Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 278606, 9 pages
http://dx.doi.org/10.1155/2013/278606

Research Article

On Kadison-Schwarz Type Quantum Quadratic Operators on $\mathbb{M}_2(\mathbb{C})$

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Received 3 January 2013; Revised 13 March 2013; Accepted 27 March 2013

Academic Editor: Natig M. Atakishiyev

We study the description of Kadison-Schwarz type quantum quadratic operators (q.q.o.) acting from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$. Note that such kind of operators is a generalization of quantum convolution. By means of such a description we provide an example of q.q.o. which is not a Kadison-Schwartz operator. Moreover, we study dynamics of an associated nonlinear (i.e., quadratic) operators acting on the state space of $\mathbb{M}_2(\mathbb{C})$.

1. Introduction

It is known that one of the main problems of quantum information is the characterization of positive and completely positive maps on C^*-algebras. There are many papers devoted to this problem (see, e.g., [1–4]). In the literature the completely positive maps have proved to be of great importance in the structure theory of C^*-algebras. However, general positive (order-preserving) linear maps are very intractable [2, 5]. It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called *Kadison-Schwarz property*, that is, a map ϕ satisfies the Kadison-Schwarz property if $\phi(a)^*\phi(a) \leq \phi(a^*a)$ holds for every a. Note that every unital completely positive map satisfies this inequality, and a famous result of Kadison states that any positive unital map satisfies the inequality for self-adjoint elements a. In [6] relations between n-positivity of a map ϕ and the Kadison-Schwarz property of certain map is established. Certain relations between complete positivity, positive, and the Kadison-Schwarz property have been considered in [7–9]. Some spectral and ergodic properties of Kadison-Schwarz maps were investigated in [10–12].

In [13] we have studied quantum quadratic operators (q.q.o.), that is, maps from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, with the Kadison-Schwarz property. Some necessary conditions for the trace-preserving quadratic operators are found to be the Kadison-Schwarz ones. Since trace-preserving maps arise naturally in quantum information theory (see, e.g., [14]) and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with which it interacts. Note that in [15, 16] quantum quadratic operators acting on a von Neumann algebra were defined and studied. Certain ergodic properties of such operators were studied in [17, 18] (see for review [19]). In the present paper we continue our investigation; that is, we are going to study further properties of q.q.o. with Kadison-Schwarz property. We will provide an example of q.q.o. which is not a Kadison-Schwarz operator and study its dynamics. We should stress that q.q.o. is a generalization of quantum convolution (see [20]). Some dynamical properties of quantum convolutions were investigated in [21].

Note that a description of bistochastic Kadison-Schwarz mappings from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C})$ has been provided in [22].

2. Preliminaries

In what follows, by $\mathbb{M}_2(\mathbb{C})$ we denote an algebra of 2×2 matrices over complex filed $\mathbb{C}$. By $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ we mean tensor product of $\mathbb{M}_2(\mathbb{C})$ into itself. We note that such a product can be considered as an algebra of 4×4 matrices $\mathbb{M}_4(\mathbb{C})$ over $\mathbb{C}$. In the sequel $\mathbb{1}$ means an identity matrix, that is, $\mathbb{1} = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. By $S(\mathbb{M}_2(\mathbb{C}))$ we denote the set of all states (i.e., linear positive functionals which take value 1 at $\mathbb{1}$) defined on $\mathbb{M}_2(\mathbb{C})$.

Definition 1. A linear operator $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ is said to be

(a) a *quantum quadratic operator (q.q.o.)* if it satisfies the following conditions:

(i) unital, that is, $\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$;

(ii) Δ is positive, that is, $\Delta x \geq 0$ whenever $x \geq 0$;

(b) a *Kadison-Schwarz operator (KS)* if it satisfies

$$\Delta\left(x^{*}x\right) \geq \Delta\left(x^{*}\right)\Delta(x), \quad \forall x \in \mathbb{M}_2(\mathbb{C}). \tag{1}$$

One can see that if Δ is unital and KS operator, then it is a q.q.o. A state $h \in S(\mathbb{M}_2(\mathbb{C}))$ is called *a Haar state* for a q.q.o. Δ if for every $x \in \mathbb{M}_2(\mathbb{C})$ one has

$$(h \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes h) \circ \Delta(x) = h(x)\mathbb{1}. \tag{2}$$

Remark 2. Note that if a quantum convolution Δ on $\mathbb{M}_2(\mathbb{C})$ becomes a $*$-homomorphic map with a condition

$$\begin{aligned}&\overline{\mathrm{Lin}}\left(\left(\mathbb{1} \otimes \mathbb{M}_2(\mathbb{C})\right)\Delta\left(\mathbb{M}_2(\mathbb{C})\right)\right)\\ &\quad= \overline{\mathrm{Lin}}\left(\left(\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{1}\right)\Delta\left(\mathbb{M}_2(\mathbb{C})\right)\right) = \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}),\end{aligned} \tag{3}$$

then a pair $(\mathbb{M}_2(\mathbb{C}), \Delta)$ is called a *compact quantum group* [20]. It is known [20] that for any given compact quantum group there exists a unique Haar state w.r.t. Δ.

Remark 3. Let $U : \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator such that $U(x\otimes y) = y\otimes x$ for all $x, y \in \mathbb{M}_2(\mathbb{C})$. If a q.q.o. Δ satisfies $U\Delta = \Delta$, then Δ is called a *quantum quadratic stochastic operator*. Such a kind of operators was studied and investigated in [17].

Each q.q.o. Δ defines a conjugate operator $\Delta^{*} : (\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}))^{*} \to \mathbb{M}_2(\mathbb{C})^{*}$ by

$$\Delta^{*}(f)(x) = f(\Delta x), \qquad f \in \left(\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})\right)^{*}, \quad x \in \mathbb{M}_2(\mathbb{C}). \tag{4}$$

One can define an operator V_Δ by

$$V_\Delta(\varphi) = \Delta^{*}(\varphi \otimes \varphi), \quad \varphi \in S\left(\mathbb{M}_2(\mathbb{C})\right), \tag{5}$$

which is called a *quadratic operator (q.c.)*. Thanks to conditions (a) (i), (ii) of Definition 1 the operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ to $S(\mathbb{M}_2(\mathbb{C}))$.

3. Quantum Quadratic Operators with Kadison-Schwarz Property on $\mathbb{M}_2(\mathbb{C})$

In this section we are going to describe quantum quadratic operators on $\mathbb{M}_2(\mathbb{C})$ and find necessary conditions for such operators to satisfy the Kadison-Schwarz property.

Recall [23] that the identity and Pauli matrices $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{6}$$

In this basis every matrix $x \in \mathbb{M}_2(\mathbb{C})$ can be written as $x = w_0\mathbb{1} + \mathbf{w}\sigma$ with $w_0 \in \mathbb{C}$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here $\mathbf{w}\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3$.

Lemma 4 (see [3]). *The following assertions hold true:*

(a) *x is self-adjoint if and only if w_0, $\mathbf{w}$ are reals;*

(b) $\mathrm{Tr}(x) = 1$ *if and only if $w_0 = 0.5$; here* Tr *is the trace of a matrix x;*

(c) *$x > 0$ if and only if $\|\mathbf{w}\| \leq w_0$, where $\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}$.*

Note that any state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$ can be represented by

$$\varphi\left(w_0\mathbb{1} + \mathbf{w}\sigma\right) = w_0 + \langle \mathbf{w}, \mathbf{f}\rangle, \tag{7}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ with $\|\mathbf{f}\| \leq 1$. Here as before $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{C}^3$. Therefore, in the sequel we will identify a state φ with a vector $\mathbf{f} \in \mathbb{R}^3$.

In what follows by τ we denote a normalized trace, that is, $\tau(x) = (1/2)\,\mathrm{Tr}(x)$, $x \in \mathbb{M}_2(\mathbb{C})$.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ. Then one has

$$\begin{aligned}\tau \otimes \tau(\Delta x) &= \tau(\tau \otimes id)(\Delta(x))\\ &= \tau(x)\tau(\mathbb{1}) = \tau(x), \quad x \in \mathbb{M}_2(\mathbb{C}),\end{aligned} \tag{8}$$

which means that τ is an invariant state for Δ.

Let us write the operator Δ in terms of a basis in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ formed by the Pauli matrices, namely,

$$\begin{aligned}&\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1},\\ &\Delta(\sigma_i) = b_i(\mathbb{1} \otimes \mathbb{1}) + \sum_{j=1}^{3} b_{ji}^{(1)}\left(\mathbb{1} \otimes \sigma_j\right)\\ &\qquad + \sum_{j=1}^{3} b_{ji}^{(2)}\left(\sigma_j \otimes \mathbb{1}\right) + \sum_{m,l=1}^{3} b_{ml,i}\left(\sigma_m \otimes \sigma_l\right), \quad i = 1,2,3,\end{aligned} \tag{9}$$

where $b_i, b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ijk} \in \mathbb{C}$ $(i, j, k \in \{1,2,3\})$.

One can prove the following.

Theorem 5 (see [13, Proposition 3.2]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ, then it has the following form:*

$$\Delta(x) = w_0\mathbb{1} \otimes \mathbb{1} + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}}\rangle \sigma_m \otimes \sigma_l, \tag{10}$$

where $x = w_0 + \mathbf{w}\sigma$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3}) \in \mathbb{R}^3$, $m, n, k \in \{1,2,3\}$.

Let us turn to the positivity of Δ. Given vector $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ put

$$\beta(\mathbf{f})_{ij} = \sum_{k=1}^{3} b_{ki,j} f_k. \tag{11}$$

Define a matrix $\mathbb{B}(\mathbf{f}) = (\beta(\mathbf{f})_{ij})_{ij=1}^{3}$.

By $\|\mathbb{B}(\mathbf{f})\|$ we denote a norm of the matrix $\mathbb{B}(\mathbf{f})$ associated with Euclidean norm in $\mathbb{C}^3$. Put

$$S = \left\{ \mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3 : p_1^2 + p_2^2 + p_3^2 \leq 1 \right\} \tag{12}$$

and denote

$$|||\mathbb{B}||| = \sup_{\mathbf{f} \in S} \|\mathbb{B}(\mathbf{f})\|. \tag{13}$$

Proposition 6 (see [13, Proposition 3.3]). *Let Δ be a q.q.o. with a Haar state τ, then $|||\mathbb{B}||| \leq 1$.*

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then due to Theorem 5 Δ has the form (10). Take arbitrary states $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ and let $\mathbf{f}, \mathbf{p} \in S$ be the corresponding vectors (see (7)). Then one finds that

$$\Delta^* (\varphi \otimes \psi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i p_j, \quad k = 1, 2, 3. \tag{14}$$

Thanks to Lemma 4 the functional $\Delta^*(\varphi \otimes \psi)$ is a state if and only if the vector

$$\mathbf{f}_{\Delta^*(\varphi,\psi)} = \left(\sum_{i,j=1}^{3} b_{ij,1} f_i p_j, \sum_{i,j=1}^{3} b_{ij,2} f_i p_j, \sum_{i,j=1}^{3} b_{ij,3} f_i p_j \right) \tag{15}$$

satisfies $\| \mathbf{f}_{\Delta^*(\varphi,\psi)} \| \leq 1$.

So, we have the following.

Proposition 7 (see [13, Proposition 4.1]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then $\Delta^*(\varphi \otimes \psi) \in S(\mathbb{M}_2(\mathbb{C}))$ for any $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ if and only if the following holds:*

$$\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 \leq 1, \quad \forall \mathbf{f}, \mathbf{p} \in S. \tag{16}$$

From the proof of Proposition 6 and the last proposition we conclude that $|||\mathbb{B}||| \leq 1$ holds if and only if (16) is satisfied.

Remark 8. Note that characterizations of positive maps defined on $\mathbb{M}_2(\mathbb{C})$ were considered in [24] (see also [25]). Characterization of completely positive mappings from $\mathbb{M}_2(\mathbb{C})$ into itself with invariant state τ was established in [3] (see also [26]).

Next we would like to recall (see [13]) some conditions for q.q.o. to be the Kadison-Schwarz ones.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator with a Haar state τ; then it has the form (10). Now we are going to find some conditions to the coefficients $\{b_{ml,k}\}$ when Δ is a Kadison-Schwarz operator. Given $x = w_0 + \mathbf{w}\sigma$ and state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$, let us denote

$$\mathbf{x}_m = (\langle \mathbf{b}_{m1}, \mathbf{w} \rangle, \langle \mathbf{b}_{m2}, \mathbf{w} \rangle, \langle \mathbf{b}_{m3}, \mathbf{w} \rangle), \qquad f_m = \varphi(\sigma_m), \tag{17}$$

$$\alpha_{ml} = \langle \mathbf{x}_m, \mathbf{x}_l \rangle - \langle \mathbf{x}_l, \mathbf{x}_m \rangle, \qquad \gamma_{ml} = [\mathbf{x}_m, \overline{\mathbf{x}}_l] + [\overline{\mathbf{x}}_m, \mathbf{x}_l], \tag{18}$$

where $m, l = 1, 2, 3$. Here and in what follows $[\cdot, \cdot]$ stands for the usual cross-product in $\mathbb{C}^3$. Note that here the numbers α_{ml} are skew symmetric, that is, $\overline{\alpha_{ml}} = -\alpha_{ml}$. By π we will denote mapping $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$ defined by $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = \pi(1)$.

Denote

$$\mathbf{q}(\mathbf{f}, \mathbf{w}) = (\langle \beta(\mathbf{f})_1, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_2, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_3, [\mathbf{w}, \overline{\mathbf{w}}] \rangle), \tag{19}$$

where $\beta(\mathbf{f})_m = (\beta(\mathbf{f})_{m1}, \beta(\mathbf{f})_{m2}, \beta(\mathbf{f})_{m3})$ (see (11)).

Theorem 9 (see [13, Theorem 3.6]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a Kadison-Schwarz operator with a Haar state τ; then it has the form (10) and the coefficients $\{b_{ml,k}\}$ satisfy the following conditions:*

$$\|\mathbf{w}\|^2 \geq i \sum_{m=1}^{3} f_m \alpha_{\pi(m),\pi(m+1)} + \sum_{m=1}^{3} \|\mathbf{x}_m\|^2, \tag{20}$$

$$\begin{aligned} &\left\| \mathbf{q}(\mathbf{f}, \mathbf{w}) - i \sum_{m=1}^{3} f_m \gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m, \overline{\mathbf{x}}_m] \right\| \\ &\quad \leq \|\mathbf{w}\|^2 - i \sum_{k=1}^{3} f_k \alpha_{\pi(k),\pi(k+1)} - \sum_{m=1}^{3} \|\mathbf{x}_m\|^2 \end{aligned} \tag{21}$$

for all $\mathbf{f} \in S$, $\mathbf{w} \in \mathbb{C}^3$. Here as before $\mathbf{x}_m = (\langle \mathbf{b}_{m1}, \mathbf{w} \rangle, \langle \mathbf{b}_{m2}, \mathbf{w} \rangle, \langle \mathbf{b}_{m3}, \mathbf{w} \rangle)$; $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, and $\mathbf{q}(\mathbf{f}, \mathbf{w})$, α_{ml}, and γ_{ml} are defined in (19), (17), and (18), respectively.

Remark 10. The provided characterization with [2, 3] allows us to construct examples of positive or Kadison-Schwarz operators which are not completely positive (see next section).

Now we are going to give a general characterization of KS operators. Let us first give some notations. For a given mapping $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, by $\Delta(\sigma)$ we denote the vector $(\Delta(\sigma_1), \Delta(\sigma_2), \Delta(\sigma_3))$, and by $\mathbf{w}\Delta(\sigma)$ we mean the following:

$$\mathbf{w}\Delta(\sigma) = w_1 \Delta(\sigma_1) + w_2 \Delta(\sigma_2) + w_3 \Delta(\sigma_3), \tag{22}$$

where $\mathbf{w} \in \mathbb{C}^3$. Note that the last equality (22), due to the linearity of Δ, can also be written as $\mathbf{w}\Delta(\sigma) = \Delta(\mathbf{w}\sigma)$.

Theorem 11. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a unital $*$-preserving linear mapping. Then Δ is a KS operator if and only if one has*

$$i [\mathbf{w}, \overline{\mathbf{w}}] \Delta(\sigma) + (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \leq \mathbb{1} \otimes \mathbb{1}, \tag{23}$$

for all $\mathbf{w} \in \mathbb{C}^3$ with $\|\mathbf{w}\| = 1$.

Proof. Let $x \in M_2(\mathbb{C})$ be an arbitrary element, that is, $x = w_0\mathbb{1} + \mathbf{w}\sigma$. Then $x^* = \overline{w_0}\mathbb{1} + \overline{\mathbf{w}}\sigma$. Therefore

$$x^*x = \left(\left|w_0\right|^2 + \|\mathbf{w}\|^2\right)\mathbb{1} + \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i\left[\mathbf{w},\overline{\mathbf{w}}\right]\right)\sigma. \quad (24)$$

Consequently, we have

$$\begin{aligned}\Delta(x) &= w_0\mathbb{1}\otimes\mathbb{1} + \mathbf{w}\Delta(\sigma),\\ \Delta(x^*) &= \overline{w_0}\mathbb{1}\otimes\mathbb{1} + \overline{\mathbf{w}}\Delta(\sigma),\end{aligned} \quad (25)$$

$$\begin{aligned}\Delta(x^*x) = &\left(\left|w_0\right|^2 + \|\mathbf{w}\|^2\right)\mathbb{1}\otimes\mathbb{1}\\ &+ \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i\left[\mathbf{w},\overline{\mathbf{w}}\right]\right)\Delta(\sigma),\end{aligned} \quad (26)$$

$$\begin{aligned}\Delta(x)^*\Delta(x) = &\left|w_0\right|^2\mathbb{1}\otimes\mathbb{1} + \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w}\right)\Delta(\sigma)\\ &+ (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)).\end{aligned} \quad (27)$$

From (26) and (27) one gets

$$\begin{aligned}&\Delta(x^*x) - \Delta(x)^*\Delta(x)\\ &\quad = \|\mathbf{w}\|^2\mathbb{1}\otimes\mathbb{1} - i\left[\mathbf{w},\overline{\mathbf{w}}\right]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)).\end{aligned} \quad (28)$$

So, the positivity of the last equality implies that

$$\|\mathbf{w}\|^2\mathbb{1}\otimes\mathbb{1} - i\left[\mathbf{w},\overline{\mathbf{w}}\right]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \geq 0. \quad (29)$$

Now dividing both sides by $\|\mathbf{w}\|^2$ we get the required inequality. Hence, this completes the proof. □

4. An Example of Q.Q.O. Which Is Not Kadison-Schwarz One

In this section we are going to study dynamics of (57) for a special class of quadratic operators. Such class operators are associated with the following matrix $\{b_{ij,k}\}$ given by

$$\begin{array}{lll} b_{11,1} = \varepsilon, & b_{11,2} = 0, & b_{11,3} = 0,\\ b_{12,1} = 0, & b_{12,2} = 0, & b_{12,3} = \varepsilon,\\ b_{13,1} = 0, & b_{13,2} = \varepsilon, & b_{13,3} = 0,\\ b_{22,1} = 0, & b_{22,2} = \varepsilon, & b_{22,3} = 0,\\ b_{23,1} = \varepsilon, & b_{23,2} = 0, & b_{23,3} = 0,\\ b_{33,1} = 0, & b_{33,2} = 0, & b_{33,3} = \varepsilon,\end{array} \quad (30)$$

and $b_{ij,k} = b_{ji,k}$.

Via (10) we define a liner operator Δ_ε, for which τ is a Haar state. In the sequel we would like to find some conditions to ε which ensures positivity of Δ_ε.

It is easy that for given $\{b_{ijk}\}$ one can find a form of Δ_ε as follows.

$$\begin{aligned}\Delta_\varepsilon(x) = &w_0\mathbb{1}\otimes\mathbb{1} + \varepsilon\omega_1\sigma_1\otimes\sigma_1 + \varepsilon\omega_3\sigma_1\otimes\sigma_2\\ &+ \varepsilon\omega_2\sigma_1\otimes\sigma_3 + \varepsilon\omega_3\sigma_2\otimes\sigma_1 + \varepsilon\omega_2\sigma_2\otimes\sigma_2\\ &+ \varepsilon\omega_1\sigma_2\otimes\sigma_3 + \varepsilon\omega_2\sigma_3\otimes\sigma_1 + \varepsilon\omega_1\sigma_3\otimes\sigma_2\\ &+ \varepsilon\omega_3\sigma_3\otimes\sigma_3,\end{aligned} \quad (31)$$

where, as before, $x = w_0\mathbb{1} + \mathbf{w}\sigma$.

Theorem 12. *A linear operator Δ_ε given by (31) is a q.q.o. if and only if $|\varepsilon| \leq 1/3$.*

Proof. Let $x = w_0\mathbb{1} + \mathbf{w}\sigma$ be a positive element from $\mathbb{M}_2(\mathbb{C})$. Let us show positivity of the matrix $\Delta_\varepsilon(x)$. To do it, we rewrite (31) as follows: $\Delta_\varepsilon(x) = w_0\mathbb{1} + \varepsilon\mathbf{B}$; here

$$\mathbf{B} = \begin{pmatrix} \omega_3 & \omega_2 - i\omega_1 & \omega_2 - i\omega_1 & \omega_1 - 2i\omega_3 - \omega_2\\ \omega_2 + i\omega_1 & -\omega_3 & \omega_1 + \omega_2 & -\omega_2 + i\omega_1\\ \omega_2 + i\omega_1 & \omega_1 + \omega_2 & -\omega_3 & -\omega_2 + i\omega_1\\ \omega_1 + 2i\omega_3 - \omega_2 & -\omega_2 - i\omega_1 & -\omega_2 - i\omega_1 & \omega_3 \end{pmatrix}, \quad (32)$$

where positivity of x yields that $w_0, \omega_1, \omega_2, \omega_3$ are real numbers. In what follows, without loss of generality, we may assume that $w_0 = 1$, and therefore $\|\mathbf{w}\| \leq 1$. It is known that positivity of $\Delta_\varepsilon(x)$ is equivalent to positivity of the eigenvalues of $\Delta_\varepsilon(x)$.

Let us first examine eigenvalues of $\mathbf{B}$. Simple algebra shows us that all eigenvalues of $\mathbf{B}$ can be written as follows:

$$\begin{aligned}\lambda_1(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &+ 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3},\\ \lambda_2(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &- 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3},\\ \lambda_3(\mathbf{w}) = &\lambda_4(\mathbf{w}) = -\omega_1 - \omega_2 - \omega_3.\end{aligned} \quad (33)$$

Now examine maximum and minimum values of the functions $\lambda_1(\mathbf{w}), \lambda_2(\mathbf{w}), \lambda_3(\mathbf{w}), \lambda_4(\mathbf{w})$ on the ball $\|\mathbf{w}\| \leq 1$.

One can see that

$$\begin{aligned}\left|\lambda_3(\mathbf{w})\right| = \left|\lambda_4(\mathbf{w})\right| \leq \sum_{k=1}^{3}\left|\omega_k\right| \leq \sqrt{3}\sum_{k=1}^{3}\left|\omega_k\right|^2\\ \leq \sqrt{3}.\end{aligned} \quad (34)$$

Note that the functions λ_3, λ_4 can reach values $\pm\sqrt{3}$ at $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Now let us rewrite $\lambda_1(\mathbf{w})$ and $\lambda_2(\mathbf{w})$ as follows:

$$\begin{aligned}\lambda_1(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &+ \frac{2}{\sqrt{2}}\sqrt{3\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right) - \left(\omega_1 + \omega_2 + \omega_3\right)^2},\end{aligned} \quad (35)$$

$$\begin{aligned}\lambda_2(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &- \frac{2}{\sqrt{2}}\sqrt{3\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right) - \left(\omega_1 + \omega_2 + \omega_3\right)^2}.\end{aligned} \quad (36)$$

One can see that

$$\lambda_k(h\omega_1, h\omega_2, h\omega_3) = h\lambda_k(\omega_1, \omega_2, w_3), \quad \text{if } h \geq 0, \quad (37)$$

$$\lambda_1(h\omega_1, h\omega_2, h\omega_3) = h\lambda_2(\omega_1, \omega_2, w_3), \quad \text{if } h \leq 0. \quad (38)$$

where $k = 1, 2$. Therefore, the functions $\lambda_k(\mathbf{w})$, $k = 1, 2$ reach their maximum and minimum on the sphere $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$

(i.e., $\|\mathbf{w}\| = 1$). Hence, denoting $t = \omega_1 + \omega_2 + \omega_3$ from (37) and (36) we introduce the following functions:

$$g_1(t) = t + \frac{2}{\sqrt{2}}\sqrt{3-t^2}, \qquad g_2(t) = t - \frac{2}{\sqrt{2}}\sqrt{3-t^2}, \quad (39)$$

where $|t| \le \sqrt{3}$.

One can find that the critical values of g_1 are $t = \pm 1$, and the critical value of g_2 is $t = -1$. Consequently, extremal values of g_1 and g_2 on $|t| \le \sqrt{3}$ are the following:

$$\begin{aligned} \min_{|t|\le\sqrt{3}} g_1(t) &= -\sqrt{3}, \qquad \max_{|t|\le\sqrt{3}} g_1(t) = 3, \\ \min_{|t|\le\sqrt{3}} g_2(t) &= -3, \qquad \max_{|t|\le\sqrt{3}} g_2(t) = \sqrt{3}. \end{aligned} \quad (40)$$

Therefore, from (37) and (38) we conclude that

$$-3 \le \lambda_k(\mathbf{w}) \le 3, \quad \text{for any } \|\mathbf{w}\| \le 1,\ k = 1,2. \quad (41)$$

It is known that for the spectrum of $\mathbb{1} + \varepsilon\mathbf{B}$ one has

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = 1 + \varepsilon Sp(\mathbf{B}). \quad (42)$$

Therefore,

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = \left\{1 + \varepsilon\lambda_k(\mathbf{w}) : k = \overline{1,4}\right\}. \quad (43)$$

So, if

$$|\varepsilon| \le \frac{1}{\max_{\|\mathbf{w}\|\le 1}|\lambda_k(\mathbf{w})|}, \quad k = \overline{1,4}, \quad (44)$$

then one can see $1 + \varepsilon\lambda_k(\mathbf{w}) \ge 0$ for all $\|\mathbf{w}\| \le 1$, $k = \overline{1,4}$. This implies that the matrix $\mathbb{1} + \varepsilon\mathbf{B}$ is positive for all $\mathbf{w}$ with $\|\mathbf{w}\| \le 1$.

Now assume that Δ_ε is positive. Then $\Delta_\varepsilon(x)$ is positive whenever x is positive. This means that $1 + \varepsilon\lambda_k(\mathbf{w}) \ge 0$ for all $\|\mathbf{w}\| \le 1 (k = \overline{1,4})$. From (34) and (41) we conclude that $|\varepsilon| \le 1/3$. This completes the proof. □

Theorem 13. *Let $\varepsilon = 1/3$ then the corresponding q.q.o. Δ_ε is not KS operator.*

Proof. It is enough to show the dissatisfaction of (21) at some values of $\mathbf{w}$ ($\|\mathbf{w}\| \le 1$) and $\mathbf{f} = (f_1, f_1, f_2)$.

Assume that $\mathbf{f} = (1,0,0)$; then a little algebra shows that (21) reduces to the following one:

$$\sqrt{A + B + C} \le D, \quad (45)$$

where

$$\begin{aligned} A = {} & \left|\varepsilon(\overline{\omega}_2\omega_3 - \overline{\omega}_3\omega_2) - i\varepsilon^2\left(2\overline{\omega}_2\omega_3 - 2|\omega_1|^2 - \overline{\omega}_2\omega_1 \right.\right. \\ & \left.\left. + \overline{\omega}_1\omega_2 - \overline{\omega}_1\omega_3 + \overline{\omega}_3\omega_1\right)\right|^2, \\ B = {} & \left|\varepsilon(\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1) - i\varepsilon^2\left(2\overline{\omega}_1\omega_2 - 2|\omega_3|^2 - \overline{\omega}_1\omega_3 \right.\right. \\ & \left.\left. + \overline{\omega}_3\omega_1 - \overline{\omega}_3\omega_2 + \overline{\omega}_2\omega_3\right)\right|^2, \\ C = {} & \left|\varepsilon(\overline{\omega}_3\omega_1 - \overline{\omega}_1\omega_3) - i\varepsilon^2\left(2\overline{\omega}_3\omega_1 - 2|\omega_2|^2 - \overline{\omega}_3\omega_2 \right.\right. \\ & \left.\left. + \overline{\omega}_2\omega_3 - \overline{\omega}_2\omega_1 + \overline{\omega}_1\omega_2\right)\right|^2, \\ D = {} & \left(1 - 3|\varepsilon|^2\right)\left(|\omega_1|^2 + |\omega_2|^2 + |\omega_3|^2\right) \\ & - i\varepsilon^2\left(\overline{\omega}_3 w_2 - \overline{\omega}_2\omega_3 + \overline{\omega}_2\omega_1 - \overline{\omega}_1\omega_2 + \overline{\omega}_1\omega_3 - \overline{\omega}_3\omega_1\right). \end{aligned} \quad (46)$$

Now choose $\mathbf{w}$ as follows:

$$\omega_1 = -\frac{1}{9}, \qquad \omega_2 = \frac{5}{36}, \qquad \omega_3 = \frac{5i}{27}. \quad (47)$$

Then calculations show that

$$\begin{aligned} A &= \frac{9594}{19131876}, \qquad B = \frac{19625}{86093442}, \\ C &= \frac{1625}{3779136}, \qquad D = \frac{589}{17496}. \end{aligned} \quad (48)$$

Hence, we find

$$\sqrt{\frac{9594}{19131876} + \frac{19625}{86093442} + \frac{1625}{3779136}} > \frac{589}{17496}, \quad (49)$$

which means that (45) is not satisfied. Hence, Δ_ε is not a KS operator at $\varepsilon = 1/3$. □

Recall that a linear operator $T : \mathbb{M}_k(\mathbb{C}) \to \mathbb{M}_m(\mathbb{C})$ is *completely positive* if for any positive matrix $(a_{ij})_{i,j=1}^n \in \mathbb{M}_k(\mathbb{M}_n(\mathbb{C}))$ the matrix $(T(a_{ij}))_{i,j=1}^n$ is positive for all $n \in \mathbb{N}$. Now we are interested when the operator Δ_ε is completely positive. It is known [1] that the complete positivity of Δ_ε is equivalent to the positivity of the following matrix:

$$\widehat{\Delta}_\varepsilon = \begin{pmatrix} \Delta_\varepsilon(e_{11}) & \Delta_\varepsilon(e_{12}) \\ \Delta_\varepsilon(e_{21}) & \Delta_\varepsilon(e_{22}) \end{pmatrix}, \quad (50)$$

here e_{ij} $(i,j = 1,2)$ are the standard matrix units in $\mathbb{M}_2(\mathbb{C})$.

From (31) one can calculate that

$$\begin{aligned} \Delta_\varepsilon(e_{11}) &= \frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \varepsilon B_{11}, \qquad \Delta_\varepsilon(e_{22}) = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} - \varepsilon B_{11}, \\ \Delta_\varepsilon(e_{12}) &= \varepsilon B_{12}, \qquad \Delta_\varepsilon(e_{21}) = \varepsilon B_{12}^*, \end{aligned} \quad (51)$$

where

$$B_{11} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -i \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ i & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 0 & 0 & 0 & \frac{1-i}{2} \\ i & 0 & \frac{1+i}{2} & 0 \\ i & \frac{1+i}{2} & 0 & 0 \\ \frac{1-i}{2} & -i & -i & 0 \end{pmatrix}. \tag{52}$$

Hence, we find that

$$2\widehat{\Delta}_\varepsilon = \mathbb{1}_8 + \varepsilon\mathbb{B}, \tag{53}$$

where $\mathbb{1}_8$ is the unit matrix in $\mathbb{M}_8(\mathbb{C})$ and

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & -2i & 0 & 0 & 0 & 1-i \\ 0 & -1 & 0 & 0 & 2i & 0 & 1+i & 0 \\ 0 & 0 & -1 & 0 & 2i & 1+i & 0 & 0 \\ 2i & 0 & 0 & 1 & 1-i & -2i & -2i & 0 \\ 0 & -2i & -2i & 1+i & -1 & 0 & 0 & 2i \\ 0 & 0 & 1-i & 2i & 0 & 1 & 0 & 0 \\ 0 & 1-i & 0 & 2i & 0 & 0 & 1 & 0 \\ 1+i & 0 & 0 & 0 & -2i & 0 & 0 & -1 \end{pmatrix}. \tag{54}$$

So, the matrix $\widehat{\Delta}_\varepsilon$ is positive if and only if

$$|\varepsilon| \le \frac{1}{\lambda_{\max}(\mathbb{B})}, \tag{55}$$

where $\lambda_{\max}(\mathbb{B}) = \max_{\lambda \in Sp(\mathbb{B})}|\lambda|$.

One can easily calculate that $\lambda_{\max}(\mathbb{B}) = 3\sqrt{3}$. Therefore, we have the following.

Theorem 14. *Let $\Delta_\varepsilon : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be given by* (31). *Then Δ_ε is completely positive if and only if $|\varepsilon| \le 1/3\sqrt{3}$.*

5. Dynamics of Δ_ε

Let Δ be a q.q.o. on $\mathbb{M}_2(\mathbb{C})$. Let us consider the corresponding quadratic operator defined by $V_\Delta(\varphi) = \Delta^*(\varphi \otimes \varphi)$, $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$. From Theorem 5 one can see that the defined operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ into itself if and only if $|||\mathbb{B}||| \le 1$ or equivalently (16) holds. From (14) we find that

$$V_\Delta(\varphi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad \mathbf{f} \in S. \tag{56}$$

Here, as before, $S = \{\mathbf{f} = (f_1, f_2 f p_3) \in \mathbb{R}^3 : f_1^2 + f_2^2 + f_3^2 \le 1\}$.

So, (56) suggests that we consider the following nonlinear operator $V : S \to S$ defined by

$$V(\mathbf{f})_k = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad k = 1, 2, 3, \tag{57}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in S$.

It is worth to mention that uniqueness of the fixed point (i.e., $(0, 0, 0)$) of the operator given by (57) was investigated in [13, Theorem 4.4].

In this section, we are going to study dynamics of the quadratic operator V_ε corresponding to Δ_ε (see (31)), which has the following form

$$\begin{aligned} V_\varepsilon(f)_1 &= \varepsilon\left(f_1^2 + 2f_2 f_3\right), \\ V_\varepsilon(f)_2 &= \varepsilon\left(f_2^2 + 2f_1 f_3\right), \\ V_\varepsilon(f)_3 &= \varepsilon\left(f_3^2 + 2f_1 f_2\right). \end{aligned} \tag{58}$$

Let us first find some condition on ε which ensures (16).

Lemma 15. *Let V_ε be given by* (58). *Then V_ε maps S into itself if and only if $|\varepsilon| \le 1/\sqrt{3}$ is satisfied.*

Proof. "If" Part. Assume that V_ε maps S into itself. Then (16) is satisfied. Take $\mathbf{f} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\mathbf{p} = \mathbf{f}$. Then from (16) one finds that

$$\sum_{k=1}^{3}\left|\sum_{i,j=1}^{3} b_{ij,k} f_i p_j\right|^2 = 3\varepsilon^2 \le 1 \tag{59}$$

which yields $|\varepsilon| \le 1/\sqrt{3}$.

"Only If" Part. Assume that $|\varepsilon| \le 1/\sqrt{3}$. Take any $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{p} = (p_1, p_2, p_3) \in S$. Then one finds that

$$\begin{aligned} &\sum_{k=1}^{3}\left|\sum_{i,j=1}^{3} b_{ij,k} f_i p_j\right|^2 \\ &\quad = \varepsilon^2 \left(|f_1 p_1 + f_3 p_2 + f_2 p_3|^2 \right. \\ &\qquad \left. + |f_3 p_1 + f_2 p_2 + f_1 p_3|^2 + |f_2 p_1 + f_1 p_2 + f_3 p_3|^2\right) \\ &\quad \le \varepsilon^2 \left(\left(f_1^2 + f_2^2 + f_3^2\right)\left(p_1^2 + p_2^2 + p_3^2\right)\right. \\ &\qquad + \left(f_3^2 + f_2^2 + f_1^2\right)\left(p_1^2 + p_2^2 + p_3^2\right) \\ &\qquad \left. + \left(p_1^2 + p_2^2 + p_3^2\right)\left(f_2^2 + f_1^2 + f_3^2\right)\right) \\ &\quad \le \varepsilon^2 (1 + 1 + 1) = 3\varepsilon^2 \le 1. \end{aligned} \tag{60}$$

This completes the proof. □

Remark 16. We stress that condition (16) is necessary for Δ to be a positive operator. Namely, from Theorem 12 and Lemma 15 we conclude that if $\varepsilon \in (1/3, 1/\sqrt{3}]$ then the operator Δ_ε is not positive, while (16) is satisfied.

In what follows, to study dynamics of V_ε we assume $|\varepsilon| \le 1/\sqrt{3}$. Recall that a vector $\mathbf{f} \in S$ is a fixed point of V_ε if $V_\varepsilon(\mathbf{f}) = \mathbf{f}$. Clearly $(0,0,0)$ is a fixed point of V_ε. Let us find others. To do it, we need to solve the following equation:

$$\begin{aligned} \varepsilon\left(f_1^2 + 2f_2f_3\right) &= f_1, \\ \varepsilon\left(f_2^2 + 2f_1f_3\right) &= f_2, \\ \varepsilon\left(f_3^2 + 2f_1f_2\right) &= f_3. \end{aligned} \tag{61}$$

We have the following.

Proposition 17. *If $|\varepsilon| < 1/\sqrt{3}$ then V_ε has a unique fixed point $(0,0,0)$ in S. If $|\varepsilon| = 1/\sqrt{3}$ then V_ε has the following fixed points: $(0,0,0)$ and $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ in S.*

Proof. It is clear that $(0,0,0)$ is a fixed point of V_ε. If $f_k = 0$, for some $k \in \{1,2,3\}$ then due to $|\varepsilon| \le 1/\sqrt{3}$, one can see that the only solution of (61) belonging to S is $f_1 = f_2 = f_3 = 0$. Therefore, we assume that $f_k \ne 0$ $(k = 1,2,3)$. So, from (61) one finds

$$\begin{aligned} \frac{f_1^2 + 2f_2f_3}{f_2^2 + 2f_1f_3} &= \frac{f_1}{f_2}, \\ \frac{f_1^2 + 2f_2f_3}{f_3^2 + 2f_1f_2} &= \frac{f_1}{f_3}, \\ \frac{f_2^2 + 2f_1f_3}{f_3^2 + 2f_1f_2} &= \frac{f_2}{f_3}. \end{aligned} \tag{62}$$

Denoting

$$x = \frac{f_1}{f_2}, \qquad y = \frac{f_1}{f_3}, \qquad z = \frac{f_2}{f_3}. \tag{63}$$

From (62) it follows that

$$\begin{aligned} x\left(\frac{x(1 + 2/xy)}{1 + 2x/z} - 1\right) &= 0, \\ y\left(\frac{y(1 + 2/xy)}{1 + 2yz} - 1\right) &= 0, \\ z\left(\frac{z(1 + 2x/z)}{1 + 2yz} - 1\right) &= 0. \end{aligned} \tag{64}$$

According to our assumption x, y, z are nonzero, so from (64) one gets

$$\begin{aligned} \frac{x(1 + 2/xy)}{1 + 2x/z} &= 1, \\ \frac{y(1 + 2/xy)}{1 + 2yz} &= 1, \\ \frac{z(1 + 2x/z)}{1 + 2yz} &= 1, \end{aligned} \tag{65}$$

where $2x \ne -z$ and $2yz \ne -1$.

Dividing the second equality of (65) to the first one of (65) we find that

$$\frac{y(1 + 2x/z)}{x(1 + 2yz)} = 1, \tag{66}$$

which with $xz = y$ yields

$$y + 2x^2 = x + 2y^2. \tag{67}$$

Simplifying the last equality one gets

$$(y - x)(1 - 2(y + x)) = 0. \tag{68}$$

This means that either $y = x$ or $x + y = 1/2$.

Assume that $x = y$. Then from $xz = y$, one finds $z = 1$. Moreover, from the second equality of (65) we have $y + 2/y = 1 + 2y$. So, $y^2 + y - 2 = 0$; therefore, the solutions of the last one are $y_1 = 1$, $y_2 = -2$. Hence, $x_1 = 1$, $x_2 = -2$.

Now suppose that $x + y = 1/2$; then $x = 1/2 - y$. We note that $y \ne 1/2$, since $x \ne 0$. So, from the second equality of (65) we find

$$y + \frac{4}{1 - 2y} = 1 + \frac{4y^2}{1 - 2y}. \tag{69}$$

So, $2y^2 - y - 1 = 0$ which yields the solutions $y_3 = -1/2$, $y_4 = 1$. Therefore, we obtain $x_3 = 1$, $z_3 = -1/2$ and $x_4 = -1/2$, $z_4 = -2$.

Consequently, solutions of (65) are the following ones:

$$(1,1,1), \qquad \left(1,-\frac{1}{2},-\frac{1}{2}\right), \qquad \left(-\frac{1}{2},1,-2\right), \qquad (-2,-2,1). \tag{70}$$

Now owing to (63) we need to solve the following equations:

$$\begin{aligned} \frac{f_1}{f_2} &= x_k, \\ \frac{f_2}{f_3} &= z_k. \end{aligned} \quad k = \overline{1,4}, \tag{71}$$

According to our assumption $f_k \ne 0$, we consider cases when $x_k z_k \ne 0$.

Now let us start to consider several cases.

Case 1. Let $x_2 = 1$, $z_2 = 1$. Then from (71) one gets $f_1 = f_2 = f_3$. So, from (61) we find $3\varepsilon f_1^2 = f_1$, that is, $f_1 = 1/3\varepsilon$. Now taking into account $f_1^2 + f_2^2 + f_3^2 \le 1$ one gets $1/3\varepsilon^2 \le 1$. From the last inequality we have $|\varepsilon| \ge 1/\sqrt{3}$. Due to Lemma 15 the operator V_ε is well defined if and only if $|\varepsilon| \le 1/\sqrt{3}$; therefore, one gets $|\varepsilon| = 1/\sqrt{3}$. Hence, in this case a solution is $(\pm 1/\sqrt{3}; \pm 1/\sqrt{3}; \pm 1/\sqrt{3})$.

Case 2. Let $x_2 = 1$, $z_2 = -1/2$. Then from (71) one finds $f_1 = f_2$, $2f_2 = -f_3$. Substituting the last ones to (61) we get $f_1 + 3f_1^2\varepsilon = 0$. Then, we have $f_1 = -1/3\varepsilon$, $f_2 = -1/3\varepsilon$, $f_3 = 2/3\varepsilon$. Taking into account $f_1^2 + f_2^2 + f_3^2 \le 1$ we find $1/9\varepsilon^2 + 4/9\varepsilon^2 + 1/9\varepsilon^2 \le 1$. This means $|\varepsilon| \ge \sqrt{2/3}$; due to Lemma 15 in this case the operator V_ε is not well defined; therefore, we conclude that there is no fixed point of V_ε belonging to S.

Using the same argument for the rest of the cases we conclude the absence of solutions. This shows that if $|\varepsilon| < 1/\sqrt{3}$ the operator V_ε has unique fixed point in S. If $|\varepsilon| = 1/\sqrt{3}$, then V_ε has three fixed points belonging to S. This completes the proof. □

Now we are going to study dynamics of operator V_ε.

Theorem 18. *Let V_ε be given by* (58). *Then the following assertions hold true:*

(i) *if $|\varepsilon| < 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

(ii) *if $|\varepsilon| = 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ with $\mathbf{f} \notin \{(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})\}$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

Proof. Let us consider the following function $\rho(\mathbf{f}) = f_1^2 + f_2^2 + f_3^2$. Then we have

$$\begin{aligned}\rho(V_\varepsilon(\mathbf{f})) &= \varepsilon^2\left(\left(f_1^2 + 2f_2f_3\right)^2 + \left(f_2^2 + 2f_1f_3\right)^2 + \left(f_3^2 + 2f_1f_2\right)^2\right)\\ &\le \varepsilon^2\left(f_1^2 + 2|f_2||f_3| + f_2^2 + 2|f_1||f_3| + f_3^2 + 2|f_1||f_2|\right)\\ &\le \varepsilon^2\left(f_1^2 + f_2^2 + f_3^2 + f_2^2 + f_1^2 + f_3^2 + f_3^2 + f_1^2 + f_2^2\right)\\ &= 3\varepsilon^2\left(f_1^2 + f_2^2 + f_3^2\right) = 3\varepsilon^2\rho(\mathbf{f}).\end{aligned} \tag{72}$$

This means

$$\rho(V_\varepsilon(\mathbf{f})) \le 3\varepsilon^2\rho(\mathbf{f}). \tag{73}$$

Due to $\varepsilon^2 \le 1/3$ from (73) one finds that

$$\rho\left(V_\varepsilon^{n+1}(\mathbf{f})\right) \le \rho\left(V_\varepsilon^n(\mathbf{f})\right), \tag{74}$$

which yields that the sequence $\{\rho(V_\varepsilon^n(\mathbf{f}))\}$ is convergent. Next we would like to find the limit of $\{\rho(V_\varepsilon^n(\mathbf{f}))\}$.

(i) First we assume that $|\varepsilon| < 1/\sqrt{3}$; then from (73) we obtain

$$\rho\left(V_\varepsilon^n(\mathbf{f})\right) \le 3\varepsilon^2\rho\left(V_\varepsilon^{n-1}(\mathbf{f})\right) \le \cdots \le \left(3\varepsilon^2\right)^n\rho(\mathbf{f}). \tag{75}$$

This yields that $\rho(V_\varepsilon^n(\mathbf{f})) \to 0$ as $n \to \infty$, for all $\mathbf{f} \in S$.

(ii) Now let $|\varepsilon| = 1/\sqrt{3}$. Then consider two distinct subcases.

Case A. Let $f_1^2 + f_2^2 + f_3^2 < 1$ and denote $d = f_1^2 + f_2^2 + f_3^2$. Then one gets

$$\begin{aligned}\rho(V_\varepsilon(\mathbf{f})) &\le \varepsilon^2\left(\left(f_1^2 + 2|f_2||f_3|\right)^2 + \left(f_2^2 + 2|f_1||f_3|\right)^2 + \left(f_3^2 + 2|f_1||f_2|\right)^2\right)\\ &\le \varepsilon^2\left(\left(f_1^2 + f_2^2 + f_3^2\right)^2 + \left(f_2^2 + f_1^2 + f_3^2\right)^2 + \left(f_3^2 + f_1^2 + f_2^2\right)^2\right)\\ &= 3\varepsilon^2 d^2 = dd = d\rho(\mathbf{f}).\end{aligned} \tag{76}$$

Hence, we have $\rho(V_\varepsilon(\mathbf{f})) \le d\rho(\mathbf{f})$. This means $\rho(V_\varepsilon^n(\mathbf{f})) \le d^n\rho(\mathbf{f}) \to 0$. Hence, $V_\varepsilon^n(\mathbf{f}) \to 0$ as $n \to \infty$.

Case B. Now take $f_1^2 + f_2^2 + f_3^2 = 1$ and assume that $\mathbf{f}$ is not a fixed point. Therefore, we may assume that $f_i \ne f_j$ for some $i \ne j$, otherwise from Proposition 17 one concludes that $\mathbf{f}$ is a fixed point. Hence, from (58) one finds

$$\begin{aligned}V_\varepsilon(\mathbf{f})_1 &= \varepsilon\left(f_1^2 + 2f_2f_3\right) = \varepsilon\left(1 - f_2^2 - f_3^2 + 2f_2f_3\right)\\ &= \varepsilon\left(1 - \left(f_2 - f_3\right)^2\right).\end{aligned} \tag{77}$$

Similarly, one gets

$$\begin{aligned}V_\varepsilon(\mathbf{f})_2 &= \varepsilon\left(1 - \left(f_1 - f_3\right)^2\right),\\ V_\varepsilon(\mathbf{f})_3 &= \varepsilon\left(1 - \left(f_1 - f_2\right)^2\right).\end{aligned} \tag{78}$$

It is clear that $|V_\varepsilon(\mathbf{f})_k| \le |\varepsilon|$ $(k = 1,2,3)$. According to our assumption $f_i \ne f_j$ $(i \ne j)$ we conclude that one of $|V_\varepsilon(\mathbf{f})_k|$ is strictly less than $1/\sqrt{3}$; this means $V_\varepsilon(\mathbf{f})_1^2 + V_\varepsilon(\mathbf{f})_2^2 + V_\varepsilon(\mathbf{f})_3^2 < 1$. Therefore, from Case A, one gets that $V_\varepsilon^n(\mathbf{f}) \to 0$ as $n \to \infty$. □

Acknowledgments

The authors acknowledge the MOHE Grant FRGS11-022-0170. The first named author acknowledges the junior associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. The authors would like to thank an anonymous referee whose useful suggestions and comments improved the content of the paper.

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