Università degli Studi di Pisa

Dipartimento di Matematica
Corso di Laurea in Matematica

Fourth-order geometric flows and integral pinching of the curvature

18 ottobre 2013

Tesi di Laurea Magistrale

Candidato
Giovanni Mascellani
Università di Pisa

Relatore
Dott. Carlo Mantegazza
Scuola Normale Superiore

Correlatore
Prof. Zindine Djadli
Université Joseph Fourier de Grenoble

Controrelatore
Dott. Roberto Frigerio
Università di Pisa

Anno Accademico 2012/2013

Contents

Foreword

Geometric flows have recently become a very important tool for studying the topology of smooth manifolds admitting Riemannian metrics satisfying certain hypotheses. A geometric flow is the evolution of the Riemannian metric g_0 of a smooth manifold according to a differential rule of the form $\partial_t g(t) = P(g(t))$, where at each time $g(t)$ is a positively defined $(2, 0)$ tensor (such that $g(0) = g_0$) on a fixed differentiable manifold M and $P(g)$ is a smooth differential operator depending on g itself and on its space derivatives, hopefully chosen in order to have the effect of increasing the "regularity" of the Riemannian manifold. Once the metric has been "enhanced" by the flow, one can study it more easily and obtain topological results that, since the flow is smooth, must also hold for the initial differentiable manifold.

The study of a geometric flow usually goes through some recurrent steps:

1. The very first point is to show that, given the initial metric, there is a (usually unique) smooth solution of the flow for at least a short interval of time.

2. The maximal time for the existence of a smooth solution can be finite or infinite: in the first case a singularity of the flow develops, so its nature must be investigated in order to possibly exclude it by a contradiction argument, or to classify it to get topological information on the manifold, or finally to fully understand its structure and possibly perform a smooth topological "surgery" in order to continue the flow after the singular time. A very remarkable example of this last situation (which by far is the most difficult case to deal with) is the success in the study of Hamilton's Ricci flow, that is, the flow $\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}$, on the 3–manifolds due to Perelman (see [Per02, Per03a, Per03b]), leading to the proof of the *Poincaré conjecture*.

   In our work we will deal only with the first situation: assuming that the flow of $g(t)$ is defined in the maximal time interval $[0, T)$ with $0 < T < \infty$ and that at time T a singularity develops, we will try to exclude this scenario by a contradiction argument (just to mention, another recent great success of the application of Ricci flow to geometric problems, the proof of the *differentiable sphere theorem* by Brendle and Schoen [BS09], follows this line). In this respect, a fundamental point of this program is to show that the Riemann curvature tensor must be unbounded as $t \to T$.

3. After obtaining the above result, the idea is to perform a blow–up analysis: we take $t_i \nearrow T$ and dilate the metric $g(t_i)$ so that the rescaled sequence of manifolds have uniformly bounded curvatures; then, we prove that they stay within a precompact

class and take a limit of such sequence. At this point, one has to study the properties of such possible limit manifold (this may require a full classification result) in order to proceed in one of the ways described above.

4. In our case, we actually want to find a contradiction in this procedure by studying the limit manifold. This would imply that the flow cannot actually be singular in finite time and the maximal time of smooth existence has to be $+\infty$.

5. Then, once we know that the flow is defined for all times, we prove again that there is a limit manifold as $t \to +\infty$ and we study its properties. For example, if the limit manifold turns out to have constant positive sectional curvature, it must be the quotient of the standard sphere. Hence, the initial manifold too is topologically a quotient of the sphere, concluding the geometric program.

Among the geometric flows, a special class is given by the ones arising as gradients of geometric functionals of the metric and the curvature. In such cases, because of the variational structure of the flow, the natural energy (the value of the functional) is decreasing in time and one can take advantage of this fact to carry out some of the arguments mentioned above.

Our work, which fits in this context, is based upon the PhD thesis of Vincent Bour [Bou12], who studied a class of geometric gradient flows of the fourth order (see also [Bou10]). To briefly describe it, we recall that the Riemann curvature tensor Riem_g of a Riemannian manifold (M^n, g) can be orthogonally decomposed as

$$\mathrm{Riem}_g = \mathcal{W}_g + \mathcal{Z}_g + \mathcal{S}_g,$$

with

$$\mathcal{S}_g = \frac{R_g}{2n(n-1)} g \owedge g$$
$$\mathcal{Z}_g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{n} g\right) \owedge g$$

where Ric_g is the Ricci tensor, R_g the scalar curvature and the remaining Weyl curvature $\mathcal{W}_g$ is a fully traceless tensor (the operation $\owedge$ indicates the Kulkarni–Nomizu product, see the next chapter for all the definitions).

Then, we define for $0 < \lambda < 1$

$$\mathcal{F}^\lambda(g) = (1-\lambda)\int_{M^n} |\mathcal{W}_g|^2 \, dv_g + \lambda \int_{M^n} |\mathcal{Z}_g|^2 \, dv_g$$

and consider the gradient flow

$$\frac{\partial g}{\partial t}(t) = -2\nabla \mathcal{F}^\lambda(g(t)). \tag{1}$$

We will follow the steps outlined above in order to prove that if we consider a compact manifold M^4 with an initial smooth metric g_0 such that (M^4, g_0) has positive scalar curvature and initial energy $\mathcal{F}^\lambda(g_0)$ sufficiently low, the flow (1) exists for all times and converges in the $\mathcal{C}^\infty$ topology to a smooth metric g_∞ on M of positive constant sectional curvature. Thanks to the Uniformization Theorem, we have that M is diffeomorphic to a quotient of the 4–sphere, thus, it can only be either the 4–sphere or the 4–dimensional real projective space.

To Anna, Bernardo
and Giuseppe.

CHAPTER 1

Preliminaries

Contents

In this chapter we set up some notations and we recall the technical tools that we'll use later on in the thesis. A standard reference for most of the material presented here is [GHL90].

1.1 Notations and conventions

Let M be a smooth differentiable manifold. As customary, we will denote the space of the p–covariant and q–contravariant tensors of class $\mathcal{C}^\infty$ with $T^{(p,q)}(M)$. The symbol $\mathcal{S}^2(M)$ will instead indicate the space of smooth symmetric 2–forms (bilinear forms) on M and $\mathcal{S}^2_+(M)$ will indicate the cone of the positively defined 2–forms in $\mathcal{S}^2(M)$ (i.e., the Riemannian metrics).

Fixed a local basis, if $g \in \mathcal{S}^2_+(M)$ is the metric of the Riemannian manifold (M, g), we will use g^{ij} to indicate the components of the matrix g^{-1}. The matrices g and g^{-1} provide canonical isomorphisms between TM and T^*M, which we'll denote with $\flat$ and $\sharp$. In coordinates, these isomorphisms work by raising and lowering indices (the convention of summing over repeated indices will be adopted in all the thesis), that is,

$$(A^\sharp)^j = A_i g^{ij} \qquad \text{and} \qquad (B_\flat)_j = B^i g_{ij}.$$

We will use $A * B$ to indicate any linear combination of tensors obtained from $A \otimes B$ by contracting some pairs of indices with the metric or its inverse. A very useful property of such $*$–product is that $|A * B| \leqslant C|A||B|$, for a certain constant C depending only on the structure of the contraction.

The *trace* of a tensor T with respect to two of its indices is a summation of the form

$$T_{ij}g^{ij}, \qquad T^{ij}g_{ij} \qquad \text{or} \qquad T^i_j\delta^j_i,$$

where δ^i_j is 1 if the two indices are equal and zero otherwise. The *norm* of a (p,q)–tensor T is

$$|T| = \Big(T^{i_1\dots i_q}_{j_1\dots j_p} \cdot T^{k_1\dots k_q}_{\ell_1\dots\ell_p} \cdot g_{i_1k_1}\dots g_{i_qk_q} \cdot g^{j_1\ell_1}\dots g^{j_p\ell_p}\Big)^{\frac{1}{2}}.$$

A Riemannian metric induces a *canonical volume density*

$$dv_g = \sqrt{\det(g_{ij})}\, dx^1 \,\dots\, dx^n,$$

such that for a certain measurable subset $A \subseteq \Omega \subseteq M$ contained into a chart $\varphi\colon \Omega \to \mathbb{R}^n$ and a measurable function $f\colon M \to \mathbb{R}$ with compact support we have

$$\int_A f\, dv_g = \int_{\varphi(A)} (f \circ \varphi^{-1})\sqrt{\det(g_{ij})}\, dx^1 \,\dots\, dx^n.$$

In particular, we define the volume $\mathrm{Vol}_g(A) = \int_A 1\, dv_g$.

The canonical density dv_g is the density associated to the *canonical volume form* $d\omega_g$ that exists locally for all manifolds and globally for oriented manifolds. The canonical volume form is characterized by the fact that it evaluates to 1 on every positive orthonormal basis for g. Its coordinate expression is

$$d\omega_g = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots \wedge dx^n.$$

We will denote the *Christoffel symbols* of the Levi–Civita connection associated to g with

$$\Gamma^k_{ij} = \frac{1}{2} g^{h\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

and the *covariant derivative* with

$$\nabla_X Y = (X^i\partial_i Y^k + X^i y^j \Gamma^k_{ij})\partial_k.$$

We also define the *Lie derivative* of a vector field Y with respect to another vector field X,

$$\mathcal{L}_X Y = [X, Y],$$

that does not depend on the metric. If instead of X we have a 1–form η, the Lie derivative is considered to be computed with respect to the associated vector field $\eta^\sharp$:

$$\mathcal{L}_\eta Y = \mathcal{L}_{\eta^\sharp} Y = [\eta^\sharp, Y].$$

Both the covariant and Lie derivatives with respect to a fixed vector field X can be extended in a unique way to tensors of all orders (covariant, contravariant and mixed), by requiring that they commute with the contraction of indices and satisfy a Leibniz rule with respect to the tensor product: $\mathcal{L}_X(S \otimes T) = \mathcal{L}_X S \otimes T + S \otimes \mathcal{L}_X T$ (and similarly for ∇_X).

We mention the two formulas $\nabla_X g = 0$ and

$$\mathcal{L}_\eta g_{ij} = \nabla_i \eta_j + \nabla_j \eta_i. \tag{1.1}$$

Moreover, thanks to the Stokes theorem, on closed Riemannian manifolds we have a formula for integrating by parts: if $\nabla_\alpha S * T$ is a fully contracted tensor, then

$$\int_M \nabla_\alpha T * S = -\int_M S * \nabla_\alpha T,$$

provided that the structure of the $*$ operator is preserved; i.e., the same indices are contracted in the same way.

The covariant derivative is not a tensor, but the *Riemann curvature operator* obtained as the difference of two double covariant derivatives with swapped indices is:

$$\mathrm{Riem}_{ijk}^{\ell} = \left(\nabla^2_{ji}\partial_k - \nabla^2_{ij}\partial_k\right)^{\ell} = \partial_i \Gamma^{\ell}_{jk} - \partial_j \Gamma^{\ell}_{ik} + \Gamma^{h}_{ik}\Gamma^{\ell}_{hj} - \Gamma^{h}_{jk}\Gamma^{\ell}_{hi}.$$

More often we will use the fully covariant Riemann tensor:

$$\mathrm{Riem}_{ijk\ell} = \mathrm{Riem}_{ijk}^{h}\, g_{h\ell}.$$

Taking the non trivial traces we obtain the *Ricci tensor* and the *scalar curvature*:

$$\begin{aligned} \mathrm{Ric}_{ik} &= \mathrm{Riem}_{ijk\ell}\, g^{j\ell} \\ R &= \mathrm{Ric}_{ik}\, g^{ik} = \mathrm{Riem}_{ijk\ell}\, g^{ik} g^{j\ell}. \end{aligned}$$

The *traceless Ricci tensor* is to be mentioned too:

$$\mathring{\mathrm{Ric}}_{ik} = \mathrm{Ric}_{ik} - \frac{R}{n} g_{ik},$$

whose square norm is given by

$$\|\mathring{\mathrm{Ric}}_g\|^2_{L^2} = \|\mathrm{Ric}_g\|^2_{L^2} - \frac{1}{n}\|R_g\|^2_{L^2}.$$

It is well known that the Riemann tensor enjoys the following symmetries:

$$\mathrm{Riem}_{ijk\ell} = \mathrm{Riem}_{k\ell ij} = -\mathrm{Riem}_{jik\ell} = -\mathrm{Riem}_{ij\ell k}.$$

Thanks to these relations, we can interpret it as a symmetric bilinear form on $\Lambda^2 TM$, the space of alternating 2–vectors on M.

More in general, we can introduce the space of (p,q)*–forms*

$$\Lambda^{(p,q)}(M) = \Lambda^p T^*M \otimes \Lambda^q T^*M.$$

The subspace of the *symmetric* (p,p)*–forms* will be denoted with

$$\mathcal{C}^{(p)}(M) = \Lambda^p T^*M \odot \Lambda^p T^*M,$$

where $\odot$ indicates the symmetric product. The *Kulkarni–Nomizu product* is the bilinear product $\mathcal{C}^{(1)}(M) \times \mathcal{C}^{(1)}(M) \to \mathcal{C}^{(2)}(M)$ that maps

$$(a \owedge b)_{ijk\ell} = a_{ik}b_{j\ell} + a_{j\ell}b_{ik} - a_{i\ell}b_{jk} - a_{jk}b_{i\ell}.$$

On the space of double forms $\Lambda^{(p,q)}(M)$, we apply a renormalization on the induced scalar product:

$$\langle S|T\rangle = \frac{1}{p!q!} g^{i_1k_1} \dots g^{i_pk_p} g^{j_1\ell_1} \dots g^{j_q\ell_q} \left\langle S_{i_1\dots i_p|j_1\dots j_q} | T_{k_1\dots k_p|\ell_1\dots\ell_q} \right\rangle.$$

The Riemann tensor admits a very important orthogonal decomposition with respect to such scalar product:

$$\mathrm{Riem}_g = \mathcal{W}_g + \mathcal{Z}_g + \mathcal{S}_g, \tag{1.2}$$

where

$$\mathcal{S}_g = \frac{R_g}{2n(n-1)} g \owedge g \qquad \text{and} \qquad \mathcal{Z}_g = \frac{1}{n-2} \mathring{\mathrm{Ric}}_g \owedge g,$$

where the remaining *Weyl curvature* $\mathcal{W}_g$ is a fully traceless tensor, i.e., it yields a zero result under all trace operations. Since the decomposition is orthogonal, the L^2 norms satisfy

$$\|\mathrm{Riem}_g\|^2_{L^2} = \|\mathcal{S}_g\|^2_{L^2} + \|\mathcal{Z}_g\|^2_{L^2} + \|\mathcal{W}_g\|^2_{L^2} = \frac{1}{2n(n-1)} \|R_g\|^2_{L^2} + \frac{1}{n-2} \|\mathring{\mathrm{Ric}}_g\|^2_{L^2} + \|\mathcal{W}_g\|^2_{L^2}. \tag{1.3}$$

Another important tensor is the *Schouten* one, which is defined as

$$A_g = \mathrm{Ric}_g - \frac{1}{2(n-1)} R_g g$$

and satisfies

$$\mathrm{Riem}_g = \mathcal{W}_g + \frac{1}{n-2} A_g \owedge g.$$

The *Laplacian* of a (p,q) tensor T will be defined as

$$\Delta T = -\nabla^\alpha \nabla_\alpha T.$$

A number of differential operators and other notations will be used. Following the notations in [Bou12] we define

$$\delta \colon \Lambda^{(p+1,q)}(M) \to \Lambda^{(p,q)}(M) \qquad (\delta T)_{i_1\dots i_p|j_1\dots j_p} = -\nabla^\alpha T_{\alpha i_1\dots i_p|j_1\dots j_q},$$

$$\tilde{\delta} \colon \Lambda^{(p,q+1)}(M) \to \Lambda^{(p,q)}(M) \qquad (\tilde{\delta} T)_{i_1\dots i_p|j_1\dots j_p} = -\nabla^\alpha T_{i_1\dots i_p|\alpha j_1\dots j_q},$$

$$\mathsf{D} \colon \Lambda^{(p,q)}(M) \to \Lambda^{(p+1,q)}(M) \qquad (\mathsf{D} T)_{i_0\dots i_p|j_1\dots j_p} = \sum_{k=0}^{p} (-1)^k \nabla_{i_k} T_{i_1\dots \hat{i}_k \dots i_p|j_1\dots j_q},$$

$$\tilde{\mathsf{D}} \colon \Lambda^{(p,q)}(M) \to \Lambda^{(p,q+1)}(M) \qquad (\tilde{\mathsf{D}} T)_{i_1\dots i_p|j_0\dots j_p} = \sum_{k=0}^{q} (-1)^k \nabla_{j_k} T_{i_1\dots i_p|j_1\dots \hat{j}_k \dots j_q}.$$

Moreover, if $T \in \mathcal{C}^{(2)}(\mathbb{R})$ and $u \in \mathcal{S}^2(M)$, we define

$$(T \vee T)_{ij} = T_{\alpha\beta\gamma i} T^{\alpha\beta\gamma}{}_j \qquad \text{and} \qquad (\mathring{T}u)_{ij} = T_{\alpha i \beta j} u^{\alpha\beta}.$$

We conclude this section with a lemma on how these operators act on the curvature tensors.

1.1 Lemma. *Let (M^n, g) be a Riemannian manifold. Then the followings identities hold:*

$$\begin{aligned}
\mathsf{D}\operatorname{Riem}_g &= 0, & \tilde{\mathsf{D}}\operatorname{Riem}_g &= 0,\\
\delta\operatorname{Riem}_g &= -\tilde{\mathsf{D}}\operatorname{Ric}_g, & \tilde{\delta}\operatorname{Riem}_g &= -\mathsf{D}\operatorname{Ric}_g,\\
\delta\operatorname{Ric}_g &= -\frac{1}{2}\tilde{\mathsf{D}}R_g, & \tilde{\delta}\operatorname{Ric}_g &= -\frac{1}{2}\mathsf{D}R_g, \qquad \textit{(Schur's lemma)}\\
\delta(R_g g) &= -\tilde{\mathsf{D}}R_g, & \tilde{\delta}(R_g g) &= -\mathsf{D}R_g,\\
\delta A_g &= -\frac{n-2}{2(n-1)}\tilde{\mathsf{D}}R_g, & \tilde{\delta}A_g &= -\frac{n-2}{2(n-1)}\mathsf{D}R_g,\\
\delta\mathcal{W}_g &= -\frac{n-3}{n-2}\tilde{\mathsf{D}}A_g, & \tilde{\delta}\mathcal{W}_g &= -\frac{n-3}{n-2}\mathsf{D}A_g.
\end{aligned}$$

In particular, if R_g is constant last two equalities become

$$\delta\mathcal{W}_g = -\frac{n-3}{n-2}\tilde{\mathsf{D}}\mathring{\operatorname{Ric}}_g \qquad \textit{and} \qquad \tilde{\delta}\mathcal{W}_g = -\frac{n-3}{n-2}\mathsf{D}\mathring{\operatorname{Ric}}_g.$$

Proof. We only prove the left column, since the right one is completely analogous.

The first three equalities are just application of the well–known differential Bianchi identity:

$$\nabla_i \operatorname{Riem}_{jk\ell m} + \nabla_\ell \operatorname{Riem}_{jkmi} + \nabla_m \operatorname{Riem}_{jki\ell} = 0.$$

Taking it as it is, we have the first equality; tracing on g^{ij} we have the second one and tracing on g^{ij} and $g^{k\ell}$ we have the third one. The fourth equality just follows from the definition and the fifth one from the previous two.

For the last one first we see that

$$(\tilde{\mathsf{D}}(R_g g))_{ijk} = \nabla_j R_g g_{ik} - \nabla_k R_g g_{ij} = (\tilde{\mathsf{D}}R_g)_j g_{ik} - (\tilde{\mathsf{D}}R_g)_k g_{ij},$$

so

$$\begin{aligned}
(\delta(A_g \owedge g))_{ijk} &= -\nabla^\alpha A_{\alpha j} g_{ik} - \nabla^\alpha A_{ik} g_{\alpha j} + \nabla^\alpha A_{\alpha k} g_{ij} + \nabla^\alpha A_{ij} g_{\alpha k}\\
&= -(\tilde{\mathsf{D}}A_g)_{ijk} - (\delta A_g)_k g_{ij} + (\delta A_g)_j g_{ik}\\
&= -(\tilde{\mathsf{D}}A_g)_{ijk} + \frac{n-2}{2(n-1)}(\tilde{\mathsf{D}}R_g)_k g_{ij} - \frac{n-2}{2(n-1)}(\tilde{\mathsf{D}}R_g)_j g_{ik}\\
&= -(\tilde{\mathsf{D}}A_g)_{ijk} - \frac{n-2}{2(n-1)}(\tilde{\mathsf{D}}(R_g g))_{ijk}.
\end{aligned}$$

At last

$$\begin{aligned}
\delta\mathcal{W}_g &= \delta\operatorname{Riem}_g - \frac{1}{n-2}\delta(A_g \owedge g)\\
&= -\tilde{\mathsf{D}}\operatorname{Ric}_g + \frac{1}{n-2}\tilde{\mathsf{D}}A_g + \frac{1}{2(n-1)}\tilde{\mathsf{D}}(R_g g)\\
&= \left(\frac{1}{n-2} - 1\right)\tilde{\mathsf{D}}A_g\\
&= -\frac{n-3}{n-2}\tilde{\mathsf{D}}A_g.
\end{aligned}$$

□

1.2 Convergence of manifolds

We will make extensive use of the notion of pointed $\mathcal{C}^\infty$ convergence, which we are going to define now. Our definition is somewhat weaker than Hamilton's in [Ham95] (since we do not want to deal with orthonormal frames at the points), but stronger than Petersen's in [Pet98, Chapter 10] (which does not require F_i to carry x to x_i in Definition 1.3 below).

1.2 Definition. Let A be a precompact set of a complete manifold M. We say that a sequence of function $(f_i)_{i\in\mathbb{N}} \subseteq \mathcal{C}^\infty(A)$ *converges in the* $\mathcal{C}^\infty$ *topology* to $f \in \mathcal{C}^\infty(A)$ when the sequence of functions converges in all the charts of a finite atlas covering A. A sequence of tensors converges if all the coordinates do.

This definition is clearly independent of the chosen atlas.

1.3 Definition. Let $(M_i, x_i, g_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete Riemannian manifolds. It is said to *converge in the pointed* $\mathcal{C}^\infty$ *topology* to the pointed Riemannian manifold (M, x, g) if for every $R > 0$ there exist a domain $\Omega \supseteq B(x, R)$ in M and differential embeddings $F_i \colon \Omega \to M_i$ for large i such that:

- $F_i(x) = x_i$ for all i;
- $F_i^* g_i \to g$ on Ω in the $\mathcal{C}^\infty$ topology.

In order to build pointed $\mathcal{C}^\infty$ limits, we make use of the following precompactness result by Hamilton.

1.4 Theorem (Theorem 2.3 in [Ham95])**.** *Let $(M_i, x_i, g_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete Riemannian manifolds. Let us suppose that for each i we have that $\operatorname{inj}_{g_i}(x_i) > \delta$ for some $\delta > 0$ independent from i. Moreover, let us suppose that for each $p \geqslant 0$ the p–th covariant derivatives of the Riemann tensor are uniformly bounded by a constant: $|\nabla^p \operatorname{Riem}_{g_i}| \leqslant B_p$.*

Then there is a subsequence that converges in the pointed $\mathcal{C}^\infty$ topology.

In order to obtain estimates on the injectivity radius, we define the concept of noncollapsing.

1.5 Definition. Let (M, g) be a Riemannian manifold and $\kappa > 0$. The manifold is said to be *κ–noncollapsed* when for each $r > 0$ and B ball of radius r with $|\operatorname{Riem}_g| \leqslant \frac{1}{r^2}$ on B we have that $\operatorname{Vol}_g(B) \geqslant \kappa r^n$.

The property of noncollapsing is related to the injectivity radius thanks to the following lemma.

1.6 Lemma (Cheeger, Lemma 51 in [Pet98])**.** *For all $C > 0$ and $\kappa > 0$ there is a constant $\delta(n, C, \kappa) > 0$ such that the following assertion holds. Let (M, g) be a complete Riemannian manifold such that $\operatorname{Vol}_g(B(x, 1)) \geqslant \kappa$ for all $x \in M$ and $\sup_M |\operatorname{Riem}_g| \leqslant C$; then $\operatorname{inj}_g(M) > \delta$.*

The property of κ–noncollapsing is scale invariant, so, after possibly rescaling the metric so that the curvature is bounded by 1, we know that a κ–noncollapsed manifold has injectivity radius bounded from below by a constant $\delta(n, \kappa)$. Moreover, trivially, a compact manifold has finite volume; the converse is not true in general, but it holds when the manifold is noncollapsed and has bounded curvature.

1.7 Lemma. *Let (M, g) be a complete Riemannian manifold such that the volume of unit balls is uniformly bounded from below:*

$$\inf_{x \in M} \mathrm{Vol}_g(B(x, 1)) > 0.$$

Then M is compact if and only if it has finite volume.

Proof. If M is compact, clearly it has finite volume. On the other hand, let $x_0 \in M$ and $B(x_0, i)$ be balls of increasing radius. If M is complete but not compact, we have that $B(x_0, i) \subsetneq B(x_0, i+1)$. So we can take, for each $i \in \mathbb{N}$, a point $y_i \in B(x_0, 3i+1) \subseteq B(x_0, 3i)$, whose unit balls are all disjoint. Thus, M has infinite volume. □

1.3 The Yamabe constant

Unless otherwise noted, the proofs for the assertions contained in this section can be found in Chapter 5 of [Aub98].

Let (M^n, g) be a compact Riemannian manifold. The *Yamabe constant* is defined as

$$Y(M^n, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_{M^n} R_{\tilde{g}} \, dv_{\tilde{g}}}{\mathrm{Vol}_{\tilde{g}}(M^n)^{1-\frac{2}{n}}},$$

where $[g]$ is the class of metrics conformally equivalent to g. Clearly the Yamabe constant is a conformal invariant.

A number of facts are known about the Yamabe constant. It is positive if and only if g is conformally equivalent to a metric with positive scalar curvature. Moreover, there is a metric $\hat{g} \in [g]$ that actually reaches the infimum in the definition (although this fact is completely non trivial). Such metric is called a *Yamabe minimizer* and is of constant scalar curvature, with

$$R_{\hat{g}} = \frac{Y(M^n, [g])}{\mathrm{Vol}_{\hat{g}}(M^n)^{\frac{2}{n}}}. \tag{1.4}$$

The definition clearly cannot be generalized immediately to complete non–compact manifolds. Indeed, we have to change viewpoint and define the Yamabe constant as

$$Y(M^n, g) = \inf_{\substack{u \in \mathcal{C}_0^\infty(M^n) \\ u \not\equiv 0}} Y(M^n, g, u)$$

where

$$Y(M^n, g, u) = \frac{\int_{M^n} \left(\frac{4(n-1)}{n-2} |du|^2 + R_g u^2 \right) dv_g}{\left(\int_{M^n} u^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}}}.$$

One can check that this new definition depends only on $[g]$ and actually coincides with the former when M^n is compact. The minimizer condition corresponds, in this new setting, to the following:

$$\int_{M^n} \left(\frac{4(n-1)}{n-2} |du|^2 + R_g u^2 \right) dv_g = Y(M^n, [g]) \qquad \text{and} \qquad \int_{M^n} u^{\frac{2n}{n-2}} \, dv_g = 1.$$

Of course, in general, the existence of the Yamabe minimizer cannot be guaranteed if M^n is not compact.

The Yamabe constant is upper semicontinuous with respect to pointed $\mathcal{C}^\infty$ convergence.

1.8 Proposition. *Let $(M_i^n, g_i, x_i)_{i\in\mathbb{N}}$ be a sequence of complete Riemannian manifolds converging to (M^n, g, x) in the pointed $\mathcal{C}^\infty$ topology. Then*

$$Y(M^n,[g]) \geqslant \limsup_{i\to\infty} Y(M_i^n,[g_i]).$$

Proof. We just have to show that each competitor in the infimum defining $Y(M^n,[g])$ is an approximate competitor in the definition of $Y(M_i^n,[g_i])$ for large values of i.

More precisely, take $u \in \mathcal{C}_0^\infty(M^n)$. Since it has compact support, take i such Ω_i contains it and define u_i as the push–forward of u along F_i (extended with 0 outside $F_i(\Omega_i)$. We have that

$$Y(M_i^n, g_i, u_i) = Y(M^n, F_i^*(g_i), u) \longrightarrow Y(M^n, g, u) \qquad \text{for } i \to \infty,$$

where, of course, $Y(M_i^n, g_i, u_i) \geqslant Y(M_i^n,[g_i])$. Taking the superior limit on i on the left and then the infimum on u on the right we have the thesis. □

We will heavily employ the Yamabe constant to control the injectivity radius of manifolds, according to the following key proposition.

1.9 Proposition. *Let (M^n, g) be a complete Riemannian manifold with positive Yamabe constant and $n \geqslant 3$. Then M^n is κ–noncollapsed with*

$$\kappa = \left(\frac{Y(M^n,[g])}{2^{n+5}n(n-1)}\right)^{\frac{n}{2}}.$$

The proof of this fact will pass through an estimate of the $W^{1,2}$–Sobolev constant. Let (M^n, g) be a complete Riemannian manifold with $n \geqslant 3$ and $U \subseteq M^n$ an open set, we will speak of *the Sobolev constant* of U as

$$s_g(U) = \inf\left\{ C \in \mathbb{R} \mid \|u\|_{L^{\frac{2n}{n-2}}} \leqslant C(\|du\|_{L^2} + \|u\|_{L^2}),\ \forall u \in \mathcal{C}_0^\infty(U) \right\},$$

with the convention that $\inf \emptyset = +\infty$.

If M^n is closed, we always have that $s_g(U) < +\infty$, but in the particular case of a positive Yamabe constant the following explicit bound is known.

1.10 Lemma. *Let (M^n, g) be a complete Riemannian manifold with positive Yamabe constant and $n \geqslant 3$. Let $U \subseteq M^n$ be an open set. Then*

$$s_g(U)^2 \leqslant \frac{1}{Y(M^n,[g])} \max\left\{\sup_U |R_g|, \frac{4(n-1)}{n-2}\right\}.$$

Proof. This is just an application of the definition, since

$$\|u\|^2_{L^{\frac{2n}{n-2}}} \leqslant \frac{1}{Y(M^n,[g])}\left(\frac{4(n-1)}{n-2}\|du\|^2_{L^2} + \sup_U |R_g| \cdot \|u\|^2_{L^2}\right).$$

□

Then, the Sobolev constant is connected to the volume of the balls via the following lemma.

1.11 Lemma (Lemma 3.2 in [Heb96])**.** *Let (M^n, g) be a complete Riemannian manifold, $x \in M^n$ and $r > 0$. Take $B = B(x, r)$, then,*

$$\mathrm{Vol}_g(B) \geqslant \min\left\{\frac{1}{2s_g(B)}, \frac{r}{2^{\frac{n}{2}+2}s_g(B)}\right\}^n.$$

Proof of 1.9. The conclusion follows by concatenating the two previous lemmas, possibly after rescaling the manifold so that $\sup_{M^n}|\mathrm{Riem}_g| = 1$ on the ball. □

1.4 The Bochner technique

The Bochner technique is a standard way to obtain "gap" results, where a sufficiently tight estimate actually implies the rigidity of an object.

We will not prove the results contained in this section, because they are out of the scope of this work. The proofs can be found in [Bou12], Section 3 of Chapter 1.

Let T be a tensor on a Riemannian manifold (M, g). Suppose that it satisfies an inequality of the form

$$\langle \nabla^*\nabla T|T\rangle + \mu R|T|^2 \leqslant a|T|^2 \tag{1.5}$$

for some $\mu \in \mathbb{R}$ and $a \in \mathcal{C}^\infty(M)$.

Any smooth tensor T satisfies the following *Kato inequality* on the points where $|T| \neq 0$

$$|d|T||^2 \leqslant |\nabla T|^2,$$

which is obtained just by applying the Cauchy–Schwarz inequality to the right-hand term of the identity $d\langle T|T\rangle = 2\langle \nabla T|T\rangle$. In many cases the Kato inequality can be refined in order to have

$$(1+\varepsilon)|d|T||^2 \leqslant |\nabla T|^2 \tag{1.6}$$

for some $0 < \varepsilon < 1$.

We then have the following theorem.

1.12 Theorem. *Let (M^n, g) be a complete Riemannian manifold without boundary and T be a tensor with satisfies conditions (1.5) and (1.6) for some $a \in \mathcal{C}^\infty(M)$, $0 \leqslant \varepsilon < 1$ and $\mu = \frac{1}{1-\varepsilon}\frac{n-2}{4(n-1)}$. Suppose moreover that*

$$\|a\|_{L^{\frac{n}{2}}} \leqslant \mu Y(M^n, [g]) \tag{1.7}$$

and for some $x_0 \in M^n$ there holds $\mathrm{Vol}_g^{\frac{\varepsilon}{2}}(B(x_0, r)) = O(r)$ for $r \to \infty$.

Then either T vanishes on M^n or equality holds in inequality (1.7).

In order to control the volume growth of the geodesic balls, we recall the well-known Bishop–Gromov inequality.

1.13 Theorem (Bishop–Gromov inequality, Lemma 36 in [Pet98])**.** *Let* (M^n, g) *be a complete Riemannian manifold with* $\mathrm{Ric}_g \geqslant (n-1)\kappa g$*, with* $\kappa \in \mathbb{R}$*. Let also* $v(n, \kappa, r)$ *be the volume of the ball of radius* r *in the* n*–dimensional space form of curvature* κ*. Then for each* $x \in M^n$

$$r \longmapsto \frac{\mathrm{Vol}_g(B(x,r))}{v(n,\kappa,r)}$$

is a non–increasing function with limit 1 *as* $r \to 0$*.*

In particular, $\mathrm{Vol}_g(B(x,r)) \leqslant v(n, \kappa, r)$*.*

CHAPTER 2

A class of fourth-order geometric flows

Contents

In this chapter we begin studying a class of fourth–order flows, that is more general than that we are really interested into; the results contained in this chapter apply in a rather wide context. As a rule of thumb, we may think them as being "independent from the specific problem", while the following chapter will fill in the details that are specific for the geometric theorem we are aiming to (Theorem 3.16 and Corollary 3.17).

For a closed Riemannian manifold (M^n, g_0), let us consider the following class of evolution equations:

$$\begin{cases} \partial_t g(t) & = P(g(t)) \\ g(0) & = g_0, \end{cases} \tag{2.1}$$

where $P\colon S^2_+(M) \to \mathcal{S}^2(M)$ is a smooth map of the form

$$P(g) = \delta\tilde{\delta}\operatorname{Riem}_g + a\Delta R_g g + b\nabla^2 R_g + \operatorname{Riem}_g * \operatorname{Riem}_g \qquad a, b \in \mathbb{R}. \tag{2.2}$$

Since the Riemann tensor carries two derivatives of the metric, this is a fourth–order system of partial differential equations. As it is usual in the theory of evolution equations, one first has to show existence and uniqueness of solutions for a small time and then study the behaviour of solutions near the maximal existence time, in order to conclude that the solution actually extends past that time.

Often, the first part can be established in great generality and depends only on the structure of the differential equation, but not on the initial datum (of course, the resulting time of existence may depend on how "good" the initial datum is). On the other hand, the possibility to carry out the extension argument at maximal time usually depends on the geometric nature of the problem and the initial datum that was fed to the equation: one has to check that the evolution of the initial datum does not lead to ill–behaved situations where the solution (or some of its derivatives) blows up, forcing the flow to halt.

We can pass immediately to show a short–time existence result for the flow defined above. We only take a moment for showing a technical lemma that will be useful in the rest of the chapter.

2.1 Lemma. *Let M be a compact manifold and $(g(t))_{t\in[0,T)}$ a family of metrics solution of (2.1). Then the variation of the volume of $g(t)$ is given by*

$$\partial_t \operatorname{Vol}_{g(t)}(M) = \frac{1}{2}\int_M \operatorname{tr}(P(g(t)))\, dv_g.$$

Proof. First we recall that, given a square matrix $B = (b_{ij})$, its *adjugate matrix* is the square matrix $\operatorname{adj} B = C = (c_{ij})$ of the same order such that c_{ij} is $(-1)^{i+j}$ times the determinant of the matrix obtained from B by removing the j–th row and the i–th column. If B is not singular, then $B^{-1} = \frac{\operatorname{adj} B}{\det B}$.

Jacobi's formula states that for a matrix $A(t)$ that depends smoothly from time the derivative of the determinant can be written as

$$\partial_t \det A(t) = \operatorname{tr}(\operatorname{adj} A(t) \cdot \partial_t A(t)).$$

Writing $\operatorname{Vol}_{g(t)}(M) = \int_M \sqrt{\det g(t)}\, dx$ and differentiating under the sign of integral, the result follows. □

2.1 Short–time existence

The flow (2.1) is a fourth–order quasi–linear parabolic system of partial differential equations. Here *quasi–linear* informally means that the equations depend linearly on the space derivatives of maximal order, with "coefficients" that depend on the lower order derivatives.

The behaviour of such kind of systems is often determined by its principal symbol, which we define.

2.2 Definition. Let $L\colon h \mapsto L_k(\nabla^k h) + \cdots + L_0$ be a linear differential operator of order k. Then, for $\xi \in T^*M$, its *principal symbol* is

$$\sigma_\xi L(h) = L_k(\xi \otimes \cdots \otimes \xi \otimes h).$$

2.3 Definition. A linear differential operator L of order k is *strongly elliptic* if k is even and $(-1)^{k/2+1}\sigma_\xi L$ is uniformly positive for all $\xi \in T^*M \setminus \{0\}$; i.e., there is $\alpha > 0$ such that for all h

$$(-1)^{k/2+1} \langle \sigma_\xi L(h) | h \rangle \geqslant \alpha |\xi|^k |h|^2.$$

If P is a non–linear differential operator, we say it is *strongly elliptic* if its linearization at every point is strongly elliptic in the sense defined above.

Informally speaking, the strong ellipticity condition is what is required in order to have short–time existence. Unfortunately, there is no systematic reference to back this assertion, so we take a brief tour to justify it.

Let us first consider the case of a single equation (instead of systems) and a linear operator. In [HP99] Huisken and Polden prove a short–time existence theorem for linear operators with a particular product structure. See also [Pol96].

In order to give their result, we first have to introduce some Sobolev spaces adapted to the parabolic problem. Take $s \in \mathbb{N}$ and $a \in \mathbb{R}_{\geqslant 0}$ and let

$$LW_a^s = \left\{ f \colon M^n \times [0, \infty) \to \mathbb{R} \;\middle|\; \int_0^\infty e^{-2at} \|f\|_{W^{s,2}(M^n)} \, dt < \infty \right\},$$

which is an Hilbert space with the scalar product

$$\langle f|g \rangle_{LW_a^s} = \int_0^\infty e^{-2at} \, \langle f|g \rangle_{W^{s,2}(M^n)} \, dt.$$

Then

$$P_a^m = \{\, f \colon M^n \times [0, \infty) \to \mathbb{R} \mid \partial_t^i f \in LW_a^{2(m-i)p} \; \forall i \leqslant m \,\},$$

which is again an Hilbert space with the scalar product

$$\langle f|g \rangle_{P_a^m} = \sum_{i \leqslant m} \langle \partial_t^i f | \partial_t^i g \rangle_{LW_a^{2(m-i)p}} .$$

For these definitions, all derivatives are to be considered in the distributional sense.

Huisken and Polden prove the following.

2.4 Theorem (Theorem 7.14 in [HP99])**.** *Let (M^n, g) be a closed Riemannian manifold and L a linear differential operator of order $2p$ having the structure*

$$Lu(x) = \sum_{0 \leqslant k \leqslant 2p} L_k^{i_1 \ldots i_k} \nabla_{i_1 \ldots i_k} u(x) \qquad u \in \mathcal{C}^\infty(M^n),$$

where the terms L_k are smooth tensors. Let us also suppose that the leading term can be factorized as

$$L_{2p}^{i_1 j_1 \ldots i_p j_p} = E^{i_1 j_1} \cdots E^{i_p j_p},$$

the tensor E being a strictly positive 2–covector (i.e., satisfying $E \geqslant \lambda g$ for some $\lambda > 0$).

Then for every $m \in \mathbb{N}$ there is $a > 0$ such that the following is a linear isomorphism of Banach spaces:

$$\begin{aligned} P_a^m &\stackrel{\Phi}{\longrightarrow} W^{p(2m-1),2}(M^n) \times P_a^{m-1} \\ u &\longmapsto (u_0, L(u)), \end{aligned}$$

where $u_0 = u(\cdot, 0)$ is the initial value of u.

In particular, thanks to standard Sobolev embedding theorems, the inverse operator Φ^{-1} maps smooth functions to smooth functions.

The proof is essentially based on a variation on the Lax–Milgram theorem (see Lemma 7.8 in [HP99]) between suitable function spaces, similar to those defined above. A priori estimates are used to give uniqueness of the solution and the continuity of the map Φ^{-1} (which, as noted, brings to the maximum regularity of the solutions).

The condition on the structure and ellipticity is by far stronger than necessary. It is enough to have a Gårding inequality (Lemma 7.7 in the same work), which establishes the coercivity condition for the Lax–Milgram theorem. Thus, in Theorem 2.4 we can remove the hypothesis on the product structure of L_{2p} and instead require that for every $\varphi \in \mathcal{C}^\infty(M^n)$ we have

$$-\int_M \varphi \cdot L_{2p}^{i_1\dots i_{2p}} \nabla^{2p}_{i_1\dots i_{2p}} \varphi \, dv_g \geqslant \sigma \|\varphi\|^2_{W^{p,2}(M^n)} - C\|\varphi\|^2_{L^2(M^n)},$$

for $\sigma > 0$ and $C > 0$ constants that depend only on n, λ and the $\mathcal{C}^{p-1}$ norms of the tensors L_k and Riem_g.

Huisken' and Polden's paper also contains a theorem for the non–linear case, but unfortunately there is a gap in their proof, as it has been pointed out by Sharples in [Sha04]. This gap has been filled by Mantegazza and Martinazzi in [MM12], who proved the following statement.

2.5 Theorem (Theorem 1.1 in [MM12])**.** *Let (M^n, g) be a closed Riemannian manifold. Let P be a smooth and quasi–linear differential operator of order $2p$ defined on $M^n \times [0,T)$ for some $T > 0$ and having the structure*

$$\begin{aligned} Pu(x,t) = {} & A^{i_1\dots i_{2p}}(x,t,u,\nabla u,\dots,\nabla^{2p-1}u)\nabla^{2p}_{i_1\dots i_{2p}} u(x,t) \\ & + b(x,t,u,\nabla u,\dots,\nabla^{2p-1}u), \end{aligned} \tag{2.3}$$

where b is a smooth function and and A is a smooth $(2p,0)$–tensor. Let us also suppose that the leading term can be factorized as

$$A^{i_1 j_1 \dots i_p j_p} = (-1)^{p-1} E_1^{i_1 j_1} \cdots E_p^{i_p j_p},$$

the tensors $E_1, \dots, E_p$ being locally elliptic in the following sense: for each $L > 0$ there must be a constant $\lambda > 0$ such that, however taken $x \in M$, $t \in [0,T)$, $|u| < L$ and $\psi_k \in \bigotimes^k T_x^* M$ with $|\psi_k| < L$ we have for each tensor E_ℓ

$$E_\ell^{ij}(x,t,u,\psi_1,\dots,\psi_{2p-1})\xi_i\xi_j \geqslant \lambda |\xi|^2 \qquad \forall \xi \in T_x^* M.$$

Then, for each $u_0 \in \mathcal{C}^\infty(M^n)$, there is $0 < T' \leqslant T$ such that the problem

$$\begin{cases} \partial_t u(x,t) & = Pu(x,t) \\ u(\cdot,0) & = u_0 \end{cases}$$

has a unique smooth solution in $[0,T')$. Such solution depends continuously on u_0 in the $\mathcal{C}^\infty$ topology.

What Mantegazza and Martinazzi do is to define the analogous of operator Φ for the quasi–linear case and proving that, for m big enough, it is locally a $\mathcal{C}^1$ diffeomorphism (the derivatives have to be intended in the Fréchet sense). Using the inverse function theorem, they see that the quasi–linear problem can be seen as the perturbation of a suitable linear problem, that is solved using the result of Huisken and Polden.

As in the linear case, the hypothesis of product structure is excessively strong. It is enough to have the following Gårding inequality: for each $u \in C^\infty(M \times [0,T))$ it must hold

$$-\int_M \varphi A^{i_1\dots i_{2p}}(u)\nabla^{2p}_{i_1\dots i_{2p}}\varphi\, dv_g \geqslant \sigma\|\varphi\|^2_{W^{p,2}(M^n)} - C\|\varphi\|^2_{L^2(M^n)} \qquad \forall\varphi \in \mathcal{C}^\infty(M), \tag{2.4}$$

where the constants $\sigma > 0$ and $C > 0$ depend continuously only on the norms of A, u, Riem_g and their derivatives (we adopted the following shorthand convention: $A^{i_1\dots i_{2p}}(u) = A^{i_1\dots i_{2p}}(x,t,u,\nabla u,\dots,\nabla^{2p-1}u)$).

What we have to do now is to show that the strong ellipticity condition actually forces the differential operator to satisfy Gårding's inequalty. This is a rather standard argument; first we introduce the following estimates.

2.6 Proposition (Gagliardo–Nirenberg inequalities, Proposition 5.1 in [Man02])**.** *Let (M^n,g) be a closed Riemannian manifold and take numbers $p,q,r \in [0,\infty)$ and $j,s \in \mathbb{N}$ such that $0 \leqslant j \leqslant s$ and*

$$\frac{1}{p} = \frac{j}{sq} + \frac{s-j}{sr}.$$

Then there is $C(n,p,q,r,s,j) > 0$ such that for tensor T on M^n there holds

$$\|\nabla^j T\|_{L^p(M^n)} \leqslant C \cdot \|\nabla^s T\|^{\frac{j}{s}}_{L^q(M^n)} \cdot \|T\|^{\frac{s-j}{s}}_{L^r(M^n)}.$$

We also recall the weighted Young's inequality, also called the Peter–Paul inequality.

2.7 Proposition (Weighted Young's inequality)**.** *Take real numbers $a,b \geqslant 0$, $\varepsilon > 0$ and $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leqslant \varepsilon^p\frac{a^p}{p} + \frac{1}{\varepsilon^q}\frac{b^q}{q}.$$

Proof. This is just the concavity inequality for the logarithm function:

$$\log(ab) = \frac{1}{p}\log a^p + \frac{1}{q}\log b^q \leqslant \log\left(\frac{a^p}{p} + \frac{b^q}{q}\right).$$

Integrating one obtains the inequality without weights. Taking εa instead of a and $\frac{b}{\varepsilon}$ instead of b, the general case appears. □

By repeatedly using the integration by parts formula, we have that

$$-\int_M \varphi A^{i_1\dots i_{2p}}(u)\nabla^{2p}_{i_1\dots i_{2p}}\varphi\, dv_g = (-1)^{p+1}\int_M \nabla^p_{i_p\dots i_1}\varphi \cdot \nabla^p_{i_{p+1}\dots i_{2p}}\varphi \cdot A^{i_1\dots i_{2p}}(u)\, dv_g$$
$$+ \sum_{k=1}^{p}\int_M \nabla^{p-k}\varphi * \nabla^p\varphi * \nabla^k A(u)\, dv_g.$$

First we estimate the higher order term: the principal symbol of the operator P'_u is exactly $A^{i_1\dots i_{2p}}(u)\xi_1\dots\xi_{2p}$, so putting in the strong ellipticity condition we have

$$(-1)^{p+1}\int_M \nabla^p_{i_p\dots i_1}\varphi\cdot\nabla^p_{i_{p+1}\dots i_{2p}}\varphi\cdot A^{i_1\dots i_{2p}}(u)\,dv_g \geqslant \alpha\|\varphi\|_{W^{p,2}(M^n)}.$$

So, what is left to do is estimating the lower order terms, both partially eroding the $\|\varphi\|_{W^{p,2}}$ estimate (but not too much!) and using the $\|\varphi\|_{L^2}$ part. Composing the inequalities enunciated above (plus other standard integral inequalities) we have (the elements "dv_g" are omitted to avoid having too long formulae):

$$\begin{aligned}
&\left|\int_M \nabla^{p-k}\varphi * \nabla^p\varphi * \nabla^k A(u)\right|\\
&\quad\leqslant \sup_M|\nabla^k A(u)|\int_M|\nabla^{p-k}\varphi||\nabla^p\varphi|\\
&\quad\leqslant \sup_M|\nabla^k A(u)|\left(\left(\int_M|\nabla^{p-k}\varphi|^2\right)^{\frac12}\left(\int_M|\nabla^p\varphi|^2\right)^{\frac12}\right)\\
&\quad\leqslant \sup_M|\nabla^k A(u)|\left(\varepsilon\int_M|\nabla^p\varphi|^2+C(\varepsilon)\int_M|\nabla^{p-k}\varphi|^2\right)\\
&\quad\leqslant \sup_M|\nabla^k A(u)|\left(\varepsilon\int_M|\nabla^p\varphi|^2+C(\varepsilon)C\left(\int_M|\nabla^p\varphi|^2\right)^{\frac{p-k}{p}}\left(\int_M|\varphi|^2\right)^{\frac{k}{p}}\right)\\
&\quad\leqslant \sup_M|\nabla^k A(u)|\left(\varepsilon\int_M|\nabla^p\varphi|^2+\varepsilon' C(\varepsilon)C\int_M|\nabla^p\varphi|^2+C(\varepsilon)C(\varepsilon')C\int_M|\varphi|^2\right),
\end{aligned}$$

where $C(\varepsilon)$ and $C(\varepsilon')$ are the coefficient produced by the application of Young's inequality.

By tuning the numbers ε and ε' in all the lower order terms, we can make sure that the total coefficient before $\int_M|\nabla^p\varphi|^2$ is not more than $\frac{\alpha}{2}$. In other words, we have obtained Gårding's inequality with $\sigma=\frac{\alpha}{2}$ and a possibly very large C given by the sum of the coefficients in front of $\int_M|\varphi|^2$ terms.

Summing everything up, we have shown the following theorem.

2.8 Theorem. *Let (M^n,g) be a closed Riemannian manifold. Let P be a smooth and quasi–linear differential operator of order $2p$ defined on $M^n\times[0,T)$ for some $T>0$ and having the structure (2.3), where b is a smooth function and and A is a smooth $(2p,0)$–tensor. Let us also suppose that Gårding's inequality holds in the sense stated above: for each $u\in\mathcal{C}^\infty(M^n\times[0,T))$ and $\varphi\in\mathcal{C}^\infty(M^n)$ there must hold inequality (2.4).*

Then, for each $u_0\in\mathcal{C}^\infty(M^n)$, there is $0<T'\leqslant T$ such that the problem

$$\begin{cases}\partial_t u(x,t) &= Pu(x,t)\\ u(\cdot,0) &= u_0\end{cases}$$

has a unique smooth solution in $[0,T')$. Such solution depends continuously on u_0 in the $\mathcal{C}^\infty$ topology.

2.2 The DeTurck trick

Unfortunately, in our case the operator (2.2) is not strictly elliptic, so the general theory doesn't apply directly. This is a common feature of all flows based on geometric functionals, i.e., functionals that are invariant by diffeomorphism. See [CK04, Section 3.2] for a more complete account on this behaviour.

Hamilton solved this problem for the Ricci flow in [Ham82] with a technical and lengthy proof based on the Nash–Moser inverse function theorem. A few years later a much simpler proof was devised by Dennis DeTurck in [DeT83]: he finds a way to modify the equation in order to make it strongly elliptic, so that the solution of the original flow may be recovered by pulling back the modified solution along a suitable vector field. This technique is now known as the "DeTurck trick" and can be adapted surprisingly well to many other types of flows.

A variation on the DeTurck trick can be used in our case. Let g be a Riemannian metric and $\xi \in T^*M$. Then we define

$$R_\xi(g) = \xi \otimes \xi - |\xi|^2 g.$$

Then let g and g_0 be two Riemannian metrics and define

$$\gamma_{g,g_0} = \frac{1}{2} g_{i\delta} g^{\alpha\beta} (\Gamma^\delta_{\alpha\beta}(g) - \Gamma^\delta_{\alpha\beta}(g_0)) dx^i.$$

What DeTurck originally did was to solve $\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)} - \mathcal{L}_{-\gamma_{g(t),g_0}}$ instead of $\partial_t g(t) = -2\,\mathrm{Ric}_{g(t)}$, for a fixed reference metric g_0. Here we do something similar, although the intervening vector field must be adapted to our problem.

2.9 Proposition. *Let P be as in (2.2) and $V\colon \mathcal{S}^2_+(M) \to T^*M$ defined by*

$$V_g = -\nabla^*\nabla\gamma_{g,g_0} + \frac{2(b-a)-1}{4} dR_g.$$

Then

$$\sigma_\xi(P - \mathcal{L}_V)'_g = -\frac{1}{2}|\xi|^4 \,\mathrm{Id}_{\mathcal{S}^2(M)} + a \langle R_\xi|\cdot\rangle\, R_\xi \tag{2.5}$$

and

- *$P - \mathcal{L}_V$ is strongly elliptic if $a < \frac{1}{2(n-1)}$;*
- *$P - \mathcal{L}_W$ is not elliptic for any $W\colon \mathcal{S}^2_+(M) \to T^*M$ if $a \geqslant \frac{1}{2(n-1)}$.*

Proof. From Lemma 1.1 it follow that $\tilde{\delta}\,\mathrm{Riem}_g = -\mathsf{D}\,\mathrm{Ric}_g$. Moreover one can easily verify that $\delta\mathsf{D}(R_g g) = \Delta R_g g + \nabla^2 R_g$ just applying the definitions. So, also recalling the identity (1.1),

$$(P - \mathcal{L}_{V_g}(g)) = -\delta\mathsf{D}\,\mathrm{Ric}_g + \mathcal{L}_{\Delta\gamma_{g,g_0} + \frac{1}{4}\nabla R_g} g + a\delta\mathsf{D}(R_g g) + \mathrm{Riem}_g * \mathrm{Riem}_g\,. \tag{2.6}$$

We have now to compute the principal symbol of the linearized operator. We begin with the scalar curvature term. The following formula for the linearized scalar curvature is Lemma 3.7 in [CK04] (although some definitions differ by a sign):

$$R'_g(h) = \delta\tilde{\delta}h + \Delta\,\mathrm{tr}\,h - \langle \mathrm{Ric}\,|h\rangle\,.$$

Thus the principal symbol is

$$\sigma_\xi R'_g(h) = \langle \xi \otimes \xi | h \rangle - |\xi|^2 \operatorname{tr} h = \langle R_\xi | h \rangle \,.$$

Recalling again the identity $\delta \mathsf{D}(R_g g) = \nabla^2 R_g + \Delta R_g g$, it follows that

$$\sigma_\xi (\delta \mathsf{D}(R \cdot))'_g(h) = \xi \otimes \xi \cdot \sigma_\xi R'_g(h) - |\xi|^2 \cdot \sigma_\xi R'_g(h) = R_\xi \langle R_\xi | h \rangle \,.$$

After that we have to work on the first two terms of identity (2.6). The linearized Lie derivative is

$$(\mathcal{L}_V)'_g(h)_{ij} = \nabla_i V'_g(h)_j + \nabla_j V'_g(h)_i + (\nabla h * V)_{ij}$$

so, since the operator V is of degree two,

$$\sigma_\xi (\mathcal{L}_V)'_g(h)_{ij} = \xi_i \sigma_\xi V'_g(h)_j + \xi_j \sigma_\xi V'_g(h)_i.$$

Then we study the principal symbol of the Ricci tensor. Taking again the linearization expression from [CK04], Lemma 3.5, we see that

$$\begin{aligned}
\operatorname{Ric}'_g(h)_{ij} &= \frac{1}{2} g^{pq} (\nabla_q \nabla_j h_{ip} + \nabla_q \nabla_i h_{jp} - \nabla_p \nabla_q h_{ij} - \nabla_i \nabla_j h_{pq}) \\
&= \frac{1}{2} \Delta h_{ij} + \frac{1}{2} g^{pq} \left(\nabla_i \left(\nabla_q h_{jp} - \frac{1}{2} \nabla_j h_{pq} \right) + \nabla_j \left(\nabla_q h_{ip} - \frac{1}{2} \nabla_i h_{pq} \right) \right) \\
&\quad + (h * \operatorname{Riem}_g)_{ij}.
\end{aligned}$$

The first term of the last expression is a Laplacian, so it exhibits a nice elliptic symbol. In order to remove the second term we take advantage of the term γ_{g,g_0}, which in turn contains the Christoffel symbols. From Lemma 3.2 in [CK04] we have that

$$\Gamma'_g(h)^k_{ij} = \frac{1}{2} g^{k\ell} (\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}),$$

hence

$$(\gamma_{\cdot, g_0})'_g(h)_m = \frac{1}{4} g_{mk} g^{ij} g^{k\ell} (\nabla_i h_{j\ell} + \nabla_j h_{i\ell} - \nabla_\ell h_{ij}) + (h * (\Gamma(g) - \Gamma(g_0)))_m.$$

Putting together the last identities and considering only the principal part we conclude that

$$\sigma_\xi (\operatorname{Ric} - \mathcal{L}_{\gamma_{\cdot, g_0}})'_g(h) = -\frac{1}{2} |\xi|^2 h.$$

At last we're ready to obtain identity (2.5). As before, we use the Lie derivative term in order to correct the discrepancy of the main term $\delta \mathsf{D} \operatorname{Ric}_g$ from being a Laplacian. More specifically, there holds the identity

$$\delta \mathsf{D} \operatorname{Ric}_g = \Delta \operatorname{Ric}_g + \frac{1}{2} \nabla^2 R_g + \operatorname{Riem} * \operatorname{Riem},$$

which follows from expanding the definition of δ and D, commuting derivatives and applying Schur's formula (see Lemma 1.1).

Going ahead, from the symbol of $\mathrm{Ric} - \mathcal{L}_\gamma$ and taking into account that $\Delta\mathcal{L}_V = \mathcal{L}_{\Delta V} + \nabla\,\mathrm{Riem} * V + \mathrm{Riem} * V$, we have that

$$\sigma_\xi(\Delta\,\mathrm{Ric} - \mathcal{L}_{\Delta\gamma_{\cdot,g_0}})'_g(h) = \sigma_\xi(\Delta(\mathrm{Ric} - \mathcal{L}_{\gamma_{\cdot,g_0}}))'_g(h) = \frac{1}{2}|\xi|^4 h.$$

On the other hand it is clear that $\mathcal{L}_{\frac{1}{4}\nabla R_g} g = \frac{1}{2}\nabla^2 R_g$, so substituting all the principal symbols obtained so far in equation (2.6) we have that identity (2.5) holds.

We easily compute that

$$|R_\xi|^2 = |\xi|^4 + n|\xi|^4 - 2\left\langle \xi \otimes \xi \middle| |\xi|^2 g\right\rangle = (n-1)|\xi|^4.$$

Moreover, for each $W\colon \mathcal{S}_2^+(M) \to T^*M$ we have that the image of $\sigma_\xi(\mathcal{L}_w)'_g$ lies in $R_\xi^\perp$:

$$\begin{aligned}\left\langle \sigma_\xi(\mathcal{L}_W)'_g \middle| R_\xi \right\rangle &= \left\langle \xi \otimes \sigma_\xi W'_g + \sigma_\xi W'_g \otimes \xi \middle| \xi \otimes \xi - |\xi|^2 g\right\rangle \\ &= 2|\xi|^2 \left\langle \xi \middle| \sigma_\xi W'_g\right\rangle - 2|\xi|^2 \left\langle \xi \middle| \sigma_\xi W'_g\right\rangle \\ &= 0.\end{aligned}$$

Then, if $a < \frac{1}{2(n-1)}$,

$$\begin{aligned}-\left\langle \sigma_\xi(P - \mathcal{L}_v)'_g(h) \middle| h\right\rangle &= \frac{1}{2}|\xi|^4|h|^2 - a\left\langle R_\xi \middle| h\right\rangle^2 \\ &\geqslant \frac{1}{2}|\xi|^4|h|^2 - a|R_\xi|^2|h|^2 \\ &= \frac{1}{2}(1 - 2a(n-1))|\xi|^4|h|^2,\end{aligned}$$

so $P - \mathcal{L}_V$ is strongly elliptic.

On the other hand, if $a \geqslant \frac{1}{2(n-1)}$ we can evaluate the symbol on R_ξ and have

$$\begin{aligned}-\left\langle \sigma_\xi(P - \mathcal{L}_W)'_g(R_\xi) \middle| R_\xi\right\rangle &= -\left\langle \sigma_\xi(P - \mathcal{L}_V)'_g(R_\xi) \middle| R_\xi\right\rangle + \left\langle \sigma_\xi(\mathcal{L}_W - \mathcal{L}_V)'_g(R_\xi) \middle| R_\xi\right\rangle \\ &= \frac{1}{2}|\xi|^4|R_\xi|^2 - a\left\langle R_\xi \middle| R_\xi\right\rangle^4 \\ &= \frac{1}{2}(1 - 2a(n-1))|\xi|^4|R_\xi|^2 \\ &\leqslant 0,\end{aligned}$$

so $P - \mathcal{L}_W$ is not strongly elliptic. □

Theorem 2.8 then gives us a solution defined on $[0, T) \times M$ for the modified flow

$$\begin{cases}\partial_t \tilde{g}(t) &= P(\tilde{g}(t)) - \mathcal{L}_{V_{\tilde{g}(t)}}\tilde{g}(t) \\ \tilde{g}(0) &= g_0.\end{cases} \tag{2.7}$$

In order to recover the solution for the original system, let us consider the flow of $V(\tilde{g}(t))$:

$$\begin{cases}\partial_t \varphi_t &= V_{\tilde{g}(t)} \circ \varphi_t \\ \varphi_0 &= \mathrm{Id}_M .\end{cases} \tag{2.8}$$

2.10 Proposition. *The family of metrics* $(g_t)_{t\in[0,T_1)}$ *is a solution of* (2.1) *if and only if* $T_1 \leqslant T$ *and* $g(t) = \varphi_t^*\tilde{g}(t)$ *for all* $t \in [0, T_1)$.

Thus, solving the modified flow and then pulling back the solution via φ_t, we can build a solution for the flow we're interested into. Before the proof, we introduce a lemma.

2.11 Lemma. *Let* M^n *be a smooth manifold,* $(g(t))_{t\in(-\varepsilon,\varepsilon)}$ *be a smooth family of metrics and* $(\varphi_t)_{t\in(-\varepsilon,\varepsilon)}$ *be a smooth family of diffeomorphisms. Then*

$$\partial_t(\varphi_t^* g(t)) = \varphi^*(\partial_t g(t) + \mathcal{L}_{V_t} g(t)),$$

where $V_t = \partial_t\varphi_t \circ \varphi_t^{-1}$.

Proof. Just a simple computation:

$$\begin{aligned}\partial_t(\varphi_t^* g(t))|_{t=t_0} &= \varphi_{t_0}^*(\partial_t g(t_0)) + \varphi_{t_0}^*(\partial_t(\varphi_t \circ \varphi_{t_0}^{-1})^* g_{t_0}|_{t_0}) \\ &= \varphi_{t_0}^*(\partial_t g(t_0) + \mathcal{L}_{\partial_t\varphi_t|_{t_0}\circ\varphi_{t_0}^{-1}} g(t_0)).\end{aligned}$$ □

Proof of Proposition 2.10. The operator $P(g)$ is *geometrical*; i.e., it is invariant under diffeomorphisms: if $\varphi\colon M \to N$ is a smooth diffeomorphism, then

$$\varphi^*(P(g)) = P(\varphi^*(g)).$$

Then, if we put $g(t) = \varphi_t^*\tilde{g}(t)$, using Lemma 2.11 we have that

$$\partial_t g(t) = \varphi_t^*(\partial_t\tilde{g}(t) + \mathcal{L}_{V_{\tilde{g}(t)}}\tilde{g}(t)) = \varphi_t^*(P(\tilde{g}(t))) = P(g(t))$$

and, of course, $g(0) = g_0$. So we have reconstructed a solution for (2.1) in the time interval $[0, T)$.

On the other hand, let $g(t)$ be a solution of (2.1) on the time interval $[0, T_1)$. We consider the flow

$$\begin{cases} \partial_t\psi_t &= -V_{g(t)} \circ \psi_t \\ \psi_0 &= \mathrm{Id}_M\,. \end{cases}$$

Then on $[0, T_1)$:

$$\begin{aligned}\partial_t(\psi_t^* g(t)) &= \psi_t^*(\partial_t g(t) - \mathcal{L}_{V_{g(t)}} g(t)) \\ &= \psi_t^*(P(g(t))) - \psi_t^*(\mathcal{L}_{V_{g(t)}} g(t)) \\ &= P(\psi_t^*(g(t))) - \mathcal{L}_{V_{\psi_t^*(g(t))}}\psi_t^*(g(t)),\end{aligned}$$

so $T_1 < T$ and for all $t \in [0, T_1)$ we have $\psi_t^* g(t) = \tilde{g}(t)$. But $(\psi_t)^{-1} = \varphi_t$, so we have the thesis. □

2.3 Bando–Bernstein–Shi estimates

Once the short–time existence has been settled, we begin considering near a maximal time of existence for the flow. The Bando–Bernstein–Shi estimates that we work out in this section provide a way to control the higher order derivatives of the Riemann tensor. First we give an L^2 control; thanks to Sobolev embedding theorem, the control is then made to be in L^∞.

2.12 Theorem. *Let M^n be a compact manifold and let $(g(t))_{t\in[0,T)}$ be a family of metrics that solves the flow (2.1) with $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$. Suppose moreover that for a certain $\beta < \infty$ and all times $t \in [0,T)$ we have $\|\mathrm{Riem}_{g(t)}\|_{L^\infty} < \beta$.*

Then for all $k \in \mathbb{N}$ there is a constant $c(n,k,\beta,P,T) > 0$ such that for all $t \in (0,T]$ we have

$$\int_M |\nabla^k \mathrm{Riem}_{g(t)}|^2 \, dv_{g(t)} \leqslant \frac{c}{t^{\frac{k}{2}}} \int_M |\mathrm{Riem}_{g(0)}|^2 \, dv_{g(0)}.$$

In order to give the proof, we first establish a formalism that will be used to control the behaviour of lower order terms. Then we give some lemmas and at last we prove Theorem 2.12 itself.

Take $j, m \in \mathbb{N}$ and some tensors T, T_1, ..., T_j; we will write:

$$\mathcal{P}_m(T_1,\dots,T_j) = \sum_{k_1+\dots+k_j=m} \nabla^{k_1}T_1 * \dots * \nabla^{k_j}T_j$$
$$\mathcal{P}_m^{(j)}(T) = \mathcal{P}_m(T,\dots,T) = \sum_{k_1+\dots+k_j=m} \nabla^{k_1}T * \dots * \nabla^{k_j}T.$$

With this notation, we will use the symbol $\textsc{lots}^{(k)}(g)$ (where $\textsc{lots}$ stands for "lower order terms") to indicate any element in the linear span of

$$\int_M \mathcal{P}^{(3)}_{2k+2}(\mathrm{Riem}_g)\, dv_g \qquad \text{and} \qquad \int_M \mathcal{P}^{(4)}_{2k}(\mathrm{Riem}_g)\, dv_g.$$

The meaning of the expression "lower order terms" is made explicit in the following lemma.

2.13 Lemma (Corollary 9.15 in [Bou10])**.** *Let (M^n, g) be a Riemannian manifold. Then there is a constant $C(n,k)$ such that*

$$|\textsc{lots}^{(k)}(g)| \leqslant \frac{1}{2}\int_M |\nabla^{k+2}\mathrm{Riem}_g|^2\, dv_g + C \cdot \sup_M |\mathrm{Riem}_g|^{k+2} \cdot \int_M |\mathrm{Riem}_g|^2\, dv_g.$$

Then we study the form of the linearization of quadratic functionals associated to the covariant derivatives of the Riemann tensor and of the scalar curvature, up to lower order terms.

2.14 Lemma. *Let (M^n, g) be a Riemannian manifold, and $k \geqslant 0$. We recall, moreover, that P is defined as in (2.2). Then the following formulae are satisfied:*

$$\left(\int_M |\nabla^k \mathrm{Riem}|^2\, dv\right)'_g (P(g)) = -\int_M \left(|\nabla^{k+2}\mathrm{Riem}_g|^2 - \frac{a}{2}|\nabla^{k+2}R_g|^2\right) dv_g + \textsc{lots}^{(k)}(g)$$
$$\left(\int_M |\nabla^k R|^2\, dv\right)'_g (P(g)) = -(1-2a(n-1))\int_M |\nabla^{k+2}R_g|^2\, dv_g + \textsc{lots}^{(k)}(g).$$

The coefficients of the lower order terms depend only on k and P.

Proof. Taken a p–covariant tensor T, let us define the 2–covariant tensor $T \veebar T$ by

$$(T \veebar T)_{ij} = \sum_{k=1}^{p} T_{\ell_1 \dots i \dots j_p} T_{m_1 \dots j \dots m_p} g^{\ell_1 m_1} \dots g^{\ell_p m_p},$$

where the indices i and j are to be put instead of ℓ_k and m_k (and the corresponding term $g^{\ell_k m_k}$ must be removed).

Distributing the derivative, and handling $\partial_t dv_g$ like in Lemma 2.1, we get

$$\begin{aligned}\left(\int_M |\nabla^k \operatorname{Riem}|^2 \, dv\right)'_g (P(g)) &= 2\int_M \langle (\nabla^k \operatorname{Riem})'_g (P(g)) | \nabla^k \operatorname{Riem}_g \rangle \, dv_g \\ &- \int_M \langle P(g) | \nabla^k \operatorname{Riem}_g \veebar \nabla^k \operatorname{Riem}_g \rangle + \frac{1}{2}\int_M |\nabla^k \operatorname{Riem}_g|^2 \operatorname{tr}(P(g)) \, dv_t.\end{aligned}$$

Since $P(g)$ carries at most two derivatives of the Riemann tensor, the last two terms are $\textsc{lots}^{(k)}(g)$. Then we have to commute the linearization with the covariant derivatives: every time we do that, we have an additional term in which a derivative of Riem_g is lost and a derivative of the Christoffel symbols appear. This new term is again $\textsc{lots}^{(k)}(g)$. So

$$\left(\int_M |\nabla^k \operatorname{Riem}|^2 \, dv\right)'_g (P(g)) = 2\int_M \langle \nabla^k \operatorname{Riem}'_g (P(g)) | \nabla^k \operatorname{Riem}_g \rangle \, dv_g + \textsc{lots}^{(k)}(g).$$

We recall the formula $\operatorname{Riem}'_g = -\frac{1}{2}\mathsf{D}\tilde{\mathsf{D}} + \operatorname{Riem}_g *\cdot$ (compare with Remark 3.4 in [CK04]). Then, the integrand of the main term in the last formula is of type $\mathcal{P}^{(2)}_{2k+4}$, so each time we commute two derivatives the error term is in $\textsc{lots}^{(k)}(g)$. So we move all the derivatives on the right–hand term of the scalar product and substitute Riem'_g and then $P(g)$ obtaining

$$\begin{aligned}&\left(\int_M |\nabla^k \operatorname{Riem}|^2 \, dv\right)'_g (P(g)) \\ &\quad = -\int_M \langle \mathsf{D}\tilde{\mathsf{D}}(P(g)) | \Delta^k \operatorname{Riem}_g \rangle \, dv_g + \textsc{lots}^{(k)}(g) \\ &\quad = -\int_M \langle \Delta^2 \operatorname{Riem}_g | \Delta^k \operatorname{Riem}_g \rangle \, dv_g - a\int_M \langle \mathsf{D}\tilde{\mathsf{D}}(\Delta R_g g) | \Delta^k \operatorname{Riem}_g \rangle \, dv_g \\ &\qquad - b\int_M \langle \mathsf{D}\tilde{\mathsf{D}}\tilde{\mathsf{D}}\mathsf{D} R_g | \Delta^k \operatorname{Riem}_g \rangle \, dv_g + \textsc{lots}^{(k)}(g) \\ &\quad = -\int_M |\nabla^{k+2} \operatorname{Riem}_g|^2 \, dv_g - a\int_M \langle \Delta R_g | \operatorname{tr}(\tilde{\delta}\delta\Delta^k \operatorname{Riem}_g) \rangle \, dv_g + \textsc{lots}^{(k)}(g),\end{aligned}$$

since $\tilde{\mathsf{D}}\tilde{\mathsf{D}} = \operatorname{Riem}_g *\cdot$, so last terms finishes in $\textsc{lots}^{(k)}(g)$. Going on, commuting again derivatives and applying Schur's formula (Lemma 1.1):

$$\begin{aligned}
&\left(\int_M |\nabla^k \operatorname{Riem}|^2\, dv\right)'_g (P(g)) \\
&\quad = -\int_M |\nabla^{k+2} \operatorname{Riem}_g|^2\, dv_g - a\int_M \langle \Delta R_g | \Delta^k \tilde{\delta}\delta \operatorname{Ric}_g)\rangle\, dv_g + \text{LOTs}^{(k)}(g) \\
&\quad = -\int_M |\nabla^{k+2} \operatorname{Riem}_g|^2\, dv_g - a\int_M \langle \Delta R_g | \Delta^{k+1} R_g)\rangle\, dv_g + \text{LOTs}^{(k)}(g) \\
&\quad = -\int_M |\nabla^{k+2} \operatorname{Riem}_g|^2\, dv_g + \frac{a}{2}\int_M |\nabla^{k+2} R_g|^2\, dv_g + \text{LOTs}^{(k)}(g),
\end{aligned}$$

which finishes the first formula we had to prove.

We proceed in a similar way for the second one.

$$\begin{aligned}
&\left(\int_M |\nabla^k R|^2\, dv\right)'_g (P(g)) \\
&\quad = 2\int_M \langle (\nabla^k R)'_g (P(g)) | \nabla^k R_g\rangle\, dv_g - \int_M \langle P(g) | \nabla^k R_g \veebar \nabla^k R_g\rangle\, dv_g \\
&\qquad + \frac{1}{2}\int_M |\nabla^k R_g|^2 \operatorname{tr}(P(g))\, dv_g \\
&\quad = 2\int_M \langle (\nabla^k R)'_g (P(g)) | \nabla^k R_g\rangle\, dv_g + \text{LOTs}^{(k)}(g) \\
&\quad = 2\int_M \langle R'_g (P(g)) | \Delta^k R_g\rangle\, dv_g + \text{LOTs}^{(k)}(g).
\end{aligned}$$

Then we use $\delta\mathsf{D}(R_g g) = \Delta R_g g + \nabla^2 R_g$ and $R'_g = \operatorname{tr}\delta\mathsf{D} + \operatorname{Riem}_g *\cdot$ (see Lemma 3.7 in [CK04]) and conclude like above.

$$\begin{aligned}
&\left(\int_M |\nabla^k R|^2\, dv\right)'_g (P(g)) \\
&\quad = 2\int_M \langle R'_g (\delta\tilde{\delta} \operatorname{Riem}_g + a\delta\mathsf{D}(R_g g) + (b-a)\mathsf{D}\tilde{\mathsf{D}} R_g) | \Delta^k R_g\rangle\, dv_g + \text{LOTs}^{(k)}(g) \\
&\quad = -\int_M \langle \Delta^2 R_g | \Delta^k R_g\rangle\, dv_g + 2a(n-1)\int_M \langle \Delta^k R_g | \Delta^k R_g\rangle\, dv_g + \text{LOTs}^{(k)}(g) \\
&\quad = -(1-2a(n-1))\int_M |\nabla^{k+2} R_g|^2\, dv_g + \text{LOTs}^{(k)}(g).
\end{aligned}$$

□

Finally we can prove Theorem 2.12.

Proof of Theorem 2.12. Let us consider this auxiliary operator:

$$\mathcal{A}_k(g) = \int_M |\nabla^k \operatorname{Riem}_g|^2\, dv_g + \frac{a_+}{1-2a_+(n-1)}\int_M |\nabla^k R_g|^2\, dv_g,$$

where $a_+ = \max\{a, 0\}$. Moreover, let us define the constant

$$c_a = \frac{1-2a_+(n-1)}{2}.$$

Applying the two lemmas above we see that

$$
\begin{aligned}
(\mathcal{A}_k)'_g(P(g)) + c_a \mathcal{A}_{k+2}(g) &= -(1-c_a)\int_M |\nabla^{k+2}\operatorname{Riem}_g|^2\, dv_g \\
&\quad - \frac{a_+ - a}{2}\int_M |\nabla^{k+2} R_g|^2\, dv_g + \textsc{lots}^{(k)}(g) \\
&\leqslant -\frac{1}{2}\int_M |\nabla^{k+2}\operatorname{Riem}_g|^2\, dv_g + \textsc{lots}^{(k)}(g) \\
&\leqslant C_k \beta^{k+2}\int_M |\operatorname{Riem}_g|^2\, dv_g \\
&\leqslant C_k \beta^{k+2}\mathcal{A}_0(g),
\end{aligned}
$$

where $C_k(n) = C(k,n)$ is given by Lemma 2.13.

Therefore, defining the polynomial

$$
f_k(t) = \sum_{j=0}^{k} \frac{c_a^j t^j}{j!}\mathcal{A}_{2j}(g(t)),
$$

we see that

$$
\begin{aligned}
f_k'(t) &\leqslant \sum_{j=1}^{k} \frac{c_a^j t^{j-1}}{(j-1)!}\mathcal{A}_{2j}(g(t)) + \sum_{j=0}^{k}\frac{c_a^j t^j}{j!}(A_{2j})'_{g(t)}(P(g(t))) \\
&= \sum_{j=0}^{k-1}\left(\frac{c_a^j t^j}{j!}((A_{2j})'_{g(t)}(P(g(t))) + c_a\mathcal{A}_{2j+2}(g(t)))\right) + \frac{c_a^k t^k}{k!}(\mathcal{A}_{2k})'_{g(t)}(P(g(t))) \\
&\leqslant \sum_{j=0}^{k}\left(\frac{c_a^j t^j}{j!}((A_{2j})'_{g(t)}(P(g(t))) + c_a\mathcal{A}_{2j+2}(g(t)))\right) \\
&\leqslant \sum_{j=0}^{k}\frac{c_a^j t^j}{j!}C_k\beta^{2j+2}\mathcal{A}_0(g(t)) \\
&\leqslant C'\beta^2(1+\beta t)^{2k}\mathcal{A}_0(g(t)) \\
&\leqslant C'\beta^2(1+\beta t)^{2k} f_k(t),
\end{aligned}
$$

where $C' = C'(k,n)$. This gives an estimate on the logarithmic derivative of f_k:

$$
(\log f_k)'(t) \leqslant C'\beta^2(1+\beta t)^{2k}
$$

which in turns implies that for $t > 0$

$$
f_k(t) \leqslant f_k(0)\exp(C'\beta^2(1+\beta t)^{2k} t).
$$

At last:

$$\begin{aligned}
\int_M |\nabla^{2k} \operatorname{Riem}_{g(t)}|^2 \, dv_{g(t)} &\leqslant \mathcal{A}_{2k}(g(t)) \\
&\leqslant \frac{k!}{c_a^k t^k} f_k(t) \\
&\leqslant \frac{k!}{c_a^k t^k} f_k(0) \exp(C' \beta^2 (1+\beta t)^{2k} t) \\
&\leqslant \frac{c'}{t^k} \mathcal{A}_0(g(0)) \\
&\leqslant \frac{c}{t^k} \int_M |\operatorname{Riem}_{g(0)}|^2 \, dv_{g(0)},
\end{aligned}$$

which is the thesis for k even.

For the case k odd we just have to apply Proposition 2.6 to the tensor $\nabla^{2k} \operatorname{Riem}$, taking $j = 1$, $s = 2$ and $p = q = r = 2$. □

From the Bando–Bernstein–Shi estimates we can recover an L^∞ control on the derivatives of the Riemann tensor using the following Sobolev embedding.

2.15 Proposition. *For each $n \in \mathbb{N}$ and $Y_0 > 0$ there is a constant $C(n, Y_0)$ such that the following proposition holds. Let (M^n, g) be a compact Riemannian manifold with $Y(M, [g]) \geqslant Y_0$. Then for each smooth tensor T there holds*

$$\|T\|_{L^\infty} \leqslant C(\|T\|_{L^2} + \cdots + \|\nabla^k T\|_{L^2}),$$

where k is the integer part of $\frac{n}{2} + 1$.

Proof. It descends from Proposition 9.21 in [Bou10]. The intervening Sobolev constant is controlled with the Yamabe constant using Proposition 5.2 in the same work. □

2.16 Theorem. *Let M^4 be a smooth manifold and let $(g(t))_{t \in [0,T)}$ be a family of metrics that solve the flow (2.1) for $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$. Suppose moreover that for a certain $\beta < \infty$ and all times $t \in [0, T)$ we have $\|\operatorname{Riem}_{g(t)}\|_{L^\infty} < \beta$ and that the Yamabe constant is uniformly bounded from below by a positive constant.*

Then for each $T' \in (0, T)$ and $k \in \mathbb{N}$ there is a constant $c(n, k, \beta, P, T, T') > 0$ such that

$$\sup_{[T', T]} |\nabla^k \operatorname{Riem}_{g(\cdot)}| \leqslant c \|\operatorname{Riem}_{g(0)}\|_{L^2}.$$

Proof. Thanks to Theorem 2.12, on $[T', T]$ the covariant derivatives of the metric are bounded in the L^2 norm, uniformly along the flow. Applying Proposition 2.15, the result follows. □

2.4 A lemma about uniform boundedness of geometric flows

The following lemma is an adaptation to higher–order flows of Lemma 2.4 in [Ham95]. It is a standard lemma required in many types of arguments concerning limits of evolving metrics. Basically it says that under very broad assumptions, if we have a collection of flows that do not blow up each one in its own metric, then they do not blow up with respect to a common fixed metric. Thus, we can use standard analytic tools, like the Ascoli–Arzelà compactness theorem.

2.17 Lemma. *Let (M^n, g) be a Riemannian manifold and take $N, k \in \mathbb{N}$. Let $K \subseteq M^n$ be a compact subset and $I \subseteq \mathbb{R}$ a compact interval that contains 0. Let $(g_i)_{i\in\mathbb{N}}$ be a sequence of metrics defined on open neighbourhoods of $I \times K$ and solutions to a differential equation of order $k+2$ of the following form:*

$$\partial_t g_i(t) = \sum_{0\leqslant i_1\leqslant\ldots\leqslant i_j\leqslant k} \nabla^{i_1}_{g_i(t)} \operatorname{Riem}_{g_i(t)} * \cdots * \nabla^{i_j}_{g_i(t)} \operatorname{Riem}_{g_i(t)} .$$

Suppose moreover that the following conditions are met.

1. *All the metrics $g_i(0)$ are uniformly equivalent to g on K; i.e., there is $C > 0$ such that*
$$e^{-C} g \leqslant g_i(0) \leqslant e^C g \qquad \forall i \in \mathbb{N}.$$

2. *For all $1 \leqslant j \leqslant N$, the j-th derivative of $g_i(0)$ with respect to g is uniformly bounded on K; i.e., there are $C_j > 0$ such that*
$$\sup_K |\nabla^j_g g_i(0)|_g \leqslant C_j \qquad \forall i \in \mathbb{N}, \forall 1 \leqslant j \leqslant N.$$

3. *For all $1 \leqslant j \leqslant N+k$, the j-th derivative of $\operatorname{Riem}_{g_i}$ with respect to g_i is bounded with respect to g_i on $I \times K$; i.e., there are $C'_j > 0$ such that*
$$\sup_{I\times K} |\nabla^j_{g_i} \operatorname{Riem}_{g_i}|_{g_i} \leqslant C'_j \qquad \forall i \in \mathbb{N}, \forall 1 \leqslant j \leqslant N+k.$$

Then the metric g_i are uniformly equivalent to g on $I \times K$ and for all $1 \leqslant j \leqslant N$ the j-th derivatives of g_i with respect to g are uniformly bounded with respect to g on $I \times K$. I.e., there are constants $c(C, C_j, C'_j, n, I), c_j(C, C_j, C'_j, n, I) > 0$ such that

$$e^{-c} g \leqslant g_i(t) \leqslant e^c g, \qquad \sup_{I\times K} |\nabla^j_g g_i(t)|_g \leqslant c_j \qquad \forall i \in \mathbb{N}, \forall 1 \leqslant j \leqslant N, \forall t \in I.$$

The following lemma will be used in the proof.

2.18 Lemma (Lemma 6.49 in [CK04]). *Let M^n be a manifold and $(g(t))_{t\in I}$ a smooth family of metrics, where $I \subseteq \mathbb{R}$ is an interval. Suppose that there exists a constant $C < \infty$ such that*

$$\int_I |\partial_t g(t)|_{g(t)}\, dt \leqslant C.$$

Then for all $t \in I$

$$e^{-C} g(0) \leqslant g(t) \leqslant e^C g(0).$$

Proof of Lemma 2.17. From condition 3 and the structure of the differential equation we obtain that $|\partial_t g_i(t)|$ are uniformly bounded, so Lemma 2.18 implies that, for a fixed $i \in \mathbb{N}$, the metrics $g_i(t)$ are uniformly equivalent for $t \in I$. Since $g_i(0)$ are uniformly equivalent to g for $i \in \mathbb{N}$, it follows that $g_i(t)$ are uniformly equivalent to g independently from i and t.

Then we have to prove that the g–derivatives of the metric are uniformly bounded with respect to g. This will be done employing an induction argument on the order of derivation. The induction base step, i.e. the fact $\sup_{I\times K} |g_i|_g \leqslant c_0$, immediately follows from the uniform equivalence shown above.

We observe that, given two metrics g and $\tilde{g}$, the difference of the Christoffel symbols $\Gamma_{\tilde{g}} - \Gamma_g$ is a tensor (although the two Christoffel symbols alone are not), so

$$\nabla_g T = \nabla_{\tilde{g}} T + (\Gamma_{\tilde{g}} - \Gamma_g) * T = \nabla_{\tilde{g}} T + \tilde{g} * \nabla_g \tilde{g} * T,$$

where the second equality is true since, in normal coordinates for g, we have that $(\nabla_g)_i \tilde{g}_{jk} = \partial_i \tilde{g}_{jk}$ and

$$(\Gamma_{\tilde{g}} - \Gamma_g)_{ij}^k = (\Gamma_{\tilde{g}})_{ij}^k = \frac{1}{2}\tilde{g}^{k\ell}(\partial_i \tilde{g}_{j\ell} + \partial_j \tilde{g}_{i\ell} - \partial_\ell \tilde{g}_{ij}).$$

Applying repeatedly the previous formula, for each $1 \leqslant j \leqslant N$ we obtain that $\nabla_g^j(\partial_t g_i(t))$ is a finite sum of terms of the following form:

$$\sum_{\substack{0\leqslant i_1,\dots,i_p\leqslant j\\ 0\leqslant j_1,\dots,j_q\leqslant k+j}} \nabla_g^{i_1} g_i(t) * \cdots * \nabla_g^{i_p} g_i(t) * \nabla_{g_i(t)}^{j_1} \mathrm{Riem}_{g_i(t)} * \cdots * \nabla_{g_i(t)}^{j_q} \mathrm{Riem}_{g_i(t)} \qquad p, q \in \mathbb{N}.$$

It follows that $|\partial_t \nabla_g^j g_i(t)|_g = |\nabla_g^j(\partial_t g_i(t))|_g$ is controlled by a polynomial in $|\nabla_g g_i(t)|_g$, $\dots$, $|\nabla_g^j g_i(t)|_g$, so using condition 2, integrating over I and inducting on j we have the thesis. □

2.5 Compactness results for flows

In order to study the behaviours of the flow approaching a singular time, we have to extend Theorem 1.4 in order to deal with flows instead of just manifolds. The following theorem makes fundamental use of Lemma 2.17.

2.19 Theorem (compare with Theorem 1.2 in [Ham95])**.** *Let M^4 be a smooth manifold and $(g_i(t), x_i)_{i\in\mathbb{N}}$ a family of pointed complete solutions of the flow (2.1) with $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$, defined on an possibly infinite interval $(-\alpha, \omega] \subseteq \mathbb{R}$ containing 0. Suppose that the curvatures of the metrics g_i are uniformly bounded on $(-\alpha, \omega] \times M^4$ and that the Yamabe constant is uniformly bounded from below on $(-\alpha, \omega] \times M^4$.*

Then there is a subsequence of $(M, g_i(t), x_i)_{i\in\mathbb{N}}$ that converges in the pointed $\mathcal{C}^\infty$ topology to $(M_\infty, g_\infty(t), x_\infty)$, which is still a complete solution of (2.1) defined on the same interval.

The proof is analogous to the one in Hamilton's paper.

Proof. First consider the case $-\alpha > -\infty$ and take an small $0 < \varepsilon < \alpha$. From Theorem 2.16 we know that all the derivatives of the Riemann tensor are uniformly bounded on $(-\alpha + \varepsilon, \omega]$. Moreover, the hypothesis on the Yamabe constant implies that the injectivity radius of the metrics $g_i(0)$ at the points x_i are bounded from below by a positive constant. Theorem 1.4 then applies and, up to passing to a subsequence, there is a manifold $(M_\infty, g_\infty(0), x_\infty)$ that is the pointed $\mathcal{C}^\infty$ limit of $(M, g_i(0), x_i)$.

Now we have to extend the built solution to the whole interval $(-\alpha + \varepsilon, \omega]$. From the definition of pointed $\mathcal{C}^\infty$ convergence, for each $j \in \mathbb{N}$ we have open sets $\Omega_j \subseteq M_\infty$ and embeddings $F_j \colon \Omega_j \to M^4$ that are diffeomorphisms onto their images, such that $\Omega_j \nearrow M_\infty$. Using the fact already mentioned that the derivatives of $\mathrm{Riem}_{g_i(t)}$ are bounded with respect to the metric $g_i(t)$ uniformly with respect to i and t and invoking Lemma 2.17, we have that all the derivatives of the metric $g_i(t)$ are bounded with respect to a fixed metric g. It follows that, again up to passing to a subsequence, for each $j \in \mathbb{N}$ and $t \in (-\alpha+\varepsilon, \omega]$ the sequence $\left(F_j^* g_i(t)\right)_{i\in\mathbb{N}}$ converges to a metric $g_\infty(t)$ defined on Ω_j. Using a diagonal argument, $g_\infty(t)$ can be defined on the whole manifold M_∞. Such metric is still a solution of the flow (2.1), because the convergence is $\mathcal{C}^\infty$.

Using again a diagonal argument, we can prove that g_∞ is actually defined over $(-\alpha, \omega]$ and also cover the case $-\alpha = -\infty$. □

2.6 Blow–up at a singular time

With all the theory built so far, we are finally able to discuss what happens to the flow near a singular time. Not only we will see that, thanks to the Bando–Bernstein–Shi estimates, the Riemannian tensor must blow up near the singular time, but we will also inspect the limit structure in order to better understand its geometry.

2.20 Theorem. *Let M^4 be a compact manifold and $(g(t))_{t\in[0,T)}$ a family of metrics that solve the flow (2.1). for $a < \frac{1}{2(n-1)}$ and $b \in \mathbb{R}$. Suppose that $T < \infty$ is the maximal time of existence of such solution and that the Yamabe constant is uniformly bounded from below along the flow.*

Then the Riemannian tensor must blow up approaching T. That is,

$$\limsup_{t\to T} \|\mathrm{Riem}_{g(t)}\|_{L^\infty} \to \infty.$$

Proof. If $\|\mathrm{Riem}_{g(t)}\|_{L^\infty}$ is bounded near T, then Theorem 2.16 applies and all the derivatives of $\mathrm{Riem}_{g(t)}$ are uniformly bounded near T. Thus, the solution must extend beyond T.

More precisely, thanks to Lemma 2.17 the metrics $(g(t))_{t\in[0,T)}$ are uniformly equivalent and all the derivatives uniformly bounded with respect to a fixed metric. So Ascoli–Arzelà theorem implies that, up to a subsequence, they converge uniformly to a symmetric 2–covector. Up to taking another subsequence and using a diagonal argument, all the derivatives converge uniformly. Moreover, the limit tensor $g(T)$ is again positively defined (i.e., is a metric), because of the uniform equivalence.

Therefore, we can take the limit metric $g(T)$ and start again a flow, obtaining $(g(t))_{t\in[T,T+\varepsilon)}$ for some $\varepsilon > 0$. The function $(g(t))_{t\in[0,T+\varepsilon)}$ resulting from gluing the pieces is smooth at every time (in particular, the convergence of spatial derivatives implies the convergences of time derivatives, since they are related by the equation of the flow) and is again a solution of the flow. This contradicts the maximality of T. □

Now, let us consider more carefully what happens near a singular time. Let M a compact manifold and $g(t)$ a solution of (2.1) on $[0, T) \times M$, with $0 \leqslant T \leqslant \infty$ such that $\limsup_{t \to T} \|\mathrm{Riem}_{g(t)}\|_{L^\infty} = \infty$. Moreover, let us suppose that the Yamabe constant is uniformly bounded from below along the flow: $\inf_{t \in [0,T)} Y(M, [g(t)]) = Y_0 > 0$.

We can pick times $t_i \nearrow T$ and points $x_i \in M$ that satisfy

$$\|\mathrm{Riem}_{g(t_i)}\|_{L^\infty} = \sup_{t \leqslant t_i} \|\mathrm{Riem}_{g(t_i)}\|_{L^\infty} \qquad \text{and} \qquad |\mathrm{Riem}_{g(t_i)}(x_i)| = \|\mathrm{Riem}_{g(t_i)}\|_{L^\infty} \nearrow \infty.$$

Let us call $\lambda_i = |\mathrm{Riem}_{g(t_i)}(x_i)|$ and define the following rescaled flows:

$$g_i(t) = \lambda_i g\left(t_i + \frac{t}{\lambda_i^2}\right).$$

Evidently, for all $i \in \mathbb{N}$ we have that g_i is a solution of (2.1) of $[-\lambda_i^2 t_i, \lambda_i^2 (T - t_i))$. Thus, for every $\alpha > 0$, we have that g_i is eventually defined on $[-\alpha, 0]$.

We now want to show that these rescaled flows have an accumulation point. Thanks to the rescaling, we see that $\|\mathrm{Riem}_{g_i(t)}\|_{L^\infty} \leqslant 1$ for all $t \in [-\alpha, 0]$ and $i \in \mathbb{N}$. Moreover, since the Yamabe constant is scale invariant, Proposition 1.9, Lemma 1.6 and the following discussion lead us to concluding that the injectivity radius of $g_i(0)$ is uniformly bounded from below for all $i \in \mathbb{N}$.

Theorem 2.19 thus applies and we can find a subsequence of $(M, g_i(t), x_i)$ converging in the pointed $\mathcal{C}^\infty$ topology to a complete solution $(M_\infty, g_\infty(t), x_\infty)$ of (2.1) for $t \in [-\alpha, 0]$. Thanks to a diagonal argument, the limit solution can actually be defined for $t \in (-\infty, 0]$. Moreover, if $T = \infty$, we can actually take $t \in \mathbb{R}$.

In any case, the limit manifold is not flat: indeed, the definition of *pointed* convergence implies that $|\mathrm{Riem}_{g_\infty(0)}(x_\infty)| = 1$. This will turn out to be a useful detail in order to carry out proofs by absurd on the blow–up model.

CHAPTER 3

Flows arising as gradient of curvature functionals

Contents

In this chapter we finally deal with the particular problem we are studying. The final theorem we want to prove (Corollary 3.17) is that a 4–manifold that verifies certain integral pinching conditions has positive constant sectional curvature. We begin with giving an intuitive idea of what we would like to achieve.

As already discussed, the Riemann tensor admits the following decomposition:

$$\mathrm{Riem}_g = \frac{R_g}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{\mathrm{Ric}}_g \owedge g + \mathcal{W}_g.$$

A metric is of constant sectional curvature if and only the Ricci and Weyl terms of this decomposition are zero (in such case, by Schur's formula, it is well known that the function R_g is constant). So we try to design our flow so as to gradually remove the Ricci and Weyl parts.

It is natural, then, to pick the L^2 energy of $\mathcal{Z}_g$ and $\mathcal{W}_g$ and require $\partial_t g(t)$ to be opposite of the gradient of a linear combination of these two energies:

$$\partial_t g(t) = -2\nabla_g((1-\lambda)\|\mathcal{W}\|^2_{L^2} + \lambda\|\mathcal{Z}\|^2_{L^2})$$

(it is customary to add a factor 2 to simplify a few formulae).

Working with gradient flows also gives the nice property of having a monotonically decreasing "energy" along the flow (i.e., the value of the functional of whose the gradient is taken; in our case it is $(1-\lambda)\|\mathcal{W}\|^2_{L^2}+\lambda\|\mathcal{Z}\|^2_{L^2})$. This energy will ultimately lead to the estimates with which we will complement the blow–up analysis already begun in the previous chapter, showing that the flow exists for all times with no singularities when the starting energy is not too high (with an explicit value for this requirement).

Once the flow exists for all times, one checks the structure of the limit manifold. Following the ideas outlined above, we have to be sure that the limit metric has zero traceless Ricci and Weyl tensors. This will be done employing the Bochner technique presented in the first chapter.

3.1 Functional derivatives and gradients

If $\mathcal{F}\colon \mathcal{S}^2_+(M)\to\mathbb{R}$ is a smooth functional, its derivative at point g is defined as the unique functional $\mathcal{F}'_g\colon \mathcal{S}^2(M)\to\mathbb{R}$ (where $\mathcal{S}^2(M)$ was taken as the tangent space of $\mathcal{S}^2_+(M)$) that satisfies

$$\mathcal{F}(g+h)=\mathcal{F}(g)+\mathcal{F}'_g(h)+o(|h|)\qquad \forall g\in\mathcal{S}^2_+(M), h\in\mathcal{S}^2(M) \text{ with } h\to 0.$$

We can represent the derivative of a smooth functional as the integration against a certain symmetric 2-covector, which we'll call the *gradient* of $\mathcal{F}$; it satisfies

$$\langle\nabla\mathcal{F}_g|h\rangle_{L^2(g)}=\mathcal{F}'_g(h)$$

and express, intuitively, the "perturbation direction" towards which the functional grows most quickly.

With these notions, we set up the following flow:

$$\begin{cases}\partial_t g(t) &= -2\nabla\mathcal{F}(g(t))\\ g(0) &= g_0.\end{cases}\tag{3.1}$$

It immediately follows that

$$\partial_t\mathcal{F}(g(t))=\mathcal{F}'_{g(t)}(\partial_t g(t))=-2\mathcal{F}'_{g(t)}(\nabla\mathcal{F}(g(t)))=-2\|\nabla\mathcal{F}(g(t))\|^2_{L^2(g)}\leqslant 0,\tag{3.2}$$

so that the functional is decreasing along the flow. This is in accord with the intuitive interpretation we have just given and the choice of the minus sign. It also shows that metrics that are critical for the functional $\mathcal{F}$ exhibit a constant flow.

3.2 Gradients of curvature functionals

Let us take an operator $T\colon\mathcal{S}^2_+(M)\to\mathcal{T}^{(p,q)}(M)$ and define the functional $\mathcal{F}_T\colon\mathcal{S}^2_+(M)\to\mathbb{R}$ by

$$\mathcal{F}_T(g)=\|T\|^2_{L^2(g)}=\int_M|T|^2_g\,dv_g.$$

As discussed in the introduction of this chapter, we will consider the case $n=\dim M=4$ and take the functional

$$\mathcal{F}^\lambda(g)=(1-\lambda)\mathcal{F}_{\mathcal{W}}(g)+\lambda\mathcal{F}_{\mathcal{Z}}(g).\tag{3.3}$$

In order to better understand this functional, we put in in slightly different forms. We define the auxiliary functional $\mathcal{F}_2$, which is the L^2 energy of the second symmetric function of the eigenvalues of the Schouten tensor (defined in Section 1.1):

$$\mathcal{F}_2(g) = \frac{1}{2}\int_M (\mathrm{tr}^2(A_g) - |A_g|^2)\, dv_g = \frac{n^2-4n+4}{4}\mathcal{F}_{\mathcal{S}}(g) - \frac{n-2}{2}\mathcal{F}_{\mathcal{Z}}(g).$$

In dimension 4 one can verify, using Theorem 3.5, that

$$\mathcal{F}_2(g) + \mathcal{F}_{\mathcal{W}}(g) = \mathcal{F}_{\mathrm{Riem}}(g) - \mathcal{F}_{\mathring{\mathrm{Ric}}}(g) = 8\pi^2\chi(M),$$

so $\nabla\mathcal{F}_2(g) = -\nabla\mathcal{F}_{\mathcal{W}}(g)$. It follows that

$$\nabla\mathcal{F}^\lambda = \nabla\mathcal{F}_{\mathcal{W}} + \lambda(\nabla\mathcal{F}_{\mathcal{Z}} + \nabla\mathcal{F}_2) = \nabla\mathcal{F}_{\mathcal{W}} + \lambda\nabla\mathcal{F}_{\mathcal{S}} = \nabla\mathcal{F}_{\mathcal{W}} + \frac{\lambda}{24}\nabla\mathcal{F}_R. \tag{3.4}$$

On the other hand, substituting $\nabla\mathcal{F}_{\mathring{\mathrm{Ric}}}$ in (1.3), we have that $\frac{1}{2}\nabla\mathcal{F}_{\mathrm{Riem}} = \nabla\mathcal{F}_{\mathcal{W}} + \frac{1}{24}\nabla\mathcal{F}_R$, from which

$$\nabla\mathcal{F}^\lambda(g) = \frac{1}{2}\mathcal{F}_{\mathrm{Riem}} + \frac{\lambda-1}{24}\mathcal{F}_R.$$

Explicit formulae for the L^2 norms of the curvature functionals are available in [Bes87, Chapter 4.H]:

$$\nabla\mathcal{F}_{\mathrm{Riem}}(g) = -\delta\tilde{\delta}\,\mathrm{Riem} - \frac{1}{2}\,\mathrm{Riem}\vee\mathrm{Riem} + \frac{1}{2}|\mathrm{Riem}|^2 g, \tag{3.5a}$$

$$\nabla\mathcal{F}_{\mathcal{W}}(g) = -\delta\tilde{\delta}\mathcal{W} - \frac{1}{n-2}\mathring{\mathcal{W}}\mathring{\mathrm{Ric}} - \frac{1}{2}(\mathcal{W}\vee\mathcal{W} - |\mathcal{W}|^2 g), \tag{3.5b}$$

$$\nabla\mathcal{F}_R(g) = 2\delta\mathsf{D}(Rg) - 2R(\mathrm{Ric} - \frac{1}{4}Rg). \tag{3.5c}$$

3.1 Remark. In dimension 4 the last term in the expansion of $\nabla\mathcal{F}_{\mathcal{W}}$ is zero. So

$$\nabla\mathcal{F}_{\mathcal{W}}(g) = -\delta\tilde{\delta}\mathcal{W} - \frac{1}{2}\mathring{\mathcal{W}}\mathring{\mathrm{Ric}}. \tag{3.6}$$

Putting everything together, we reach the following theorem.

3.2 Theorem. *Let (M^4, g_0) be a closed Riemannian manifold. The flow (3.1) with the functional (3.3) has the following form:*

$$\partial_t g(t) = \delta\tilde{\delta}\,\mathrm{Riem}_{g(t)} + \frac{1-\lambda}{6}(\Delta R_{g(t)}g(t) + \nabla^2 R_{g(t)}) + \mathrm{Riem}_{g(t)} * \mathrm{Riem}_{g(t)}\,.$$

Thanks to Theorem 2.8 and Proposition 2.10, if $\lambda > 0$ there is a unique evolving metric $(g(t))_{t\in[0,T)}$ which is a maximal time solution of the flow, for a certain $0 < T \leqslant \infty$.

Using the formulae above we immediately start getting useful pieces of information about the flow. For example, we see that the volume of the evolving metric is constant, a result that will be useful in the continuation.

3.3 Lemma. *Let (M^4, g) be a compact Riemannian manifold. Then*

$$\mathrm{tr}(\nabla\mathcal{F}^\lambda(g)) = \frac{\lambda}{4}\Delta R_g.$$

Proof. Write $\nabla\mathcal{F}^\lambda(g) = \mathcal{F}_\mathcal{W}(g) + \frac{\lambda}{24}\mathcal{F}_\mathbb{R}(g)$. Substituting equalities (3.5), we immediately see that the Weyl part vanishes and that the scalar part gives rise to the Laplacian of the scalar curvature. □

3.4 Corollary. *Let M^n be a closed manifold and $(g(t))_{t\in[0,T)}$ a family of metrics solution of (3.1) with the functional (3.3) and $\lambda > 0$. Then $\mathrm{Vol}_{g(t)}(M^n)$ is constant.*

Proof. From Lemma 2.1 and Lemma 3.3 we know that

$$\partial_t \mathrm{Vol}_{g(t)}(M^n) = \frac{\lambda}{8}\int_M \Delta R_g \, dv_g.$$

By Stokes' theorem, such integral in zero. □

3.3 Estimates on the Yamabe constant and curvatures

One of the niceties of the functional $\mathcal{F}^\lambda$ is that it gives an easy estimate for the Yamabe constant. First we mention the Chern–Gauss–Bonnet theorem for 4 dimensions, which is the analogue for what the Gauss–Bonnet theorem is in 2 dimensions (the Chern–Gauss–Bonnet formula is actually available for all even–dimensional closed Riemannian manifolds, but enunciating it in full generality requires far more theory construction than we need).

3.5 Theorem (Chern–Gauss–Bonnet theorem). *Let (M^4, g) be a closed Riemannian manifold. Then*

$$8\pi^2\chi(M) = \mathcal{F}_\mathcal{S}(g) - \mathcal{F}_\mathcal{Z}(g) + \mathcal{F}_\mathcal{W}(g).$$

The proof can be found in [Spi99a, Spi99b].

3.6 Lemma. *Let (M^4, g) be a closed Riemannian manifold. The for all $\lambda \geqslant 0$*

$$Y(M,[g])^2 \geqslant 24((1-\lambda)8\pi^2\chi(M) - \mathcal{F}^\lambda(g)). \tag{3.7}$$

Proof. Take a Yamabe minimizer $\hat{g} \in [g]$, which has constant scalar curvature. From (1.4) we know that

$$Y(M,[\hat{g}])^2 = \int_M R_{\hat{g}}^2 \, dv_g \geqslant 24\left(\frac{1}{24}\mathcal{F}_R(\hat{g}) - \frac{1}{2}\mathcal{F}_{\mathring{\mathrm{Ric}}}(\hat{g})\right) = 24\mathcal{F}_2(\hat{g}).$$

Now, we already know that the Yamabe constant is a conformal invariant. Since the energy of the Weyl tensor is conformally invariant in dimension 4, from (3.6) we know that $\mathcal{F}_2$ is a conformal invariant too. Thus the same inequality is true for g. So

$$\begin{aligned} Y(M,[g])^2 &\geqslant 24\mathcal{F}_2(g) \\ &= 24(\lambda\mathcal{F}_2(g) + (1-\lambda)(8\pi^2\chi(M) - \mathcal{F}_\mathcal{W}(g))) \\ &\geqslant 24((1-\lambda)8\pi^2\chi(M) - \mathcal{F}^\lambda(g)). \end{aligned}$$ □

A corollary follows immediately.

3.7 Corollary. *Let M^4 be a closed manifold and $(g(t))_{t\in[0,T)}$ a family of metrics solution of (3.1), with $0 < T \leqslant \infty$. Suppose that $Y(M,[g(0)]) = Y_0 > 0$ and that for $g(0)$ the right-hand side of (3.7) is non-negative. Then the Yamabe constant is uniformly bounded from below by Y_0 along the flow.*

Finally we mention two other lemmas concerning estimates that we will use in the following sections.

3.8 Lemma. *Let M^4 be a closed manifold and $(g(t))_{t\in[0,T)}$ a family of metrics solution of (3.1) on the functional defined by equation (3.3), with $0 < T \leqslant \infty$ and $0 < \lambda < 1$. Then the quantities*

$$\mathcal{F}_{\mathcal{S}}(g(t)), \quad \mathcal{F}_{\mathcal{Z}}(g(t)), \quad \mathcal{F}_{\mathcal{W}}(g(t)) \quad \text{and} \quad \mathcal{F}_{\mathrm{Riem}}(g(t))$$

are bounded from above along the flow by constants depending only on λ, $\chi(M)$ and $\mathcal{F}^\lambda(g(0))$.

Proof. As already remarked, from inequality (3.2) we know that $\mathcal{F}^\lambda(g(t)) \leqslant \mathcal{F}^\lambda(g(0))$ for all times $t \in [0,T)$. From the definition of $\mathcal{F}^\lambda$ in equation (3.3) we have

$$\mathcal{F}_{\mathcal{Z}}(g(t)) \leqslant \frac{1}{\lambda}\mathcal{F}^\lambda(g(t)) \qquad \text{and} \qquad \mathcal{F}_{\mathcal{Z}}(g(t)) \leqslant \frac{1}{1-\lambda}\mathcal{F}^\lambda(g(t)),$$

so the lemma is proved for $\mathcal{F}_{\mathcal{Z}}$ and $\mathcal{F}_{\mathcal{W}}$.

In order to have the result for $\mathcal{F}_{\mathcal{S}}$, we exploit the Chern–Gauss–Bonnet formula (Theorem 3.5) and have

$$\mathcal{F}_{\mathcal{S}}(g(t)) = 8\pi^2\chi(M) + \mathcal{F}_{\mathcal{Z}}(g(t)) - \mathcal{F}_{\mathcal{W}}(g(t)) \leqslant 8\pi^2\chi(M) + \mathcal{F}_{\mathcal{Z}}(g(t)).$$

Finally, $\mathcal{F}_{\mathrm{Riem}}$ follows, being the sum of the other three. □

3.9 Lemma. *Let (M^4, g) be a closed Riemannian manifold and $\lambda \in (0,1)$ such that, for a certain $\varepsilon > 0$,*

$$\begin{cases} \lambda & \leqslant \frac{4}{13} \\ \mathcal{F}^\lambda(g) & \leqslant 2\lambda(\pi^2\chi(M) - \varepsilon) \end{cases} \qquad \text{or} \qquad \begin{cases} \lambda & \geqslant \frac{4}{13} \\ \mathcal{F}^\lambda(g) & \leqslant \frac{8}{9}(1-\lambda)(\pi^2\chi(M) - \varepsilon). \end{cases} \tag{3.8}$$

Then

$$\mathcal{F}_{\mathcal{W}}(g) + \frac{1}{2}\mathcal{F}_{\mathcal{Z}}(g) \leqslant \frac{1}{8\times 24}Y(M,[g])^2 - \varepsilon.$$

Proof. If $\lambda \leqslant \frac{4}{13}$, then $1-\lambda \geqslant \frac{9}{13}$ and $\frac{9}{13(1-\lambda)} \leqslant 1 \leqslant \frac{4}{13\lambda}$, so

$$\frac{9}{13}\mathcal{F}_{\mathcal{W}} + \frac{4}{13}\mathcal{F}_{\mathcal{Z}} = \frac{9}{13(1-\lambda)}(1-\lambda)\mathcal{F}_{\mathcal{W}} + \frac{4}{13\lambda}\lambda\mathcal{F}_{\mathcal{Z}} \leqslant \frac{4}{13\lambda}\mathcal{F}^\lambda \leqslant \frac{8}{13}(\pi^2\chi(M) - \varepsilon).$$

On the other hand, if $\lambda \geqslant \frac{4}{13}$, then $1-\lambda \leqslant \frac{9}{13}$ and $\frac{4}{13\lambda} \leqslant 1 \leqslant \frac{9}{13(1-\lambda)}$, so

$$\frac{9}{13}\mathcal{F}_{\mathcal{W}} + \frac{4}{13}\mathcal{F}_{\mathcal{Z}} = \frac{9}{13(1-\lambda)}(1-\lambda)\mathcal{F}_{\mathcal{W}} + \frac{4}{13\lambda}\lambda\mathcal{F}_{\mathcal{Z}} \leqslant \frac{9}{13(1-\lambda)}\mathcal{F}^\lambda \leqslant \frac{8}{13}(\pi^2\chi(M) - \varepsilon).$$

In both cases, multiplying by $\frac{13}{8}$ and applying Lemma 3.6 with $\lambda = 0$ we get the thesis:

$$\begin{aligned}\mathcal{F}_{\mathcal{W}}(g) + \frac{1}{2}\mathcal{F}_{\mathcal{Z}}(g) &\leqslant \pi^2\chi(M) - \frac{1}{8}\mathcal{F}_{\mathcal{W}}(g) - \varepsilon \\ &= \frac{1}{8\times 24}Y(M,[g])^2 - \varepsilon.\end{aligned}$$

□

3.4 Critical metrics for the functional

3.10 Theorem. *Let (M^4, g) be a complete Riemannian manifold with positive Yamabe constant and $\lambda \in [0,1]$. Suppose that $R_g \in L^2(M)$ and, if $\lambda = 0$, suppose it is constant. Moreover, suppose that g is critical for $\mathcal{F}^\lambda$ (i.e., $\nabla\mathcal{F}^\lambda(g) = 0$) and that the following integral pinching holds:*

$$\|\mathcal{W}_g\|^2_{L^2} + \frac{1}{2}\|\mathcal{Z}_g\|^2_{L^2} < \frac{1}{8\times 24}Y(M,[g])^2.$$

Then g is of constant sectional curvature.

The proof will be carried out using the Bochner technique outlined in Section 1.4 to prove that the traceless Ricci tensor and the Weyl tensor must be zero in the hypotheses of the theorem.

First we discuss the case of the traceless Ricci tensor.

3.11 Lemma. *Let (M^4, g) be a Riemannian manifold with constant scalar curvature. Then*

$$\nabla\mathcal{F}^\lambda(g) = \frac{1}{2}\nabla^*\nabla\mathring{\mathrm{Ric}}_g - \overline{\mathring{(\mathcal{W}_g + \mathcal{Z}_g)}}\mathring{\mathrm{Ric}}_g + \frac{1}{4}|\mathring{\mathrm{Ric}}|^2 g + \frac{2-\lambda}{12}R_g\mathring{\mathrm{Ric}}. \tag{3.9}$$

Proof. Putting together equations (3.5) and (3.4) and remark 3.1 we have that

$$\nabla\mathcal{F}^\lambda(g) = -\delta\tilde{\delta}\mathcal{W}_g - \frac{1}{2}\mathring{\mathcal{W}}_g\mathring{\mathrm{Ric}}_g + \frac{\lambda}{12}\delta\mathsf{D}(R_g g) - \frac{\lambda}{12}R_g\mathring{\mathrm{Ric}}.$$

The third term vanishes, since R_g is constant. Moreover we substitute $\tilde{\delta}\mathcal{W}_g = -\frac{1}{2}\mathsf{D}\mathring{\mathrm{Ric}}_g$ (which is Lemma 1.1, to arrive to the following form:

$$\nabla\mathcal{F}^\lambda(g) = \frac{1}{2}\delta\mathsf{D}\mathring{\mathrm{Ric}}_g - \frac{1}{2}\mathring{\mathcal{W}}_g\mathring{\mathrm{Ric}}_g - \frac{\lambda}{12}R_g\mathring{\mathrm{Ric}}. \tag{3.10}$$

So the only thing left to do is manipulate this formula in order to find equality (3.9).

Expanding the definition, commuting derivatives and applying Schur's formula (Lemma 1.1) we have

$$\begin{aligned}
(\delta\mathsf{D}\mathring{\mathrm{Ric}}_g)_{ij} &= -\nabla^\alpha\nabla_\alpha\mathring{\mathrm{Ric}}_{ij} + \nabla^\alpha\nabla_i\mathring{\mathrm{Ric}}_{\alpha j}\\
&= -\nabla^\alpha\nabla_\alpha\mathring{\mathrm{Ric}}_{ij} + \nabla_i\nabla^\alpha\mathring{\mathrm{Ric}}_{\alpha j} - \mathrm{Riem}_i{}^\alpha{}_\alpha{}^\ell\cdot\mathring{\mathrm{Ric}}_{\ell j} - \mathrm{Riem}_i{}^\alpha{}_j{}^\ell\cdot\mathring{\mathrm{Ric}}_{\alpha\ell}\\
&= -\nabla^\alpha\nabla_\alpha\mathring{\mathrm{Ric}}_{ij} + \frac{1}{2}\nabla_i\nabla_j R_g - \mathrm{Riem}_i{}^\alpha{}_\alpha{}^\ell\cdot\mathring{\mathrm{Ric}}_{\ell j} - \mathrm{Riem}_i{}^\alpha{}_j{}^\ell\cdot\mathring{\mathrm{Ric}}_{\alpha\ell}\\
&= -\nabla^\alpha\nabla_\alpha\mathring{\mathrm{Ric}}_{ij} + (\mathrm{Ric}_g\circ\mathring{\mathrm{Ric}}_g)_{ij} - (\mathring{\mathrm{Riem}}_g\mathring{\mathrm{Ric}}_g)_{ij}.
\end{aligned}$$

We have the easy formula

$$\overline{\mathring{(g \owedge u)}}v = \langle u|v\rangle\, g + \operatorname{tr} v\cdot u - u\circ v - v\circ u,$$

from which we obtain

$$\begin{gathered}
(n-2)\mathring{\mathcal{Z}}_g\mathring{\mathrm{Ric}}_g = \overline{\mathring{(g\owedge \mathring{\mathrm{Ric}}_g)}}\mathring{\mathrm{Ric}}_g = |\mathring{\mathrm{Ric}}_g|^2 g - 2\mathring{\mathrm{Ric}}_g\circ\mathring{\mathrm{Ric}}_g,\\
\mathring{\mathrm{Riem}}_g\mathring{\mathrm{Ric}}_g = \mathring{\mathcal{W}}_g\mathring{\mathrm{Ric}}_g + \mathring{\mathcal{Z}}_g\mathring{\mathrm{Ric}}_g - \frac{1}{n(n-1)}R_g\mathring{\mathrm{Ric}}_g.
\end{gathered}$$

Putting together last formulae we find that

$$\delta \mathrm{D}\mathring{\mathrm{Ric}}_g = \nabla^*\nabla\mathring{\mathrm{Ric}}_g + \frac{1}{n-1}R_g\mathring{\mathrm{Ric}}_g - \overline{\left(\mathcal{W}_g + \frac{n}{2}\mathcal{Z}_g\right)}^{\circ}\mathring{\mathrm{Ric}}_g + \frac{1}{2}|\mathring{\mathrm{Ric}}_g|^2 g$$

and substituting in identity (3.10) we find identity (3.9), which is the thesis. □

3.12 Proposition. *Let (M^4, g) be a complete Riemannian manifold with positive Yamabe constant and constant scalar curvature, and take $\lambda \in [0,1]$. Suppose that g is critical for $\mathcal{F}^\lambda$ and that the following integral pinching holds:*

$$\|\mathcal{W}_g\|^2_{L^2} + \frac{1}{2}\|\mathcal{Z}_g\|^2_{L^2} < \frac{1}{8\times 24}Y(M,[g])^2.$$

Then we have $\mathcal{Z}_g = \mathring{\mathrm{Ric}}_g = 0$.

Proof. Since $\nabla\mathcal{F}^\lambda(g) = 0$ and using Lemma 3.11 we get

$$\begin{aligned} 0 &= \langle \nabla\mathcal{F}^\lambda(g)|\mathring{\mathrm{Ric}}_g\rangle \\ &= \frac{1}{2}\langle \nabla^*\nabla\mathring{\mathrm{Ric}}_g|\mathring{\mathrm{Ric}}_g\rangle - \langle \mathcal{W}_g + \frac{1}{2}\mathring{\mathrm{Ric}}_g \owedge g|\mathring{\mathrm{Ric}}_g \owedge \mathring{\mathrm{Ric}}_g\rangle + \frac{2-\lambda}{12}R_g|\mathring{\mathrm{Ric}}_g|^2, \end{aligned}$$

where the second equality follows from the trivial formula $\langle \mathring{T}u|v\rangle = \langle T|u \owedge v\rangle$. Thus, for each $\lambda \in [0,1]$,

$$\begin{aligned} \langle \nabla^*\nabla\mathring{\mathrm{Ric}}_g|\mathring{\mathrm{Ric}}_g\rangle + \frac{1}{6}R_g|\mathring{\mathrm{Ric}}_g|^2 &\leqslant 2\left\langle \mathcal{W}_g + \frac{1}{2}\mathring{\mathrm{Ric}}_g \owedge g \,\middle|\, \mathring{\mathrm{Ric}}_g \owedge \mathring{\mathrm{Ric}}_g \right\rangle \\ &\leqslant 2\cdot\left|\mathcal{W}_g + \frac{1}{2}\mathring{\mathrm{Ric}}_g \owedge g\right| \cdot \left|\mathring{\mathrm{Ric}}_g \owedge \mathring{\mathrm{Ric}}_g\right| \\ &= 2\cdot\left(|\mathcal{W}_g|^2 + \frac{1}{4}|\mathring{\mathrm{Ric}}|^2\right)^{\frac{1}{2}} \cdot \left|\mathring{\mathrm{Ric}}_g \owedge \mathring{\mathrm{Ric}}_g\right| \\ &\leqslant 2\cdot\left(|\mathcal{W}_g|^2 + \frac{1}{4}|\mathring{\mathrm{Ric}}|^2\right)^{\frac{1}{2}} \cdot \frac{2}{\sqrt{3}}|\mathring{\mathrm{Ric}}_g|^2. \end{aligned}$$

So we can apply Theorem 1.12. We have to put $\mu = \frac{1}{6}$, which gives $\varepsilon = 0$, so Kato inequality is automatically satisfied (and the volume growth requirement in the complete case is void). The pinching condition gives the inequality for the function a (and, in particular, it guarantees that the equality case cannot hold). Thus $\mathring{\mathrm{Ric}}_g = \mathcal{Z}_g = 0$. □

We have now to prove that the Weyl tensor too vanishes using the pinching hypotheses we are requiring. The partial result $\mathring{\mathrm{Ric}} = 0$, together with the constant scalar curvature, helps us: the formula $\delta\mathcal{W}_g = -\frac{1}{2}\mathrm{D}\mathring{\mathrm{Ric}}_g$ (see Lemma 1.1) implies that the Weyl tensor is harmonic, i.e., has zero divergence. This will be used in the following.

Before going on, we have to discuss an improvement of the decomposition (1.2) when the manifold is 4 dimensional and orientable. A proof of the facts that follow can be found in [Bes87, Section 1.H]. Fixed an orientation of (M^n, g), consider the associated volume form $d\omega_g$. We can then define the *Hodge operator* $*\colon \Lambda^p T^*M \to \Lambda^{n-p}T^*M$, which verifies the identity

$$\beta \wedge (*\alpha) = \langle \alpha|\beta\rangle\, d\omega_g \qquad \forall \alpha, \beta \in \Lambda^p T^*M.$$

When $n = 4$ and $p = n - p = 2$ and choosing a point $x \in M$, the Hodge operator is an autoadjoint involutive automorphism of the 6–dimensional space $\Lambda^2 T_x^* M$. Thus $\Lambda^2 T_x^* M$ splits orthogonally as the sum of $\Lambda_+^2 T_x^* M$ and $\Lambda_-^2 T_x^* M$, respectively the two eigenspaces associated to the eigenvalues 1 and -1. They are both of dimension 3. The Weyl tensor, seen as an endomorphism of $\Lambda^2 T^* M$, leaves invariant these two subspaces, so may be decomposed as $\mathcal{W}_g = \mathcal{W}_g^+ + \mathcal{W}_g^-$, where $\mathcal{W}_g^\pm \in \operatorname{End}(\Lambda_\pm^2 T^* M)$. Thus we can rewrite equation (1.2) in this form

$$\mathrm{Riem}_g = \mathcal{S}_g + \mathcal{Z}_g + \mathcal{W}_g^+ + \mathcal{W}_g^- .$$

Of course, since the decomposition is orthogonal, $|\mathcal{W}_g|^2 = |\mathcal{W}_g^+|^2 + |\mathcal{W}_g^-|^2$.

3.13 Proposition. *Let (M^4, g) be a complete and oriented Riemannian manifold with positive Yamabe constant, constant scalar curvature and $\mathring{\mathrm{Ric}} = 0$, and take $\lambda \in [0, 1]$. Suppose that g is critical for $\mathcal{F}^\lambda$ and the following integral pinching holds:*

$$\|\mathcal{W}_g^+\|_{L^2}^2 < \frac{1}{24} Y(M, [g])^2 .$$

Then we have $\mathcal{W}_g^+ = 0$.

Proof. The hypothesis $\mathring{\mathrm{Ric}} = 0$ implies (as already discussed above) $\delta \mathcal{W}_g = 0$, which in turn implies $\delta \mathcal{W}_g^+ = 0$, since we have the decomposition $|\delta \mathcal{W}_g|^2 = |\delta \mathcal{W}_g^+|^2 + |\delta \mathcal{W}_g^-|^2$ (equation (14) in [Der83]). The argument in the proof of Lemma 4 of [GL99] shows such assumption the following refined Kato inequality holds:

$$\frac{5}{3} |d|\mathcal{W}_g^+||^2 \leqslant |\nabla \mathcal{W}_g^+|^2$$

(which is equation (11) in the cited paper).

In order to use the Bochner technique, we have to work out an estimate of type (1.5) for $\mathcal{W}_g^+$. Number 16.73 in [Bes87] gives the following Weitzenböck formula:

$$\frac{1}{2} \Delta |\mathcal{W}_g^+|^2 + |\nabla \mathcal{W}_g^+|^2 = -\frac{1}{2} R_g |\mathcal{W}_g^+|^2 + 18 \det \mathcal{W}_g^+ ,$$

where the determinant is to be computed considering $\mathcal{W}_g^+$ as a 3×3 matrix acting on $\Lambda_+^2 T^* M$ as discussed above.

We have to estimate the determinant of $\mathcal{W}_g^+$, which is a symmetric and traceless. Putting it in diagonal form with eigenvalues $\lambda_{1,2,3}$, we have that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = |\mathcal{W}_g^+|^2 .$$

One easily shows, using standard constrainted optimization theory, that

$$\det \mathcal{W}_g^+ = \lambda_1 \lambda_2 \lambda_3 \leqslant \frac{1}{3\sqrt{6}} |\mathcal{W}_g^+|^3 .$$

Putting this estimate in the Weitzenböck we have:

$$\langle \nabla^* \nabla \mathcal{W}_g^+ | \mathcal{W}_g^+ \rangle = \frac{1}{2} \Delta |\mathcal{W}_g^+|^2 + |\nabla \mathcal{W}_g^+|^2 \leqslant -\frac{1}{2} R_g |\mathcal{W}_g^+|^2 + \sqrt{6} |\mathcal{W}_g^+|^3 ,$$

so we are going to use Theorem 1.12 with $\varepsilon = \frac{2}{3}$, $\mu = \frac{1}{2}$ and $a = \sqrt{6}|\mathcal{W}_g^+|$. The only missing detail is to show the assumption on the growth of the balls what is required in the complete case. This is easily set using the Bishop–Gromov inequality (Theorem 1.13): since $\mathring{\mathrm{Ric}} = 0$, we have that $\mathrm{Ric} = \frac{1}{n} Rg \geqslant 0$, so the volume growth is at most quadratic. □

3.14 Remark. We notice that reversing the orientation put on M^4 changes to Hodge operator to its opposite, thus swapping $\mathcal{W}_g^+$ and $\mathcal{W}_g^-$. Thus, reading the proof in the mirror, Proposition 3.13 is also true when substituting all "$\mathcal{W}_g^+$" with "$\mathcal{W}_g^-$".

At last, we have all the tools to prove the theorem.

Proof of Theorem 3.10. If $\lambda \neq 0$ and g is a critical point for $\mathcal{F}^\lambda$, then Lemma 3.3 implies that the scalar curvature is harmonic. If M is compact, then R_g is a positive constant (since the Yamabe constant is positive). If M is not compact, then R_g must be a non–negative constant, because it is harmonic and L^2 (see Theorem 3 in [Yau76]). So, in any case, the scalar curvature is constant. From Proposition 3.12 we then know that $\mathring{\mathrm{Ric}} = 0$.

It follows that the curvature (and in particular the Weyl tensor) is harmonic, so we may also apply Proposition 3.13 and have $\mathcal{W}_g^+ = 0$, perhaps after passing to a two-fold covering if M^4 is not orientable. As noted in remark 3.14, the same argument applies to $\mathcal{W}_g^-$, so we actually see that $\mathcal{W}_g = 0$.

Hence, looking at the decomposition of the Riemann tensor, the only surviving component is the scalar one, which is constant. So M^4 has constant sectional curvature. □

3.5 Finishing the blow–up analysis

We will now extend the analysis carried in Section 2.6 when the flow is of the type described above. First, we will see that, provided that the initial energy is not too high, the blow–up of the Riemann tensor is not possible, so the flow must exist for all times with bounded derivatives. Then we check that Theorem 3.10 actually apply to our case.

Let $(M, g_i(t), x_i) \to (M_\infty, g_\infty(t), x_\infty)$ be the pointed $\mathcal{C}^\infty$ limit resulting from the blow–up of the flow (3.1) near a singular time $0 < T \leqslant \infty$. We know that the Yamabe constant is bounded from below by a positive constant thanks to Corollary 3.7.

If the limit manifold M_∞ was compact, by the definition of pointed $\mathcal{C}^\infty$ convergence, it would without boundary and thus diffeomorphic to M. But then then metrics $g_i(t)$ should converge to $g_\infty(t)$, which is impossible since from Corollary 3.4 we know that $\mathrm{Vol}_{g(t)}(M)$ is constant and

$$\mathrm{Vol}_{g_i(t)}(M) = \lambda_i^{\frac{n}{2}} \mathrm{Vol}_{g(t_i + t/\lambda_i^2)}(M)$$

tends to infinity.

Thanks to Proposition 1.8, the limit metric g_∞ has positive Yamabe constant. Moreover, it has Riemannian curvature bounded by 1. Then Proposition 1.9 and Lemma 1.7 imply that it has infinite volume.

3.15 Lemma. *The limit flow $g_\infty(t)$ is constant; i.e., $g_\infty(t)$ is critical for the functional $\mathcal{F}^\lambda$. If $T = \infty$, the lemma is valid also without the hypothesis that $\lambda_i \to \infty$, provided that they're bounded from below by a positive constant.*

Proof. Using equation (3.2) we see that for $0 \leqslant t < T$

$$\int_0^t \|\nabla \mathcal{F}^\lambda(g(s))\|_{L^2}^2 \, ds = \frac{1}{2}(\mathcal{F}^\lambda(g_0) - \mathcal{F}^\lambda(g(t))),$$

so

$$\int_0^T \|\nabla \mathcal{F}^\lambda(g(s))\|_{L^2}^2 \, ds \leqslant \frac{1}{2}\mathcal{F}^\lambda(g_0) < \infty.$$

With a change of variables we have, for any $\alpha > 0$,

$$\int_{-\alpha}^0 \|\nabla \mathcal{F}^\lambda(g_i(s))\|_{L^2}^2 \, ds = \int_{t_i - \frac{\alpha}{\lambda_i^2}}^{t_i} \|\nabla \mathcal{F}^\lambda(g(s))\|_{L^2}^2 \, ds \longrightarrow 0$$

for $t_i \to T$ and $\lambda_i \to \infty$. The second hypothesis is not required when $T = \infty$ if the λ_i are bounded from below.

Applying Fatou's lemma for $i \to \infty$ we have that $\|\nabla \mathcal{F}^\lambda(g_\infty(t))\|_{L^2}^2$ is zero almost everywhere, so it is actually zero. □

Since $\nabla \mathcal{F}^\lambda(g_\infty(0)) = 0$, it follows from Lemma 3.3 that the scalar curvature of g_∞ is harmonic. But from Lemma 3.8, using $\mathcal{F}_\mathcal{S}$ scale invariance in dimension 4, and Fatou's lemma we know that it has bounded L^2 norm. Thus it must be constant, by Theorem 3 in [Yau76]. Since g_∞ has infinite volume, it must then be zero. Thus, the limit manifold is scalar–flat.

Then, comparing with formulae (3.5), we have that $\nabla \mathcal{F}^\lambda(g_\infty(0)) = 0$ degenerates to $\nabla \mathcal{F}_W(g_\infty(0))$, which means that the limit manifold is also Bach–flat.

3.16 Theorem. *Let (M^4, g_0) be a closed Riemannian manifold with positive Yamabe constant and $\lambda \in (0,1)$ such that*

$$\begin{cases} \lambda & \leqslant \frac{4}{13} \\ \mathcal{F}^\lambda(g_0) & < 2\lambda\pi^2\chi(M) \end{cases} \quad \text{or} \quad \begin{cases} \lambda & \geqslant \frac{4}{13} \\ \mathcal{F}^\lambda(g_0) & < \frac{8}{9}(1-\lambda)\pi^2\chi(M). \end{cases} \tag{3.11}$$

Then the flow (3.1) for the functional $\mathcal{F}^\lambda$ in (3.3) exists for all non-negative times and converges in the $\mathcal{C}^\infty$ topology to a metric of constant positive curvature.

In particular, M^4 is diffeomorphic to the sphere S^4 or to the real projective plane $\mathbb{PR}^4$.

Proof. The flow exists for at least a small time because of Theorem 3.2.

Suppose that the flow has a singularity for $0 < T \leqslant \infty$: we'll obtain an absurd from this, proving that the flow is actually defined for all non-negative times and has all derivatives bounded on $[0, \infty)$.

Since $\mathcal{F}^\lambda$ is decreasing along the flow, the condition (3.8) holds for all the times where the flow is defined. Such condition is scale–invariant, so it is satisfied also by the rescaled manifolds intervening in Section 2.6. Thus, by Lemma 3.9, for all rescaled metrics and all times t where they're defined we have

$$\mathcal{F}_W(g_i(t)) + \frac{1}{2}\mathcal{F}_\mathcal{Z}(g_i(t)) \leqslant \frac{1}{8 \times 24} Y(M, [g_i(t)])^2 - \varepsilon.$$

Now, the left–hand side is lower semicontinuous with respect to the pointed $\mathcal{C}^\infty$ convergence (thanks to Fatou's lemma), while the right–hand side is upper semicontinuous (see Lemma 1.8). So the same inequality passes to the limit manifold (M_∞, g_∞). We also recall that we obtained earlier in this section that the limit metric is Bach–flat and scalar–flat.

Then we apply Theorem 3.10 with $\lambda = 0$ (exploiting the Bach–flatness), which means that M_∞ has constant sectional curvature, so it must be completely flat. But this is not possible, because at least on point x_∞ we have $|\mathrm{Riem}_{g_\infty}| = 1$: as discussed above, this contradiction shows that the flow is defined on $[0, \infty)$ and it has all derivatives bounded on such domain.

On the other hand, let us see what happens near $+\infty$ when the flow has no singularities. The analysis of Section 2.6 and the first part of this section is still valid, with the following differences: since the Riemann tensor doesn't blow up, we just take $t_i \nearrow \infty$ and $\lambda_i = 1$. From the definition of pointed $\mathcal{C}^\infty$ convergence we easily obtain that the limit manifold M_∞ has finite volume, so it is compact. Thus, it is diffeomorphic to M. As before, it is a critical point for $\mathcal{F}^\lambda$, although differently from before it's not scalar–flat and Bach–flat (since those results required infinite volume).

At last we apply Theorem 3.10 again, proving that the limit metric has constant sectional curvature. The curvature must be positive because the Yamabe constant is positive and the manifold is compact. So the proof is complete. □

This last corollary is nothing more than the interpretation of Theorem 3.16 in a wholly metric way.

3.17 Corollary. *Let (M^4, g) be a closed Riemannian manifold with positive Yamabe constant with the following integral pinching on the curvature:*

$$\|\mathcal{W}_g\|^2_{L^2} + \frac{5}{8}\|\mathcal{Z}_g\|^2_{L^2} < \frac{1}{8}\|\mathcal{S}_g\|^2_{L^2}. \tag{3.12}$$

Then it is diffeomorphic to the sphere S^4 or to the real projective space $\mathbb{PR}^4$.

Proof. We apply Theorem 3.16 with $\lambda = \frac{4}{13}$. The pinching condition is

$$\frac{9}{13}\mathcal{F}_{\mathcal{W}}(g) + \frac{4}{13}\mathcal{F}_{\mathcal{Z}}(g) < \frac{8}{13}\pi^2\chi(M) = \frac{1}{13}(\mathcal{F}_{\mathcal{S}}(g) - \mathcal{F}_{\mathcal{Z}}(g) + \mathcal{F}_{\mathcal{W}}(g)),$$

using Chern–Gauss–Bonnet formula (Theorem 3.5). □

Bibliography

[Aub98] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer–Verlag, 1998.

[Bes87] A. L. Besse, *Einstein manifolds*, Springer–Verlag, Berlin, 1987.

[Bou10] V. Bour, *Fourth order curvature flows and geometric applications*, ArXiv Preprint Server – http://arxiv.org, 2010.

[Bou12] ______, *Flots géométriques d'ordre quatre et pincement intégral de la courbure*, Ph.D. thesis, Université Joseph Fourier – Grenoble, 2012, http://tel.archives-ouvertes.fr/docs/00/77/17/20/PDF/Bour.pdf.

[BS09] S. Brendle and R. Schoen, *Manifolds with* $1/4$*–pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), 287–307.

[CK04] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004.

[Der83] A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), no. 3, 405–433.

[DeT83] D. M. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Diff. Geom. **18** (1983), no. 1, 157–162.

[GHL90] S. Gallot, D. Hulin, and J. Lafontaine, *Riemannian geometry*, Springer–Verlag, 1990.

[GL99] M. J. Gursky and C. Lebrun, *On Einstein manifolds of positive sectional curvature*, Ann. Global Anal. Geom. **17** (1999), no. 4, 315–328.

[Ham82] R. S. Hamilton, *Three–manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), no. 2, 255–306.

[Ham95] ______, *A compactness property for solutions of the Ricci flow*, Amer. J. Math. **117** (1995), no. 3, 545–572.

[Heb96] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics, vol. 1635, Springer–Verlag, Berlin, 1996.

[HP99] G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces*, Calculus of variations and geometric evolution problems (Cetraro, 1996), Springer–Verlag, Berlin, 1999, pp. 45–84.

[Man02] C. Mantegazza, *Smooth geometric evolutions of hypersurfaces*, Geom. Funct. Anal. **12** (2002), no. 1, 138–182.

[MM12] C. Mantegazza and L. Martinazzi, *A note on quasilinear parabolic equations on manifolds*, Ann. Sc. Norm. Sup. Pisa **11 (5)** (2012), 857–874.

[Per02] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv Preprint Server – http://arxiv.org, 2002.

[Per03a] ______, *Finite extinction time for the solutions to the Ricci flow on certain three–manifolds*, ArXiv Preprint Server – http://arxiv.org, 2003.

[Per03b] ______, *Ricci flow with surgery on three–manifolds*, ArXiv Preprint Server – http://arxiv.org, 2003.

[Pet98] P. Petersen, *Riemannian geometry*, Springer–Verlag, 1998.

[Pol96] A. Polden, *Curves and Surfaces of Least Total Curvature and Fourth–Order Flows*, Ph.D. thesis, Mathematisches Institut, Univ. Tübingen, 1996, Arbeitsbereich Analysis Preprint Server – Univ. Tübingen, http://poincare.mathematik.uni-tuebingen.de/mozilla/home.e.html.

[Sha04] J. J. Sharples, *Linear and quasilinear parabolic equations in Sobolev space*, J. Diff. Eqs. **202** (2004), no. 1, 111–142.

[Spi99a] M. Spivak, *A comprehensive introduction to differential geometry. Vol. IV*, third ed., Publish or Perish Inc., Houston, Texas, 1999.

[Spi99b] ______, *A comprehensive introduction to differential geometry. Vol. V*, third ed., Publish or Perish Inc., Houston, Texas, 1999.

[Yau76] S.-T. Yau, *Some function–theoretic properties of complete Riemannian manifold and their applications to geometry*, Indiana Univ. Math. J. **25** (1976), no. 7, 659–670.