood inequality 3.3.4, and let $f : N \to M$ be a degree 1 map, i.e. a map satisfying $f_*[N] = [M]$ (for example, we can take the map collapsing the P summands to a point). Then the pull-back $\pi_{E'} : E' \to N$ of $\pi_E \times \pi_E : E \times E \to M$ by f is flat and, thanks to Lemma 4.3.1 and to Remark 4.3.2, it is almost trivial too. Furthermore we have $\langle e(E'), [N]\rangle = \langle e(E \times E), [M]\rangle = \langle e(E), [\Sigma_3]\rangle^2 = 4$. Therefore, thanks to Proposition 4.3.7, $\pi_{E'} : E' \to N$ is isomorphic to the tangent bundle $\pi_T : TN \to N$, which in particular must be flat.

Finally, if we consider the 6-manifold Q given by $R \times \Sigma_3$ where $R := N\#P\#P\#P$, a calculation gives $\chi(Q) = 8$. Therefore, if $h : R \to M$ is a degree 1 map and $\pi_{E''} : E'' \to Q$ is the direct product of the pull-back of $\pi_E \times \pi_E : E \times E \to M$ by h with $\pi_E : E \to \Sigma_3$, then, since

$$\langle e(E''), [Q]\rangle = \langle e(h^*(E \times E) \times E), [Q]\rangle = \langle e((E \times E)), [M]\rangle \cdot \langle e(E), [\Sigma_3]\rangle = 8,$$

we have that $\pi_T : TN \to N$ must be flat.

Remark 4.3.8. Since the connected sum of two closed manifolds of dimension $n \geqslant 3$ which are not homotopy equivalent to S^n is never aspherical, we have that neither N and Q nor their products are aspherical.

Chapter 5

Milnor-Wood inequality for $(\mathbb{H}^2)^n$-manifolds

Milnor-Wood inequality has been recently generalized to all manifolds whose Riemannian universal cover is isometric to a product of hyperbolic planes $\mathbb{H}^2$. This result, which is the first substantial extension of the analog inequality for closed oriented surfaces, confirms both Conjectures 1 and 2 for all manifolds which are locally isometric to products of surfaces of constant curvature.

5.1 Bounded cohomology

The space of singular chains with real coefficients $C_k(X;\mathbb{R})$ on a given topological space X can be equipped with the ℓ^1-norm with respect to the standard basis of singuar simplices:

$$\left\|\sum_{i=1}^{r} a_i\sigma_i\right\| = \sum_{i=1}^{r} |a_i|$$

This norm induces a seminorm on homology groups which assigns to each homology class the infimum of the norms of its representatives:

$$\|\alpha\| = \inf\left\{\sum_{i=1}^{r} |a_i| \;\middle|\; \left[\sum_{i=1}^{r} a_i\sigma_i\right] = \alpha\right\}$$

The *simplicial volume* $\|M\|$ of a closed oriented manifold M is the seminorm of its fundamental class $[M]$.

The space of real valued cochains $C^k(X;\mathbb{R})$ can be equipped with the operator norm

$$\|c\| = \sup\left\{|c(\alpha)| \;\middle|\; \alpha \in C_k(X;\mathbb{R}), \|\alpha\| = 1\right\}$$

which takes the name of *Gromov norm*. The subspaces of bounded cochains $C_b^*(X) \subset C^*(X;\mathbb{R})$ form a cochain complex because the boundary operator is bounded. The *bounded cohomology groups* $H_b^*(X)$ of X are the cohomology groups associated with this cocomplex. Both standard and bounded cohomology groups can be given seminorms analogous to the one defined previously for homology groups, allowing infinite seminorms in the standard case.

Remark 5.1.1. Every continuous map $f : X \to Y$ induces homomorphisms $f_* : H_*(X) \to H_*(Y)$, $f^* : H^*(Y) \to H^*(X)$ and $f^* : H_b^*(Y) \to H_b^*(X)$ which satisfy $\|f_*(\alpha)\| \leqslant \|\alpha\|$ for all $\alpha \in H_*(X)$ and $\|f^*(a)\| \leqslant \|a\|$ for all $a \in H^*(Y)$ and all $a \in H_b^*(Y)$.

The homomorphism $c : H_b^*(X) \to H^*(X;\mathbb{R})$ induced by the inclusion of the cocomplexes $C_b^*(X) \hookrightarrow C^*(X;\mathbb{R})$, which is called the *comparison map*, is natural, i.e. for every continuous map $f : X \to Y$ we have a commutative diagram

$$\begin{tikzcd} H^*(Y) \arrow[r, "f^*"] & H^*(X) \\ H_b^*(Y) \arrow[u, "c"] \arrow[r, "f^*"'] & H_b^*(X) \arrow[u, "c"'] \end{tikzcd}$$

Remark 5.1.2. Bounded cohomology groups do not yield a proper cohomology theory, as the Excision Axiom does not hold. Indeed explicit computations of such groups are usually very hard to perform.

Lemma 5.1.3. *For all cohomology classes $a \in H^k(X;\mathbb{R})$ the seminorm satisfies the equality*

$$\|a\| = \inf \left\{ \|a_b\| \,\middle|\, a_b \in H_b^k(X) : c(a_b) = a \right\} \cup \{\infty\}$$

Proof. We obviously have $\|a\| \leqslant \|a_b\|$. If we take a representative φ for a satisfying $\|\varphi\| \leqslant \|a\| + \varepsilon$, then φ determines a bounded cohomology class a_b satisfying $\|a_b\| \leqslant \|a\| + \varepsilon$ and $c(a_b) = a$. $\square$

Lemma 5.1.4. *Let M be an oriented closed manifold of dimension n. Then for all cohomology classes $a \in c(H_b^n(M))$ we have*

$$|\langle a, [M]\rangle| = \|a\|\,\|M\|$$

Proof. For all cocycles $\psi \in C_b^n(M)$ representing a and all cycles $z \in C_n(M;\mathbb{R})$ representing $[M]$ we have $|\langle a, [M]\rangle| = |\psi(z)| \leqslant \|\psi\|\,\|z\|$. Therefore by taking the infimum on all representatives we get the inequality $|\langle a, [M]\rangle| \leqslant \|a\|\,\|M\|$. On the other hand we can fix a cycle $z \in C_n(M;\mathbb{R})$ representing $[M]$ and then consider the linear functional φ defined on $\mathrm{Span}(z) \oplus B_k(X;\mathbb{R})$ by $\varphi(z) = \|a\|\,\|M\|$ and by $\varphi\big|_{B_k(X;\mathbb{R})} \equiv 0$. Now by the Hanh-Banach Theorem there exists a cochain $\psi \in C^k(X;\mathbb{R})$ having norm $\|a\| \frac{\|M\|}{\|z\|} \leqslant \|a\|$ which extends φ (in particular it must be a cocycle). This cocycle defines a cohomology class $b \in c(H_b^n(M))$ satisfying $\|b\| \leqslant \|a\|$ and $\langle b, [M]\rangle = \|a\|\,\|M\| \geqslant |\langle a, [M]\rangle|$. But since $H^n(M;\mathbb{R})$ is one-dimensional b must equal $\lambda \cdot a$ for some $\lambda \in \mathbb{R}$, and thus we get $|\lambda| \leqslant 1$ and $|\lambda| \geqslant 1$. This gives the other inequality. $\square$

5.2 Continuous group cohomology

If G is a topological group we can define for each k the space of continuous k-cochains

$$C_c^k(G) = \left\{ f : G^{k+1} \to \mathbb{R} \,\middle|\, f \text{ continuous} \right\}.$$

Since G acts on the left of these spaces by

$$g \cdot f(g_0, \ldots, g_k) = f(gg_0, \ldots, gg_k)$$

we can consider the subspaces of G-invariant cochains $C_c^*(G)^G$. These form a cocomplex where the coboundary operator is defined by

$$\delta f(g_0, \ldots, g_k) = \sum_{i=0}^{k} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_k).$$

This cocomplex yields *continuous cohomology groups* $H_c^*(G)$.

Remark 5.2.1. For a discrete group Γ the continuity condition is empty, and thus continuous cohomology groups are simply denoted by $H^*(\Gamma)$. These groups naturally coincide with the (simplicial) cohomology groups with real coefficients of the Eilenberg-MacLane space $K(\Gamma, 1)$, which is a classifying space for principal Γ-bundles. With each manifold M we can associate a characteristic map $f : M \to K(\pi_1(M, x_0), 1)$ such that the universal cover $\pi : \tilde{M} \to M$ is isomorphic to the pull-back by f of the universal principal $\pi_1(M, x_0)$-bundle over $K(\pi_1(M, x_0), 1)$. If M is aspherical then $f^* : H^*(\pi_1(M, x_0)) = H^*(K(\pi_1(M, x_0), 1); \mathbb{R}) \to H^*(M; \mathbb{R})$ is an isomorphism.

Continuous cohomology of topological groups admits a bounded version too. Indeed we can endow the spaces $C_c^k(G)^G$ with the norm

$$\|f\| = \sup\left\{ |f(g_0, \ldots, g_k)| \;\middle|\; (g_0, \ldots, g_k) \in G^{k+1} \right\}$$

Consider the subspaces of bounded G-invariant cochains

$$C_{c,b}^k(G)^G = \left\{ f \in C_c^k(G)^G \;\middle|\; \|f\| < +\infty \right\}.$$

Then, since coboundary operators are bounded with respect to this norm, $C_{c,b}^*(G)^G$ defines a cocomplex which yields *continuous bounded cohomology groups* $H_{c,b}^*(G)$. Both continuous and bounded continuous cohomology groups can be given the usual seminorms

$$\|a\| = \inf\left\{ \|f\| \;\middle|\; [f] = a \right\},$$

allowing infinite values in the unbounded case.

Remark 5.2.2. For a discrete group Γ the seminorm defined for $H^*(\Gamma)$ coincides with the seminorm induced by the Gromov norm on $H^*(K(\Gamma, 1); \mathbb{R})$.

Remark 5.2.3. Every continuous homomorphism $\varphi : G \to G'$ between topological groups induces homomorphisms $\varphi^* : H_c^*(G') \to H_c^*(G)$ and $\varphi^* : H_{c,b}^*(G') \to H_{c,b}^*(G)$ satisfying $\|\varphi^*(a)\| \leqslant \|a\|$ for all $a \in H_c^*(G')$ and all $a \in H_{c,b}^*(G')$.

The inclusion $C_{c,b}^*(G)^G \hookrightarrow C_c^*(G)^G$ induces a *comparison map* $c : H_{c,b}^*(G) \to H_c^*(G)$ which is natural, i.e. for every continuous homomorphism $\varphi : G \to G'$ we have a commutative diagram

$$\begin{array}{ccc} H_c^*(G') & \xrightarrow{\varphi^*} & H_c^*(G) \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle c} \\ H_{c,b}^*(G') & \xrightarrow[\varphi^*]{} & H_{c,b}^*(G) \end{array}$$

Remark 5.2.4. Lemma 5.1.3 immediately extends to continuous group cohomology.

Theorem 5.2.5 (Gromov)**.** *If $f : M \to K(\pi_1(M, x_0), 1)$ denotes the characteristic map of the universal cover $\pi : \tilde{M} \to M$ then $f^* : H_b^*(K(\pi_1(M, x_0), 1); \mathbb{R}) \to H_b^*(M; \mathbb{R})$ is an isometric isomorphism.*

Corollary 5.2.6. *If M is aspherical then $f^* : H^*(K(\pi_1(M, x_0), 1); \mathbb{R}) \to H^*(M; \mathbb{R})$ is an isomorphism which preserves seminorms.*

If G is a Lie group its continuous cohomology can be computed in terms of its Lie algebra $\mathfrak{g}$ in the following way: let us consider the space of left invariant differential forms on G

$$\Omega^*(G)^G = \left\{ \omega \in \Omega^*(G) \;\middle|\; \mathrm{d}L_a^* \omega = \omega \;\; \forall\, a \in G \right\}$$

with coboundary map given by the exterior derivative of differential forms. This cocomplex is naturally equivalent to the cocomplex $\bigwedge^* \mathfrak{g}^*$ endowed with the Koszul differential

$$\mathrm{d}\omega(X_0,\ldots,X_p) = \sum_{0\leqslant i<j\leqslant p} (-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p)$$

The *cohomology groups $H^*(\mathfrak{g})$ of the Lie algebra* $\mathfrak{g}$ are the cohomology groups associated with this cocomplex. Now let

$$\begin{array}{rccc} i_X: & \Omega^{p+1}(G) & \to & \Omega^p(G) \\ & \omega & \mapsto & [(X_1,\ldots,X_p)\mapsto \omega(X,X_1,\ldots,X_p)] \end{array}$$

be the *interior product*. If K is a closed subgroup of G, with Lie algebra $\mathfrak{k}$, let us consider the cocomplex

$$\Omega^*(G)^G_K = \left\{\omega\in\Omega^*(G)^G \;\middle|\; \mathrm{d}R_a^*\omega=\omega\ \ \forall\, a\in K,\ i_X(\omega)=0\ \ \forall\, X\in\mathfrak{k}=\mathrm{Lie}(K)\right\}$$

Remark 5.2.7. This is indeed a cocomplex since for all $X\in\mathfrak{k}$ we have

$$\begin{aligned} i_X\mathrm{d}\omega(X_1,\ldots,X_p) &= \mathrm{d}\omega(X,X_1,\ldots,X_p) = \sum_{i=1}^{p}(-1)^i\omega([X,X_i],X_1,\ldots,\hat{X}_i,\ldots,X_p) = \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\omega(\mathrm{Ad}_{\exp tX}X_1,\ldots,\mathrm{Ad}_{\exp tX}X_p)\right]_{t=0} = 0 \end{aligned}$$

The *relative cohomology groups $H^*(\mathfrak{g},K)$ of the Lie algebra $\mathfrak{g}$ with respect to the subgroup* K are the cohomology groups associated with this cocomplex.

Remark 5.2.8. If $\pi: G\to G/K$ denotes the standard projection, we have that $\pi^*(\Omega^*(G/K)^G) = \Omega^*(G)^G_K\subset\Omega^*(G)^G$. Thus every element in $H^*(\mathfrak{g},K)$ is represented by some G-invariant closed differential form defined on G/K.

Theorem 5.2.9 (Van Est). *Let G be a Lie group and let K be a maximal compact subgroup. Then there exists a natural isomorphism $\Phi: H^*_c(G)\to H^*(\mathfrak{g},K)$ which is multiplicative, i.e. it preserves the cup products.*

One of the immediate advantages of the Van Est isomorphism is that continuous group cohomology inherits from Lie algebra cohomology a Künneth formula: indeed if G_1,G_2 are Lie groups and $K_i<G_i$ are closed subgroups for $i=1,2$ then we have natural isomorphisms $H^p(\mathfrak{g}_1\oplus\mathfrak{g}_2,K_1\times K_2)\simeq\bigoplus_{i=0}^p H^i(\mathfrak{g}_1,K_1)\otimes H^{p-i}(\mathfrak{g}_2,K_2)$ for every p. Thus we get:

Corollary 5.2.10. *If G_1 and G_2 are Lie groups then $H^p_c(G_1\times G_2)\simeq\bigoplus_{i=0}^p H^i_c(G_1)\otimes H^{p-i}_c(G_2)$.*

Another useful feature of continuous cohomology of Lie groups is its behaviour with respect to cocompact subgroups:

Lemma 5.2.11. *Let G be a unimodular Lie group, let $H<G$ be a cocompact subgroup such that the G-invariant measure induced on G/H is finite and let $\iota: H\hookrightarrow G$ denote the inclusion homomorphism. Then the induced map $\iota^*: H^*_c(G)\to H^*_c(H)$ is isometrically injective.*

Example 5.2.12. Consider a closed oriented surface Σ_g of genus $g>1$. A hyperbolic structure on Σ_g induces a representation $\rho:\pi_1(\Sigma_g,x_0)\hookrightarrow\mathrm{Isom}^+(\mathbb{H}^2)\simeq\mathrm{PSL}(2,\mathbb{R})$ which embeds $\pi_1(\Sigma_g,x_0)$ as a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})$. This representation can always be lifted in 2^g ways to a representation $\tilde{\rho}:\pi_1(\Sigma_g,x_0)\hookrightarrow\mathrm{SL}(2,\mathbb{R})$ whose image is a cocompact lattice in $\mathrm{SL}(2,\mathbb{R})$. Therefore both $H^*_c(\mathrm{PSL}(2,\mathbb{R}))$ and $H^*_c(\mathrm{SL}(2,\mathbb{R}))$ embed isometrically into $H^*(\pi_1(\Sigma_g,x_0))\simeq H^*(\Sigma_g;\mathbb{R})$. In particular we get $H^k_c(\mathrm{PSL}(2,\mathbb{R}))=H^k_c(\mathrm{SL}(2,\mathbb{R}))=0$ for $k>2$.

For bounded cohomology groups we have even better properties:

Lemma 5.2.13. *Let G be a unimodular Lie group, let $H < G$ be a subgroup such that the G-invariant measure induced on G/H is finite and let $\iota : H \hookrightarrow G$ denote the inclusion homomorphism. Then the induced map $\iota^* : H^*_{c,b}(G) \to H^*_{c,b}(H)$ is isometrically injective.*

Finally, bounded continuous cohomology is blind to amenable normal subgroups:

Lemma 5.2.14. *Let G be a Lie group and let $N \lhd G$ be a normal Lie subgroup. If N is amenable then the projection $\pi : G \to G/N$ induces an isometric isomorphism $\pi^* : H^*_{c,b}(G/N) \to H^*_{c,b}(G)$.*

As a corollary in the unbounded case we get the following result:

Corollary 5.2.15. *Let G be a Lie group and let $N \lhd G$ be a normal Lie subgroup. If N is amenable then the projection $\pi : G \to G/N$ induces a homomorphism $\pi^* : H^*_c(G/N) \to H^*_c(G)$ which preserves the seminorms.*

5.3 Continuous Euler class

Let G be a Lie group, let BG denote the classifying space for principal G-bundles and let $\pi_{PG} : PG \to BK$ denote the universal principal G-bundle. Then the real Euler class of rank-$2m$ vector bundles with structure group G can be realized as an element in $H^{2m}(BG;\mathbb{R})$: indeed there exists a unique $e_{2m}(BG) \in H^{2m}(BG;\mathbb{R})$ such that for every oriented vector bundle $\pi_E : E \to M$ with structure group G if the pull back of $\pi_{PG} : PG \to BK$ under the map $f : M \to BG$ is isomorphic to the principal G-bundle $\pi_P : P \to M$ associated with $\pi_E : E \to M$, then we have $e_{\mathbb{R}}(E) = f^*(e_{2m}(BG))$.

Remark 5.3.1. The vector bundle $\pi_E : E \to M$ is flat if and only if its structure group can be reduced to G^δ (that is G endowed with the discrete topology). This happens if and only if the map $f : M \to BG$ factorizes through BG^δ.

The following Theorem of Gromov puts the Euler class of flat bundles in the context of bounded cohomology:

Theorem 5.3.2 (Gromov)**.** *Let $G < \mathrm{GL}(2m,\mathbb{C})$ be a linear algebraic $\mathbb{R}$-group and let $G(\mathbb{R})_+$ denote $G \cap \mathrm{GL}^+(2m,\mathbb{R})$. Then the real Euler class of flat oriented rank-$2m$ vector bundles with structure group $G(\mathbb{R})_+$ is realized by a bounded cohomology class $e_{2m}(BG(\mathbb{R})^\delta_+)$ inside $c(H^{2m}_b(BG(\mathbb{R})^\delta_+))$ whose norm equals 2^{-2m}.*

If G is a Lie group and $K < G$ is a maximal compact subgroup then the inclusion $\iota_K : K \hookrightarrow G$ is a homotopy equivalence, and thus induces a homotopy equivalence $\iota_{BK} := (\iota_K)^B_* : BK \to BG$ between classifying spaces. The Chern-Weil homomorphism introduced for smooth manifolds in Remark 2.3.5 can be defined also for classifying spaces of principal bundles (and more in general for simplicial manifolds, compare with [14]), and we have the following result:

Theorem 5.3.3. *Let K be a compact Lie group and let $\pi_{PK} : PK \to BK$ be the universal principal K-bundle. Then the Chern-Weil homomorphism $w_{PK} : I^h(K) \to H^{2h}(BK;\mathbb{R})$ is an isomorphism.*

The projection $\pi_G : G \to G/K$ gives a principal K-bundle which can be endowed with the standard connection Γ mapping each $a \in G$ to the subspace $d_e R_a(\mathfrak{k})$ where R_a denotes the right translation on G and $\mathfrak{k}$ denotes the Lie algebra of K. Then, since Γ is invariant with respect to the right action of the whole G, the Chern-Weil homomorphism $w_G : I^h(K) \to H^{2h}_{\mathrm{dR}}(G/K;\mathbb{R})$ maps every $\mathrm{Ad}(K)$-invariant form in $I^h(K)$ to a closed differential form on G/K which is actually G-invariant. Therefore the image of w_G is actually contained in $H^{2h}(\mathfrak{g}, K)$, which is isomorphic to $H_c^{2h}(G)$. We have therefore a well-defined map $\varphi : H^{2h}(BG) \to H_c^{2h}(G)$ given by the composition

$$H^{2h}(BG;\mathbb{R}) \xrightarrow[\iota_{BK}^*]{\sim} H^{2h}(BK;\mathbb{R}) \xrightarrow[w_{PK}^{-1}]{\sim} I^h(K) \xrightarrow[w_G]{} H^{2h}(\mathfrak{g}, K) \xrightarrow[\Phi]{\sim} H_c^{2h}(G)$$

Proposition 5.3.4. *For every Lie group G we have the following commutative diagram:*

$$\begin{tikzcd}
H^*(BG;\mathbb{R}) \arrow[r, "\iota_{BG^\delta}^*"] \arrow[d, "\varphi"'] & H^*(BG^\delta;\mathbb{R}) \arrow[d, equal] \\
H_c^*(G) \arrow[r, "\iota_{G^\delta}^*"'] & H^*(G^\delta)
\end{tikzcd}$$

Therefore we have a *continuous Euler class* $\varepsilon_{2m}(G) := \varphi(e(BG)) \in H_c^{2m}(G)$ which generates Euler classes of flat oriented rank-$2m$ vector bundles with structure group G in the following way: if the universal cover of a manifold M is the pull-back by $f : M \to B\pi_1(M, x_0)$ of the universal principal $\pi_1(M, x_0)$-bundle over $B\pi_1(M, x_0)$ then every flat oriented rank-$2m$ vector bundle $\pi : E \to M$ with structure group G determines a representation

$$\begin{tikzcd}
\pi_1(M, x_0) \arrow[r, "\rho^\delta"'] \arrow[rr, bend left, "\rho"] & G^\delta \arrow[r, hook, "\iota_{G^\delta}"'] & G
\end{tikzcd}$$

such that the associated principal G-bundle is obtained by pulling back the universal principal G-bundle by the map

$$M \xrightarrow{f} B\pi_1(M, x_0) \xrightarrow{\rho_{B\pi_1}^\delta} BG^\delta \xrightarrow{\iota_{BG^\delta}} BG.$$

Therefore we have the commutative diagram

$$\begin{tikzcd}
H^*(BG;\mathbb{R}) \arrow[r, "\iota_{BG^\delta}^*"] \arrow[d, "\varphi"'] & H^*(BG^\delta;\mathbb{R}) \arrow[r, "(\rho_{B\pi_1}^\delta)^*"] \arrow[d, equal] & H^*(B\pi_1(M, x_0);\mathbb{R}) \arrow[r, "f^*"] \arrow[d, equal] & H^*(M;\mathbb{R}) \\
H_c^*(G) \arrow[r, "\iota_{G^\delta}^*"'] & H^*(G^\delta) \arrow[r, "(\rho^\delta)^*"'] & H^*(\pi_1(M, x_0))
\end{tikzcd}$$

and the real Euler class $e_{\mathbb{R}}(E)$ equals $f^*(\rho^*(\varepsilon_{2m}(G)))$.

Remark 5.3.5. Let G be a reductive algebraic $\mathbb{R}$-group. Then there exists a cocompact lattice $\iota_\Gamma : \Gamma \hookrightarrow G(\mathbb{R})_+$ and the induced homomorphism $\iota_\Gamma^* : H_c^*(G(\mathbb{R})_+) \hookrightarrow H_c^*(\Gamma)$ is isometrically injective thanks to Lemma 5.2.11. But since ι_Γ factorizes like

$$\Gamma \hookrightarrow G(\mathbb{R})_+^\delta \xhookrightarrow{\iota_{G(\mathbb{R})_+^\delta}} G(\mathbb{R})_+$$

we have that the homomorphism $\iota_{G(\mathbb{R})_+^\delta}^* : H_c^*(G(\mathbb{R})_+) \hookrightarrow H^*(G(\mathbb{R})_+^\delta)$ is isometrically injective too. Therefore the continuous Euler class $\varepsilon_{2m}(G(\mathbb{R})_+)$ is a bounded class of seminorm 2^{-2m}.

Remark 5.3.6. From now on ε_{2m} will denote $\varepsilon_{2m}(\mathrm{GL}^+(2m, \mathbb{R}))$ and thus for every closed subgroup $G < \mathrm{GL}^+(2m, \mathbb{R})$ if $\iota_G : G \hookrightarrow \mathrm{GL}^+(2m, \mathbb{R})$ denotes the inclusion we will have $\varepsilon_{2m}(G) = \iota_G^*(\varepsilon_{2m})$.

5.4 Properties of the continuous Euler class

The continuous Euler class allows us to study flat bundles directly using continuous group cohomology and groups representations. We begin by recovering the standard properties of the Euler class in this context. Let G_1 and G_2 be reductive algebraic $\mathbb{R}$-groups, let $G_i(\mathbb{R})$ denote their group of $\mathbb{R}$-rational points, consider representations $\rho_i : G_i(\mathbb{R}) \to \mathrm{GL}^+(m_i, \mathbb{R})$ and set $m = m_1 + m_2$. Now consider the homomorphism

$$\begin{array}{rccc} \rho_\triangle : & G_1(\mathbb{R}) \times G_2(\mathbb{R}) & \to & \mathrm{GL}^+(m, \mathbb{R}) \\ & (a_1, a_2) & \mapsto & \begin{pmatrix} \rho_1(a_1) & 0 \\ 0 & \rho_2(a_2) \end{pmatrix} \end{array}$$

and denote with π_i the i-th projection $G_1(\mathbb{R}) \times G_2(\mathbb{R}) \to G_i(\mathbb{R})$. Then the continuous Euler class satisfies the Whitney sum property:

Lemma 5.4.1. $\rho_\triangle^*(\varepsilon_m) = \pi_1^*(\rho_1^*(\varepsilon_{m_1})) \smile \pi_2^*(\rho_2^*(\varepsilon_{m_2}))$

Proof. Since the product $G := G_1 \times G_2$ is reductive too, $G(\mathbb{R})$ admits cocompact lattices. Since finite index subgroups of cocompact lattices are cocompact lattices, Selberg's Lemma gives us a torsion-free cocompact lattice $\Gamma < G(\mathbb{R})$. Now if $K < G(\mathbb{R})$ is a maximal compact subgroup then the quotient $G(\mathbb{R})/K$ is contractible, and since the right action of Γ onto $G(\mathbb{R})/K$ is free and properly discontinuous we get an aspherical manifold $M = \Gamma\backslash(G(\mathbb{R})/K)$ with fundamental group Γ. In particular the characteristic map $f : M \to B\Gamma$ inducing the universal cover of M gives isomorphisms $f^* : H^*(\Gamma) \hookrightarrow H^*(M; \mathbb{R})$ in every dimension. Now the representation

$$\Gamma \xhookrightarrow{\iota_\Gamma} G(\mathbb{R}) \xrightarrow{\rho_\triangle} \mathrm{GL}^+(m, \mathbb{R})$$

induces a flat oriented rank-m vector bundle $\pi : E \to M$ such that $E \simeq E_1 \oplus E_2$ where E_i is induced by the representation

$$\Gamma \xhookrightarrow{\iota_\Gamma} G(\mathbb{R}) \xrightarrow{\pi_i} G_i(\mathbb{R}) \xrightarrow{\rho_i} \mathrm{GL}^+(m_i, \mathbb{R}).$$

Therefore we have

$$\begin{aligned} f^*(\iota_\Gamma^*(\rho_\triangle^*(\varepsilon_{2m}))) = e_\mathbb{R}(E) = e_\mathbb{R}(E_1) \smile e_\mathbb{R}(E_2) = f^*(\iota_\Gamma^*(\pi_1^*(\rho_1^*(\varepsilon_{m_1})))) \smile f^*(\iota_\Gamma^*(\pi_2^*(\rho_2^*(\varepsilon_{m_2})))) = \\ = f^*(\iota_\Gamma^*(\pi_1^*(\rho_1^*(\varepsilon_{m_1})) \smile \pi_2^*(\rho_2^*(\varepsilon_{m_2})))) \end{aligned}$$

and we conclude thanks to the injectivity of uniqueness of $f^* \circ \iota_\Gamma^*$. □

Now let a be an element in $\mathrm{GL}(2m, \mathbb{R})$ and let ρ_a denote the conjugation

$$\begin{array}{rccc} \rho_a : & \mathrm{GL}^+(2m, \mathbb{R}) & \to & \mathrm{GL}^+(2m, \mathbb{R}) \\ & b & \mapsto & aba^{-1} \end{array}$$

Then an inversion of the orientation switches the sign of the continuous Euler class:

Lemma 5.4.2. $\rho_a^*(\varepsilon_{2m}) = (-1)^{\det a} \varepsilon_{2m}$

Proof. If $\rho : \pi_1(M, x_0) \to \mathrm{GL}^+(2m, \mathbb{R})$ is a representation inducing a flat oriented vector bundle E, let ρ' denote the composition $\rho_a \circ \rho$ and E' be the induced flat oriented vector bundle. An isomorphism between E and E' is equivalent to a $\pi_1(M, x_0)$-equivariant automorphism of $\tilde{M} \times \mathbb{R}^{2m}$ where the action of $\pi_1(M, x_0)$ is induced by ρ on the domain and by ρ' on the range. Such a map is given by

$$\begin{array}{rccc} \varphi : & \tilde{M} \times \mathbb{R}^{2m} & \to & \tilde{M} \times \mathbb{R}^{2m} \\ & (\tilde{x}, \xi) & \mapsto & (\tilde{x}, a\xi) \end{array}$$

Now obviously the orientation of the bundle is preserved or reversed depending on $\det a$. □

Now let us move on to establishing some vanishing results for the continuous Euler class. First of all let us consider the *tensor representation* $(\mathbb{R}^4, \rho_\otimes)$ for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which is obtained by identifying $\mathbb{R}^2 \otimes \mathbb{R}^2$ with $\mathbb{R}^4$ via the homomorphism which sends the ordered basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ to the standard basis of $\mathbb{R}^4$. In other words, we have

$$\begin{array}{rccc} \rho_\otimes : & \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \to & \mathrm{GL}^+(4,\mathbb{R}) \\ & (a,b) & \mapsto & \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \end{array}$$

Lemma 5.4.3. *$\rho_\otimes^*(\varepsilon_4) = 0$.*

Proof. Let $\tau : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ denote the homomorphism mapping $\tau : (a,b) \mapsto (b,a)$. Let $e_{(23)}$ denote the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and let $\rho_{(23)}$ denote the conjugation $a \mapsto e_{(23)} \cdot a \cdot e_{(23)}^{-1}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\tau} & \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\ \rho_\otimes \downarrow & & \downarrow \rho_\otimes \\ \mathrm{GL}^+(4,\mathbb{R}) & \xrightarrow[\rho_{(23)}]{} & \mathrm{GL}^+(4,\mathbb{R}) \end{array}$$

and therefore we get

$$\tau^*(\rho_\otimes^*(\varepsilon_4)) = \rho_\otimes^*(\rho_{(23)}^*(\varepsilon_4)) = -\rho_\otimes^*(\varepsilon_4)$$

Now let $e_{(13)(24)}$ denote the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and let $\rho_{(13)(24)}$ denote the conjugation $a \mapsto e_{(13)(24)} \cdot a \cdot e_{(13)(24)}^{-1}$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\tau} & \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\ \rho_\triangle \downarrow & & \downarrow \rho_\triangle \\ \mathrm{GL}^+(4,\mathbb{R}) & \xrightarrow[\rho_{(13)(24)}]{} & \mathrm{GL}^+(4,\mathbb{R}) \end{array}$$

and therefore we get

$$\tau^*(\rho_\triangle^*(\varepsilon_4)) = \rho_\triangle^*(\rho_{(13)(24)}^*(\varepsilon_4)) = \rho_\triangle^*(\varepsilon_4)$$

Now the group $H_c^4(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}))$ is one dimensional, generated by the cup product $\pi_1^*(\varepsilon_2(\mathrm{SL}(2,\mathbb{R}))) \smile \pi_2^*(\varepsilon_2(\mathrm{SL}(2,\mathbb{R})))$. Indeed, thanks to Example 5.2.12, $H_c^2(\mathrm{SL}(2,\mathbb{R}))$ embeds into $H^2(\Sigma_g;\mathbb{R}) \simeq \mathbb{R}$ and $\varepsilon_2(\mathrm{SL}(2,\mathbb{R}))$, which is non-zero, must generate. Thus the Künneth formula gives $H_c^4(\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})) \simeq H_c^2(\mathrm{SL}(2,\mathbb{R}) \otimes H_c^2(\mathrm{SL}(2,\mathbb{R}) \simeq \mathbb{R}$ and the generator must be $\pi_1^*(\varepsilon_2(\mathrm{SL}(2,\mathbb{R}))) \smile \pi_2^*(\varepsilon_2(\mathrm{SL}(2,\mathbb{R})))$. Therefore Lemma 5.4.1 gives $\rho_\otimes^*(\varepsilon_4) = \lambda \cdot \rho_\triangle^*(\varepsilon_4)$ for some $\lambda \in \mathbb{R}$. This yields

$$\lambda \cdot \rho_\triangle^*(\varepsilon_4) = \lambda \cdot \tau^*(\rho_\triangle^*(\varepsilon_4)))) = \tau^*(\lambda \cdot \rho_\triangle^*(\varepsilon_4)) = -\lambda \cdot \rho_\triangle^*(\varepsilon_4)$$

$\square$

Notation: Whenever dealing with product groups $G = G_1 \times \ldots \times G_n$ we will treat each factor G_i as a subgroup of G via the identification with the image of the i-th inclusion $\iota_i : G_i \hookrightarrow G$. Analogously for almost direct products $G = (G_1 \times \ldots \times G_n)/C$ with $C \triangleleft G_1 \times \ldots \times G_n$ discrete and central we will consider each almost direct factor G_i as a subgroup of G via the identification with the image of the composition

$$G_i \xhookrightarrow{\iota_i} G_1 \times \ldots \times G_n \xrightarrow{\pi} G$$

Lemma 5.4.4. *Let M_1 and M_2 be closed aspherical oriented manifolds of dimension $m_1, m_2 \geqslant 1$ respectively and set $m = m_1 + m_2$. Let $\rho : \pi_1(M_1, x_1) \times \pi_1(M_2, x_2) \to \mathrm{GL}^+(m, \mathbb{R})$ be a representation. If $\Gamma_1 = \rho(\pi_1(M_1, x_1))$ is amenable then $\rho^*(\varepsilon_m) = 0 \in H^m(\pi_1(M_1, x_1) \times \pi_1(M_2, x_2))$.*

Proof. By Tit's alternative (Theorem 6.5.13) Γ_1 must be virtually solvable, otherwise it would contain a non-amenable subgroup. Therefore its Zariski closure G_1 is virtually solvable, and thus it is an extension of a solvable group (which is amenable) by a finite group (which is amenable). Hence G_1 must be amenable. Now Lemma 5.2.14 gives an isometric isomorphism between $H_{c,b}^*(G_1 \times G_2)$ and $H_{c,b}^*(G_2)$ which fits into the following commutative diagram:

$$\begin{tikzcd}
H_c^*(\mathrm{GL}^+(m,\mathbb{R})) \arrow[r, "\varphi^*"] \arrow[rrr, bend left, "\rho^*"] & H_c^*(G_1 \times G_2) \arrow[rr, "(\rho_1,\rho_2)^*"] & & H^*(\pi_1(M_1,x_1) \times \pi_1(M_2,x_2)) & \\
 & & H_c^*(G_2) \arrow[ul] \arrow[rr, "\rho_2^*"] & & H^*(\pi_1(M_2,x_2)) \arrow[ul] \\
H_{c,b}^*(\mathrm{GL}^+(m,\mathbb{R})) \arrow[r, "\varphi^*"] \arrow[uu, "c"] & H_{c,b}^*(G_1 \times G_2) \arrow[uu, "c"] & & & \\
 & & H_{c,b}^*(G_2) \arrow[ul, leftrightarrow] \arrow[uu, "c"] & &
\end{tikzcd}$$

Since ε_m lies in the image of $c : H_{c,b}^*(\mathrm{GL}^+(m,\mathbb{R})) \to H_c^*(\mathrm{GL}^+(m,\mathbb{R}))$ then $\rho^*(\varepsilon_m)$ lies in the image of the rightmost diagonal map $H^*(\pi_1(M_2,x_2)) \to H^*(\pi_1(M_1,x_1) \times \pi_1(M_2,x_2))$. But $H^m(\pi_1(M_2,x_2)) \simeq H^m(M_2;\mathbb{R}) = 0$ because M_2 is aspherical of dimension strictly less than m. $\square$

Lemma 5.4.5. *Let G be a Lie group and let $\rho : G \to \mathrm{GL}^+(m,\mathbb{R})$ be a representation. Suppose G decomposes as a semi-direct product $S \ltimes A$ where A is a closed amenable normal subgroup. If $p : S \ltimes A \to S$ denotes the projection and $\iota_S : S \hookrightarrow S \ltimes A$ denotes the inclusion then $\rho^*(\varepsilon_m) = p^*(\iota_S^*(\rho^*(\varepsilon_m)))$.*

Proof. The equality $p \circ \iota_S = \mathrm{id}_S$ gives $\iota_S^* \circ p^* = \mathrm{id}$ on all cohomology groups. But it may very well be that $p^* \circ \iota_S^* \neq \mathrm{id}$ on continuous cohomology groups. Nevertheless we have $p^* \circ \iota_S^* = \mathrm{id}$

for bounded continuous cohomology groups because, thanks to Lemma 5.2.14, in this case p^* is an isometric isomorphism. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
 & & H^m_c(S) \arrow[dr, "p^*"] & \\
H^m_c(\mathrm{GL}^+(m,\mathbb{R})) \arrow[r, "\rho^*"] & H^m_c(G) \arrow[ur, "\iota_S^*"] & & H^m_c(G) \\
 & & H^m_{c,b}(S) \arrow[uu, "c"] \arrow[dr, "p^*"] & \\
H^m_{c,b}(\mathrm{GL}^+(m,\mathbb{R})) \arrow[uu, "c"] \arrow[r, "\rho^*"] & H^m_{c,b}(G) \arrow[uu, "c"] \arrow[ur, "\iota_S^*"] \arrow[rr, leftrightarrow, "\mathrm{id}"] & & H^m_{c,b}(G) \arrow[uu, "c"]
\end{tikzcd}$$

□

Remark 5.4.6. As a corollary we get that, since $\mathrm{GL}^+(2,\mathbb{R})$ is isomorphic to $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}_+$, if $p : \mathrm{GL}^+(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ denotes the associated projection then $\varepsilon_2 = p^*(\varepsilon_2(\mathrm{SL}(2,\mathbb{R})))$.

5.5 The simplicial volume of $(\mathbb{H}^2)^n$-manifolds

Let $M_1, \ldots, M_n$ be Riemannian manifolds and let ∇^i denote the Levi-Civita connection of M_i. The Levi-Civita connection ∇ of the product $M := M_1 \times \ldots \times M_n$ is then uniquely determined by the condition

$$\nabla_{X_1+\ldots+X_n}(Y_1 + \ldots + Y_n) = \nabla^1_{X_1} Y_1 + \ldots + \nabla^n_{X_n} Y_n$$

for all $X_i, Y_i \in \mathfrak{X}(M_i)$ for $i = 1, \ldots, n$.

Proposition 5.5.1. *Suppose the Riemannian manifolds $M_1, \ldots, M_n$ are oriented, endow the product M with the induced orientation and let $p_i : M \to M_i$ denote the i-th projection. Then we have*

$$[\mathrm{Pf}(\Omega)] = p_1^*[\mathrm{Pf}(\Omega^1)] \smile \ldots \smile p_n^*[\mathrm{Pf}(\Omega^1)]$$

Proof. If we consider a local positive orthonormal frame for each TM_i their ordered union is a local positive orthonormal frame for TM. Then the local connection form matrix ω of ∇ with respect to this local frame is given by

$$\begin{pmatrix} p_1^*\omega^1 & & \\ & \ddots & \\ & & p_n^*\omega^n \end{pmatrix}$$

Therefore the structure equation $\Omega = \mathrm{d}\omega - \omega \wedge \omega$ together with the fact that pull-backs commute with exterior derivatives and wedge products gives

$$\begin{pmatrix} p_1^*\Omega^1 & & \\ & \ddots & \\ & & p_n^*\Omega^n \end{pmatrix}$$

Now Remark 2.1.3 allows us to conclude. □

If we consider the half-space model for $\mathbb{H}^2$ we can realize $(\mathbb{H}^2)^n$ as the set

$$(\Pi^2)^n := \left\{ (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} \middle| x_i > 0 \text{ if } i \text{ is even} \right\}$$

with Riemannian metric tensor given by

$$g := \sum_{i=1}^{n} \frac{1}{x_{2i}^2}(\mathrm{d}x_{2i-1} \otimes \mathrm{d}x_{2i-1} + \mathrm{d}x_{2i} \otimes \mathrm{d}x_{2i})$$

The Riemannian volume form $\nu_{(\mathbb{H}^2)^n}$ is then given by

$$\nu_{(\mathbb{H}^2)^n} := \frac{1}{\prod_{i=1}^n x_{2i}^2} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{2n}$$

If we consider the i-th projection

$$\begin{array}{cccc} p_i : & (\mathbb{H}^2)^n & \to & \mathbb{H}^2 \\ & (x_1, \ldots, x_{2n}) & \mapsto & (x_{2i-1}, x_{2i}) \end{array}$$

then we have $\nu_{(\mathbb{H}^2)^n} = p_1^*\nu_{\mathbb{H}^2} \wedge \ldots \wedge p_1^*\nu_{\mathbb{H}^2}$.

Theorem 5.5.2. *Let M be a closed oriented $2n$-dimensional Riemannian manifold whose universal cover is isometric to $(\mathbb{H}^2)^n$. Then* $\mathrm{Vol}(M) = (-2\pi)^n\chi(M)$.

Proof. In the half-space model for $\mathbb{H}^2$ a positive orthonormal frame is given by $X_1 := x_2\frac{\partial}{\partial x_1}$ and $X_2 := x_2\frac{\partial}{\partial x_2}$. With respect to this frame the local curvature form matrix $\Omega_{\mathbb{H}^2}$ is given by

$$\Omega_{\mathbb{H}^2} := \begin{pmatrix} 0 & \nu_{\mathbb{H}^2} \\ -\nu_{\mathbb{H}^2} & 0 \end{pmatrix}$$

Therefore, thanks to Proposition 5.5.1 we can choose othonormal vector fields for $(\mathbb{H}^2)^n$ yielding a local curvature form matrix $\Omega_{(\mathbb{H}^2)^n}$ of the form

$$\begin{pmatrix} p_1^*\Omega_{\mathbb{H}^2} & & \\ & \ddots & \\ & & p_n^*\Omega_{\mathbb{H}^2} \end{pmatrix}$$

Thus we have

$$\mathrm{Pf}(\Omega_{(\mathbb{H}^2)^n}) = \mathrm{Pf}(p_1^*\Omega_{\mathbb{H}^2}) \wedge \ldots \wedge \mathrm{Pf}(p_n^*\Omega_{\mathbb{H}^2}) = (p_1^*\nu_{\mathbb{H}^2}) \wedge \ldots \wedge (p_n^*\nu_{\mathbb{H}^2}) = \nu_{(\mathbb{H}^2)^n}$$

and the Chern-Gauss-Bonnet Theorem gives $\mathrm{Vol}(M) = (-2\pi)^n\chi(M)$. □

Let P denote the group $\mathrm{PSL}(2,\mathbb{R})$, let S denote $\mathrm{SL}(2,\mathbb{R})$ and $\pi : S \to P$ denote the standard projection. Consider the groups $P^n = \prod_{i=1}^n P$ and $S^n = \prod_{i=1}^n S$ with projection $\pi^n : S^n \to P^n$ and let $\pi_i : P^n \to P$ denote the projection onto the i-th factor. The group of orientation-preserving isometries of the hyperbolic plane $\mathrm{Isom}^+(\mathbb{H}^2)$ is isomorphic to P and acts transitively on $\mathbb{H}^2$. This gives $P/\mathrm{Stab}(x_0) \simeq \mathbb{H}^2$ for any point $x_0 \in \mathbb{H}^2$. In the half-plane model the stabilizer of the point $e_2 = (0,1)$ corresponds to $K = \mathrm{PSO}(2)$, which is a maximal compact subgroup of P, and therefore Theorem 5.2.9 gives an isomorphism $\Phi_P : H_c^2(P) \to H^2(\mathrm{Lie}(P), K)$. Now since the volume form $\nu_{\mathbb{H}^2}$ is P-invariant it determines, thanks to Remark 5.2.8, a cohomology class in $H^2(\mathrm{Lie}(P), K)$. We denote by ω_P its image in $H_c^2(P)$ under the isomorphism Φ_P^{-1}. Analogously the connected component of the identity in $\mathrm{Isom}^+((\mathbb{H}^2)^n)$ is isomorphic to P^n and acts transitively on $(\mathbb{H}^2)^n$. The stabilizer of the point $\sum_{i=1}^n e_{2i}$ corresponds to $K^n := \prod_{i=1}^n K$, which is a maximal compact subgroup of P^n. Once again since the volume form $\nu_{(\mathbb{H}^2)^n}$ is P^n-invariant it determines a cohomology class in $H^{2n}(\mathrm{Lie}(P^n), K^n)$. We denote by ω_{P^n} its image in $H_c^{2n}(P^n)$ under the Van Est isomorphism $\Phi_{P^n}^{-1} : H^{2n}(\mathrm{Lie}(P^n), K^n) \to H_c^{2n}(P^n)$. Finally we denote by ω_S and ω_{S^n} respectively the images of ω_P and ω_{P^n} under the homomorphisms $\pi^* : H_c^2(P) \to H_c^2(S)$ and $(\pi^n)^* : H_c^{2n}(P^n) \to H_c^{2n}(S^n)$ induced by projections.

Remark 5.5.3.

$$\begin{aligned}\omega_{P^n} &= \Phi_{P^n}^{-1}\left([p_1^*\omega_{\mathbb{H}^2}] \smile \ldots \smile [p_n^*\omega_{\mathbb{H}^2}]\right) = \Phi_{P^n}^{-1}\left([p_1^*\omega_{\mathbb{H}^2}]\right) \smile \ldots \smile \Phi_{P^n}^{-1}\left([p_n^*\omega_{\mathbb{H}^2}]\right) = \\ &= \pi_1^*\left(\Phi_P^{-1}\left([\omega_{\mathbb{H}^2}]\right)\right) \smile \ldots \smile \pi_n^*\left(\Phi_P^{-1}\left([\omega_{\mathbb{H}^2}]\right)\right) = \pi_1^*(\omega_P) \smile \ldots \smile \pi_n^*(\omega_P)\end{aligned}$$

Lemma 5.5.4. $\omega_{S^n} = (-4\pi)^n(\pi_1^*(\varepsilon_2(S)) \smile \ldots \smile \pi_n^*(\varepsilon_2(S))) \in H_c^{2n}(S^n)$

Proof. We have

$$\begin{aligned}\omega_{S^n} &= (\pi^n)^*(\omega_{P^n}) = (\pi^n)^*(\pi_1^*(\omega_P) \smile \ldots \smile \pi_n^*(\omega_P)) = \pi_1^*(\pi^*(\omega_P)) \smile \ldots \smile \pi_n^*(\pi^*(\omega_P)) = \\ &= \pi_1^*(\omega_S) \smile \ldots \smile \pi_n^*(\omega_S)\end{aligned}$$

Therefore it suffices to prove the equality $\omega_S = -4\pi\varepsilon_2(S) \in H_c^2(S)$. Now consider a compact oriented surface Σ_g of genus $g > 1$ endowed with a complete hyperbolic structure which yields an embedding $\rho : \pi_1(\Sigma_g, x_0) \hookrightarrow P$. Recall that, since Σ_g is aspherical, the characteristic map $f : \Sigma_g \to K(\pi_1(\Sigma_g, x_0), 1)$ induces an isometric isomorphism $f^* : H^*(\pi_1(\Sigma_g, x_0)) \to H^*(\Sigma_g; \mathbb{R})$. As is shown in [25] (pages 312-314) ρ lifts to a representation $\tilde{\rho} : \pi_1(\Sigma_g, x_0) \hookrightarrow S$ which determines an oriented plane bundle $E_{\tilde{\rho}} \to \Sigma_g$ satisfying

$$\langle e_{\mathbb{R}}(E_{\tilde{\rho}}), [\Sigma_g]\rangle = 1 - g = \frac{\chi(\Sigma_g)}{2}.$$

Furthermore we have

$$\langle f^*(\tilde{\rho}^*(\omega_S)), [\Sigma_g]\rangle = \langle f^*(\rho^*(\omega_P)), [\Sigma_g]\rangle = \mathrm{Vol}(\Sigma_g) = -2\pi\chi(\Sigma_g)$$

which, together with the fact that $H_c^2(S)$ is one-dimensional, yields the equality

$$\tilde{\rho}^*(\varepsilon_2(S)) = -4\pi\tilde{\rho}^*(\omega_S).$$

Now $\tilde{\rho}(\pi_1(\Sigma_g, x_0))$ is a cocompact lattice in S and therefore, thanks to Lemma 5.2.11, $\tilde{\rho}^*$ is injective. □

The following Theorem is a special case of Theorem 2 in [10].

Theorem 5.5.5. *Let M be a closed oriented $2n$-dimensional Riemannian manifold whose universal cover is isometric to $(\mathbb{H}^2)^n$. Then*

$$\|M\| = \frac{\mathrm{Vol}(M)}{\|\omega_{P^n}\|}$$

Lemma 5.5.6. *Let M be a closed oriented $2m$-dimensional Riemannian manifold whose universal cover is isometric to $(\mathbb{H}^2)^n$. Then*

$$\|M\| = \frac{\chi(M)}{(-2)^n\,\|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\|}$$

Proof. Set $G = \mathrm{GL}^+(2, \mathbb{R})$ and $S = \mathrm{SL}(2, \mathbb{R})$, let $p : G \to S$ be the projection associated with the isomorphism $\mathrm{GL}^+(2, \mathbb{R}) \simeq \mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}_+$ and $p^n : G^n \to S^n$ be the projection of the n-fold product. If $\rho_\triangle : S^n \hookrightarrow \mathrm{GL}^+(2n, \mathbb{R})$ denotes the diagonal representation then Lemma 5.4.1 gives

$$\rho_\triangle^*(\varepsilon_{2n}) = \pi_1^*(\varepsilon_2(S)) \smile \ldots \smile \pi_n^*(\varepsilon_2(S))$$

Therefore we have

$$(p^n)^*(\rho_\triangle^*(\varepsilon_{2n})) = \pi_1^*(p^*(\varepsilon_2(S))) \smile \ldots \smile \pi_n^*(p^*(\varepsilon_2(S))) = \pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)$$

where the last equality uses Remark 5.4.6. Now since $\ker(p^n)$ is amenable Corollary 5.2.15 gives

$$\|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\| = \|\pi_1^*(\varepsilon_2(S)) \smile \ldots \smile \pi_n^*(\varepsilon_2(S))\| = \frac{1}{(4\pi)^n}\,\|\omega_{S^n}\|$$

Again Corollary 5.2.15 gives $\|\omega_{S^n}\| = \|\omega_{P^n}\|$. Now Lemma 5.5.4, Theorem 5.5.2 and Theorem 5.5.5 allow us to conclude. □

5.6 Representations of direct products of groups

The present section is devoted to the study of representations of (almost) direct products of groups. We begin with a straightforward consequence of Schur's Lemma:

Lemma 5.6.1. *Let V be a finite dimensional complex vector space, let G_1, G_2 be Lie groups and let $\rho : G_1 \times G_2 \to \mathrm{GL}(V)$ be an irreducible representation. Then there exist two representations $r_i : G_i \to W_i$ for $i = 1, 2$ such that $(V, \rho) \simeq (W_1 \otimes W_2, r_1 \otimes r_2)$. Moreover every linear transformation of V commuting with $\rho(G_1)$ corresponds to a linear transformation of $W_1 \otimes W_2$ of the form $\mathrm{id}_{W_1} \otimes B$ for some $B \in \mathrm{End}(W_2)$*

Proof. Let ρ_i denote the restriction $\rho\big|_{G_i} : G_i \to \mathrm{GL}(V)$ and observe that, since G_1 and G_2 are commuting subgroups in $G_1 \times G_2$, $\rho_2(b)$ is G_1-equivariant for every $b \in G_2$. Consider a non-trivial irreducible subrepresentation $U \subset V$ for (V, ρ_1). If $U = V$ then Schur's Lemma implies $\mathrm{im}\, \rho_2 \subset \mathbb{C} \cdot \mathrm{id}_V$, and we can choose $W_1 = V$, $r_1 = \rho_1$, $W_2 = \mathbb{C}$ and r_2 such that $\rho_2(b) = r_2(b) \cdot \mathrm{id}_V$ for all $b \in G_2$. Therefore suppose $U \neq V$. Since V is irreducible U cannot be a subrepresentation of (V, ρ_2) too, and thus there must exist some element $b_1 \in G_2$ such that $\rho_2(b_1)(U) \not\subset U$. If we set $U_0 = U$ and $U_1 = \rho_2(b_1)(U_0)$ then also U_1 is a subrepresentation of (V, ρ_1) and, since U_0 is G_1-irreducible, they must be disjoint. Now, since U_1 is G_1-irreducible too (being the image of a G_1-irreducible subspace with respect to an invertible equivariant endomorphism), we can find $b_2 \in G_2$ such that $\rho_2(b_2)(U_0)$ is disjoint from both U_0 and U_1. Thus iterating we get a finite set $b_1, \ldots, b_{k-1} \in G_2$ such that $U_j = \rho_2(b_j)(U_0)$ and $V = \bigoplus_{j=0}^{k-1} U_j$. If we choose an ordered basis $\mathscr{B}_0$ of U_0 we can extend it to an ordered basis $\mathscr{B}$ of V by adding $\rho_2(b_j)(\mathscr{B}_0)$ for $j = 1, \ldots, k-1$ to it. With respect to the basis $\mathscr{B}$ the elements of $\mathrm{im}\, \rho_1$ are block diagonal matrices with k identical blocks. In order to prove the Lemma we must show that if T is a G_1-equivariant linear transformation then each of the k^2 square blocks of T determined by the basis $\mathscr{B} = \mathscr{B}_0 \cup \ldots \cup \mathscr{B}_{k-1}$ is a scalar matrix. In other words, denoting by $\pi_j : V \to U_j$ the associated projections (which are G_1-equivariant because they are projections associated to the splitting of a representation of G_1), we have to show that $\rho_2(b_j)^{-1} \circ \pi_j \circ T \circ \rho_2(b_i) : U \to U$ is a scalar transformation $b_{ij}(T) \cdot \mathrm{id}_U$ for every $i, j = 0, \ldots, k-1$. Since all four transformations involved are G_1-equivariant and since U is G_1-irreducible, this is a consequence of Schur's Lemma. Then we can choose $W_1 = U_0$ and $W_2 = \mathbb{C}^k$ with $r_2(b)_{ij} = b_{ij}(\rho_2(b))$. $\square$

Lemma 5.6.2. *Let $S = S_1 \times \ldots \times S_n$ be a product of reductive algebraic groups, and let $\rho : S \to \mathrm{GL}(m, \mathbb{C})$ be a representation such that $H_i = \rho(S_i) < \mathrm{GL}(m, \mathbb{C})$ is non-abelian for all $i = 1, \ldots, n$. Then $m \geqslant 2n$, and if $m = 2n$ there exists a unique natural number $0 \leqslant t(\rho) \leqslant n$ and a unique splitting*

$$\mathbb{C}^{2n} = \bigoplus_{i=1}^{t(\rho)/2} V_i \oplus \bigoplus_{i=t(\rho)+1}^{n} V_i$$

such that:

(i) $\dim V_i = 4$ *for* $1 \leqslant i \leqslant t(\rho)/2$ *and* $\dim V_i = 2$ *for* $t(\rho) < i \leqslant n$*;*

(ii) every representation $\rho_i := \rho\big|_{S_i} : S_i \to \mathrm{GL}(2n, \mathbb{C})$ with $i > t(\rho)$ is irreducible when restricted to V_i while it is scalar when restricted to V_j for $j \neq i$;

(iii) every representation $\rho_{(2i-1,2i)} := \rho\big|_{S_{2i-1} \times S_{2i}} : S_{2i-1} \times S_{2i} \to \mathrm{GL}(2n, \mathbb{C})$ with $i \leqslant t(\rho)/2$ is isomorphic to the tensor product $r_{2i-1} \otimes r_{2i}$ of two 2-dimensional irreducible representations $r_{2i-1} : S_{2i-1} \to \mathrm{GL}(W_{2i-1})$ and $r_{2i} : S_{2i} \to \mathrm{GL}(W_{2i})$ when restricted to V_i while it is scalar when restricted to V_j for $j \neq i$.

Proof. We proceed by induction on n. When $n = 1$ then clearly $\mathrm{GL}(1, \mathbb{C}) \simeq \mathbb{C}^*$ does not admit non-abelian subgroups. Moreover if $m = 2$ then S_1 must act irreducibly on $\mathbb{C}^2$ otherwise, being reductive, it would split $\mathbb{C}^2$ into two S_1-invariant complex lines. But this is impossible because H_1 would be abelian.

Now suppose $n > 1$. Since S is reductive $\mathbb{C}^m$ splits uniquely as $\mathbb{C}^m = \bigoplus_{j=1}^{\ell} U_j$ where each U_j is an irreducible subrepresentation for S. Now, since H_i is non-abelian, its restriction to one of the subspaces U_j must be non-abelian. Therefore if we define F_j as the subset of $\{1, \ldots, n\}$ consisting of those i for which H_i is non-abelian when restricted to U_j, we have two cases: if F_j is strictly contained in $\{1, \ldots, n\}$ for all j then the inductive hypothesis yields

$$m = \sum_{j=1}^{\ell} \dim U_j \geqslant 2 \sum_{j=1}^{\ell} |F_j| \geqslant 2n.$$

On the other hand, if $F_j = \{1, \ldots, n\}$ for some j we can write $S = S' \times S_n$ where $S' = S_1 \times \ldots \times S_{n-1}$ and, thanks to Lemma 5.6.1, $(U_j, \pi_{U_j} \circ \rho)$ is isomorphic to a tensor representation $(W' \otimes W_n, r' \otimes r_n)$. Now, since both $\pi_{U_j}(\rho(S'))$ and $\pi_{U_j}(\rho(S_n))$ are non abelian, $r'(S')$ and $r_n(S_n)$ must be non-abelian too. Therefore the inductive hypothesis yields $\dim W' \geqslant 2(n-1)$ and $\dim W_n \geqslant 2$, which gives $\dim U_j \geqslant 4(n-1)$. Since $n > 1$ in particular we have $\dim U_j \geqslant 2n$.

Let's suppose now that $m = 2n$ and let us distiguish between two cases: if $\ell > 1$ then $\dim U_j < 2n$ for all j, and therefore $F_j \neq \{1, \ldots, n\}$ for all j. Therefore $\sum_{j=1}^{\ell} |F_j| = n$ and thus $F_j \cap F_h = \varnothing$ for all $j \neq h$. Then $\dim U_j = 2|F_j|$ for all j and the inductive hypothesis applies. Moreover, since every H_i restricts to a non-abelian group on U_j for exactly one j while its restriction to U_h for $h \neq j$ is abelian and commutes with all H_k for $k \neq i$, Schur's Lemma gives the desired result. Conversely, if $\ell = 1$ then the system

$$\begin{cases} m \geqslant 4n - 4 \\ m = 2n \\ n > 1 \end{cases}$$

gives $n = 2$ and $m = 4$. Therefore $\rho : S_1 \times S_2 \to \mathrm{GL}(4, \mathbb{C})$ must be isomorphic to the tensor product of two irreducible 2-dimensional representations $r_i : S_i \to \mathrm{GL}(2, \mathbb{C})$ for $i = 1, 2$.

□

Corollary 5.6.3. *If $S < \mathrm{GL}(2n, \mathbb{C})$ is a semisimple algebraic group then it has no more than n almost simple factors.*

Proof. Let $S_1, \ldots, S_k$ denote the almost simple factors of S. Then the natural representation $\varphi : S_1 \times \ldots \times S_k \to \mathrm{GL}(2n, \mathbb{C})$ induced by inclusions satisfies the hypotheses of Lemma 5.6.2, which gives $k \leqslant n$. □

Proposition 5.6.4. *Let $G = G_1 \times \ldots \times G_n$ be a product Lie group, and let $\rho : G \to \mathrm{GL}^+(m, \mathbb{R})$ be a representation such that $\rho(G_i)$ is non-amenable for all $i = 1, \ldots, n$. Then $m \geqslant 2n$. If $m = 2n$ the identity component of the Zariski closure $\overline{\rho(G)}$ in $\mathrm{GL}(2n, \mathbb{C})$ is reductive. Moreover in this case there exists a commutative diagram*

$$\begin{array}{ccc} \prod_{i=1}^{n} G_i & \xrightarrow{\ \rho\ } & \mathrm{GL}^+(2n, \mathbb{R}) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\iota^+} \\ \prod_{i=1}^{n} \mathrm{GL}(2, \mathbb{R}) & \xrightarrow[\ \psi\ ]{} & \mathrm{GL}(2n, \mathbb{R}) \end{array}$$

a unique natural number $0 \leqslant t(\rho) \leqslant n$ and a unique splitting

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{t(\rho)/2} X_i \oplus \bigoplus_{i=t(\rho)+1}^{n} X_i$$

such that:

(*i*) $\dim X_i = 4$ *for* $1 \leqslant i \leqslant t(\rho)/2$ *and* $\dim X_i = 2$ *for* $t(\rho) < i \leqslant n$*;*

(*ii*) *every representation*

$$\varphi_{ij} : G_i \xrightarrow{\varphi|_{G_i}} \prod_{i=1}^{n} \mathrm{GL}(2,\mathbb{R}) \xrightarrow{\pi_j} \mathrm{GL}(2,\mathbb{R})$$

is irreducible if $j = i$ *and is scalar if* $j \neq i$*;*

(*iii*) *every representation*

$$\psi_i : \mathrm{GL}(2,\mathbb{R}) \xhookrightarrow{\iota_i} \prod_{i=1}^{n} \mathrm{GL}(2,\mathbb{R}) \xrightarrow{\psi} \mathrm{GL}(2n,\mathbb{R})$$

with $i > t(\rho)$ *is the standard representation when restricted to* X_i *while it is trivial when restricted to* X_j *for* $j \neq i$*;*

(*iv*) *every representation*

$$\psi_{(2i-1,2i)} : \mathrm{GL}(2,\mathbb{R}) \times \mathrm{GL}(2,\mathbb{R}) \xhookrightarrow{(\iota_{2i-1},\iota_{2i})} \prod_{i=1}^{n} \mathrm{GL}(2,\mathbb{R}) \xrightarrow{\quad\psi\quad} \mathrm{GL}(2n,\mathbb{R})$$

with $i \leqslant t(\rho)/2$ *is isomorphic to the standard tensor representation when restricted to* X_i *while it is scalar when restricted to* X_j *for* $j \neq i$*.*

Finally, if $t(\rho) = 0$ *then* $\rho(G)$ *is conjugated in* $\mathrm{GL}^+(2n,\mathbb{R})$ *to a subgroup of the diagonal subgroup* $\rho_\triangle(\prod_{i=1}^{n} \mathrm{GL}^+(2,\mathbb{R}))$*.*

Proof. Consider $\mathrm{GL}(m,\mathbb{R})$ as a subgroup of $\mathrm{GL}(m,\mathbb{C})$ and let H_i denote the connected component of the identity in the Zariski closure of $\rho(G_i)$ in $\mathrm{GL}(m,\mathbb{C})$. Note that all these groups commute because so do the groups $\rho(G_i)$. Let $H_i = S_i \ltimes U_i$ be a Levi decomposition with S_i reductive and U_i the unipotent radical of H_i. Now S_i is non-abelian because otherwise by Tit's alternative it would have a finite-index solvable subgroup, which would make it into an amenable group. But then H_i would be amenable too because it is an extension of U_i (which is solvable and hence amenable) by S_i. Then $\rho(G_i)$ could not be non-amenable, because it would have $\rho(G_i) \cap H_i$ as a finite-index amenable subgroup. Therefore we can apply Lemma 5.6.2 to the obvious representation $\sigma : S_1 \times \ldots \times S_n \to \mathrm{GL}(m,\mathbb{C})$ induced by inclusions and obtain $m \geqslant 2n$.

Now suppose $m = 2n$ and consider the decomposition

$$\mathbb{C}^{2n} = \bigoplus_{i=1}^{t(\sigma)/2} V_i \oplus \bigoplus_{i=t(\sigma)+1}^{n} V_i.$$

associated with σ. We rename the groups in the following way:

$$F_i = \begin{cases} H_{2i-1} \times H_{2i} \text{ if } i \leqslant \frac{t}{2} \\ H_i \text{ if } i > t \end{cases} \qquad R_i = \begin{cases} S_{2i-1} \times S_{2i} \text{ if } i \leqslant \frac{t}{2} \\ S_i \text{ if } i > t \end{cases} \qquad T_i = \begin{cases} U_{2i-1} \times U_{2i} \text{ if } i \leqslant \frac{t}{2} \\ U_i \text{ if } i > t \end{cases}$$

Any $u \in T_i$ commutes with R_j for all $j \neq i$ and thus by Schur's Lemma it acts scalarly on V_j (more precisely, being unipotent, it must act trivially on V_j). To show that V_i is u-invariant suppose by contradiction that there exists some $v \in V_i$ such that $u(v) \notin V_i$ and consider $j \neq i$ such that $\pi_j(v) \neq 0$. Now choose $s \in R_j$ such that $s(u(v)) \notin \mathrm{Span}(u(v))$, which exists since R_j acts irreducibly on V_j. Now we have

$$2 = \dim \mathrm{Span}(s(u(v)), u(v)) = \dim \mathrm{Span}(u^{-1}(s(u(v))), v) = \dim \mathrm{Span}(s(v), v) = 1$$

because s and u commute and s acts scalarly on $\mathrm{Span}(v)$. Therefore the decomposition is H_i-invariant for all i. Moreover if

$$K_i = \begin{cases} \overline{\rho(\Gamma_{2i-1})} \times \overline{\rho(\Gamma_{2i})} \text{ if } i \leqslant \frac{t}{2} \\ \overline{\rho(\Gamma_i)} \text{ if } i > t \end{cases}$$

then by repeating the reasoning with K_i in place of T_i we can prove that the decomposition is actually K_i-invariant for all i.

Now let us prove that $H_1 \cdots H_n$ is reductive. Let's begin by taking into account T_i for $i > t(\sigma)$, i.e. $T_i = U_i$. The space $V_i^{U_i}$ of U_i-invariant vectors in V_i is non-empty because U_i is unipotent. But since U_i is normal in H_i then $s^{-1}us \in U_i$ for all $s \in S_i$ and $u \in U_i$. Therefore $V_i^{U_i}$ must be also S_i-invariant and hence it must coincide with the whole V_i, which is S_i-irreducible. Thus U_i acts trivially on $\mathbb{C}^{2n}$, which means $U_i = \{\mathrm{id}\}$. Now let us consider T_i for $i \leqslant t(\sigma)/2$, for which we have

$$F_i = H_{2i-1} \times H_{2i} = (S_{2i-1} \ltimes U_{2i-1}) \times (S_{2i} \ltimes U_{2i}) \simeq (S_{2i-1} \times S_{2i}) \ltimes (U_{2i-1} \times U_{2i}) = R_i \ltimes T_i.$$

Every $u \in U_{2i}$ commutes with S_{2i-1} and thus by Lemma 5.6.1 it is of the form $\mathrm{id}_{W_{2i-1}} \otimes B$ for some $B \in \mathrm{End}(W_{2i})$. Therefore the space of U_{2i}-invariant vectors in W_{2i} is non-empty and, as before, it must coincide with the whole W_{2i}. This shows that U_{2i} is trivial, and the same can be done for U_{2i-1}. Thus we have that $H_1 \cdots H_n = S_1 \cdots S_n$ is reductive.

Finally, since K_i is closed under complex conjugation for every i and since the above decomposition for $\mathbb{C}^{2n}$ is unique, each V_j is of the form $X_j \otimes \mathbb{C}$ for some $X_j \in \mathbb{R}^{2n}$. Moreover if K_i acts irreducibly or scalarly on V_j then the same holds for G_i (or $G_{2i-1} \times G_{2i}$) on X_i. Clearly if $t = 0$ then every G_i must preserve the orientation on X_i, because it acts sclalarly on all X_j with $j \neq i$ and it globally preserves the orientation of $\mathbb{R}^{2n}$. This gives the last statement. □

Remark 5.6.5. Suppose G is isomorphic to $(G_1 \times \ldots \times G_n)/C$ for some discrete central subgroup C of $G_1 \times \ldots \times G_n$ and $\rho : G \to \mathrm{GL}^+(m, \mathbb{R})$ is a representation such that the image $\rho(G_i)$ is non-amenable for every i. Then by precomposing ρ with the projection $\pi : G_1 \times \ldots \times G_n \to G$ we get a representation $\tilde{\rho} : G_1 \times \ldots \times G_n \to \mathrm{GL}^+(m, \mathbb{R})$ satisfying the hypotheses of Lemma 5.6.4. Therefore we have $m \geqslant 2n$ and if $m = 2n$ then $\iota \circ \tilde{\rho} = \tilde{\psi} \circ \tilde{\varphi}$. Since C is central its image under φ must act scalarly on any irreducible subrepresentation of the factors G_i, and thus $\tilde{\varphi}(C)$ is central too. Moreover, since it is also contained in $\ker \tilde{\psi}$, we get the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\varphi"'] & \mathrm{GL}^+(2n, \mathbb{R}) \arrow[d, hook, "\iota^+"] \\ \left(\prod_{i=1}^n \mathrm{GL}(2, \mathbb{R})\right) / \tilde{\varphi}(C) \arrow[r, "\psi"'] & \mathrm{GL}(2n, \mathbb{R}) \end{tikzcd}$$

If the restriction of $\tilde{\psi}$ to a factor (or a pair of factors) of $\prod_{i=1}^n \mathrm{GL}(2, \mathbb{R})$ is irreducible or scalar (or a tensor product of irreducible representations) on some subspace of $\mathbb{R}^{2n}$ then the same holds for ψ restricted to the image of the corresponding factor (or pair of factors) in $\left(\prod_{i=1}^n \mathrm{GL}(2, \mathbb{R})\right) / \tilde{\varphi}(C)$. We can then set $t(\rho) = t(\tilde{\rho})$ and if $t(\rho) = 0$ then $\rho(G)$ is conjugated in $\mathrm{GL}^+(2n, \mathbb{R})$ to a subgroup of the diagonal subgroup $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}))$.

Terminology: Whenever a Lie group G is isomorphic to $(G_1 \times \ldots \times G_n)/C$ for some discrete central subgroup C of $G_1 \times \ldots \times G_n$ we say that G is an *almost direct product of* $G_1, \ldots, G_n$. If $G < \mathrm{GL}^+(m, \mathbb{R})$ is an almost direct product and all of its factors are non-amenable we write $t(G)$ to indicate $t(\iota)$ for the inclusion $\iota : G \hookrightarrow \mathrm{GL}^+(m, \mathbb{R})$.

Lemma 5.6.6. *Let $G_1, \ldots, G_n$ be Lie groups, let G be an almost direct product of $G_1, \ldots, G_n$ and let $\rho : G \to \mathrm{GL}^+(2n, \mathbb{R})$ be a representation such that $\rho(G_i)$ is non-amenable for every i. If $t(\rho) > 0$ then*

$$\rho^*(\varepsilon_{2n}) = 0$$

Proof. Set $\mathrm{GL}^+ := \prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R})$ and $\mathrm{GL} := \prod_{i=1}^n \mathrm{GL}(2, \mathbb{R})$. Let us suppose at first that G is a direct product and that the image of the homomorphism $\varphi : G \to \mathrm{GL}$ given by Lemma 5.6.4 is contained in GL^+. Now it follows from the definition of $\psi : \mathrm{GL} \to \mathrm{GL}(2n, \mathbb{R})$ that $\psi(\mathrm{GL}^+)$ is contained in $\mathrm{GL}^+(2n, \mathbb{R})$ and therefore we have

$$\begin{tikzcd} G \arrow[r, "\varphi"'] \arrow[rr, bend left, "\rho"] & \mathrm{GL}^+ \arrow[r, "\psi'"'] & \mathrm{GL}^+(2n, \mathbb{R}) \end{tikzcd}$$

where $\psi' := \psi|_{\mathrm{GL}^+}$. Set $\mathrm{SL} := \prod_{i=1}^n \mathrm{SL}(2, \mathbb{R})$ and consider the inclusion $\iota_{\mathrm{SL}} : \mathrm{SL} \hookrightarrow \mathrm{GL}^+$ and the projection $\pi_{\mathrm{GL}^+} : \mathrm{GL}^+ \to \mathrm{SL}$. Then, since $t(\rho) > 0$, we can suppose, up to conjugation by an element in $\mathrm{GL}^+(2n, \mathbb{R})$, that $\psi' \circ \iota_{\mathrm{SL}}$ has the form

$$\begin{array}{rccc} \psi' \circ \iota_{\mathrm{SL}} : & \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times S & \to & \mathrm{GL}^+(2n, \mathbb{R}) \\ & (a, b, c) & \mapsto & \begin{pmatrix} \rho_\otimes(a,b) & 0 \\ 0 & \psi''(c) \end{pmatrix} \end{array}$$

where $S = \prod_{i=3}^n \mathrm{SL}(2, \mathbb{R})$ and where $\psi'' : S \to \mathrm{GL}^+(2n-4, \mathbb{R})$ is a suitable representation. Now ψ' and $\psi' \circ \iota_{\mathrm{SL}} \circ \pi_{\mathrm{GL}^+}$ lie in the same path-connected component of the space of representations $\mathrm{Rep}(\mathrm{GL}^+, \mathrm{GL}^+(2n, \mathbb{R}))$ because the curve $\sigma : t \mapsto \sigma_t$ defined by

$$\sigma_t : (a_1, \ldots, a_n) \mapsto \psi' \left((\det a_1)^{-t/2} a_1, \ldots, (\det a_n)^{-t/2} a_n \right)$$

satisfies $\sigma_0 = \psi'$ and $\sigma_1 = \psi' \circ \iota_{\mathrm{SL}} \circ \pi_{\mathrm{GL}^+}$. Therefore, thanks to Lemma 1.2.14, we have

$$(\psi')^*(\varepsilon_{2n}) = \pi^*_{\mathrm{GL}^+}((\psi' \circ \iota_S)^*(\varepsilon_{2n})) = \pi^*_{\mathrm{GL}^+}(\pi_1^*(\rho_\otimes^*(\varepsilon_4)) \smile \pi_2^*((\psi'')^*(\varepsilon_{2n-2})))$$

Then, thanks to Lemma 5.4.1, we have $\rho^*(\varepsilon_{2n}) = 0$.

Now if G is an almost direct product then, using the notation of Remark 5.6.5, we have that $\tilde{\varphi}(C)$ is central in GL, and therefore it must be contained in GL^+. Therefore we have well defined homomorphisms

$$\begin{tikzcd} G_i \arrow[r, hook, "\iota_i"'] \arrow[rrrr, bend left, "\overline{\varphi}_i"] & G_1 \times \ldots \times G_n \arrow[r, "\pi"'] & G \arrow[r, "\varphi"'] & \mathrm{GL}/\tilde{\varphi}(C) \arrow[r] & \mathrm{GL}/\mathrm{GL}^+ \end{tikzcd}$$

and $H_i := \ker \overline{\varphi}_i$ is a finite index subgroup of G_i. Thus $H := H_1 \times \ldots \times H_n$ is a finite index subgroup of $G_1 \times \ldots \times G_n$ which is mapped into GL^+ by $\tilde{\varphi}$, and $\rho(H_i)$ is a finite index subgroup of $\rho(G_i)$ (in particular it must be non-amenable). Thus the homomorphism

$$\begin{tikzcd} H_1 \times \ldots \times H_n \arrow[r, hook, "\iota"'] \arrow[rrr, bend left, "\tilde{\rho}"] & G_1 \times \ldots \times G_n \arrow[r, "\pi"'] & G \arrow[r, "\rho"'] & \mathrm{GL}^+(2n, \mathbb{R}) \end{tikzcd}$$

gives $\tilde{\rho}^*(\varepsilon_{2m}) = 0$ thanks to the previous case. Now $\tilde{\rho}^*(\varepsilon_{2m}) = \iota^*(\pi^*(\rho^*(\varepsilon_{2m})))$ and $\iota^* \circ \pi^*$ is injective thanks to Lemmas 5.2.11 and 5.2.14. □

5.7 Representations of lattices in $\mathrm{Isom}^+((\mathbb{H}^2)^n)$

We focus now on the study of representations of lattices in $\mathrm{Isom}^+((\mathbb{H}^2)^n)$, and in particular those contained in the connected component of the identity, which is isomorphic to $\prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$. The main ingredient of our discussion is Margulis' super-rigidity Theorem, which can be found in the Appendices (Theorem 6.5.23) or in Margulis' book [23].

Lemma 5.7.1. *Let Γ be an irreducible lattice in $G := \prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$ with $n \geqslant 2$ and let $\rho : \Gamma \to \mathrm{GL}^+(m,\mathbb{R})$ be a representation. If S denotes the connected component of the identity in the Zariski closure of $\rho(\Gamma)$ in $\mathrm{GL}(m,\mathbb{C})$ then S is semisimple. Moreover, if S has k non-compact factors whose product is denoted by S^{nc}, then $k \leqslant n$ and $S^{nc}(\mathbb{R})^0$ covers $\prod_{i=1}^k \mathrm{PSL}(2,\mathbb{R})$.*

Proof. The semisimplicity of S is an immediate consequence of Theorem 6.5.21. If S^c and S^{nc} denote the subgroups of compact and non-compact factors of S respectively then, since $S^c \cap S^{nc}$ is central in both subgroups, we have a well defined surjective homomorphism $S/S^c \to \mathrm{Ad}(S^{nc})$. If Γ' denotes $\rho^{-1}(S)$ then Γ' is irreducible in G and the homomorphism

$$\begin{tikzcd} \Gamma' \arrow[r, "\rho|_{\Gamma'}"'] \arrow[rrr, bend left, "\sigma"] & S \arrow[r] & S/S^c \arrow[r] & \mathrm{Ad}(S^{nc}) \end{tikzcd}$$

has dense image. Therefore, since $\mathrm{Ad}(S^{nc})(\mathbb{R})^0 = \mathrm{Ad}(S^{nc}(\mathbb{R})^0)$, Margulis' super-rigidity Theorem 6.5.23 gives a surjective homomorphism $\tilde{\sigma} : G \to \mathrm{Ad}(S^{nc}(\mathbb{R})^0)$ which extends σ. Therefore $\mathrm{Ad}(S^{nc}(\mathbb{R})^0)$ is a quotient of G and thus it must be of the form $\prod_{i=1}^k \mathrm{PSL}(2,\mathbb{R})$ for $k \leqslant n$. Then, since every factor of S^{nc} determines a different almost simple normal subgroup of $\mathrm{Ad}(S^{nc}(\mathbb{R})^0)$, we conclude. $\square$

Now let us consider $G = \prod_{i=1}^n \mathrm{PSL}(2,\mathbb{R})$ and a general lattice $\Gamma < G$. Since $\mathrm{PSL}(2,\mathbb{R})$ is simple Theorem 6.5.17 gives a realization $G = G_1 \times \ldots \times G_m$ where $G_i = \prod_{j=1}^{h_i} \mathrm{PSL}(2,\mathbb{R})$ and $\Gamma_i := \Gamma \cap G_i$ is an irreducible lattice in G_i. Moreover, up to replacing Γ by a finite index subgroup, we can suppose that $\Gamma = \Gamma_1 \times \ldots \times \Gamma_m$. We say a factor Γ_i of Γ is *rigid* if $h_i > 1$. Then Γ is *rigid* if all of its factors are rigid, while it is *completely reducible* if none of its factors is rigid. We break up the study of a generic representation $\rho : \Gamma \to \mathrm{GL}^+(2n,\mathbb{R})$ into three cases, depending on the decomposition of Γ.

Case 1 - Γ is completely reducible: if $m = n$ then each G_i equals $\mathrm{PSL}(2,\mathbb{R})$ and no factor of Γ is rigid. Therefore Proposition 5.6.4 gives the following possibilities:

(*i*) $\rho(\Gamma_i)$ is amenable for some $i \in \{1,\ldots,n\}$;

(*ii*) $t(\rho) > 0$;

(*iii*) $\rho(\Gamma)$ conjugated in $\mathrm{GL}^+(2n,\mathbb{R})$ to a subgroup of $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R}))$.

Case 2 - Γ is rigid: if $h_i > 1$ for all $i = 1,\ldots,m$ then all factors G_i of G have real rank at least 2.

Proposition 5.7.2. *Up to replacing Γ by a finite index subgroup ρ admits a factorization*

$$\begin{tikzcd} \Gamma \arrow[r, "\varphi"'] \arrow[rr, bend left, "\rho"] & H \arrow[r, "\psi"'] & \mathrm{GL}^+(2n,\mathbb{R}) \end{tikzcd}$$

such that one of the following conditions holds:

(*i*) *there exists an amenable normal subgroup* $N \lhd H$ *such that* H/N *covers* $\prod_{i=1}^k \mathrm{PSL}(2,\mathbb{R})$ *with* $k < n$*;*

(*ii*) H *is an almost direct product of* n *Lie groups* $H_1, \ldots, H_n$ *such that* $\psi(H_i)$ *is non-amenable for every* $i = 1, \ldots, n$*. In particular either* $t(\psi) > 0$ *or* $\psi(H)$ *is conjugated to a subgroup of the diagonal group* $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R}))$.

Proof. The Zariski closure S_j of $\rho(\Gamma_j)$ in $\mathrm{GL}(2n,\mathbb{C})$ has finitely many connected components. Hence, up to replacing Γ by a finite index subgroup, we may assume that S_j is connected for each j. We set $S := \overline{\rho(\Gamma)} = S_1 \cdots S_m$ and $H := S_1(\mathbb{R})^0 \times \ldots \times S_m(\mathbb{R})^0$ so that, up to further replacing Γ by a finite index subgroup if necessary, ρ factorizes as

$$\Gamma \xrightarrow[\varphi]{} H \xrightarrow[\psi]{} \mathrm{GL}^+(2n,\mathbb{R})$$

where ψ is induced by the inclusions of the factors. Now S is a semisimple subgroup of $\mathrm{GL}(2n,\mathbb{C})$ and thus, thanks to Corollary 5.6.3, it has no more than n factors. Therefore we have two cases: if for some j the number of non-compact factors of S_j is strictly less than h_j then we can set $N := S_1^c(\mathbb{R})^0 \times \ldots \times S_m^c(\mathbb{R})^0$, and we are in the first case. On the other hand if every S_j has exactly h_j non-compact factors, then $S_j = S_j^{nc}$ for all j. Therefore Remark 5.6.5 applies to $\psi : H \to \mathrm{GL}^+(2n,\mathbb{R})$ and, since the factors of H are exactly n, either $t(\psi) > 0$ or $\psi(H)$ is conjugated to $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R}))$. □

Case 3 - The mixed case: some but not all of the factors Γ_i are rigid. Up to reordering them we may write $\Gamma = \Gamma^r \times \Gamma^{nr} < G^r \times G^{nr} = G$ where $\Gamma^r < G^r$ is the rigid part and $\Gamma^{nr} < G^{nr}$ is the non-rigid part. Then $G^r = \prod_{i=1}^k \mathrm{PSL}(2,\mathbb{R})$ with $k < n$ and $G^{nr} = \prod_{i=1}^{n-k} \mathrm{PSL}(2,\mathbb{R})$. The rigid part decomposes like $\Gamma_1^r \times \ldots \times \Gamma_\ell^r$ with $\ell < k$ while the non-rigid part decomposes like $\Gamma_1^{nr} \times \ldots \times \Gamma_{n-k}^{nr}$.

Proposition 5.7.3. *Up to replacing each rigid factor by a finite index subgroup (without changing the non-rigid factors) we have thet either* $\rho(\Gamma_j^{nr})$ *is amenable for some non-rigid factor or* ρ *admits a factorization*

$$\begin{tikzcd} \Gamma^r \times \Gamma^{nr} \arrow[r, "\varphi\times\mathrm{id}"'] \arrow[rr, bend left, "\rho"] & H \times \Gamma^{rn} \arrow[r, "\psi"'] & \mathrm{GL}^+(2n,\mathbb{R}) \end{tikzcd}$$

such that one of the following conditions holds:

(*i*) *there exists an amenable normal subgroup* $N \lhd H$ *such that* H/N *covers* $\prod_{i=1}^h \mathrm{PSL}(2,\mathbb{R})$ *with* $h < k$*;*

(*ii*) $H \times \Gamma^{rn}$ *is an almost direct product of* n *Lie groups* $H_1, \ldots, H_k, \Gamma_1^{nr}, \ldots, \Gamma_{n-k}^{nr}$ *such that* $\psi(H_i)$ *is non-amenable for every* $i = 1, \ldots, n$*. In particular either* $t(\psi) > 0$ *or* $\psi(H \times \Gamma^{rn})$ *is conjugated to a subgroup of the diagonal group* $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R}))$.

Proof. Once again, up to replacing Γ_j^r by a finite index subgroup, we may assume that the Zariski closure S_j^r of $\rho(\Gamma_j^r)$ in $\mathrm{GL}(2n,\mathbb{C})$ is connected for each rigid factor. Then we can set $H := S_1^r(\mathbb{R})^0 \times \ldots \times S_\ell^r(\mathbb{R})^0$ so that, up to further replacing Γ by a finite index subgroup if necessary, ρ factorizes as

$$\Gamma^r \times \Gamma^{nr} \xrightarrow{\varphi\times\mathrm{id}} H \times \Gamma^{rn} \xrightarrow{\psi} \mathrm{GL}^+(2n,\mathbb{R})$$

where ψ is induced by the inclusions on H and is given by the restriction of ρ on Γ^{rn}. Now the identity component S^{nr} of the Zariski closure of $\rho(\Gamma^{nr})$ in $\mathrm{GL}(2n,\mathbb{C})$ is reductive thanks to Proposition 5.6.4. Therefore, if we apply Lemma 5.6.2 to the representation induced by the inclusions of all the almost simple factors of the group $S^r \times S^{nr}$ we obtain that H cannot have more than k factors. Therefore, repeating the argument of the previous Proposition, we conclude. □

5.8 Milnor-Wood inequality for $(\mathbb{H}^2)^n$-manifolds

Theorem 5.8.1. *Let Γ be a cocompact lattice in $\mathrm{Isom}^+((\mathbb{H}^2)^n)$ and $\rho : \Gamma \to \mathrm{GL}^+(2n, \mathbb{R})$ a representation. Then*

$$\|\rho^*(\varepsilon_{2n})\| \leqslant \|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\|$$

Proof. The inclusion of a finite index subgroup $\Gamma' \hookrightarrow \Gamma$ induces isometric embeddings of both continuous and bounded cohomology groups $H^*(\Gamma) \to H^*(\Gamma')$ and $H_b^*(\Gamma) \to H_b^*(\Gamma')$ thanks to Lemmas 5.2.11 and 5.2.13. As a result, we can replace Γ by a finite index subgroup which is contained in the identity connected component $\mathrm{Isom}^+((\mathbb{H}^2)^n)^0 \simeq \prod_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ and which decomposes as in the previous Section. We will argue case by case by showing that if $\rho^*(\varepsilon_{2n}) \neq 0$ then, up to replacing once again Γ by a finite index subgroup, $\rho(\Gamma)$ is conjugated to a subgroup of the diagonal group $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}))$. By Lemma 5.4.4 we have that if for some j the image $\rho(\Gamma_j)$ is amenable then, since Γ_j is the fundamental group of an aspherical closed oriented manifold, $\rho^*(\varepsilon_{2n}) = 0$. Therefore we shall assume below that this is not the case.

Case 1: Γ is completely reducible. Then, assuming $\rho^*(\varepsilon_{2n}) \neq 0$, we have by Lemma 5.6.6 that $t(\rho) = 0$, and thus we deduce that $\rho(\Gamma)$ is conjugated to a subgroup of $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}))$, as discussed in Case 1 of the previous Section.

Case 2: Γ is rigid. Then, up to replacing Γ by a further finite index subgroup if necessary, we get from Proposition 5.7.2 that ρ^* factors like

$$H_c^{2n}(\mathrm{GL}^+(2n, \mathbb{R})) \xrightarrow{\psi^*} H_c^{2n}(H) \xrightarrow{\varphi^*} H^{2n}(\Gamma)$$

Now if $k < n$ and we set $P^k := \prod_{i=1}^k \mathrm{PSL}(2, \mathbb{R})$ then $H_c^{2n}(P^k) = 0$ thanks to the Künneth formula in Lemma 5.2.10 and to the Remark 5.2.12. Therefore in case (i) Lemma 5.2.14 gives the following commutative diagram:

$$\begin{tikzcd}
0 = H_c^{2n}(P^k) \arrow[r] & H_c^{2n}(H/N) \arrow[r] & H_c^{2n}(H) \\
H_{c,b}^{2n}(P^k) \arrow[u, "c"] \arrow[r, leftrightarrow] & H_{c,b}^{2n}(H/N) \arrow[u, "c"] \arrow[r, leftrightarrow] & H_{c,b}^{2n}(H) \arrow[u, "c"]
\end{tikzcd}$$

Hence there are no bounded elements in $H_c^{2n}(H)$, and thus $\rho^*(\varepsilon_{2n}) = 0$. On the other hand, in case (ii) either $t(\psi) > 0$, in which case $\rho^*(\varepsilon_{2n}) = 0$ by Lemma 5.6.6, or $\rho(\Gamma)$ is conjugated to a subgroup of $\rho_\triangle(\prod_{i=1}^n \mathrm{GL}^+(2, \mathbb{R}))$.

Case 3: The mixed case. Up to replacing Γ by a further finite index subgroup, if necessary, we get from Proposition 5.7.3 that ρ^* factors like

$$H_c^{2n}(\mathrm{GL}^+(2n, \mathbb{R})) \xrightarrow{\psi^*} H_c^{2n}(H \times \Gamma^{nr}) \xrightarrow{(\varphi \times \mathrm{id})^*} H^{2n}(\Gamma)$$

Now since every non-rigid factor Γ_j^{nr} is a cocompact lattice in $\mathrm{PSL}(2, \mathbb{R})$ Lemma 5.2.11 gives $H^*(\Gamma_j^{nr}) \simeq H_c^*(\mathrm{PSL}(2, \mathbb{R}))$. Therefore, just like before, we have that case (i) gives $\rho^*(\varepsilon_{2n}) = 0$, and the same holds for case (ii) with $t(\psi) > 0$.

Thus we reduced orselves to the situation where, up to conjugating the image, ρ factors through a map ρ_0 into $\prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R})$:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho_0"'] \arrow[rr, bend left, "\rho"] & \prod_{i=1}^n \mathrm{GL}^+(2,\mathbb{R}) \arrow[r, "\rho_\triangle"'] & \mathrm{GL}^+(2n,\mathbb{R}) \end{tikzcd}$$

Now Lemma 5.4.1 gives $\rho_\triangle^*(\varepsilon_{2n}) = \pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)$ and therefore

$$\|\rho^*(\varepsilon_{2n})\| = \|\rho_0^*(\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2))\| \leqslant \|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\|$$

□

Theorem 5.8.2. *Let M be an oriented closed manifold whose universal Riemannian cover is $(\mathbb{H}^2)^n$ and let $\pi : E \to M$ be an oriented rank-$2n$ flat vector bundle. Then*

$$|\langle e_\mathbb{R}(E), [M]\rangle| \leqslant \frac{1}{(-2)^n}\chi(M)$$

Proof. Let Γ be the fundamental group of M embedded as a cocompact lattice in $\mathrm{Isom}^+((\mathbb{H}^2)^n)$ acting on $(\mathbb{H}^2)^n$ by deck transformations. Let $\rho : \Gamma \to \mathrm{GL}^+(2n,\mathbb{R})$ be the representation inducing the flat bundle $\pi : E \to M$. Then we have $e_\mathbb{R}(E) = f^*(\rho^*(\varepsilon_m(G)))$ where $f : M \to B\Gamma$ denotes the characteristic map for the universal cover $p : \tilde{M} \to M$. Since f^* is an isometry, Theorem 5.8.1 gives

$$\|e_\mathbb{R}(E)\| = \|\rho^*(\varepsilon_{2n})\| \leqslant \|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\|$$

Now, thanks to (ii) in Lemma 5.1.4 and to Lemma 5.5.6, we have

$$|\langle e_\mathbb{R}(E), [M]\rangle| = \|e_\mathbb{R}(E)\| \cdot \|M\| = \frac{\|e_\mathbb{R}(E)\| \cdot \chi(M)}{(-2)^n \cdot \|\pi_1^*(\varepsilon_2) \smile \ldots \smile \pi_n^*(\varepsilon_2)\|} \leqslant \frac{1}{(-2)^n}\chi(M)$$

□

5.9 Chern's conjecture for manifolds locally isomorphic to a product of surfaces of constant curvature

Thanks to Theorem 5.8.2 both Conjectures 1 and 2 can be confirmed for all closed oriented manifolds which are locally isomorphic to a product of surfaces of constant curvature. Indeed if M is such a manifold consider its Riemannian universal cover $\tilde{M} \simeq (\mathbb{R}^2)^{n_1} \times (S^2)^{n_2} \times (\mathbb{H}^2)^{n_3}$. Then, if $n_1 > 0$, the equality

$$[\mathrm{Pf}(\Omega_M)] = [\mathrm{Pf}(\Omega_{\mathbb{R}^2})]^{n_1} \smile [\mathrm{Pf}(\Omega_{S^2})]^{n_2} \smile [\mathrm{Pf}(\Omega_{\mathbb{H}^2})]^{n_3},$$

together with Theorem 2.4.3 gives $\chi(M) = 0$. On the other hand, if $n_2 > 0$, then, up to replacing M by some finite cover, we may suppose that M is the total space of an $(S^2)^{n_2}$-bundle $\pi_M : M \to N$ where N is apsherical of dimension strictly less than $m = 2(n_1+n_2+n_3)$. Therefore the exact sequence of homotopy groups for a fibration gives $\pi_1(M, x_0) \simeq \pi_1(N, \pi_M(x_0))$. Thus $H^m(\pi_1(M, x_0)) \simeq H^m(\pi_1(N, \pi_M(x_0))) \simeq H^m(N; \mathbb{R}) = 0$. Therefore we may suppose thet $n_1 = n_2 = 0$ and apply Theorem 5.8.2 to get the desired result.

Chapter 6

Appendices

This chapter is devoted to the presentation (without proofs) of the main properties of the objects of our studies. Detailed expositions of the subjects can be found in [25] and [29] as well as in [21] (for principal bundles and connections), in [13] (for obstruction theory) and in [5] and [23] (for lattices and algebraic groups). We will deal mostly with real and complex vector bundles, and we will denote with $\mathbb{K}$ the scalar field in arguments working for both $\mathbb{R}$ and $\mathbb{C}$. All topological spaces are to be considered second-countable, locally compact and Hausdorff and all maps between them are to be considered continuous. Analogously, all maps between smooth manifolds are to be considered $\mathscr{C}^\infty$.

6.1 Fiber bundles

Let B and F be topological spaces. An *F-bundle over B* is a topological space E endowed with a surjective map $\pi : E \to B$, called the *projection*, such that there exists an open covering $\{U_\alpha\}_{\alpha\in J}$ of B which yields homeomorphisms, called *local trivializations*, of the following form:

$$\begin{array}{rccc} \chi_\alpha : & \pi^{-1}(U_\alpha) & \to & U_\alpha \times F \\ & u & \mapsto & (\pi(u), \psi_\alpha(u)) \end{array}$$

E is called the *total space*, B is called the *base*, F is called the *fiber* and $\{(U_\alpha, \chi_\alpha)\}_{\alpha\in J}$ is called the *bundle atlas*.

Whenever $U_\alpha \cap U_\beta = \varnothing$ we get a homeomorphism

$$\begin{array}{rccc} \chi_\beta \circ \chi_\alpha^{-1} : & U_\alpha \cap U_\beta \times F & \to & U_\alpha \cap U_\beta \times F \\ & (x, \xi) & \mapsto & (x, \varphi_{\alpha\beta}(x)(\xi)) \end{array}$$

where maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ are called *transition functions*.

Proposition 6.1.1. *An open covering $\{U_\alpha\}$ of B and a set of maps $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ define an F-bundle over B if and only if the maps $\varphi_{\alpha\beta}$ satisfy the cochain conditions:*

(*i*) $\varphi_{\alpha\alpha} \equiv \mathrm{id}_F$

(*ii*) $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} \equiv \mathrm{id}_F$

A *bundle map* between an F-bundle $\pi : E \to B$ and an F'-bundle $\pi' : E' \to B'$ is a pair of maps $f_E : E \to E'$ and $f_B : B \to B'$ which makes the following into a commutative diagram:

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f_B"'] & B' \end{tikzcd}$$

Remark 6.1.2. Smooth bundles are obtained by requiring the spaces B, F and E to be smooth manifolds and all homeomorphisms to be diffeomorphisms. Bundle maps between smooth bundles are given by commutative diagrams of smooth maps.

Let $\pi : E \to B$ be an F-bundle and let $f : B' \to B$ be a map. The *pull-back by f* of $\pi : E \to B$ is the F-bundle over B' of total space

$$f^*E := \{(y, u) \in B' \times E \mid f(y) = \pi(u)\}$$

and projection $\pi_{f^*E} := \pi_1\big|_{f^*E}$. The bundle atlas is given by the open covering $\{V_\alpha = f^{-1}(U_\alpha)\}$ with local trivializations:

$$\begin{array}{rccc} f^*\chi_\alpha : & (\pi_{f^*E})^{-1}(V_\alpha) & \to & V_\alpha \times F \\ & (y, u) & \mapsto & (y, \psi_\alpha(u)) \end{array}$$

The map

$$\begin{array}{rccc} \tilde{f} := \pi_2\big|_{f^*E} : & f^*E & \to & E \\ & (y, u) & \mapsto & u \end{array}$$

together with $f : B' \to B$ gives a bundle map.

Remark 6.1.3. Every section $\sigma \in \Gamma(E)$ can be pulled back to a section $f^*\sigma \in \Gamma(f^*E)$ simply by defining

$$f^*\sigma(y) = (y, \sigma(f(y))) \in f^*E_y$$

Theorem 6.1.4. *Let $\pi : E \to B$ be an F-bundle. If $f_0, f_1 : B' \to B$ are homotopic maps then the F-bundles $\pi_{f_0^*E} : f_0^*E \to B'$ and $\pi_{f_1^*E} : f_1^*E \to B'$ are isomorphic.*

Corollary 6.1.5. *If B is contractible, then all F-bundles over B are equivalent to the trivial bundle $\pi_1 : B \times F \to B$.*

For an F-bundle $\pi : E \to B$ we have a long exact sequence of homotopy groups defined as follows: let $x_0 \in B$ be a basepoint and $u_0 \in E_{x_0}$ be a basepoint in the fiber of x_0. Every map $f : (I^k, \partial I^k) \to (B, x_0)$ can be lifted to a map $\tilde{f} : (I^k, \partial I^k, J^{k-1}) \to (E, E_{x_0}, u_0)$, where J^{k-1} denotes the closure of $\partial I^k \smallsetminus (I^{k-1} \times \{0\})$ in ∂I^k . This map in turn restricts to a map $\tilde{f}\big|_{I^{k-1}\times\{0\}} : (I^{k-1}, \partial I^{k-1}) \to (E_{x_0}, u_0)$. Therefore we can define:

$$\begin{array}{rccc} \partial : & \pi_k(B, x_0) & \to & \pi_{k-1}(E_{x_0}, u_0) \\ & [f] & \mapsto & \left[\tilde{f}\big|_{I^{k-1}\times\{0\}}\right] \end{array}$$

Theorem 6.1.6. *The sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k(E_{x_0}, u_0) \xrightarrow{\iota_*} \pi_k(E, u_0) \xrightarrow{\pi_*} \pi_k(B, x_0) \xrightarrow{\partial} \pi_{k-1}(E_{x_0}, u_0) \longrightarrow \cdots$$

is exact.

G-structures

Let G be a topological group acting on the left of a F via a continuous homomorphism $\varphi : G \to \mathrm{Homeo}(F)$. A *$G$-atlas* for an F-bundle $\pi : E \to B$ is a bundle atlas whose transition functions factorize through φ, that is

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{g_{\alpha\beta}} G \xrightarrow{\varphi} \mathrm{Homeo}(F)$$

A *G-structure* for $\pi : E \to B$ is a maximal G-atlas. A G-structure makes an F-bundle into an *(F, G)-bundle*, and G is called the *structure group*.

Remark 6.1.7. If G is a Lie group a smooth G-structure on a smooth F-bundle is a maximal smooth G-atlas with respect to a Lie group morphism $\varphi : G \to \mathrm{Diff}(F)$.

Example 6.1.8. A rank-m $\mathbb{K}$-vector bundle $\pi : E \to B$ with structure group G can be seen as a $(\mathbb{K}^m, G)$-bundle where the left action of G onto $\mathbb{K}^m$ is determined by a $\mathbb{K}$-linear representation. Indeed, since transition function are $\mathrm{GL}(m, \mathbb{K})$-valued, each fiber E_x is automatically endowed with a well-defined m-dimensional $\mathbb{K}$-vector space structure.

Let $h : G' \to G$ be a continuous homomorphism of topological groups. If we can find a G-atlas whose transition functions have the form

$$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \xrightarrow{g'_{\alpha\beta}} G' \xrightarrow{h} G \xrightarrow{\varphi} \mathrm{Homeo}(F)$$

we can define a G' structure on $\pi : E \to B$, and we say the structure group can be *reduced* from G to G'.

Example 6.1.9. If a real rank-m vector bundle is oriented the choice of an orientation yields the reduction of the structure group from $\mathrm{GL}(m, \mathbb{R})$ to $\mathrm{GL}^+(m, \mathbb{R})$. A $\mathrm{GL}^+(m, \mathbb{R})$-atlas is obtained by composing the local trivializations in the $\mathrm{GL}(m, \mathbb{R})$-structure which do not preserve the orientations of fibers with a reflection on the $\mathbb{R}^m$ component. Analogously the choice of a Riemannian (Hermitian) metric g for a smooth rank-m $\mathbb{K}$-vector bundle yields the reduction of the structure group from $\mathrm{GL}(m, \mathbb{K})$ to $\mathrm{O}(m)$ ($\mathrm{U}(m)$) and an atlas is obtained by applying the Gram-Schmidt process to all local frames inducing the trivializations.

A bundle map between (F, G)-bundles $\pi : E \to B$ and $\pi' : E' \to B'$ is an (F, G)-*bundle map* if, for all (U_α, χ_α) in the G-atlas of E and (U'_β, χ'_β) in the G-atlas of E' with $f_B(U_\alpha) \cap U'_\beta \neq \varnothing$, there exist maps $f_{\alpha\beta} : U_\alpha \cap f_B^{-1}(U'_\beta) \to G$ such that:

$$\chi'_\beta \circ f_E \circ \chi_\alpha^{-1} : (x, \xi) \mapsto (f_B(x), \varphi(f_{\alpha\beta}(x))(\xi))$$

For any map $f : B' \to B$ and any (F, G)-bundle $\pi : E \to B$ the pull-back by f of $\pi : E \to B$ is automatically an (F, G)-bundle over B' and $\tilde{f}$ gives an (F, G)-bundle map. Moreover, if f_0 and f_1 are homotopic maps, then the pull-backs by f_0 and f_1 are isomorphic as (F, G)-bundles.

Principal bundles

A *principal G-bundle* $P(B, G)$ is a (G, G)-bundle $\pi : P \to B$ with left action of G onto itself given by the left translation

$$\begin{array}{rccc} \varphi : & G & \to & \mathrm{Homeo}(G) \\ & a & \mapsto & [L_a : b \mapsto ab] \end{array}$$

On every principal G-bundle there is a naturally defined right action of G. Indeed, for all $a \in G$, we have local right translations given by:

$$\begin{array}{rccc} R_a : & \pi^{-1}(U_\alpha) & \to & \pi^{-1}(U_\alpha) \\ & u & \mapsto & ua := \chi_\alpha^{-1}(\pi(u), \psi_\alpha(u)a) \end{array}$$

These homeomorphisms globalize because transition functions appear as left translations, which commute with right translations, and we immediatly have $R_{ab} = R_b \circ R_a$. The right action is clearly free and the orbits coincide with the fibers of π.

Remark 6.1.10. A bundle map f between principal G-bundles is a (G, G)-bundle map if and only if it is equivariant with respect to the right action of G, i.e. $f_P(ua) = f_P(u)a$. Such a map will be called a *principal G-bundle map*.

Proposition 6.1.11. *Let $\pi : P \to M$ be a surjective submersion of smooth manifolds and let G be a Lie group. If G acts freely on P on the right so that the orbits of the action coincide with the fibers of π, then $P(M, G)$ is a smooth principal bundle.*

Remark 6.1.12. A principal G-bundle admits a global section if and only if it is isomorphic to the trivial bundle. Indeed, if $\sigma : B \to P$ is a global section, then the map:

$$\begin{array}{rccc} f_{B\times G}: & B \times G & \to & P \\ & (x, a) & \mapsto & \sigma(x)a \end{array}$$

together $f_B = \mathrm{id}_B$ defines a principal bundle isomorphism.

Associated bundles

Let $\pi_E : E \to B$ be an (F, G)-bundle. Since the transition functions of its G-atlas satisfy the cochain conditions, we can use them to construct a principal G-bundle *associated* with $\pi_E : E \to B$ simply by letting G act on itself by left translation.

Conversely, let $\pi_P : P \to B$ be a principal G-bundle and F be a space on which G acts on the left via:

$$\begin{array}{rccc} \varphi: & G & \to & \mathrm{Homeo}(F) \\ & a & \mapsto & [\xi \mapsto a\xi := \varphi(a)(\xi)] \end{array}$$

We can make G act on the right of $P \times F$ by defining $(u, \xi) \cdot a := (ua, a^{-1}\xi)$ for all $a \in G$. Thus we can construct an (F, G)-bundle $\pi_E : E \to B$ *associated* with $P(B, G)$ having total space $E := (P \times F)/G = \{u\xi := [(u, \xi)] \big| u \in P, \xi \in F\}$ and projection $\pi_E : u\xi \mapsto \pi_P(u)$. To show the local triviality we consider a local trivialization $(U_\alpha, \chi_\alpha^P)$ in the G-structure of P. Then we have $\pi_E^{-1}(U_\alpha) = (\pi_P^{-1}(U_\alpha) \times F)/G \simeq (U_\alpha \times G \times F)/G$ via the map $u\xi \mapsto [(\pi_P(u), \psi_\alpha(u), \xi)]$ and we get trivializations:

$$\begin{array}{rccc} \chi_\alpha^E: & \pi_E^{-1}(U_\alpha) & \to & U_\alpha \times F \\ & u\xi & \mapsto & (\pi_P(u), \psi_\alpha(u)\xi) \end{array}$$

Proposition 6.1.13. *These associations are inverse with each other. Two (F, G)-bundles are isomorphic if and only if their associated principal G-bundles are isomorphic.*

Example 6.1.14. If $h : G' \to G$ is a continuous homomorphism of topological groups then G' acts on G by left translation via h. The above construction yields, for each principal bundle $P'(B, G')$, an associated principal bundle $P(B, G)$ endowed with a G'-structure. In this situation we say the structure group of P can be *reduced* from G to G'.

Let $\pi_P : P \to B$ be a principal G-bundle and G' be a closed subgroup of G. Then G' acts freely on the right of P and the quotient P/G' gives a fiber bundle $\pi_{P/G'} : P/G' \to B$ with fiber G/G' and structure group G which is associated with $P(B, G)$ via the obvious action of G on the left of G/G'. Moreover the projection $p : P \to P/G'$ defines a principal G'-bundle over P/G'.

Proposition 6.1.15. *There is a one-to-one correspondence between sections $\sigma : B \to P/G'$ and principal G'-bundles over B obtained by reduction of the structure group. The correspondence associates each σ with the principal G'-bundle obtained from $p : P \to P/G'$ via pull-back by σ.*

Proposition 6.1.16. *Every reduction of the structure group for an (F, G)-bundle induces an associated principal bundle obtained by reduction of the structure group.*

The positive interplay between (F, G)-bundles and their associated principal G-bundles turns out to be very useful when dealing with smooth vector bundles. In this case we have an alternative description for the associated principal bundle: indeed, if the linear representation of a Lie group G inducing a G-structure on $\pi_E : E \to M$ is faithful, the associated principal

G-bundle can be identified with a set of ordered bases of fibers, and is therefore called a *G-frame bundle*. More precisely, every local trivialization (U_α, χ_α) in the G-structure of $\pi_E : E \to M$ defines a *local G-frame* $\mathscr{F}_\alpha = \{V_1^\alpha, \ldots, V_m^\alpha\} \subset \Gamma(E\big|_{U_\alpha})$ obtained as the inverse image of the standard basis of $\mathbb{K}^m$, i.e. given by $V_i^\alpha(x) := \chi_\alpha^{-1}(x, e_i)$. Then for every $x \in U_\alpha$ we can define the induced *G-frame at x* as the ordered basis of E_x given by $\mathscr{B}_x^\alpha = \{V_1^\alpha(x), \ldots, V_m^\alpha(x)\}$. The total space of the G-frame bundle is the set $L_G(E)$ given by the disjoint union of these oredered bases. The projection $\pi_L : L_G(E) \to M$ is given by $\pi_L : \mathscr{B}_x^\alpha \mapsto x$. Now, the group $\mathrm{GL}(m, \mathbb{K})$ acts freely and transitively on the right of the set of ordered bases of any m-dimensional $\mathbb{K}$-vector space V in the following way:

$$\mathscr{B} \cdot a := \{a_{j1} v_j, \ldots, a_{jm} v_j\} \qquad \forall \;\; a = a_{ij} \in \mathrm{GL}(m, \mathbb{K}), \quad \mathscr{B} = \{v_1, \ldots, v_m\}$$

The matrix of change of basis from $\mathscr{B}$ to $\mathscr{B} \cdot a$ is then a^{-1}. Therefore we can make G act freely on the right of $L_G(E)$ by defining $\mathscr{B}_x^\alpha \cdot a := \mathscr{B}_x^\alpha \cdot \varphi(a)$. Note that $L_G(E)$ is closed under this action because, given any local trivialization (U_α, χ_α) and any $a \in G$, there exists another local trivialization (U_β, χ_β) such that $\mathscr{B}_x^\beta = \mathscr{B}_x^\alpha \cdot a$ (for example we can take $U_\beta = U_\alpha$ and $\psi_\beta = \varphi(a)^{-1} \circ \psi_\alpha$). Then π_L coincides with the projection $L_G(E) \to L_G(E)/G$. We can endow $L_G(E)$ with a smooth structure by considering for each local trivialization (U_α, χ_α) in the G-structure the following map:

$$\begin{array}{ccc} \pi_L^{-1}(U_\alpha) & \to & U_\alpha \times G \\ \mathscr{B}_x^\beta & \mapsto & (x, g_{\alpha\beta}(x)) \end{array}$$

Note that, since $\varphi(g_{\alpha\beta}(x)) \in \mathrm{GL}(m, \mathbb{K})$ is the matrix of change of basis from $\mathscr{B}_x^\alpha$ to $\mathscr{B}_x^\beta$, we have $\mathscr{B}_x^\beta = \mathscr{B}_x^\alpha \cdot g_{\alpha\beta}(x)$. Then, since $g_{\gamma\beta}(x) = g_{\gamma\alpha}(x) g_{\alpha\beta}(x)$, changes of coordinates are smooth and thus these local trivializations define a smooth structure such that both the projection π_L and the right action of G are smooth: indeed, in local trivializations, we have $\pi_L(x, b) = x$ and $R_a((x, b)) = (x, a^{-1} b)$. Thanks to Proposition 6.1.11 $L_G(E)(M, G)$ is indeed a principal bundle.

Remark 6.1.17. The rank-m vector bundle with structure group G associated with $L_G(E)$ is naturally isomorphic to E. Then $L_G(E)$ must be the principal G-bundle associated with $\pi_E : E \to M$.

From now on the omission of the structure group from $L_G(E)$ will stand for $G = \mathrm{GL}(m, \mathbb{K})$. The frame bundle associated with the tangent bundle TM will be called the *linear frame bundle* and will be denoted with $L(M)$ instead of $L(TM)$.

Whitney sums and flag manifolds

Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be two vector bundles and let $\triangle : B \hookrightarrow B \times B$ denote the diagonal embedding of B. The *Whitney sum* of $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ is the vector bundle denoted by $\pi_1 \oplus \pi_2 : E_1 \otimes E_2 \to B$ and defined as the pull-back of $\pi_1 \times \pi_2 : E_1 \times E_2 \to B \times B$ via $\triangle$.

Now let W be an m-dimensional $\mathbb{K}$-vector space. A *flag* in W is a sequence of subspaces

$$A_1 \subset A_2 \subset \ldots \subset A_m$$

such that $\dim A_j = j$. Let $F(W)$ be the set of flags in W and consider the subgroup T of $\mathrm{GL}(m, \mathbb{K})$ consisting of upper-triangular matrices. Since T acts freely on the right of $V_m(W)$ we can identify the quotient $V_m(\mathbb{K}^m)/T$ with $F(W)$ by considering the bijection

$$[\{v_1, \ldots, v_m\}] \mapsto Span(v_1) \subset \mathrm{Span}(v_1, v_2) \subset \ldots \subset \mathrm{Span}(v_1, \ldots, v_m)$$

This makes $F(W)$ into an $\frac{m(m-1)}{2}$-dimensional manifold diffeomorphic to $\mathrm{GL}(m, \mathbb{K})/T$ and called the *flag manifold*.

Let $\pi_E : E \to B$ be a rank-m $\mathbb{K}$-vector bundle. We define the *associated flag manifold bundle* $\pi_F : F(E) \to B$ as the $(F(\mathbb{K}^m), \mathrm{GL}(m,\mathbb{K}))$-bundle given by

$$F(E) := \bigsqcup_{x\in B} F(E_x), \qquad \pi_F(F(E_x)) = x$$

A bundle atlas $\{U_\alpha, \chi^E_\alpha\}$ for E gives rise to local trivializations:

$$\begin{array}{rccc} \xi^F_\alpha : & \pi_F^{-1}(U_\alpha) & \to & U_\alpha \times F(\mathbb{K}^m) \\ & u & \mapsto & (\pi_V(u), \psi_\alpha(u)) \end{array}$$

where, if $u = \{A_1 \subset \ldots \subset A_m\} \in F(E_x)$ then $\psi_\alpha(u) = \{\psi_\alpha(A_1) \subset \ldots \subset \psi_\alpha(A_m)\} \in F(\mathbb{K}^m)$.

Proposition 6.1.18 (Splitting principle)**.** *Let $\pi : E \to B$ be a rank-m $\mathbb{K}$-vector bundle and let $\pi_F : F(E) \to B$ be its associated flag bundle.*

(*i*) *If $\mathbb{K} = \mathbb{R}$ then the pull-back by $\pi_F : F(E) \to B$ of the bundle $\pi : E \to B$ is isomorphic to an m-fold Whitney sum of real rank-1 vector bundles, and the induced homomorphism $\pi_F^* : H^k(B;\mathbb{Z}_2) \to H^k(F(E);\mathbb{Z}_2)$ is injective.*

(*ii*) *If $\mathbb{K} = \mathbb{C}$ then the pull-back by $\pi_F : F(E) \to B$ of the bundle $\pi : E \to B$ is isomorphic to an m-fold Whitney sum of complex rank-1 vector bundles, and the induced homomorphism $\pi_F^* : H^k(B;\mathbb{Z}) \to H^k(F(E);\mathbb{Z})$ is injective.*

Universal vector bundles

Let $G_r(\mathbb{K}^m)$ be the set of r-dimensional subspaces of $\mathbb{K}^m$. The injective homomorphism

$$\begin{array}{rccc} \iota : & \mathrm{GL}(r,\mathbb{K}) \times \mathrm{GL}(m-r,\mathbb{K}) & \to & \mathrm{GL}(m,\mathbb{K}) \\ & (a,b) & \mapsto & \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \end{array}$$

induces a free right action of $\mathrm{GL}(r,\mathbb{K}) \times \mathrm{GL}(m-r,\mathbb{K})$ onto $V_m(\mathbb{K}^m)$. Therefore and we can identify the quotient $V_m(\mathbb{K}^m)/(\mathrm{GL}(r,\mathbb{K}) \times \mathrm{GL}(m-r,\mathbb{K}))$ with $G_r(\mathbb{K}^m)$ by considering the bijection $[\{v_1,\ldots,v_m\}] \mapsto \mathrm{Span}(v_1,\ldots,v_r)$. This makes $G_r(\mathbb{K}^m)$ into an $r(m-r)$-dimensional manifold diffeomorphic to $\mathrm{GL}(m,\mathbb{K})/(\mathrm{GL}(r,\mathbb{K}) \times \mathrm{GL}(m-r,\mathbb{K}))$ and called the *Grassmann manifold*.

There exists a *tautological rank-r $\mathbb{K}$-vector bundle* $\pi_\gamma : \gamma_r(\mathbb{K}^m) \to G_r(\mathbb{K}^m)$ with total space

$$\gamma_r(\mathbb{K}^m) = \{(X,x) \in G_r(\mathbb{K}^m) \times \mathbb{K}^m \mid x \in X\}$$

topologized as a subspace of $G_r(\mathbb{K}^m) \times \mathbb{K}^m$ and with projection π_γ given by the restriction $\pi_1\big|_{\gamma_r(\mathbb{K}^m)}$ of the projection onto the first component. To see the local triviality let us consider an arbitrary $X \in G_r(\mathbb{K}^m)$ and the orthogonal projection $p_X : \mathbb{K}^m \to X$ with respect to the standard scalar (or Hermitian) product. Then we can consider the open neighborhood of X defined by

$$U_X = \left\{Y \in G_r(\mathbb{K}^m) \mid \ker(p_X\big|_Y) = 0\right\}$$

If $\varphi_X : X \to \mathbb{K}^r$ is a fixed linear isomorphism, we can define local trivializations

$$\begin{array}{rccc} \chi_X : & \pi_\gamma^{-1}(U_X) & \to & U_X \times \mathbb{K}^r \\ & (Y,y) & \mapsto & (Y, \varphi_X(p_X(y))) \end{array}$$

Proposition 6.1.19. *If $\pi : E \to B$ is a rank-m $\mathbb{K}$-vector bundle then there exists a sufficiently large N and a characteristic map $f : B \to G_m(\mathbb{K}^N)$ such that $\pi_{f^*\gamma} : f^*\gamma_m(\mathbb{K}^N) \to B$ is isomorphic to $\pi : E \to B$.*

Now let $\mathbb{K}^\infty$ denote the infinite direct sum $\bigoplus_{n\geqslant 1}\mathbb{K}$. Its realization as an infinite union $\bigcup_{n\geqslant 1}\mathbb{K}^n$, where each $\mathbb{K}^n$ is contained in $\mathbb{K}^{n+1}$ as the set $\{x_{n+1}=0\}$, induces inclusions

$$\ldots \subset G_k(\mathbb{K}^n) \subset G_k(\mathbb{K}^{n+1}) \subset \ldots$$

We can define the *infinite Grassmann manifold* as $G_k := G_k(\mathbb{R}^\infty) = \bigcup_{n\geqslant k} G_k(\mathbb{R}^n)$ and topologize it as the direct limit of $G_k(\mathbb{K}^n)$, i.e. declare a subset of G_k open if and only if it's intersection with each $G_k(\mathbb{K}^n)$ is open in $G_k(\mathbb{K}^n)$. As before we can define a tautological rank-r K-vector bundle $\pi_\gamma : \gamma_r(\mathbb{K}^\infty) \to G_r(\mathbb{K}^\infty)$ with total space

$$\gamma_r(\mathbb{K}^\infty) = \{(X,x) \in G_r(\mathbb{K}^\infty) \times \mathbb{K}^\infty \mid x \in X\}$$

and projection given by the restriction of the projection onto the first component.

Theorem 6.1.20. *For every rank-m K-vector bundle $\pi : E \to B$ there exists a characteristic map $f : B \to G_m(\mathbb{K}^\infty)$ such that $\pi_{f^*\gamma} : f^*\gamma_m(\mathbb{K}^\infty) \to B$ is isomorphic to $\pi : E \to B$.*

For this reason $\pi_\gamma : \gamma_m(\mathbb{K}^\infty) \to G_m(\mathbb{K}^\infty)$ is called the *universal vector bundle of rank-m*.

In the real oriented case we can define the *real oriented Grassmann manifold* $\tilde{G}_r(\mathbb{R}^m)$ as the set of oriented r-planes in $\mathbb{R}^m$, that can be identified with $\mathrm{GL}(m,\mathbb{R})/(\mathrm{GL}^+(r,\mathbb{R})\times\mathrm{GL}(m-r,\mathbb{R}))$, and we can define the *real oriented tautological rank-r vector bundle* $\pi_{\tilde{\gamma}} : \tilde{\gamma}_r(\mathbb{R}^m) \to \tilde{G}_r(\mathbb{R}^m)$.

Proposition 6.1.21. *If $\pi : E \to B$ is a real oriented rank-m vector bundle then there exists a sufficiently large N and a map $f : B \to \tilde{G}_m(\mathbb{R}^N)$ such that $\pi_{f^*\tilde{\gamma}} : f^*\tilde{\gamma}_m(\mathbb{R}^N) \to B$ is isomorphic to $\pi : E \to B$. Moreover $\tilde{\gamma}_m(\mathbb{R}^N)$ induces on $f^*\tilde{\gamma}_m(\mathbb{R}^N)$ the orientation of E.*

As before we can also define the *infinite real oriented Grassmann manifold* $\tilde{G}_r(\mathbb{R}^\infty)$ with *universal real oriented vector bundle of rank-r* $\pi_{\tilde{\gamma}} : \tilde{\gamma}_r(\mathbb{R}^\infty) \to \tilde{G}_r(\mathbb{R}^\infty)$.

Theorem 6.1.22. *For every real oriented rank-m vector bundle $\pi : E \to B$ there exists a characteristic map $f : B \to \tilde{G}_m(\mathbb{R}^\infty)$ such that the isomorphism between $\pi_{f^*\tilde{\gamma}} : f^*\tilde{\gamma}_m(\mathbb{R}^\infty) \to B$ and $\pi : E \to B$ is orientation-preserving.*

Universal principal bundles

Let G be a topological group, let I denote the interval $[0,1]$, consider the space

$$\left(\prod_{i\geqslant 0} G\right) \times \left(\bigoplus_{i\geqslant 0} I\right)$$

and let $\mathscr{P}G$ be the subspace of elements $(x,t) = ((x_0,x_1,\ldots),(t_0,t_1,\ldots))$ such that $\sum_{i\geqslant 0} t_i = 1$. Then we can define PG as the quotient of $\mathscr{P}G$ by the equivalence relation

$$(x,t) \sim (y,t) \;\text{ if }\; t_i > 0 \Rightarrow x_i = y_i$$

Let $f_i : PG \to I$ be the function defined by $f_i(x,t) := t_i$ and let $g_i : f_i^{-1}((0,1]) \to G$ the function defined by $g_i(x,t) := x_i$. We can equip PG with the smallest topology making the functions f_i and g_i into continuous functions (where the domains of the functions g_i are topologized as subspaces of PG). There is a well-defined continuous right action of G onto PG given by

$$R_a((x_0,x_1,\ldots),(t_0,t_1,\ldots)) := ((x_0a,x_1a,\ldots),(t_0,t_1,\ldots)) \quad \forall\, a \in G$$

and we can set $BG := PG/G$.

Proposition 6.1.23. *The projection $\pi_{PG} : PG \to BG$ defines a principal G-bundle. This bundle is universal: if $\pi : P \to B$ is a principal G-bundle then there exists a characteristic map $f : B \to BG$ such that the pull-back $\pi_{f^*PG} : f^*PG \to B$ is isomorphic to $\pi : P \to B$.*

Remark 6.1.24. The above construction is functorial. Indeed each continuous homomorphism $\varphi : G \to G'$ induces a bundle map

$$\begin{array}{ccc} PG & \xrightarrow{\varphi_*^P} & PG' \\ {\scriptstyle \pi_{PG}}\downarrow & & \downarrow{\scriptstyle \pi_{PG'}} \\ BG & \xrightarrow[\varphi_*^B]{} & BG' \end{array}$$

and we have $(\mathrm{id}_G)_*^P = \mathrm{id}_{PG}$, $(\mathrm{id}_G)_*^B = \mathrm{id}_{BG}$, $(\psi \circ \varphi)_*^P = \psi_*^P \circ \varphi_*^P$ and $(\psi \circ \varphi)_*^B = \psi_*^B \circ \varphi_*^B$

Proposition 6.1.25. *If the continuous homomorphisms $\varphi_0, \varphi_1 : G \to G'$ are homotopic then also the induced maps $(\varphi_0)_*^P, (\varphi_1)_*^P$ and $(\varphi_0)_*^B, (\varphi_1)_*^B$ are homotopic.*

6.2 Connections

Throughout this section all fiber bundles and all maps are to be considered smooth.

Covariant derivatives

A *covariant derivative* on a vector bundle $\pi : E \to M$ is a map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying:

(*i*) $\nabla_{fX+gY}V = f\nabla_X V + g\nabla_Y V$ for all $f, g \in \mathscr{C}^\infty_{\mathbb{K}}(M)$;

(*ii*) $\nabla_X(\lambda V + \mu W) = \lambda\nabla_X V + \mu\nabla_X W$ for all $\lambda, \mu \in \mathbb{K}$;

(*iii*) $\nabla_X(fV) = Xf \cdot V + f\nabla_X V$ for all $f \in \mathscr{C}^\infty_{\mathbb{K}}(M)$.

Proposition 6.2.1. *Every vector bundle $\pi : E \to M$ admits a covariant derivative.*

Proposition 6.2.2. (*i*) *If $X_1(x) = X_2(x)$ then $\nabla_{X_1}V(x) = \nabla_{X_2}V(x)$.*

(*ii*) *If $\gamma : (-\varepsilon, \varepsilon) \to M$ satisfies $\gamma(0) = x$ and $\gamma'(0) = X(x)$ and if $V_1 \circ \gamma \equiv V_2 \circ \gamma$ then $\nabla_X V_1(x) = \nabla_X V_2(x)$.*

If U trivializes both TM and E via local vector fields $X_1, \dots, X_n$ and sections $V_1, \dots, V_m$, the *Christoffel symbols of ∇ with respect to the local frames $X_1, \dots, X_n$ and $V_1, \dots, V_m$* are functions $\Gamma_{ij}^k \in \mathscr{C}^\infty_{\mathbb{K}}(U)$ such that:

$$\nabla_{X_i}V_j = \Gamma_{ij}^k V_k$$

Let $\gamma : [a, b] \to M$ be a smooth curve. A *section over* γ is a section of the pull-back bundle γ^*E. There exists a unique operator $D_\gamma : \Gamma(\gamma^*E) \to \Gamma(\gamma^*E)$, called the *covariant derivative along* γ, satisfying:

(*i*) $D_\gamma(\lambda V + \mu W) = \lambda D_\gamma V + \mu D_\gamma W$ for all $\lambda, \mu \in \mathbb{K}$;

(*ii*) $D_\gamma(fV) = f'V + fD_\gamma V$ for all $f \in \mathscr{C}^\infty_{\mathbb{K}}([a, b])$;

(*iii*) $D_\gamma V = \nabla_{\gamma'}\tilde{V}$ for all $\tilde{V} \in \Gamma(E)$ such that $\tilde{V} \circ \gamma = V$.

A section V over γ is *parallel along* γ if $D_\gamma V \equiv 0$.

Proposition 6.2.3. *For all $u \in E_{\gamma(t)}$ there exists a unique section V_u parallel along γ such that $V_u(t) = u$.*

The *parallel transport along* γ is the family of operators $\tau_\gamma^{t,s} : E_{\gamma(t)} \to E_{\gamma(s)}$ mapping each $u \in E_{\gamma(t)}$ to $V_u(s)$ where V_u is the unique parallel extension of u along γ. For every $t, s \in [a, b]$ the map $\tau_\gamma^{t,s}$ is a linear isomorphism.

If $\pi : E \to M$ has structure group G a covariant derivative ∇ is said to be *G-compatible* if the parallel transports are $(\mathbb{K}^m, G)$-bundle maps, i.e. if for every pair of G-frames at $\gamma(t)$ and $\gamma(s)$ the induced linear isomorphism of $\mathbb{K}^m$ is an element of $\varphi(G)$.

Remark 6.2.4. A covariant derivative is G-compatible if and only if the parallel extension of any G-frame in $L_G(E)$ along a curve γ determines a curve entirely contained in $L_G(E)$.

Horizontal distributions

Let $\pi : P \to M^n$ be a principal G-bundle and let $G_u := \ker d_u\pi \subset T_uP$ be the *vertical subspace at* u, i.e. the subspace of vectors tangent to the fiber $P_{\pi(u)}$. An *invariant horizontal distribution* Γ on $P(M, G)$ is a smooth distribution of n-planes $u \mapsto Q_u \subset T_uP$ such that:

(*i*) $T_uP = G_u \oplus Q_u$

(*ii*) $Q_{ua} = d_uR_a(Q_u)$

An invariant horizontal distribution yields operators $v, h : \mathfrak{X}(P) \to \mathfrak{X}(P)$ associated with the direct sum $T_uP = G_u \oplus Q_u$ such that every $X \in \mathfrak{X}(P)$ decomposes as $X = vX + hX$ with $vX_u \in G_u$ and $hX_u \in Q_u$ for all $u \in P$. An association $u \mapsto Q_u \subset T_uP$ satisfying (*i*) and (*ii*) is a smooth distribution if and only if vX and hX are smooth vector fields for all $X \in \mathfrak{X}(P)$.

Proposition 6.2.5. *Every principal bundle $P(M, G)$ admits an invariant horizontal distribution.*

An invariant horizontal distribution Γ on a principal G-bundle $\pi : P \to M$ gives rise to isomorphisms $d_u\pi\big|_{Q_u} : Q_u \to T_{\pi(u)}M$ for all $u \in P$. If U trivializes both P and TM then Γ induces a trivialization $T(P\big|_U) \simeq (P\big|_U) \times \mathbb{R}^n \times \mathfrak{g}$ where Q_u corresponds to $\{u\} \times \mathbb{R}^n \times \{0\}$. Therefore every vector field $X \in \mathfrak{X}(M)$ admits a unique *horizontal lift* $\tilde{X} \in \mathfrak{X}(P)$ defined by $\tilde{X}_u := (d_u\pi\big|_{Q_u})^{-1}(X_{\pi(u)})$ which is clearly smooth, as can be seen in local trivializations. Moreover every horizontal lift of a vector field is right invariant under the action of G, thanks to the invariance of Γ, and every right invariant horizontal vector field on P covers a vector field on M.

Let $\gamma : [a, b] \to M$ be a smooth curve. A *horizontal lift of* γ is a smooth curve $\tilde{\gamma} : [a, b] \to P$ such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}'(t) \in Q_{\gamma(t)}$.

Proposition 6.2.6. *For all $t \in [a, b]$ and $u \in P_{\gamma(t)}$ there exists a unique horizontal lift $\tilde{\gamma}_u$ of γ such that $\tilde{\gamma}_u(t) = u$.*

For a smooth curve $\gamma : [a, b] \to M$ the *parallel transport along* γ is the family of maps $\tau_\gamma^{t,s} : P_{\gamma(t)} \to P_{\gamma(s)}$ sending every $u \in P_{\gamma(t)}$ to $\tilde{\gamma}_u(s)$ where $\tilde{\gamma}_u$ is the unique horizontal lift of γ such that $\tilde{\gamma}_u(t) = u$.

Proposition 6.2.7. *For all $t, s \in [a, b]$ the map $\tau_\gamma^{t,s}$ is a diffeomorphism between $P_{\gamma(t)}$ and $P_{\gamma(s)}$.*

Covariant derivatives induced by horizontal distributions

An invariant horizontal distribution Γ on $P(M, G)$ allows us to construct a G-compatible parallel transport on every (F, G)-bundle $\pi_E : E \to M$ associated with $P(M, G)$. Indeed we can induce a horizontal distribution on E by considering the map $\varphi_\xi : P \to E$ sending $v \mapsto v\xi$ and by defining $Q_w := d_u\varphi_\xi(Q_u)$. The distribution is well-defined since, if $w = v\eta$ with $v = ua$ and

$\eta = a^{-1}\xi$, then $\varphi_\eta = \varphi_\xi \circ R_{a^{-1}}$ and so $d_v\varphi_\eta(Q_v) = d_u\varphi_\xi(d_{ua}R_{a^{-1}}(d_uR_a(Q_u))) = d_u\varphi_\xi(Q_u)$. Moreover $\pi_E \circ \varphi_\xi = \pi_P$, and therefore $T_wE = Q_w \oplus \ker d_w\pi_E$. Just like we discussed before, horizontal lifts of vector fields and curves exist and are unique, and allow us to define parallel transport along curves. If $\gamma : [a,b] \to M$ is a curve, u is a point in $P_{\gamma(t)}$ and $\tilde{\gamma}_u$ is the unique horizontal lift of γ in P such that $\tilde{\gamma}_u(t) = t$, we have:

$$\begin{array}{rccc} \tau_\gamma^{t,s}: & E_{\gamma(t)} & \to & E_{\gamma(s)} \\ & u\xi & \mapsto & \tilde{\gamma}_u(s)\xi \end{array}$$

These are clearly (F,G)-bundle maps since they induce diffeomorphisms of F having the form $\xi \mapsto \psi_\alpha^{-1}(u)\psi_\beta(\tilde{\gamma}_u(s))\xi$.

Let $P(M,G)$ be a principal bundle and $\pi_E : E \to M$ be the associated rank-m vector bundle with structure group G. Let Γ be an invariant horizontal distribution on $P(M,G)$ which induces a G-compatible parallel transport τ on E.

If $\gamma : [a,b] \to M$ is a smooth curve we can define the covariant derivative along γ as the operator $D_\gamma : \Gamma(\gamma^*E) \to \Gamma(\gamma^*E)$ given by:

$$V \mapsto \left[D_\gamma V : t \mapsto \frac{\mathrm{d}}{\mathrm{d}h}\left[\tau_\gamma^{t+h,t}(V(t+h))\right]_{h=0} \right]$$

It is easily verified that D_γ is indeed $\mathbb{K}$-linear and satisfies the Leibniz rule:

$$D_\gamma(fV) = f'V + fD_\gamma V$$

Lemma 6.2.8. *If $\gamma, \mu : (-\varepsilon, \varepsilon) \to M$ are smooth curves such that $\gamma(0) = \mu(0) = x$ and $\gamma'(0) = \mu'(0) = X \in T_xM$, then $D_\gamma(V \circ \gamma)(0) = D_\mu(V \circ \mu)(0)$ for all sections $V \in \Gamma(E)$.*

We can define the G-compatible covariant derivative $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ given by:

$$(\nabla_X V)_x = D_\gamma V(0) \quad \text{for } \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that } \gamma(0) = x \text{ and } \gamma'(0) = X_x$$

Thanks to Lemma 6.2.8 ∇ is well defined, and it satisfies:

(i) $\nabla_{fX}V = f\nabla_X V$ for all $f \in \mathscr{C}^\infty_{\mathbb{K}}(M)$;

(ii) $\nabla_X(fV) = X(f)V + f\nabla_X V$ for all $f \in \mathscr{C}^\infty_{\mathbb{K}}(M)$.

Remark 6.2.9. Every curve σ in E is horizontal if and only if it is parallel with respect to ∇, i.e. $\sigma'(t) \in Q_{\sigma(t)}\ \forall\, t \;\Leftrightarrow\; D_{(\pi_E \circ \sigma)}\sigma(t) \equiv 0$.

Horizontal distributions induced by covariant derivatives

With each covariant derivative ∇ on a rank-m vector bundle $\pi : E \to M$ we can associate a horizontal distribution on E as follows: for all $u \in E$ there exists a natural linear isomorphism $i_u : E_{\pi(u)} \to \ker d_u\pi \subset T_uE$. Indeed, if a local frame $V_1, \ldots, V_r$ induces a local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{K}^m$ with coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_m)$, then these coordinates induce a local frame for $T(E\big|_U)$ given by sections:

$$\partial_i = \frac{\partial}{\partial x_i}, \quad \dot{\partial}_j = \frac{\partial}{\partial \xi_j} \quad \text{for } i = 1, \ldots, n, \quad j = 1, \ldots, m$$

Then i_u is defined by $V_j(\pi(u)) \mapsto \dot{\partial}_j\Big|_u$, which is independent of the choice of the local frame.

Lemma 6.2.10. *Let $V, W \in \Gamma(E)$ be sections of E such that $V(x) = W(x) = u$. Then for all $X \in T_xM$ we have:*

$$d_xV(X) - i_u(\nabla_X V(x)) = d_xW(X) - i_u(\nabla_X W(x))$$

Therefore we can define, for all $x \in M$ and $u \in E_x$, the linear map:

$$\begin{array}{rccc} \mathscr{Q}_u : & T_xM & \to & T_uE \\ & X & \mapsto & d_xV(X) - i_u(\nabla_X V(x)) \end{array}$$

where V is any section of E satisfying $V(x) = u$. In local coordinates we have:

$$\mathscr{Q}_u(X) = X_i\partial_i\big|_u + X_i\frac{\partial \xi_j}{\partial x_i}(x)\dot{\partial}_j\Big|_u - X_i\frac{\partial \xi_j}{\partial x_i}(x)\dot{\partial}_j\Big|_u - X_i\xi_j(x)\Gamma^k_{ij}(x)\dot{\partial}_k\Big|_u = X_i\partial_i\big|_u - X_iu_j\Gamma^k_{ij}(x)\dot{\partial}_k\Big|_u$$

Thus we have $\mathscr{Q}_u(T_xM) \oplus \ker d_u\pi = T_uE$ and the map $\mathscr{Q} : u \mapsto \mathscr{Q}_u(T_{\pi(u)}M)$ is smooth.

Remark 6.2.11. Every curve σ in E is parallel if and only if it is horizontal with respect to $\mathscr{Q}$, i.e. $D_{(\pi\circ\sigma)}\sigma(t) \equiv 0 \Leftrightarrow \sigma'(t) \in Q_{\sigma(t)}\ \forall\, t$.

Now let us consider a vector bundle $\pi_E : E \to M$ with structure group G and frame bundle $\pi_L : L_G(E) \to M$. Let ∇ be a G-compatible covariant derivative on E. The vector bundle $\pi^m : E^m = E \oplus \ldots \oplus E \to M$ can be endowed with a G-structure and with the G-compatible covariant derivative $\nabla^m = \nabla \oplus \ldots \oplus \nabla$ given by:

$$\nabla^m_{(X_1,\ldots,X_m)}(V_1, \ldots, V_m) = (\nabla_{X_1}V_1, \ldots, \nabla_{X_m}V_m)$$

We can clearly realize $L_G(E)$ as a sub-bundle of E^m. Let $\mathscr{Q}^m$ be the horizontal distribution on E^m associated with the covariant derivative ∇^m. For all $\mathscr{B}^\alpha_x \in L_G(E)$ the horizontal subspace at $\mathscr{B}^\alpha_x$ is entirely contained in $T_{\mathscr{B}^\alpha_x}(L_G(E))$, and therefore we can define $Q_{\mathscr{B}^\alpha_x} := \mathscr{Q}^m(\mathscr{B}^\alpha_x)$. The invariance under the right action of G is easy to prove.

Equivalence

We showed that with each G-compatible covariant derivative on a vector bundle with structure group G we can associate a G-invariant horizontal distribution on the associated principal G-bundle and vice versa.

Theorem 6.2.12. *These associations are inverse with each other.*

Therefore from now on we will use the term *connection* to refer to both these concepts and we will use interchageably both characterizations according to the features we will need to highlight. We will call *linear connection* every connection defined on the tangent bundle to a manifold or on the linear frame bundle.

6.3 Curvature

The concept of curvature of a connection arises naturally as a tensor field for covariant derivatives and as a differential form for invariant horizontal distributions. These different characterizations are equivalent, and we will present both of them as we will need all of their features.

Curvature tensor of a covariant derivative

Let $\pi : E \to M$ be a rank-m vector bundle, let ∇ be a connection on E. The *curvature tensor* of ∇ is the map

$$\begin{array}{rccc} R_\nabla : & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) & \to & \Gamma(E) \\ & (X, Y, V) & \mapsto & \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V \end{array}$$

Remark 6.3.1. It can be easily verified that R_∇ is $\mathscr{C}^\infty_{\mathbb{K}}(M)$-linear in all variables. Therefore it is a tensor field, i.e. a section of the vector bundle $T^0_2(M) \otimes \mathrm{Hom}(E, \mathbb{K}) \otimes E$.

If U trivializes both TM and E via local vector fields $X_1, \ldots, X_n$ and sections $V_1, \ldots, V_m$ the *curvature symbols of* ∇ *with respect to* $X_1, \ldots, X_n$ *and* $V_1, \ldots V_m$ are functions $(R_\nabla)^h_{ijk} \in \mathscr{C}^\infty_{\mathbb{K}}(U)$ such that:

$$(R_\nabla)_{X_i,X_j}V_k = (R_\nabla)^h_{ijk}V_h$$

Curvature form of an invariant horizontal distribution

Let $P(M,G)$ be a principal bundle. Using the right action of G on P we can realize $\mathfrak{g} = \mathrm{Lie}(G)$ as a subalgebra of vertical vector fields in $\mathfrak{X}(P)$. Indeed we can define, for all $A \in \mathfrak{g}$, the vector field $A^* \in \mathfrak{X}(P)$ given by:

$$A^*_u := \frac{\mathrm{d}}{\mathrm{d}t}\left[u \exp(tA)\right]_{t=0}$$

The map

$$\begin{array}{rccc} \sigma : & \mathfrak{g} & \to & \mathfrak{X}(P) \\ & A & \mapsto & A^* \end{array}$$

is an injective Lie algebra homomorphism which sends every non-zero vector to a nowhere-vanishing vector field. Therefore we get, for all $u \in P$, an isomorphism:

$$\begin{array}{rccc} \Psi_u : & \mathfrak{g} & \to & G_u \\ & A & \mapsto & A^*_u \end{array}$$

Thus given a principal connection Γ we can define its *connection form* ω_Γ as the $\mathfrak{g}$-valued 1-form given by $(\omega_\Gamma)_u(X_u) := \Psi_u^{-1}(vX_u)$.

The *curvature form* of Γ is the $\mathfrak{g}$-valued 2-form Ω_Γ defined by:

$$\Omega_\Gamma(X,Y) := (\mathrm{d}\omega_\Gamma)(hX, hY)$$

Theorem 6.3.2 (Structure equation for curvature)**.**

$$\mathrm{d}\omega_\Gamma(X,Y) = -[\omega_\Gamma(X), \omega_\Gamma(Y)] + \Omega_\Gamma(X,Y)$$

Curvature form of a covariant derivative

Let $\pi : E \to M$ be a rank-m vector bundle, let ∇ be a connection on E. Let's fix a local frame $V_1, \dots, V_m$ for E and a local coordinate system $x_1, \dots, x_n$ for $U \subset M$ yielding Christoffel symbols Γ^k_{ij}. The *local connection forms* with respect to $V_1, \dots, V_m$ are the 1-forms $(\omega_\nabla)_{ij}$ defined on U by:

$$(\omega_\nabla)_{ij} := \Gamma^j_{ki}\mathrm{d}x_k$$

Remark 6.3.3. $\nabla_X V_i = (\omega_\nabla)_{ij}(X)V_j$

Remark 6.3.4. The definition is independent of the choice of the local coordinate system $x_1, \dots, x_n$.

We can arrange these local forms into a matrix $\omega_\nabla = ((\omega_\nabla)_{ij})$ with entries in $\Omega^1(U)$, called the *matrix of local connection forms.*

If $\pi : E \to M$ has structure group G, ∇ is G-compatible and $V_1, \dots, V_m$ is a local G-frame then for all $X \in \mathfrak{X}(M)$ the matrix $\omega_\nabla(X)$ is $\mathrm{d}\varphi(\mathfrak{g})$-valued, where $\mathrm{d}\varphi : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{K})$ is the representation of the Lie algebra of G associated with $\varphi : G \to \mathrm{GL}(m, \mathbb{K})$. Indeed the parallel transport along any curve $\gamma : [a,b] \to M$ can be written in the following form:

$$\begin{array}{rccc} \tau^{t,s}_\gamma : & E_{\gamma(t)} & \to & E_{\gamma(s)} \\ & V_i(t) & \mapsto & \varphi(f(s))_{ji}V_j(s) \end{array}$$

where $f : [a,b] \to G$ satisfies $f(t) = e$. Therefore we can use:

$$\nabla_{\gamma'(t)} V_i = \frac{\mathrm{d}}{\mathrm{d}h}\left[\tau^{t+h,t}_\gamma(V_i(t+h))\right]_{h=0}$$

Proposition 6.3.5. *A connection ∇ on a vector bundle of structure group G is G-compatible if and only if its matrices of local connection forms with respect to local G-frames are $\mathrm{d}\varphi(\mathfrak{g})$-valued local 1-forms.*

Now we can define a $d\varphi(\mathfrak{g})$-valued 1-form $\tilde{\omega}_\nabla$ on $L_G(E)$ in the following way: for each vector field $X \in \mathfrak{X}(L_G(E))$ we consider the integral curve σ of X starting from $\mathscr{B}_x^\alpha \in L_G(E)$. Then any local G-frame $V_1, \dots, V_m$ for E extending σ on a neighborhood U yields local connection forms $(\omega_\nabla)_{ij}$. Thus if $\bar{X} = d\pi_L(X) \in \mathfrak{X}(M)$ we can define the 1-forms

$$((\tilde{\omega}_\nabla)_{ij})_{\mathscr{B}_x^\alpha}(X_{\mathscr{B}_x^\alpha}) := ((\omega_\nabla)_{ij})_x(\bar{X}_x) \quad \forall\, X \in \mathfrak{X}(L_G(E))$$

which determine a $d\varphi(\mathfrak{g})$-valued 1-form. This definition does not depend on the chosen G-frame $V_1, \dots, V_m$ extending σ.

The *local curvature forms* with respect to $V_1, \dots, V_m$ are the 1-forms $(\Omega_\nabla)_{ij}$ defined on U by:

$$(\Omega_\nabla)_{ij} := d(\omega_\nabla)_{ij} - (\omega_\nabla)_{ir} \wedge (\omega_\nabla)_{rj} = \left(\frac{\partial \Gamma_{ki}^j}{\partial x_h} - \Gamma_{hi}^r \Gamma_{kr}^j\right) dx_h \wedge dx_k = \sum_{h<k} (R_\nabla)_{hki}^j dx_h \wedge dx_k$$

Remark 6.3.6. $(R_\nabla)_{X,Y} V_i = (\Omega_\nabla)_{ij}(X,Y) V_j$

We can arrange these forms into a matrix $\Omega_\nabla = ((\Omega_\nabla)_{ij})$ with entries in $\Omega^2(U)$, called the *matrix of local curvature forms.* If ∇ is G-compatible and $V_1, \dots, V_m$ is a local G-frame then again for all $X \in \mathfrak{X}(M)$ the matrix $\Omega_\nabla(X)$ is $d\varphi(\mathfrak{g})$-valued.

Now we can define a $d\varphi(\mathfrak{g})$-valued 2-form $\tilde{\Omega}_\nabla$ on $L_G(E)$ whose entries are the standard 2-forms

$$(\tilde{\Omega}_\nabla)_{ij} := d(\tilde{\omega}_\nabla)_{ij} - (\tilde{\omega}_\nabla)_{ir} \wedge (\tilde{\omega}_\nabla)_{rj}$$

Curvature tensor of an invariant horizontal distribution

Let $P(M,G)$ be a principal bundle, let Γ be a connection on P and let $\pi_E : E \to M$ be an associated rank-m vector bundle with structure group G. For all vector fields $X, Y \in \mathfrak{X}(M)$ and all sections $Z \in \Gamma(E)$ we can define the section:

$$x \mapsto ((R_\Gamma)_{X,Y}(Z))_x := \varphi_u((\Omega_\Gamma)_u(\tilde{X}_u, \tilde{Y}_u)(\varphi_u^{-1}(Z))) \in E_x$$

where u is any point in P_x, $\varphi_u : \mathbb{K}^m \to E_{\pi_P(u)}$ maps $\xi \mapsto u\xi$ and $\tilde{X}, \tilde{Y}$ are any lifts of X and Y, i.e. vector fields in $\mathfrak{X}(P)$ such that $d\pi_P(\tilde{X}) = X$ and $d\pi_P(\tilde{Y}) = Y$. It is well-defined because two different lifts differ only for the vertical component, on which Ω_Γ vanishes, and because the curvature form satisfies:

$$R_a^* \Omega_\Gamma = \mathrm{Ad}_{a^{-1}} \circ \Omega_\Gamma \quad \forall\, a \in G$$

The map $R_\Gamma : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E)) \to \Gamma(E)$ is $\mathscr{C}_{\mathbb{K}}^\infty(M)$-multilinear, and therefore it defines a tensor field.

Equivalence

The concepts of curvature defined for covariant derivatives and for horizontal distributions coincide:

Theorem 6.3.7. *(i)* $\tilde{\omega}_\nabla = d\varphi \circ \omega_\Gamma$

(ii) $\tilde{\Omega}_\nabla = d\varphi \circ \Omega_\Gamma$

(iii) $R_\Gamma = R_\nabla$

Therefore from now on we will drop all subscripts and simply write ω, Ω and R for connection and curvature forms and tensors.

Holonomy groups

Let's consider a vector bundle $\pi_E : E \to M$ with structure group G and a principal G-bundle $\pi_P : P \to M$ associated with each other. Let ∇ and Γ be associated connections on E and P respectively. The parallel transport induced by these connections is independent on the parametrization of the curves along which we transport the fibers. Hence we can always suppose curves to be parametrized on $I = [0, 1]$ and we will write τ_γ, τ_γ^{-1} for $\tau_\gamma^{0,1}$, $\tau_\gamma^{1,0}$ respectively.

Remark 6.3.8. (*i*) If $\gamma^{-1}(t) := \gamma(1-t)$ then $\tau_{\gamma^{-1}} = \tau_\gamma^{-1}$;

(*ii*) If $\gamma(1) = \mu(0)$ then $\tau_{\gamma*\mu} = \tau_\mu \circ \tau_\gamma$.

For all $x \in M$ let us consider the set:

$$C(x) = \{\gamma : I \to M \text{ piecewise smooth such that } \gamma(0) = \gamma(1) = x\}$$

Then $\Phi_\nabla(x) := \{\tau_\gamma \in \mathrm{GL}(E_x) \mid \gamma \in C(x)\}$ is a group of linear isomorphisms, called *holonomy group of* ∇ *in* x. If we fix a G-frame at x we can identify $\Phi_\nabla(x)$ with a subgroup $\Phi_\nabla^\alpha(x)$ of G, and changing the G-frame yields conjugated subgroups. Analogously for Γ the *holonomy group of* Γ *in* x can be defined by $\Phi_\Gamma(x) := \{\tau_\gamma \in \mathrm{Diff}(P_x) \mid \gamma \in C(x)\}$, and for each choice of a point $u \in P_x$ we can identify $\Phi_\Gamma(x)$ with a subgroup $\Phi_\Gamma(u)$ of G via the homomorphism

$$\begin{array}{rcll} \Phi_\Gamma(x) & \to & G & \\ \tau_\gamma & \mapsto & a & \text{such that } \tau_\gamma(u) = ua \end{array}$$

Again, changing the point u in the fiber of P yields conjugated subgroups. The groups $\Phi_\nabla^\alpha(x)$ and $\Phi_\Gamma(u)$ are in the same conjugacy class in G, which we will denote simply with $\Phi(x)$.

If we apply the same construction to the set

$$C^0(x) = \{\gamma \in C(x) \text{ null-homotopic }\}$$

we obtain the *reduced holonomy groups* $\Phi_\nabla^0(x)$ and $\Phi_\Gamma^0(x)$ and the conjugacy class $\Phi^0(x)$ in G.

Theorem 6.3.9. *For every $x \in M$ we have:*

(*i*) *$\Phi^0(x)$ is a connected closed subgroup of G;*

(*ii*) *$\Phi^0(x)$ is a normal subgroup of $\Phi(x)$ and $\Phi(x)/\Phi^0(x)$ is countable.*

Remark 6.3.10. $\Phi(x)$ is a Lie group and $\Phi^0(x)$ is the component of e in $\Phi(x)$.

Pull-back of connections

Connections can be pulled back to pull-back bundles preserving all their features.

Proposition 6.3.11. *Let $\pi : E \to M$ be a vector bundle with structure group G and let $f : N \to M$ be a smooth map. Let ∇ be a G-compatible connection on E. Then:*

(*i*) *there exists a unique G-compatible connection $f^*\nabla$ on f^*E such that $((f^*\nabla)_X(f^*V))_y = f^*((\nabla_{d_y f(X_y)}V)_{f(y)})$;*

(*ii*) *its curvature tensor satisfy $((f^*R)_{X,Y}(f^*V))_y = f^*((R_{d_y f(X_y), d_y f(Y_y)}V)_{f(y)})$;*

(*iii*) *$\Phi_{f^*\nabla}(y) = \Phi_\nabla(f(y))$ and $\Phi^0_{f^*\nabla}(y) = \Phi^0_\nabla(f(y))$.*

Proposition 6.3.12. *Let $P(M, G)$ be a principal bundle and let $f : N \to M$ be a smooth map. Let Γ be a principal connection on P. Then:*

(*i*) *there exists a unique $f^*\Gamma$ on f^*P such that $\mathrm{d}\tilde{f} \circ f^*\Gamma = \Gamma \circ \tilde{f}$;*

(*ii*) *its connection and curvature forms are $\tilde{f}^*\omega$ and $\tilde{f}^*\Omega$;*

(*iii*) *$\Phi_{f^*\Gamma}(y) = \Phi_\Gamma(f(y))$ and $\Phi^0_{f^*\Gamma}(y) = \Phi^0_\Gamma(f(y))$.*

Reduction Theorem and Holonomy Theorem

Theorem 6.3.13 (Reduction Theorem). *Let $P(M,G)$ be a principal bundle with connected base M and let Γ be a connection on P. Define*

$$P(u) = \{v \in P \mid \text{ there exists a horizontal curve joining } u \text{ and } v\ \}$$

Then:

(i) $P(u)$ *is a $\Phi_\Gamma(u)$-principal bundle obtained from $P(M,G)$ by reduction of the structure group;*

(ii) Γ *reduces to a connection on $P(u)$.*

$P(u)$ is the *holonomy bundle* in u.

Theorem 6.3.14 (Holonomy Theorem). *Let $P(M,G)$ be a principal bundle with connected base M and let Γ be a connection on P. Then*

$$\mathrm{Lie}(\Phi_\Gamma(u)) = \mathrm{Span}\{\Omega_v(X,Y) \in \mathfrak{g} \mid v \in P(u),\ X,Y \in Q_v\}$$

6.4 Obstructions

Since manifolds have the homotopy type of CW-complexes the construction of a map between two of them can be carried out by progressive extensions of maps defined on skeletons. The obstruction one faces in doing this is encoded by a cohomology class defined in a suitable cohomology theory with twisted coefficients. We present the special case of the construction of a section in a fiber bundle.

k-Connectedness and k-simplicity

Let M be a smooth manifold. Every curve $\gamma : I \to M$ induces isomorphisms on homotopy groups $\gamma_\# : \pi_h(M,\gamma(0)) \to \pi_h(M,\gamma(1))$ for all h, where we define $\pi_0(M,x_0) = \tilde{H}_0(M;\mathbb{Z})$ for all $x_0 \in M$.

M is *k-simple* if $\pi_0(M,x_0) = 0$ and if for every pair of points $x_1, x_2 \in M$ and every pair of curves γ_1, γ_2 from x_1 to x_2 the induced isomorphisms

$$(\gamma_1)_\# : \pi_k(M,x_1) \to \pi_k(M,x_2)$$
$$(\gamma_2)_\# : \pi_k(M,x_1) \to \pi_k(M,x_2)$$

coincide.

M is *k-connected* if $\pi_h(M,x_0) = 0$ for all $h = 0,\dots,k$.

Remark 6.4.1. *(i)* M is 1-simple if and only if it is connected and $\pi_1(M,x_0)$ is abelian;

(ii) If M is 1-connected then it is k-simple for every k.

Remark 6.4.2. For a k-simple manifold M every map from an oriented sphere S^k to M determines uniquely an element of $\pi_k(M,x_0)$. Therefore the group $\pi_k(M,x_0)$ coincides with the group $[S^k, M]$ of homotopy classes of maps from S^k to M, with the usual operation obtained collapsing an equator of S^k and redifining each function on one of the spheres in the bouquet. Thus there is no need to specify a base point.

Proposition 6.4.3. *Every connected Lie group is k-simple for every k.*

Proof. Let γ be a curve from a_1 to a_2 in G. Then $\gamma_\# = (L_{a_2 a_1^{-1}})_*$. Indeed, define curves $\gamma_s : I \to G$ mapping $t \mapsto \gamma(1-s+ts)$ for all $s \in I$. Then γ_0 is the constant curve a_2 and $\gamma_1 = \gamma$. Then a homotopy between $L_{a_2 a_1^{-1}} \circ f$ and $\gamma_\#(f)$ is given by $h(s,x) = (\gamma_s)_\#(L_{\gamma(1-s)a^{-1}} \circ f)(x)$. □

Homology and cohomology with twisted coefficients

Let M be a smooth manifold and x_0 be a base point. Let $\mathbb{Z}[\pi_1(M, x_0)]$ be the free $\mathbb{Z}$-module generated by $\pi(M, x_0)$:

$$\mathbb{Z}[\pi_1(M, x_0)] = \left\{ \sum_{i=1}^{k} m_i \alpha_i \,\middle|\, m_i \in \mathbb{Z}, \alpha_i \in \pi_1(M, x_0) \right\}$$

$\mathbb{Z}[\pi_1(M, x_0)]$ can be made into a (possibly non-commutative) ring by defining the product:

$$\left(\sum_{i=1}^{k} m_i \alpha_i \right) \left(\sum_{j=1}^{h} n_j \beta_j \right) = \sum_{i,j} (m_i n_j)(\alpha_i \beta_j)$$

Let Λ be a $\mathbb{Z}$-module. Any homomorphism $\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\Lambda)$ makes Λ into a left $\mathbb{Z}[\pi_1(M, x_0)]$-module via the scalar multiplication:

$$\left(\sum_{i=1}^{k} m_i \alpha_i \right) \lambda := \sum_{i=1}^{k} m_i \rho(\alpha_i)(\lambda)$$

Now consider the singular k-chain module $C_k(\tilde{M})$. There is a natural structure of right $\mathbb{Z}[\pi_1(M, x_0)]$-module over $C_k(\tilde{M})$ induced by the monodromy action of $\pi_1(M, x_0)$ on $\tilde{M}$: indeed, if $\sigma : \triangle^k \to \tilde{M}$ is a singular k-simplex in $\tilde{M}$, we can define

$$\sigma \cdot \alpha : \triangle^k \xrightarrow{\sigma} \tilde{M} \xrightarrow{\alpha} \tilde{M}$$

for all $\alpha \in \pi_1(M, x_0)$ and then extend by linearity. Thus we can define the *module of k-chains with Λ coefficients twisted by ρ*:

$$C_k(M; \Lambda_\rho) := C_k(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M, x_0)]} \Lambda$$

These give a complex of $\mathbb{Z}$-modules with boundary operator

$$\partial : \sigma \otimes \lambda \mapsto (\partial \sigma) \otimes \lambda$$

and we can define *homology groups with twisted coefficients* $H_k(M; \Lambda_\rho)$.

If we define a left $\mathbb{Z}[\pi_1(M, x_0)]$-module structure over $C_k(\tilde{M})$ given by $\alpha \cdot \sigma := \sigma \cdot \alpha^{-1}$ we can consider the $\mathbb{Z}$-modules of *k-cochains with twisted coefficients*:

$$C^k(M; \Lambda_\rho) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(M, x_0)]}(C_k(\tilde{M}), \Lambda)$$

The coboundary operator

$$\delta = \partial^* : \varphi \mapsto [\sigma \mapsto \varphi(\partial \sigma)]$$

gives *cohomology groups with twisted coefficients* $H^k(M; \Lambda_\rho)$.

If we have homomorphisms $\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\Lambda)$ and $\rho' : \pi_1(M, x_0) \to \mathrm{Aut}(\Lambda')$ which make the $\mathbb{Z}$-modules Λ and Λ' into left $\mathbb{Z}[\pi_1(M, x_0)]$-modules, then each $\mathbb{Z}[\pi_1(M, x_0)]$-equivariant homomorphism $f : \Lambda \to \Lambda'$ induces natural homomorphisms $f_\# : H^*(M; \Lambda_\rho) \to H^*(M; \Lambda'_{\rho'})$ which map each class $[\ C_*(\tilde{M}) \xrightarrow{\varphi} \Lambda\ ]$ to $[\ C_*(\tilde{M}) \xrightarrow{\varphi} \Lambda \xrightarrow{f} \Lambda'\ ]$. Clearly if f is an isomorphism then $f_\#$ is an isomorphism too.

Remark 6.4.4. If the twisting homomorphism $\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\Lambda)$ is trivial we recover the standard homology and cohomology groups. Indeed $C_k(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M, x_0)]} \Lambda$ can be naturally identified with $C_k(M; \Lambda)$ and $\mathrm{Hom}_{\mathbb{Z}[\pi_1(M, x_0)]}(C_k(\tilde{M}), \Lambda)$ with $\mathrm{Hom}_{\mathbb{Z}}(C_k(M), \Lambda)$.

Cellular homology and cohomology with twisted coefficients

A CW-complex structure on M induces a CW-complex structure on $\tilde{M}$ since all characteristic maps of cells lift to $\tilde{M}$. If a cell $\tilde{e}$ of $\tilde{M}$ projects onto a cell e of M, then any other cell of $\tilde{M}$ projects onto e if and only if it is obtained from $\tilde{e}$ by the action of $\pi_1(M, x_0)$. Then the cellular chain modules $C_k^{\mathrm{CW}}(\tilde{M})$ are $\mathbb{Z}[\pi_1(M, x_0)]$-modules and we can define

$$C_k^{\mathrm{CW}}(M; \Lambda_\rho) := C_k^{\mathrm{CW}}(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M,x_0)]} \Lambda$$

$$C^k_{\mathrm{CW}}(M; \Lambda_\rho) := \mathrm{Hom}_{\mathbb{Z}[\pi_1(M,x_0)]}(C_k^{\mathrm{CW}}(\tilde{M}), \Lambda)$$

with boundary and coboundary operators obtained by cellular ones just like before. Therefore we obtain callular homology and cohomology groups with twisted coefficents $H_k^{\mathrm{CW}}(M; \Lambda_\rho)$ and $H^k_{\mathrm{CW}}(M; \Lambda_\rho)$.

Proposition 6.4.5. *For each CW-complex structure on M there exist natural isomorphisms $H_k^{\mathrm{CW}}(M; \Lambda_\rho) \simeq H_k(M; \Lambda_\rho)$ and $H^k_{\mathrm{CW}}(M; \Lambda_\rho) \simeq H^k(M; \Lambda_\rho)$. In particular cellular (co)homology groups are independent of the CW-complex structure chosen for M.*

Remark 6.4.6. If we specify a lifting for each k-cell in M we get a basis for $C_k^{\mathrm{CW}}(\tilde{M})$. Therefore a k-cochain with twisted coefficients on M is obtained by assigning to each k-cell in the basis of $C_k^{\mathrm{CW}}(\tilde{M})$ an element of Λ and then extending by $\mathbb{Z}[\pi_1(M, x_0)]$-linearity.

Twisted coefficients associated with a bundle of simple fiber

Let F be a k-simple manifold and $\pi : E \to M$ be an F-bundle. We can define a homomorphism $\rho : \pi_1(M, x_0) \to \mathrm{Aut}(\pi_k(E_{x_0}))$ as follows: let $\gamma : I \to M$ be a curve in M. Then γ is homotopic, without fixing endpoints, to the constant map $\gamma(0)$. Thus Theorem 6.1.4 gives a bundle isomorphism

$$\begin{tikzcd} I \times E_{\gamma(0)} \arrow[r, "\Psi"] \arrow[d] & \gamma^* E \arrow[d] \\ I \arrow[r, "\mathrm{id}"'] & I \end{tikzcd}$$

of the form

$$\begin{array}{rccc} \Psi: & I \times E_{\gamma(0)} & \to & \gamma^* E \\ & (t, u) & \mapsto & (t, \psi_t(u)) \end{array}$$

where ψ_t is a diffeomorphism between $E_{\gamma(0)}$ and $E_{\gamma(t)}$ for all $t \in I$. Therefore we get a diffeomorphism $h_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ defined as $\psi_1 \circ \psi_0^{-1}$

Proposition 6.4.7. *The homotopy class of h_γ depends only on the homotopy class with endpoints fixed of γ, and not on the chosen bundle isomorphism Ψ.*

Therefore if α is a loop in x_0 representing an element in $\pi_1(M, x_0)$ we get a well-defined homomorphism:

$$\begin{array}{rccc} \rho: & \pi_1(M, x_0) & \to & \mathrm{Aut}(\pi_k(E_{x_0})) \\ & \alpha & \mapsto & [[f] \mapsto [h_\alpha^{-1} \circ f]] \end{array}$$

Thus we can define homology and cohomology groups for M with twisted coefficients $H_h(M; \pi_k(E_{x_0})_\rho)$ and $H^h(M; \pi_k(E_{x_0})_\rho)$.

Primary obstructions

Let M be a smooth manifold and let $\pi_{\tilde{M}} : \tilde{M} \to M$ denote its universal cover. Any CW-complex structure for M induces a CW-complex structure for $\tilde{M}$ and the choice of base points $x_0 \in M$ and $\tilde{x}_0 \in \pi_{\tilde{M}}^{-1}(x_0) \subset \tilde{M}$ yields a free right action of $\pi_1(M, x_0)$ onto $\tilde{M}$ which permutes the cells of the same dimensions. Let F be a k-simple manifold, let $\pi_E : E \to M$ be an F-bundle and let f be a section defined on the $(k-1)$-skeleton M^{k-1} of M. Suppose f admits an extension g defined on M^k and consider the characteristic map $\tilde{\Phi} : D^{k+1} \to \tilde{M}$ of the $(k+1)$-cell $\tilde{e}$. The composite $\Phi := \pi_{\tilde{M}} \circ \tilde{\Phi} : D^{k+1} \to M$ is then the characterstic map of the $(k+1)$-cell $e = \pi_{\tilde{M}}(\tilde{e})$. The composition of the attaching map $\varphi = \Phi\big|_{\partial D^{k+1}}$ with the section g induces a map

$$\begin{array}{rccc} g^{\#}: & \partial D^{k+1} & \to & \Phi^* E \\ & y & \mapsto & (y, g(\varphi(y))) \end{array}$$

If we fix $y_0 \in D^{k+1}$ we get a bundle isomorphism

$$\begin{tikzcd} D^{k+1} \times E_{\Phi(y_0)} \arrow[r, "\Psi"] \arrow[d] & \Phi^* E \arrow[d] \\ D^{k+1} \arrow[r, "\mathrm{id}"'] & D^{k+1} \end{tikzcd}$$

of the form

$$\begin{array}{rccc} \Psi: & D^{k+1} \times E_{\Phi(y_0)} & \to & \Phi^* E \\ & (y, u) & \mapsto & (y, \psi_y(u)) \end{array}$$

where ψ_y is a diffeomorphism between $E_{\Phi(y_0)}$ and $E_{\Phi(y)}$ for all $y \in D^{k+1}$. We can then define the map

$$\begin{array}{rccc} \Psi_{y_0}: & \Phi^* E & \to & E_{\Phi(y_0)} \\ & (y, u) & \mapsto & \psi_{y_0}(\psi_y^{-1}(u)) \end{array}$$

Now consider any curve $\tilde{\gamma}$ from $\tilde{\Phi}(y_0)$ to $\tilde{x}_0$ in $\tilde{M}$, which projects onto a curve $\gamma = \pi_{\tilde{M}} \circ \tilde{\gamma}$ from $\Phi(y_0)$ to x_0, and set

$$g_{\tilde{e}} : \partial D^{k+1} \xrightarrow{g^{\#}} \Phi^* E \xrightarrow{\Psi_{y_0}} E_{\Phi(y_0)} \xrightarrow{h_\gamma} E_{x_0}$$

Therefore we can define a k-cochain with twisted coefficients $o(g) \in C^{k+1}(M; \pi_k(E_{x_0})_\rho)$ which associates each $(k+1)$-cell $\tilde{e}$ in $\tilde{M}$ with the homotopy class $[g_{\tilde{e}}] \in [\partial D^{k+1}, E_{x_0}] = \pi_k(E_{x_0})$ constructed above.

Proposition 6.4.8. *The k-cochain $o(g)$ is well-defined, i.e. it depends neither on the choice of $y_0 \in D^{k+1}$ nor on the choice of the bundle isomorphism $\Psi : D^{k+1} \times E_{\Phi(y_0)} \to \Phi^* E$, and it satisfies $o(g)(\tilde{e} \cdot \alpha) = \rho(\alpha^{-1})(o(g)(\tilde{e}))$.*

Proposition 6.4.9. *The $(k+1)$-cochain $o(g)$ is a cocycle.*

Proposition 6.4.10. *If g and g' are two different extensions of f on M^k then the cocycle $o(g) - o(g')$ is a coboundary.*

Therefore we have a well-defined cohomology class $o(f) \in H^{k+1}(M; \pi_k(E_{x_0})_\rho)$ given by $[o(g)]$ for any extension g of f to M^k.

Theorem 6.4.11. *The section f can be extended to M^{k+1} if and only if $o(f) = 0$.*

Now suppose the fiber F is k-simple and $(k-1)$-connected but not k-connected. Then we can always define sections on the k-skeleton of M without facing obstructions.

Proposition 6.4.12. *Let f, f' be sections defined on the k-skeleton of M. Then $o(f) - o(f')$ is a coboundary.*

Therefore we have a well-defined homology class $o(E) \in H^{k+1}(M; \pi_k(E_{x_0})_\rho)$ called the *primary obstruction* of the bundle $\pi : E \to M$ which is given by $o(f)$ for any section f defined on the k-skeleton of M.

Example 6.4.13. Let $P(M, G)$ be a principal bundle and let G' be a closed subgroup of G. Suppose G' is homotopy equivalent to G. Then Theorem 6.1.6 applied to $p : G \to G/G'$ gives $\pi_k(G/G') = 0$ for all k. Therefore there is no obstruction to defining a section for the bundle $\pi_{P/G'} : P/G' \to M$, and thus it is always possible to reduce the structure group of $P(M, G)$ to G'.

Suppose we have homotopy equivalent manifolds F and F' both k-simple and $(k-1)$-connected but not k-connected. Let $\pi : E \to M$ be an F-bundle and $\pi' : E' \to M$ be an F'-bundle, and suppose we have bundle maps

$$\begin{tikzcd} E \arrow[r, "f_E"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\mathrm{id}"'] & M \end{tikzcd} \qquad \begin{tikzcd} E \arrow[d, "\pi"'] & E' \arrow[l, "g_{E'}"'] \arrow[d, "\pi'"] \\ M & M \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

such that f_E and $g_{E'}$ restrict to homotopy equivalences on every fiber. Let f_{x_0} and g_{x_0} denote the restrictions $f_E\big|_{E_{x_0}}$ and $g_{E'}\big|_{E'_{x_0}}$ respectively. These maps induce isomorphisms between homotopy groups $(f_{x_0})_* : \pi_k(E_{x_0}) \to \pi_k(E'_{x_0})$ and $(g_{x_0})_* : \pi_k(E'_{x_0}) \to \pi_k(E_{x_0})$.

Proposition 6.4.14. *The isomorphisms $(f_{x_0})_*$ and $(g_{x_0})_*$ are $\mathbb{Z}[\pi_1(M, x_0)]$-equivariant.*

Thus $(f_{x_0})_*$ and $(g_{x_0})_*$ induce isomorphisms between cohomology groups with twisted coefficients $H^*(M; \pi_k(E_{x_0})_\rho)$ and $H^*(M; \pi_k(E'_{x_0})_{\rho'})$ which will be denoted by $(f_{x_0})_\#$ and $(g_{x_0})_\#$ respectively.

Proposition 6.4.15. *The primary obstructions of $\pi : E \to M$ and $\pi' : E' \to M$ are equivalent, i.e. we have equalities $(f_{x_0})_\#(o(E)) = o(E')$ and $(g_{x_0})_\#(o(E')) = o(E)$.*

6.5 Lattices

This section collects properties and results about Lie groups and algebraic groups. In particular we focus on amenability and on the existence and rigidity of lattices.

Semisimple Lie groups and reductive algebraic groups

The *derived series* of a Lie algebra $\mathfrak{g}$ is the descending sequence of ideals

$$D^0\mathfrak{g} \supset D^1\mathfrak{g} \supset D^2\mathfrak{g} \supset \dots$$

defined recursively by $D^0\mathfrak{g} := \mathfrak{g}$ and $D^{i+1}\mathfrak{g} := [D^i\mathfrak{g}, D^i\mathfrak{g}]$. Each quotient $D^i\mathfrak{g}/D^{i+1}\mathfrak{g}$ is abelian. A Lie algebra $\mathfrak{g}$ is *solvable* if $D^n\mathfrak{g} = 0$ for some sufficiently large n. A Lie algebra $\mathfrak{g}$ is *semisimple* if it contains no non-zero solvable ideals, while it is *simple* if it is not abelian and if it contains no non-trivial ideals.

A connected Lie group G is *semisimple* if its Lie algebra is semisimple, while it is *simple* if its Lie algebra is simple.

Theorem 6.5.1. *Every semisimple Lie algebra $\mathfrak{g}$ can be written uniquely as a direct sum of its simple ideals.*

A semisimple Lie algebra $\mathfrak{g}$ is *compact* if its Killing form is negative definite. If none of the simple ideals of a semisimple Lie algebra $\mathfrak{g}$ is compact then $\mathfrak{g}$ is *without compact factors.* A semisimple Lie group whose Lie algebra is without compact factors is *without compact factors.*

The *derived series* of a group G is the descending sequence of groups

$$G^{(0)} \rhd G^{(1)} \rhd G^{(2)} \rhd \dots$$

defined recursively by $G^{(0)} := G$ and $G^{(i+1)} := [G^{(i)}, G^{(i)}]$. Each quotient $G^{(i)}/G^{(i+1)}$ is abelian. A group G is *solvable* if $G^{(n)} = 0$ for some sufficiently large n. A group G is *virtually solvable* if it admits a finite-index subgroup which is solvable.

Lemma 6.5.2. (*i*) *Every subgroup of a solvable group is solvable;*

(*ii*) *Every quotient group of a solvable group is solvable;*

(*iii*) *Every extension of a solvable group by a solvable group is solvable;*

(*iv*) *If $G < \mathrm{GL}(n, \mathbb{C})$ is solvable then the Zariski closure $\overline{G}$ is solvable.*

Corollary 6.5.3. *If $G < \mathrm{GL}(n, \mathbb{C})$ is virtually solvable then the Zariski closure $\overline{G}$ is virtually solvable.*

Proof. Every finite-index subgroup $H < G$ admits a finite-index subgroup $H' < H$ which is normal in G. Indeed it suffices to consider the kernel of the homomorphism from G to the group of permutations of left cosets of H wich maps each $a \in G$ to the left translation by a. Therefore we can always choose a finite-index normal solvable subgroup $H \lhd G$. The Zariski closure $\overline{H}$ is normal in $\overline{G}$ and the projection $\pi : \overline{G} \to \overline{G}/\overline{H}$ defines a homomorphism of algebraic groups. Now since the image $\pi(G)$ must be finite (and in particular closed) then $\pi(\overline{G})$, which equals $\overline{\pi(G)} = \pi(G)$, must be finite too. Therefore $\overline{H}$ is a finite-index solvable subgroup of $\overline{G}$. □

A linear group $G < \mathrm{GL}(n, \mathbb{C})$ is algebraic if it is closed with respect to the Zariski topology. It is an algebraic $\mathbb{R}$-*group* if the ideal of polynomials vanishing on G is generated by polynomials with real coefficients. The group of $\mathbb{R}$-*rational points* of G is $G(\mathbb{R}) := G \cap \mathrm{GL}(n, \mathbb{R})$.

Lemma 6.5.4. *In every algebraic group G there exists a unique maximal normal connected solvable subgroup $R(G)$, and this subgroup is algebraic.*

The subgroup $R(G)$ is called the *radical* of G, and the group G is *semisimple* if its radical is trivial. If a linear algebraic group $G < \mathrm{GL}(n, \mathbb{C})$ is semisimple then its group of $\mathbb{R}$-rational points $G(\mathbb{R})$ is a semisimple Lie group. A connected algebraic group G is *simple* if it is non-abelian and has no proper normal algebraic subgroups apart from the trivial one. It is *almost simple* if it is non-abelian and has no proper normal algebraic subgroups except for finite subgroups.

Theorem 6.5.5. *Every connected semisimple algebraic group G can be written uniquely as an almost direct product of its minimal connected non-trivial normal algebraic subgroups, i.e. of its almost simple normal subgroups $G_1, \dots, G_n$. In particular, there are only finitely many such subgroups. Every connected normal algebraic subgroup of G is a product of those G_i it contains, and commutes with the other ones.*

If none of the minimal connected non-trivial normal algebraic subgroups of a semisimple algebraic group G is compact then G is *without compact factors.*

An element a in a linear algebraic group $G < \mathrm{GL}(n, \mathbb{C})$ is *unipotent* if $a - I$ is a nilpotent endomorphism of $\mathbb{C}^n$. Since sums and products of nilpotent elements of a ring are nilpotent, products of unipotent elements are unipotent. The group G is *unipotent* if all its elements are unipotent. The *unipotent radical* $R_u(G)$ of G is the maximal unipotent subgroup of $R(G)$, and G is *reductive* if its unipotent radical is trivial.

Theorem 6.5.6. *Every finite dimensional rational representation of a reductive algebraic group is completely reducible.*

Theorem 6.5.7 (Levi's decomposition)**.** *Let $G < \mathrm{GL}(n, \mathbb{C})$ be a connected linear algebraic group. Then there exists a reductive subgroup $H < G$ such that $G \simeq H \ltimes R_u(G)$.*

Proposition 6.5.8. *Let G be a reductive group. Then the commutator subgroup $[G, G]$ is semisimple and every semisimple subgroup of G is contained in $[G, G]$.*

Amenable groups

Definition 6.5.9. Let G be a Lie group and let μ be a left Haar measure for G. Let $L^\infty(G, \mu)$ be the Banach space of (classes of) real-valued measurable functions on G with finite L^∞-norm and let $L^\infty(G, \mu)^*$ be its continuous dual space endowed with the dual norm. A continuous linear functional $\Lambda \in L^\infty(G, \mu)^*$ is a *mean* if $\|\Lambda\| = 1$ and if it is positive, i.e. if $f \geqslant 0 \Rightarrow \Lambda(f) \geqslant 0$. The left translation of G onto itself defines a left action of G onto $L^\infty(G, \mu)$ given by $a \cdot f(b) := f(ab)$ for all $a, b \in G$. A mean Λ is *left-invariant* if $\Lambda(a \cdot f) = \Lambda(f)$ for all $a \in G$ and $f \in L^\infty(G, \mu)$. The Lie group G is *amenable* if it admits a left-invariant mean.

Lemma 6.5.10. (i) *Every closed subgroup of an amenable group is amenable;*

(ii) *Every quotient group of an amenable group is amenable;*

(iii) *Every extension of an amenable group by an amenable group is amenable;*

(iv) *Every direct limit of amenable groups is amenable;*

(v) *Every abelian group is amenable;*

(vi) *Every compact group is amenable.*

Corollary 6.5.11. *All solvable and virtually solvable groups are amenable.*

Proof. It is a direct consequence of (v), (iii) and (vi) of the previous Lemma. □

Example 6.5.12. A non-commutative free group is not amenable. The groups $\mathrm{SL}(n, \mathbb{K})$ (and thus $\mathrm{GL}(n, \mathbb{K})$ and $\mathrm{PSL}(n, \mathbb{K})$ too) are non-amenable for $n \geqslant 2$.

Theorem 6.5.13 (Tit's alternative)**.** *Let G be a subgroup of $\mathrm{GL}(n, \mathbb{R})$. Then G containes either a solvable subgroup of finite index or a non-abelian free subgroup.*

Lattices

A Lie group G is *unimodular* if its Haar measure is both left and right-invariant.

Example 6.5.14. Abelian groups, compact groups, semisimple Lie groups, reductive algebraic groups and the their groups of $\mathbb{R}$-rational points are all unimodular.

Lemma 6.5.15. *Let G be a unimodular Lie group and let $H < G$ be a subgroup. Then the quotient space G/H admits a unique G-invariant measure induced by the Haar measure of G.*

A discrete subgroup Γ of a unimodular Lie group G is a *lattice* if the G-invariant measure on G/Γ is finite. A subgroup $H < G$ is *cocompact* (or *uniform*) if the quotient space G/H is compact.

Theorem 6.5.16 (Borel)**.** *Semisimple Lie groups, reductive algebraic $\mathbb{R}$-groups and their groups of $\mathbb{R}$-rational points admit cocompact lattices.*

A lattice Γ in a connected semisimple Lie group G without compact factors is *reducible* if there exist two non-trivial connected closed normal subgroups $H_1, H_2 \lhd G$ such that:

(*i*) $H_1 \cdot H_2 = H$;

(*ii*) $H_1 \cap H_2$ is discrete;

(*iii*) $(\Gamma \cap H_1) \cdot (\Gamma \cap H_2)$ is a subgroup of finite index in Γ.

The lattice Γ is *irreducible* if it is not reducible.

Theorem 6.5.17. *Let G be a connected semisimple Lie group without compact factors and let $\Gamma < G$ be a lattice. Then there exists a finite family of connected normal closed subgroups $H_1, \ldots, H_m$ such that:*

(*i*) *if $\hat{H}_i := \prod_{j \neq i} H_j$ then $H_i \cap \hat{H}_i$ is discrete;*

(*ii*) *G is isomorphic to $\prod_{i=1}^m H_i$;*

(*iii*) *$\Gamma_i := \Gamma \cap H_i$ is an irreducible lattice in H_i;*

(*iv*) *$\Gamma_1 \times \ldots \times \Gamma_m$ is a normal subgroup of finite index of Γ.*

Let G be a connected semisimple Lie group. A connected commutative subgroup $T < G$ is an $\mathbb{R}$-*split torus* if all the elements of $\mathrm{Ad}(T)$ are diagonalizable in $\mathrm{Aut}(\mathfrak{g})$.

Proposition 6.5.18. *All maximal $\mathbb{R}$-split tori inside a connected semisimple Lie group have the same dimension.*

The *real rank* $\mathrm{rank}_{\mathbb{R}} G$ of a connected semisimple Lie group G is the dimension of a maximal $\mathbb{R}$-split torus in G.

Remark 6.5.19. If G_1 and G_2 are connected semisimple Lie groups then $\mathrm{rank}_{\mathbb{R}}(G_1 \times G_2)$ equals $\mathrm{rank}_{\mathbb{R}} G_1 + \mathrm{rank}_{\mathbb{R}} G_2$.

Example 6.5.20. The groups $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{R})$ have real rank 1.

Theorem 6.5.21. *Let G be a connected semisimple Lie group of real rank at least 2 without compact factors and with finite center. Let $\Gamma < G$ be an irreducible lattice. Then for each algebraic linear $\mathbb{R}$-group H and every homomorphism $\rho : \Gamma \to H(\mathbb{R})$ the Zariski closure $\overline{\mathrm{im}\, \rho}$ is semisimple.*

A group is *adjoint* if its center is trivial.

Lemma 6.5.22. *If G is a connected semisimple algebraic linear group then* $\mathrm{Ad}(G) < \mathrm{Aut}(\mathfrak{g})$ *is an adjoint group.*

Theorem 6.5.23 (Margulis' super-rigidity)**.** *Let G be a connected semisimple Lie group of real rank at least 2 without compact factors. Let $\Gamma < G$ be an irreducible lattice, let H be a connected adjoint semisimple algebraic $\mathbb{R}$-group without compact factors and let $\rho : \Gamma \to H(\mathbb{R})$ be homomorphism whose image is dense in H with respect to the Zariski topology. Then ρ extends uniquely to a surjective homomorphism $\tilde{\rho} : G \to H(\mathbb{R})^0$.*

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