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The constant term of the minimal polynomial of $\cos(2\pi/n)$ over $\mathbb{Q}$

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Abstract
Let $H(\lambda_q)$ be the Hecke group associated to $\lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ integer. In this paper, we determine the constant term of the minimal polynomial of λ_q denoted by $P_q^*(x)$.
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1 Introduction

The Hecke groups $H(\lambda)$ are defined to be the maximal discrete subgroups of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number.

Hecke [1] showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ is an integer, or $\lambda \geq 2$. In this paper, we only consider the former case and denote the corresponding Hecke groups by $H(\lambda_q)$. It is well known that $H(\lambda_q)$ has a presentation as follows (see [2]):

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle. \tag{1}$$

These groups are isomorphic to the free product of two finite cyclic groups of orders 2 and q.

The first few Hecke groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear from the above that $H(\lambda_q) \subset PSL(2,\mathbb{Z}[\lambda_q])$, but unlike in the modular group case (the case $q = 3$), the inclusion is strict and the index $[PSL(2,\mathbb{Z}[\lambda_q]) : H(\lambda_q)]$ is infinite as $H(\lambda_q)$ is discrete, whereas $PSL(2,\mathbb{Z}[\lambda_q])$ is not for $q \geq 4$.

On the other hand, it is well known that ζ, a primitive nth root of unity, satisfies the equation

$$x^n - 1 = 0. \tag{2}$$

In [3], Cangul studied the minimal polynomials of the real part of ζ, *i.e.*, of $\cos(2\pi/n)$ over the rationals. He used a paper of Watkins and Zeitlin [4] to produce further results.

Also, he made use of two classes of polynomials called Chebycheff and Dickson polynomials. It is known that for $n \in \mathbb{N} \cup \{0\}$, the nth Chebycheff polynomial, denoted by $T_n(x)$, is defined by

$$T_n(x) = \cos(n \cdot \arccos x), \quad x \in \mathbb{R}, |x| \leq 1, \tag{3}$$

or

$$T_n(\cos\theta) = \cos n\theta, \quad \theta \in \mathbb{R}\ (\theta = \arccos x + 2k\pi, k \in \mathbb{Z}). \tag{4}$$

Here we use Chebycheff polynomials.

For $n \in \mathbb{N}$, Cangul denoted the minimal polynomial of $\cos(2\pi/n)$ over Q by $\Psi_n(x)$. Then he obtained the following formula for the minimal polynomial $\Psi_n(x)$.

Theorem 1 ([3, Theorem 1]) *Let $m \in \mathbb{N}$ and $n = [|m/2|]$. Then*

(a) *If $m = 1$, then $\Psi_1(x) = x - 1$, and if $m = 2$, then $\Psi_2(x) = x + 1$.*

(b) *If m is an odd prime, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^n(x-1)}. \tag{5}$$

(c) *If $4 \mid m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n/2}(T_{\frac{n}{2}+1}(x) - T_{\frac{n}{2}-1}(x)) \prod_{d|m, d\neq m, d|\frac{m}{2}}^{q-1} \Psi_d(x)}. \tag{6}$$

(d) *If m is even and $m/2$ is odd, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x)) \prod_{d|m, d\neq m, d\ even}^{q-1} \Psi_d(x)}, \tag{7}$$

where $n' = \frac{\frac{m}{2}-1}{2}$.

(e) *Let m be odd and let p be a prime dividing m. If $p^2 \mid m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x))}, \tag{8}$$

where $n' = \frac{\frac{m}{p}-1}{2}$. If $p^2 \mid m$, then

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x))\Psi_p(x)}, \tag{9}$$

where $n' = \frac{\frac{m}{p}-1}{2}$.

For the first four Hecke groups Γ, $H(\sqrt{2})$, $H(\lambda_5)$, and $H(\sqrt{3})$, we can find the minimal polynomial, denoted by $P_q^*(x)$, of λ_q over Qas $\lambda_3 - 1$, $\lambda_4^2 - 2$, $\lambda_5^2 - \lambda_5 - 1$, and $\lambda_6^2 - 3$, respectively. However, for $q \geq 7$, the algebraic number $\lambda_q = 2\cos\frac{\pi}{q}$ is a root of a minimal polynomial of degree ≥ 3. Therefore, it is not possible to determine λ_q for $q \geq 7$ as nicely as in the first four cases. Because of this, it is easy to find and study with the minimal polynomial of λ_q instead of λ_q itself. The minimal polynomial of λ_q has been used for many aspects in the literature (see [5–8] and [9]).

Notice that there is a relation

$$P_q^*(x) = 2^{\varphi(2q)/2} \cdot \Psi_{2q}\left(\frac{x}{2}\right)$$

between $P_q^*(x)$ and $\Psi_m(x)$.

In [10], when the principal congruence subgroups of $H(\lambda_q)$ for $q \geq 7$ prime were studied, we needed to know whether the minimal polynomial of λ_q is congruent to 0 *modulo p* for prime p and also the constant term of it *modulo p*.

In this paper, we determine the constant term of the minimal polynomial $P_q^*(x)$ of λ_q. We deal with odd and even q cases separately. Of course, this problem is easier to solve when q is odd.

2 The constant term of $P_q^*(x)$

In this section, we calculate the constant term for all values of q. Let c denote the constant term of the minimal polynomial $P_q^*(x)$ of λ_q, *i.e.*,

$$c = P_q^*(0). \tag{10}$$

We know from [4, Lemma, p.473] that the roots of $P_q^*(x)$ are $2\cos\frac{h\pi}{q}$ with $(h,q) = 1$, h odd and $1 \leq h \leq q-1$. Being the constant term, c is equal to the product of all roots of $P_q^*(x)$:

$$c = \prod_{\substack{h=1 \\ (h,q)=1 \\ h \text{ odd}}}^{q-1} 2\cos\frac{h\pi}{q}. \tag{11}$$

Therefore we need to calculate the product on the right-hand side of (11). To do this, we need the following result given in [11].

Lemma 1 $\prod_{h=0}^{q-1} 2\sin(\frac{h\pi}{q} + \theta) = 2\sin q\theta$.

We now want to obtain a similar formula for cosine. Replacing θ by $\frac{\pi}{2} - \theta$, we get

$$\prod_{h=0}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = 2\sin q\left(\frac{\pi}{2} - \theta\right). \tag{12}$$

Let now μ denote the Möbius function defined by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square-free,} \\ 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors,} \end{cases} \tag{13}$$

for $n \in \mathbb{N}$. It is known that

$$\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases} \tag{14}$$

Using this last fact, we obtain

$$\begin{aligned}
&\ln \prod_{h=0,(h,q)=1}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) \\
&= \sum_{h=0}^{q-1} \ln\left(2\cos\left(\frac{h\pi}{q} - \theta\right)\right) \sum_{d|(h,q)} \mu(d) \\
&= \sum_{d|q} \mu(d) \sum_{k=0}^{\frac{q}{d}-1} \ln\left(2\cos\left(\frac{kd\pi}{q} - \theta\right)\right) \\
&= \sum_{d|q} \mu(d) \left(\ln \prod_{k=0}^{\frac{q}{d}-1} 2\cos\left(\frac{kd\pi}{q} - \theta\right)\right) \\
&= \sum_{d|q} \mu(d) \cdot \left(\ln 2\sin\frac{q}{d}\left(\frac{\pi}{2} - \theta\right)\right) \quad \text{by (12)} \\
&= \ln \prod_{d|q} \sin d\left(\frac{\pi}{2} - \theta\right)^{\mu(q/d)}.
\end{aligned} \tag{15}$$

Therefore

$$\prod_{\substack{h=0 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = \prod_{d|q} \left(\sin d\left(\frac{\pi}{2} - \theta\right)\right)^{\mu(q/d)}. \tag{16}$$

Finally, as $(0,q) \neq 1$, we can write (16) as

$$\prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = \prod_{d|q} \left(\sin d\left(\frac{\pi}{2} - \theta\right)\right)^{\mu(q/d)}. \tag{17}$$

Note that if q is even, then

$$\prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q}\right) = \prod_{\substack{h=1 \\ (h,q)=1 \\ h \text{ odd}}}^{q-1} 2\cos\frac{h\pi}{q} = c, \tag{18}$$

while if q is odd, then

$$\left|\prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q}\right)\right| = c^2, \tag{19}$$

as $\cos(h-i)\frac{\pi}{q} = -\cos\frac{i\pi}{q}$. Also note that

$$\sin d\left(\frac{\pi}{2}-\theta\right) = \begin{cases} \cos d\theta & \text{if } d \equiv 1 \bmod 4, \\ \sin d\theta & \text{if } d \equiv 2 \bmod 4, \\ -\cos d\theta & \text{if } d \equiv 3 \bmod 4, \\ -\sin d\theta & \text{if } d \equiv 0 \bmod 4. \end{cases} \tag{20}$$

To compute c, we let $\theta \to 0$ in (17). If d is odd, then $\sin d(\frac{\pi}{2}-\theta) \to \pm 1$ as $\theta \to 0$ by (20). So, we are only concerned with even d. Indeed, if q is odd, then the left-hand side at $\theta = 0$ is equal to ± 1. Therefore we have the following result.

Theorem 2 *Let q be odd. Then*

$$|c| = 1. \tag{21}$$

Proof It follows from (19) and (20). □

Let us now investigate the case of even q. As $(h,q) = 1$, h must be odd. So, by a similar discussion, we get the following.

Theorem 3 *Let q be even. Then*

$$c = \lim_{\theta\to 0}\prod_{d|q}\left(\sin d\left(\frac{\pi}{2}-\theta\right)\right)^{\mu(q/d)}. \tag{22}$$

Proof Note that by (20), the right-hand side of (22) becomes a product of $\pm(\cos d\theta)^{\pm 1}$'s and $\pm(\sin d\theta)^{\pm 1}$'s. Above we saw that we can omit the former ones as they tend to ± 1 as θ tends to 0. Now, as $\sum_{d|n}\mu(d) = 0$, there are equal numbers of the latter kind factors in the numerator and denominator, *i.e.*, if there is a factor $\sin d\theta$ in the numerator, then there is a factor $\sin d'\theta$ in the denominator. Then using the fact that

$$\lim_{\theta\to 0}\frac{\sin k\theta}{\sin l\theta} = \frac{k}{l}, \tag{23}$$

we can calculate c.

In fact the calculations show that there are three possibilities:

(i) Let $q = 2^{\alpha_0}$, $\alpha_0 \geq 2$. Then the only divisors of q such that $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0}$ and $2^{\alpha_0 - 1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin 2^{\alpha_0}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1}(\frac{\pi}{2}-\theta)} \\ &= \begin{cases} 2 & \text{if } \alpha_0 > 2, \\ -2 & \text{if } \alpha_0 = 2. \end{cases} \end{aligned} \tag{24}$$

(ii) Secondly, let $q = 2p^{\alpha}$, $\alpha \geq 1$, p odd prime. Then the only divisors of q such that $\mu(q/d) \neq 0$ are $d = 2p^{\alpha}, 2p^{\alpha-1}, p^{\alpha}$ and $p^{\alpha-1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2p^{\alpha}(\frac{\pi}{2}-\theta)\cdot \sin p^{\alpha-1}(\frac{\pi}{2}-\theta)}{\sin p^{\alpha}(\frac{\pi}{2}-\theta)\cdot \sin 2p^{\alpha-1}(\frac{\pi}{2}-\theta)} \\ &= \lim_{\theta\to 0} \epsilon \cdot \frac{\sin 2p^{\alpha}\theta \cdot \cos p^{\alpha-1}\theta}{\cos p^{\alpha}\theta \cdot \sin 2p^{\alpha-1}\theta} \\ &= \epsilon \cdot p, \end{aligned} \tag{25}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ -1 & \text{if } p \equiv -1 \bmod 4. \end{cases} \tag{26}$$

(iii) Let q be different from above. Then q can be written as

$$q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \tag{27}$$

where p_i are distinct odd primes and $\alpha_i \geq 1$, $0 \leq i \leq k$.

Here we consider the first two cases $k = 1$ and $k = 2$.

Let $k = 1$, *i.e.*, let $q = 2^{\alpha_0} p_1^{\alpha_1}$. We have already discussed the case $\alpha_0 = 1$. Let $\alpha_0 > 1$. Then the only divisors d of q with $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0} p_1^{\alpha_1}, 2^{\alpha_0-1} p_1^{\alpha_1}, 2^{\alpha_0} p_1^{\alpha_1-1}$ and $2^{\alpha_0-1} p_1^{\alpha_1-1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0} p_1^{\alpha_1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1-1}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1} p_1^{\alpha_1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1-1}(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned} \tag{28}$$

Now let $k = 2$, *i.e.*, let $q = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$, $(p_1 < p_2)$. Similarly, all divisors d of q such that $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$, $2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2}$, $2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2}$, $2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2-1}$, $2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2-1}$, $2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1}$, $2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2-1}$ and $2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)} \\ &\quad \times \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned} \tag{29}$$

Finally, $k \geq 3$, i.e., let

$$q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k} \quad \text{with } p_1 < p_2 < \cdots < p_k.$$

In this case the proof is similar, but rather more complicated. In fact, the number of all divisors d of q such that $\mu(q/d) \neq 0$ is 2^{k+1}. There is $\binom{k+1}{0} = 1$ divisor of the form

$$d = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}.$$

There are $\binom{k+1}{1} = k+1$ divisors of the form

$$d = 2^{\alpha_0-1}p_1^{\alpha_1}\cdots p_k^{\alpha_k}, 2^{\alpha_0}p_1^{\alpha_1-1}\cdots p_k^{\alpha_k}, \ldots, 2^{\alpha_0}p_1^{\alpha_1}\cdots p_k^{\alpha_k-1}.$$

There are $\binom{k+1}{2} = \frac{k(k+1)}{2}$ divisors of the form

$$\begin{aligned} d &= 2^{\alpha_0-1}p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, 2^{\alpha_0-1}p_1^{\alpha_1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k}, \ldots, 2^{\alpha_0-1}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k-1}, \\ &\quad 2^{\alpha_0}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k}, \ldots, 2^{\alpha_0}p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_k^{\alpha_k-1}, \ldots, 2^{\alpha_0}p_1^{\alpha_1}\cdots p_{k-1}^{\alpha_{k-1}-1}p_k^{\alpha_k-1}. \end{aligned}$$

If we continue, we can find other divisors d of q, similarly. Finally, there is $\binom{k+1}{k+1} = 1$ divisor of the form $2^{\alpha_0-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}$. Thus, the product of all coefficients d in the factors $\sin d(\frac{\pi}{2}-\theta)$ in the numerator is equal to the product of all coefficients e in the factors $\sin e(\frac{\pi}{2}-\theta)$ in the denominator implying $c = 1$. Therefore the proof is completed. □

Now we give an example for all possible even q cases.

Example 1 (i) Let $q = 8 = 2^3$. The only divisors of 8 such that $\mu(8/d) \neq 0$ are $d = 8$ and 4. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin 8(\frac{\pi}{2}-\theta)}{\sin 4(\frac{\pi}{2}-\theta)} \\ &= 2. \end{aligned}$$

(ii) Let $q = 14 = 2\cdot 7$. The only divisors of 14 such that $\mu(14/d) \neq 0$ are $d = 14, 2, 7$ and 1. Therefore

$$\begin{aligned} c &= \epsilon\cdot\lim_{\theta\to 0}\frac{\sin 14(\frac{\pi}{2}-\theta)\cdot\sin(\frac{\pi}{2}-\theta)}{\sin 7(\frac{\pi}{2}-\theta)\cdot\sin 2(\frac{\pi}{2}-\theta)} \\ &= -7, \end{aligned}$$

since $p \equiv -1 \bmod 4$.

(iii) Let $q = 24 = 2^3\cdot 3$. The only divisors of 24 such that $\mu(24/d) \neq 0$ are $d = 24, 12, 8$ and 4. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin 24(\frac{\pi}{2}-\theta)\cdot\sin 4(\frac{\pi}{2}-\theta)}{\sin 12(\frac{\pi}{2}-\theta)\cdot\sin 8(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned}$$

(iv) Let $q = 30 = 2\cdot 3\cdot 5$. The only divisors of 30 such that $\mu(30/d) \neq 0$ are $d = 30, 15, 10, 6, 5, 3, 2$ and 1. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin(\frac{\pi}{2}-\theta)\cdot\sin 6(\frac{\pi}{2}-\theta)\cdot\sin 10(\frac{\pi}{2}-\theta)\cdot\sin 15(\frac{\pi}{2}-\theta)}{\sin 2(\frac{\pi}{2}-\theta)\cdot\sin 3(\frac{\pi}{2}-\theta)\cdot\sin 5(\frac{\pi}{2}-\theta)\cdot\sin 30(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper alone and they read and approved the final manuscript.

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