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THE BEHAVIOR OF THE RENEWAL SEQUENCE IN CASE THE TAIL OF THE WAITING-TIME DISTRIBUTION IS REGULARLY VARYING WITH INDEX −1

J. B. G. FRENK,* *Erasmus University, Rotterdam*

Abstract

A second-order asymptotic result for the probability of occurrence of a persistent and aperiodic recurrent event is given if the tail of the distribution of the waiting time for this event is regularly varying with index −1.

RENEWAL THEORY; REGULAR VARIATION

1. Introduction and results

Suppose ε is a persistent and aperiodic recurrent event (for definition see [6], p. 308) and define:

$$f_n \triangleq \Pr\{\varepsilon \text{ occurs for the first time at the } n\text{th trial}\}.$$
$$u_n \triangleq \Pr\{\varepsilon \text{ occurs at the } n\text{th trial}\}.$$
$$u_0 \triangleq 1, \qquad f_0 \triangleq 0.$$

($\{u_n\}_{n\in\mathbb{N}}$ is the so-called renewal sequence.)

Using the probabilistic interpretation this yields $u_n = \sum_{k=1}^{n} f_k u_{n-k}$ for $n \geqq 1$. Kolmogorov [10] and independently Erdös, Pollard and Feller [4] proved

$$(1.1) \qquad \lim_{n\to\infty} u_n = \frac{1}{\mu} \quad \text{for} \quad \mu < \infty$$
$$= 0 \quad \text{for} \quad \mu = \infty$$

with $\mu \triangleq \sum_{n=1}^{\infty} nf_n$. Garsia and Lamperti [8] obtained a stronger result when $\mu = \infty$ for a certain class of lattice probability distributions F. They proved with

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* Postal address: Erasmus Universiteit Rotterdam, Faculteit der Economische Wetenschappen, Postbus 1738, 3000 DR Rotterdam, The Netherlands.

$F(n) \triangleq \sum_{m=0}^{n} f_m$ and $m(n) \triangleq \sum_{m=0}^{n} 1-F(m)$ $(n \in \mathbb{N})$

$$(1.2)\qquad \begin{aligned} &\text{(a)}\ \lim_{n\to\infty} m(n)u_n = \frac{\sin \pi\alpha}{\pi(1-\alpha)} \quad \text{for} \quad \tfrac{1}{2} < \alpha < 1 \\ &\qquad \text{if} \quad 1-F(n) \in \mathrm{RVS}_{-\alpha}^{\infty}; \\ &\text{(b)}\ \liminf_{n\to\infty} m(n)u_n = \frac{\sin \pi\alpha}{\pi(1-\alpha)} \quad \text{for} \quad 0 < \alpha \leqq \tfrac{1}{2} \\ &\qquad \text{if} \quad 1-F(n) \in \mathrm{RVS}_{-\alpha}^{\infty}. \end{aligned}$$

(For the definition of $\mathrm{RVS}_{-\alpha}^{\infty}$ the reader is referred to the next section.)

Erickson [5] considered the case $\alpha = 1$ and proved

$$(1.3)\qquad \lim_{n\to\infty} m(n)u_n = 1 \quad \text{if} \quad 1-F(n) \in \mathrm{RVS}_{-1}^{\infty}.$$

In this paper we are going to prove among some other results the following statement which is stronger then Erickson's.

Theorem.

$$(1.4)\qquad 1-F(n) \in \mathrm{RVS}_{-1}^{\infty} \Leftrightarrow -u_{[t]} \in \Pi^{\infty}.$$

($[t] \triangleq$ integral part of t: for the definition of Π^{∞} the reader is referred to the next section.)

Both relations imply

$$\lim_{n\to\infty} \frac{u_n - \dfrac{1}{m(n)}}{n(1-F(n))m^{-2}(n)} = 0.$$

2. Proofs

Using the theory of Banach algebras we provide a proof of the theorem in the case $\mu < \infty$. This method of proving the theorem for this special case will be given because of its brevity. It is not possible to use the same method when μ is infinite and we therefore give a proof of this case using the Fourier representation of μ_n. (This proof also applies to the case $\mu < \infty$.) However, before starting we need some definitions and lemmas.

Definition 1. A sequence of eventually positive numbers $\{c(n)\}_{n\in\mathbb{N}}$ is called a regularly varying sequence of index ρ if

$$\lim_{n\to\infty} \frac{c([\lambda n])}{c(n)} = \lambda^{\rho} \quad \forall \lambda > 0 \quad (\triangleq: c(n) \in \mathrm{RVS}_{\rho}^{\infty}).$$

An ultimately positive function R on $(0,\infty)$ is called regularly varying with index ρ if $\lim_{t\to\infty}(R(\lambda t)/R(t))=\lambda^\rho\ \forall\lambda>0$. ($\triangleq$: $R(t)\in \mathrm{RVF}_\rho^\infty$).

The following lemma shows that the theory of regularly varying functions also applies to regularly varying sequences.

Lemma 1. If $\{c(n)\}_{n\in\mathbb{N}}$ is a regularly varying sequence of index ρ, the function R defined on $[0,\infty)$ by $R(t)\triangleq c([t])$ is a regularly varying function of index ρ.

Proof. See [14].

Definition 2. A sequence $\{c(n)\}_{n\in\mathbb{N}}$ belongs to the class ΠS^∞ if there exists a sequence $L(n)\in \mathrm{RVS}_0^\infty$ such that $\lim_{n\to\infty}(c([nx])-c(n))/L(n)=\log x\ \forall x>0$ ($\triangleq$: $c(n)\in\Pi S^\infty$). A function R on $(0,\infty)$ belongs to the class Π^∞ if there exists a function $L(t)$ such that $\lim_{t\to\infty}(R(tx)-R(t))/L(t)=\log x\ \forall x>0$ ($\triangleq$: $R(t)\in\Pi^\infty$). ($L(t)$ is then automatically in RVF_0^∞.) The following lemma shows that the theory of the class Π^∞ also applies to the class ΠS^∞.

Lemma 2. If $\{c(n)\}_{n\in\mathbb{N}}\in\Pi S^\infty$ the function R defined on $(0,\infty)$ by $R(t)\triangleq c([t])$ is in Π^∞.

Proof. Using the definition of ΠS^∞ we obtain

$$\lim_{n\to\infty}\frac{c([[nx]z])-c(n)}{L(n)}$$
$$=\lim_{n\to\infty}\frac{c([[nx]z])-c([nx])}{L([nx])}\cdot\frac{L([nx])}{L(n)}+\lim_{n\to\infty}\frac{c([nx])-c(n)}{L(n)}=\log z+\log x \quad \forall x,z>0.$$

This implies (take $x=\frac{1}{2}$; $z=2$; $n=2k+1$ $(k\in\mathbb{N})$) $\lim_{n\to\infty}[(c(n+1)-c(n))/L(n)]=0$. Hence for all $x>0$

$$\lim_{t\to\infty}\frac{c([tx])-c([t])}{L([t])}=\lim_{t\to\infty}\frac{c([tx])-c([[t]x])}{L([tx])}\cdot\frac{L([tx])}{L([t])}+\lim_{t\to\infty}\frac{c([[t]x])-c([t])}{L([t])}$$
$$=\log x \text{ since } [tx]-[[t]x] \text{ is bounded.}$$

Case $\mu<\infty$. Define $\hat{U}(s)\triangleq\sum_{n=0}^\infty u_n s^n$ and $\hat{F}(s)\triangleq\sum_{n=0}^\infty f_n s^n$ for $|s|<1$. Before proving the theorem we recall the following result.

Lemma 3. If $\hat{F}(e^{it_0})=1$ for some $t_0\neq 0$ and $\hat{F}(e^{it})$ is the characteristic function of F, F should be a lattice distribution with the point spectrum contained in the set $\{2k\pi/t_0; k=0\ 1,\cdots\}$.

Proof. [9], p. 94.

Proof of the theorem (in the case $\mu<\infty$). We have $\hat{U}(s)=1/(1-\hat{F}(s))$ for

$|s|<1$ ([6], p. 311). Define

$$\hat{M}(s) \triangleq \frac{1-\hat{F}(s)}{1-s} = \sum_{n=0}^{\infty} (1-F(n))s^n \quad \text{for} \quad |s| \leqq 1.$$

Since $\mu \triangleq \sum_{k=1}^{\infty} kf_k < \infty$ we obtain using the monotone convergence theorem $\lim_{s \uparrow 1} \hat{M}(s) = \mu > 0$. Since ε is aperiodic we have, by Lemma 3, $1-\hat{F}(e^{it_0}) \neq 0$ for $t_0 \neq 2k\pi$ $(k \in Z)$. Obviously $1-\hat{F}(s)>0$ for $|s|<1$. It follows $|\hat{M}(s)|>0$ for $|s| \leqq 1$ and hence using a theorem of Wiener ([13], p. 665) we obtain $1/\hat{M}(s) = \sum_{n=0}^{\infty} \lambda_n s^n$ with $\sum_{n=0}^{\infty} |\lambda_n| < \infty$ and $|s| \leqq 1$. This implies together with $1-F(n) \in \mathrm{RVS}_{-1}^{\infty}$ ([1], p. 258)

$$\lambda_n \sim -\frac{1}{\mu^2}(1-F(n)) \quad (n \to \infty) \tag{2.1}$$

(take $d_n = 1-F(n)$ and $\Lambda(x) = 1/x$).

Since $\hat{U}(s) = 1/((1-s)\hat{M}(s))$ we get $u_n = \sum_{p=0}^{n} \lambda_p$ for all $n \geqq 0$. Consequently $\sum_{n=0}^{p} u_n - (p+1)u_p = -\sum_{n=1}^{p} n\lambda_n$. Using (2.1) and Lemma 1 we get

$$\lim_{p\to\infty} \frac{\sum_{n=0}^{p} u_n - (p+1)u_p}{p^2(1-F(p))} = -\lim_{p\to\infty} \frac{\sum_{n=1}^{p} n\lambda_n}{p^2(1-F(p))} = \frac{1}{\mu^2}.$$

Combining this with

$$\lim_{p\to\infty} \frac{\frac{1}{p+1}\sum_{n=0}^{p} u_n - \frac{1}{m(p)}}{p(1-F(p))} = \frac{1}{\mu^2}$$

([7], Theorem 3) we get

$$\lim_{p\to\infty} \frac{u_p - \frac{1}{m(p)}}{p(1-F(p))} = 0.$$

Clearly this implies $-u_{[t]} \in \Pi^{\infty}$ since $-(1/m([t])) \in \Pi^{\infty}$. The converse statement $(-u_{[t]} \in \Pi^{\infty} \Rightarrow 1-F(n) \in \mathrm{RVS}_{-1}^{\infty})$ will be proved at the end of this section.

We remark that the renewal theorem of Kolmogorov (for the case $\mu < \infty$) can be proved easily using Wiener's theorem. Using the same method we can also prove a second-order asymptotic result for the case $\alpha > 1$.

Lemma 4.

$$1-F(n) \in \mathrm{RVS}_{-\alpha}^{\infty} (\alpha > 1) \Leftrightarrow u_n - \frac{1}{\mu} \in \mathrm{RVS}_{1-\alpha}^{\infty}.$$

Both imply

$$\lim_{n\to\infty}\frac{u_n-(1/\mu)}{n(1-F(n))}=\frac{1}{\mu^2(\alpha-1)}.$$

Proof. Since

$$u_n-\frac{1}{\mu}=\sum_{k=0}^{n}\lambda_k-\sum_{k=0}^{\infty}\lambda_k=-\sum_{k=n+1}^{\infty}\lambda_k;$$

$\lambda_n\sim(-1/\mu^2)(1-F(n))$ if $1-F(n)\in\mathrm{RVS}_{-\alpha}^{\infty}$ ([1], p. 258) and Lemma 1 we obtain the desired result.

To prove the converse statement we consider the following cases.

(a) $1<\alpha<2\Rightarrow\sum_{p=0}^{n}u_p-(n/\mu)\in\mathrm{RV}_{2-\alpha}^{\infty}$ and applying [11], Theorem A, yields $1-F(n)\in\mathrm{RVS}_{-\alpha}^{\infty}$.

(b) $\alpha=2\Rightarrow\sum_{p=0}^{n}u_p-(n/\mu)\in\Pi\mathrm{S}^{\infty}$ and applying [7], Theorem 2, yields $1-F(n)\in\mathrm{RVS}_{-2}^{\infty}$.

(c) $\alpha>2\Rightarrow\sum_{p=n}^{\infty}(u_p-(1/\mu))\in\mathrm{RVS}_{2-\alpha}^{\infty}$ and applying [7], Theorem 1, yields $1-F(n)\in\mathrm{RVS}_{-\alpha}^{\infty}$.

The case $\mu=\infty$. Define $\phi(\theta)\triangleq\int_0^\infty e^{i\theta x}\,dF(x)$ with F some probability distribution on $(0,\infty)$. Hence in our case $\phi(\theta)=\sum_{n=1}^{\infty}e^{i\theta n}f_n$.

Before we start with the proof of the theorem we state the following lemma.

Lemma 5. If the tail of the distribution F is regularly varying with index -1 and $0<\varepsilon<1$ is some chosen number we can find $A_1, A_2, A_3>0$ such that

$$\forall n\geqq A_1\ \forall\theta\in[A_2,\varepsilon n]\frac{\left|\mathrm{Re}\,\phi\left(\frac{\theta-\pi}{n}\right)-\mathrm{Re}\,\phi\left(\frac{\theta}{n}\right)\right|}{1-F(n)}\leqq A_3\,.\,\theta^{\gamma}\quad\text{with}\quad\gamma<1.$$

Proof. The definition of ϕ yields

$$\begin{aligned}\mathrm{Re}\,\phi\left(\frac{\theta-\pi}{n}\right)-\mathrm{Re}\,\phi\left(\frac{\theta}{n}\right)&=-\int_0^\infty\int_{(\theta-\pi)x/n}^{\theta x/n}\sin z\,dz\,dF(x)\\&=\int_0^\infty\left(F\left(\frac{nz}{\theta}\right)-F\left(\frac{nz}{\theta-\pi}\right)\right)\sin z\,dz.\end{aligned}$$

Hence

$$\frac{1}{\theta}\left(\mathrm{Re}\,\phi\left(\frac{\theta-\pi}{n}\right)-\mathrm{Re}\,\phi\left(\frac{\theta}{n}\right)\right)=\int_0^\infty\left(F(nw)-F\left(nw\left(\frac{\theta}{\theta-\pi}\right)\right)\right)\sin\theta w\,dw. \tag{2.2}$$

Divide $\int_0^\infty(F(nw)-F(nw(\theta/(\theta-\pi))))\sin\theta w\,dw$ into two parts, the first part

$$I_1(\theta,n,\eta)\triangleq\int_0^{\theta-\eta}\left(F(nw)-F\left(nw\left(\frac{\theta}{\theta-\pi}\right)\right)\right)\sin\theta w\,dw$$

and the second

$$I_2(\theta, n, \eta) \triangleq \int_{\theta^{-\eta}}^{\infty} \left(F(nw) - F\left(nw\left(\frac{\theta}{\theta-\pi}\right)\right)\right) \sin \theta w \, dw$$

with $\eta \in (0, \frac{1}{2}(\sqrt{5}-1))$.

Consider $I_1(\theta, n, \eta)$. Using Fubini's theorem we get

$$I_1(\theta, n, \eta) = \theta \int_0^{\theta^{-\eta}} \int_p^{\theta^{-\eta}} \left(F\left(nw\left(1+\frac{\pi}{\theta-\pi}\right)\right) - F(nw)\right) dw \cos \theta p \, dp.$$

Since

$$\int_p^{\theta^{-\eta}} 1 - F\left(nw\left(1+\frac{\pi}{\theta-\pi}\right)\right) dw = \frac{\theta-\pi}{\theta} \int_{p(1+\pi/(\theta-\pi))}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} 1 - F(nw) \, dw$$

we obtain

$$\begin{aligned}
&\int_p^{\theta^{-\eta}} F\left(nw\left(1+\frac{\pi}{\theta-\pi}\right)\right) - F(nw) \, dw \\
&\quad = \frac{\pi}{\theta} \int_{p(1+\pi/(\theta-\pi))}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw + \int_p^{p(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw \\
&\quad\quad - \int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw.
\end{aligned}$$

Combining these relations we have

$$I_1(\theta, n, \eta) = \theta I_{11}(\theta, n, \eta) + \theta I_{12}(\theta, n, \eta) - \theta I_{13}(\theta, n, \eta)$$

with

$$I_{11}(\theta, n, \eta) \triangleq \frac{\pi}{\theta} \int_0^{\theta^{-\eta}} \int_{p(1+\pi/(\theta-\pi))}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw \cos \theta p \, dp$$

$$I_{12}(\theta, n, \eta) \triangleq \int_0^{\theta^{-\eta}} \int_p^{p(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw \cos \theta p \, dp$$

$$I_{13}(\theta, n, \eta) \triangleq \int_0^{\theta^{-\eta}} \int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} (1-F(nw)) \, dw \cos \theta p \, dp.$$

Using Lemma 1 we can apply the result stated by Pitman [12], Lemma 2;

$$\forall h, c > 0 \; \exists A_1(c, h) \; 0 \leqq \frac{1-F(nw)}{1-F(n)} \leqq \frac{A}{w^{1-h}}$$

for all $n \geqq A_1(c, h)$, $0 < w < c$ and A some constant if $1-F \in \mathrm{RVF}_{-1}^{\infty}$ and $t^{1+h}(1-F(t))$ is bounded in the neighbourhood of 0. (The condition $t^{1+h}(1-F(t))$ is bounded in the neighbourhood of 0 is omitted in Pitman's lemma. One can easily construct a counterexample if the condition is not fulfilled.) Using this inequality we find for $h=\eta$ and θ sufficiently large that for all $n \geqq A_1(3,\eta)$

$$\text{(a)}\quad \frac{|I_{11}(\theta,n,\eta)|}{1-F(n)} \leqq A\int_0^{\theta^{-\eta}}\int_{p(1+\pi/(\theta-\pi))}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} w^{-1-\eta}\,dw\,dp \leqq \frac{C_1}{\theta}\,.\,\theta^{\eta^2-\eta}$$

and C_1 some constant.

$$\text{(b)}\quad \frac{|I_{12}(\theta,n,\eta)|}{1-F(n)} \leqq A\int_0^{\theta^{-\eta}}\int_{p}^{p(1+\pi/(\theta-\pi))} w^{-1-\eta}\,dw\,dp$$

$$=\frac{A\,.\,\pi}{(\theta-\pi)\eta}\left(\frac{1-\left(1+\dfrac{\pi}{\theta-\pi}\right)^{-\eta}}{\dfrac{\pi}{\theta-\pi}}\right)\int_0^{\theta^{-\eta}} p^{-\eta}\,dp \leqq \frac{C_2}{\theta}\,.\,\theta^{\eta^2-\eta}$$

and C_2 some constant.

$$\text{(c)}\quad \frac{|I_{13}(\theta,n,\eta)|}{1-F(n)} \leqq A\int_{\theta^{-\eta}}^{\theta^{-\eta}(1+\pi/(\theta-\pi))} w^{-1-\eta}\,dw\left|\int_0^{\theta^{-\eta}}\cos\theta p\,dp\right|$$

$$=\frac{A\,.\,\pi}{\eta(\theta-\pi)}\left(\frac{1-\left(1+\dfrac{\pi}{\theta-\pi}\right)^{-\eta}}{\dfrac{\pi}{\theta-\pi}}\right)\theta^{\eta^2}\frac{1}{\theta}\left|\int_0^{\theta^{1-\eta}}\cos z\,dz\right|$$

$$\leqq\frac{C_3}{\theta^2}\,.\,\theta^{\eta^2}$$

and C_3 some constant.

Hence using these inequalities we get $\forall n \geqq A_1(3,\eta)$.

$$\frac{|I_1(\theta,n,\eta)|}{1-F(n)} \leqq C_1\,.\,\theta^{\eta^2-\eta}+C_2\,.\,\theta^{\eta^2-\eta}+C_3\,.\,\theta^{\eta^2-1} \leqq C_4\,.\,\theta^{\eta^2-\eta} \tag{2.3}$$

with C_4 some constant and θ sufficiently large.

Consider $I_2(\theta,n,\eta)$. Since

$$I_2(\theta,n,\eta) \triangleq \int_{\theta^{-\eta}}^{\infty}\left(1-F\left(nw\left(\frac{\theta}{\theta-\pi}\right)\right)\right)-(1-F(nw))\sin\theta w\,dw$$

and $1-F$ a positive non-increasing function we can apply Bonnet's form of the second mean-value theorem ([15], p. 17) to get

$$|I_2(\theta,n,\eta)| \leqq \frac{2}{\theta}\left(1-F\left(n\theta^{-\eta}\left(\frac{\theta}{\theta-\pi}\right)\right)\right)+\frac{2}{\theta}(1-F(n\theta^{-\eta})).$$

Hence using Pitman's lemma we have

$$\frac{|I_2(\theta,n,\eta)|}{1-F(n)} \leqq \frac{C_5}{\theta}(\theta^{-\eta})^{-1-\eta}=\frac{C_5}{\theta}\,.\,\theta^{\eta^2+\eta}. \tag{2.4}$$

Combining (2.2), (2.3), (2.4) we obtain

$$\frac{\left|\operatorname{Re}\phi\left(\frac{\theta-\pi}{n}\right)-\operatorname{Re}\phi\left(\frac{\theta}{n}\right)\right|}{1-F(n)} \leqq C_4 \,.\, \theta^{\eta^2-\eta+1}+C_5 \,.\, \theta^{\eta^2+\eta} \leqq C_6 \,.\, \theta^{\gamma}$$

with $\gamma \triangleq \max(\eta^2-\eta+1, \eta^2+\eta)$ for all $n \geqq A_1(3, \eta)$ and $\theta \in [A_2, \varepsilon n]$. Since $0<\eta<\frac{1}{2}(\sqrt{5}-1)$ we have $\gamma<1$.

Proof of the theorem. For u_n the following representation is well known ([5], p. 266 or [8], p. 226):

$$u_n=\frac{2}{\pi}\int_0^{\pi} W(\theta)\cos n\theta\, d\theta \quad \text{and} \quad W(\theta) \triangleq \operatorname{Re}\left(\frac{1}{1-\phi(\theta)}\right).$$

Hence for all $p>1$

$$\begin{aligned}\frac{\pi}{2}(u_n-u_{[np]}) &= \left(\int_0^{B/n} \cos n\theta W(\theta)\, d\theta - \int_0^{B/[np]} \cos([np]\theta) W(\theta)\, d\theta\right) \\ &\quad + \left(\int_{B/n}^{\varepsilon} \cos n\theta W(\theta)\, d\theta - \int_{B/[np]}^{\varepsilon} \cos([np]\theta) W(\theta)\, d\theta\right) \\ &\quad + \int_{\varepsilon}^{\pi} (\cos n\theta - \cos([np]\theta)) W(\theta)\, d\theta.\end{aligned}$$

We shall consider these three parts separately and prove

(a) $$\lim_{n\to\infty} \frac{2}{\pi} \frac{\left(\int_0^{B/n} \cos n\theta W(\theta)\, d\theta - \int_0^{B/[np]} \cos([np]\theta) W(\theta)\, d\theta\right)}{n(1-F(n))m^{-2}(n)} = \log p;$$

(b) $$\limsup_{n\to\infty} \frac{\left|\frac{2}{\pi}\int_{\varepsilon}^{\pi} (\cos n\theta - \cos([np]\theta)) W(\theta)\, d\theta\right|}{n(1-F(n))m^{-2}(n)} = 0;$$

(c) $$\limsup_{n\to\infty} \frac{\left|\frac{2}{\pi}\int_{B/n}^{\varepsilon} \cos n\theta W(\theta)\, d\theta - \int_{B/[np]}^{\varepsilon} \cos([np]\theta) W(\theta)\, d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{\frac{1}{2}(\gamma-1)}).$$

In order to prove (c) and (b) it is sufficient to prove

(c′) $$\limsup_{n\to\infty} \frac{\left|\frac{2}{\pi}\int_{B/n}^{\varepsilon} \cos n\theta W(\theta)\, d\theta\right|}{n(1-F(n))m^{-2}(n)} = O(B^{\frac{1}{2}(\gamma-1)});$$

(b′) $$\limsup_{n\to\infty} \frac{\left|\frac{2}{\pi}\int_{\varepsilon}^{\pi} \cos n\theta W(\theta)\, d\theta\right|}{n(1-F(n))m^{-2}(n)} = 0.$$

We shall first provide the proof of (a) and (b′) since the proof of (c′) is lengthy and rather technical.

Proof of (a). Using partial integration we obtain for every $p \geqq 1$ and $B > 0$

$$\int_0^{B/[np]} \cos([np]\theta) W(\theta)\, d\theta$$
$$= \cos B \int_0^{B/[np]} W(\theta)\, d\theta + [np] \int_0^{B/[np]} \sin([np]\theta) \int_0^{\theta} W(z)\, dz\, d\theta.$$

Hence

$$\int_0^{B/n} \cos n\theta W(\theta)\, d\theta - \int_0^{B/[np]} \cos([np]\theta) W(\theta)\, d\theta$$
$$= \frac{\cos B}{n} \int_{nB/[np]}^{B} W\left(\frac{\theta}{n}\right) d\theta + \frac{1}{n} \int_0^{B} \sin\theta \int_{n\theta/[np]}^{\theta} W\left(\frac{s}{n}\right) ds\, d\theta.$$

Since $1 - F(n) \in \mathrm{RVS}_{-1}^{\infty}$ we also have using Lemma 1 ([15], p. 271)

$$W\left(\frac{1}{n}\right) \in \mathrm{RVF}_{-1}^{\infty} \quad \text{and} \quad W\left(\frac{1}{n}\right) \sim \frac{\pi}{2} \cdot \frac{n^2(1-F(n))}{m^2(n)}.$$

Combining the last two results it is easy to deduce

$$\lim_{n\to\infty} \frac{2}{\pi} \frac{\left(\int_0^{B/n} \cos n\theta W(\theta)\, d\theta - \int_0^{B/[np]} \cos([np]\theta) W(\theta)\, d\theta\right)}{n(1-F(n)) m^{-2}(n)} = \log p.$$

Proof of (b′). Since $\cos n\theta = -\cos n(\theta + \pi/n)$ we obtain

$$2\int_{\varepsilon}^{\pi} \cos(n\theta) W(\theta)\, d\theta$$
$$= \int_{\varepsilon}^{\pi} \cos n\theta \left(W(\theta) - W\left(\theta - \frac{\pi}{n}\right)\right) d\theta + \int_{\varepsilon}^{\varepsilon + (\pi/n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) d\theta$$
$$- \int_{\pi}^{\pi + (\pi/n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) d\theta.$$

Because $W(\theta)$ is bounded on $[A, B]$ with $0 < A < B \leqq \pi$ we get

$$\text{(d)} \qquad \limsup_{n\to\infty} \frac{\left| \int_{\varepsilon}^{\varepsilon + (\pi/n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) d\theta \right|}{\frac{1}{n} W\left(\frac{1}{n}\right)} \leqq \limsup_{n\to\infty} C_{\varepsilon} \cdot \frac{\pi}{W\left(\frac{1}{n}\right)} = 0.$$

$$\text{(e)} \qquad \limsup_{n\to\infty} \frac{\left| \int_{\pi}^{\pi + (\pi/n)} \cos n\theta W\left(\theta - \frac{\pi}{n}\right) d\theta \right|}{\frac{1}{n} W\left(\frac{1}{n}\right)} \leqq \limsup_{n\to\infty} C_{\pi} \cdot \frac{\pi}{W\left(\frac{1}{n}\right)} = 0.$$

Erickson proved ([5], Lemma 5)

$$|\phi(\theta_1)-\phi(\theta_2)|\leqq\frac{2}{|\theta_1-\theta_2|}\cdot m\left(\frac{1}{|\theta_1-\theta_2|}\right)\quad\forall\theta_1\neq\theta_2$$

and thus using the definition of $W(\theta)$

$$\left|W(\theta)-W\left(\theta-\frac{\pi}{n}\right)\right|\leqq\frac{\frac{\pi}{n}\cdot m\left(\frac{n}{\pi}\right)}{|1-\phi(\theta)|\left|1-\phi\left(\theta-\frac{\pi}{n}\right)\right|}\leqq\frac{\pi}{n}C_{\varepsilon,\pi}m\left(\frac{n}{\pi}\right)$$

for $\theta\in[\varepsilon,\pi]$ and

$$C_{\varepsilon,\pi}=\max_{\theta\in[\varepsilon,\pi]}\left(\frac{1}{|1-\phi(\theta)|\left|1-\phi\left(\theta-\frac{\pi}{n}\right)\right|}\right).$$

(This is possible since ε is not periodic.) Hence

$$\text{(f)}\qquad\limsup_{n\to\infty}\frac{\left|\int_\varepsilon^\pi\cos n\theta(W(\theta)-W(\theta-\pi/n)\,d\theta\right|}{\frac{1}{n}W\left(\frac{1}{n}\right)}=0.$$

Using (d), (e), (f) and

$$\frac{1}{n}W\left(\frac{1}{n}\right)\sim\frac{\pi}{2}\frac{n(1-F(n))}{m^2(n)}$$

we thus get

$$\limsup_{n\to\infty}\frac{\left|\frac{2}{\pi}\int_\varepsilon^\pi\cos n\theta W(\theta)\,d\theta\right|}{n(1-F(n))m^{-2}(n)}=0.$$

Proof of (c′). We write

$$2\int_{B/n}^{\varepsilon}\cos n\theta W(\theta)\,d\theta=\frac{1}{n}\int_B^{B+\pi}\cos\theta W\left(\frac{\theta-\pi}{n}\right)d\theta-\frac{1}{n}\int_{\varepsilon n}^{\varepsilon n+\pi}\cos\theta W\left(\frac{\theta-\pi}{n}\right)d\theta$$
$$+\frac{1}{n}\int_B^{\varepsilon n}\cos\theta\left(W\left(\frac{\theta}{n}\right)-W\left(\frac{\theta-\pi}{n}\right)\right)d\theta\triangleq\frac{1}{n}(I_1(n)-I_2(n)+I_3(n)).$$

Obviously

$$\lim_{n\to\infty}\frac{\int_B^{B+\pi}\cos\theta W\left(\frac{\theta-\pi}{n}\right)d\theta}{W\left(\frac{1}{n}\right)}=\int_B^{B+\pi}\frac{\cos\theta}{\theta-\pi}\,d\theta$$

and hence

$$\limsup_{n\to\infty} \frac{\frac{1}{n}|I_1(n)|}{\frac{1}{n} W\left(\frac{1}{n}\right)} \leqq \delta \text{ for } B \text{ sufficiently large and } \delta \text{ sufficiently small.}$$

Since $W(1/n) \to \infty (n \to \infty)$ and $W((\theta-\pi)/n)$ bounded on $\theta \in [\varepsilon n, \varepsilon n + \pi]$ we find

$$\limsup_{n\to\infty} \frac{\frac{1}{n}|I_2(n)|}{\frac{1}{n} W\left(\frac{1}{n}\right)} = 0.$$

Finally we have to consider $1/n \,.\, I_3(n)$. Using the definition of $W(\theta)$ we get

$$W\left(\frac{\theta}{n}\right) - W\left(\frac{\theta-\pi}{n}\right) = \frac{\operatorname{Re}\phi\left(\frac{\theta-\pi}{n}\right) - \operatorname{Re}\phi\left(\frac{\theta}{n}\right)}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2}$$

$$+\left(1-\operatorname{Re}\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(\frac{1}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2} - \frac{1}{\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2}\right)$$

$$\triangleq \text{Integrand}_{31}(\theta, n) + \text{Integrand}_{32}(\theta, n).$$

Hence

$$\frac{1}{n} I_3(n) = \frac{1}{n}\int_B^{\varepsilon n} \cos\theta \,.\, \text{Integrand}_{31}(\theta, n)\, d\theta$$
$$+\frac{1}{n}\int_B^{\varepsilon n} \cos\theta \,.\, \text{Integrand}_{32}(\theta, n)\, d\theta.$$

We first consider $1/n \int_B^{\varepsilon n} \cos\theta \,.\, \text{Integrand}_{31}(\theta, n)\, d\theta$. Erickson proved ([5], Lemma 5)

$$\theta m\left(\frac{1}{\theta}\right) \leqq k\,|1-\phi(\theta)| \quad \text{for all} \quad \theta \in (0, 2\pi) \text{ and } k \text{ some constant.}$$

Using this inequality, Pitman's result and Lemma 5 we find for $h = (1-\gamma)/4 > 0$

$$\frac{\left|\frac{1}{n}\int_B^{\varepsilon n} \cos\theta \,.\, \text{Integrand}_{31}(\theta, n)\, d\theta\right|}{n(1-F(n))m^{-2}(n)} \leqq A_3 \,.\, k^2 \int_B^{\varepsilon n} \frac{\theta^\gamma m^2(n)\, d\theta}{\theta^2 m^2(n/\theta)}$$
$$= O\left(\int_B^{\varepsilon n} \theta^{\gamma+2h-2}\, d\theta\right) \leqq O(B^{\frac{1}{2}(\gamma-1)}).$$

Consider the second part of $(1/n) . I_3(n)$. Since

$$|1-\phi(\theta)|^2=(1-\text{Re }\phi(\theta))^2+\text{Im}^2\,\phi(\theta)$$

and

$$\frac{1}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2}-\frac{1}{\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2}=\frac{\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2-\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2}$$

we get the following relation:

$\text{Integrand}_{32}\,(\theta, n)=$

$$\frac{\left(1-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(\text{Re }\phi\left(\frac{\theta}{n}\right)-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(1-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)+1-\text{Re }\phi\left(\frac{\theta}{n}\right)\right)}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2}$$

$$+\frac{\left(1-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(\text{Im }\phi\left(\frac{\theta-\pi}{n}\right)-\text{Im }\phi\left(\frac{\theta}{n}\right)\right)\left(\text{Im }\phi\left(\frac{\theta-\pi}{n}\right)+\text{Im }\phi\left(\frac{\theta}{n}\right)\right)}{\left|1-\phi\left(\frac{\theta}{n}\right)\right|^2\left|1-\phi\left(\frac{\theta-\pi}{n}\right)\right|^2}.$$

In a similar way as Erickson provides the proof of [5], Lemma 5, we get

$$\left|\text{Im }\phi\left(\frac{\theta-\pi}{n}\right)-\text{Im }\phi\left(\frac{\theta}{n}\right)\right|\leqq\frac{\pi}{n}m\left(\frac{n}{\pi}\right)$$

and

$$\left|\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)-\text{Re }\phi\left(\frac{\theta}{n}\right)\right|\leqq\frac{\pi}{n}m\left(\frac{n}{\pi}\right)\quad\text{for all}\quad\theta\in[B,\varepsilon n].$$

Using the above relations and the mentioned inequality for $|1-\phi(\theta)|$ we find

$|\text{Integrand}_{32}\,(\theta, n)|$

$$\leqq\frac{k^4n^3m\left(\frac{n}{\pi}\right)\pi\left(1-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(2-\text{Re }\phi\left(\frac{\theta}{n}\right)-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)}{m^2\left(\frac{n}{\theta-\pi}\right)m^2\left(\frac{n}{\theta}\right)\theta^2(\theta-\pi)^2}$$

$$+\frac{k^4n^3\pi m\left(\frac{n}{\pi}\right)\left(1-\text{Re }\phi\left(\frac{\theta-\pi}{n}\right)\right)\left(\left|\text{Im }\phi\left(\frac{\theta}{n}\right)+\text{Im }\phi\left(\frac{\theta-\pi}{n}\right)\right|\right)}{m^2\left(\frac{n}{\theta-\pi}\right)m^2\left(\frac{n}{\theta}\right)\theta^2(\theta-\pi)^2}$$

$$\triangleq\text{Integrand}_{321}\,(\theta, n)+\text{Integrand}_{322}\,(\theta, n).$$

First we consider $\text{Integrand}_{321}(\theta, n)$. Since $1-\text{Re}\,\phi(1/n) \sim \frac{1}{2}\pi(1-F(n))$ $(n \to \infty)$ and $w^{1+h}(1-\text{Re}\,\phi(1/w))$ is bounded in the neighbourhood of 0 we can apply Pitman's lemma ([12], Lemma 2) to the following integral and find for $0<\eta<\frac{1}{6}$ and n sufficiently large

$$\frac{\left|\frac{1}{n}\int_B^{\varepsilon n}\cos\theta\,.\,\text{Integrand}_{321}(\theta,n)\,d\theta\right|}{n^2(1-F(n))^2m^{-3}(n)}$$
$$\leqq C_7\int_B^{\varepsilon n}\frac{(\theta-\pi)^{1+\eta}(\theta^{1+\eta}+(\theta-\pi)^{1+\eta})\theta^{2\eta}(\theta-\pi)^{2\eta}\,d\theta}{\theta^2(\theta-\pi)^2}\leqq O(B^{-(1-6\eta)}).$$

Hence

$$\limsup_{n\to\infty}\frac{\left|\frac{1}{n}\int_B^{\varepsilon n}\cos\theta\,.\,\text{Integrand}_{321}(\theta,n)\,d\theta\right|}{n(1-F(n))m^{-2}(n)}\leqq O(B^{-(1-6\eta)})\limsup_{n\to\infty}\frac{n(1-F(n))}{m(n)}$$
$$=0.$$

Since $\text{Im}\,\phi(1/n)\sim m(n)/n$ and η sufficiently small we find analogously

$$\limsup_{n\to\infty}\frac{\left|\frac{1}{n}\int_B^{\varepsilon n}\cos\theta\,.\,\text{Integrand}_{322}(\theta,n)\,d\theta\right|}{n(1-F(n))m^{-2}(n)}=O(B^{-1})$$

Combination of the above results yields

$$\limsup_{n\to\infty}\frac{|I_3(n)|}{n^2\,.\,(1-F(n))m^{-2}(n)}=O(B^{\frac{1}{2}(\gamma-1)})$$

and hence

$$\limsup_{n\to\infty}\frac{\left|\frac{2}{\pi}\int_{B/n}^{\varepsilon}\cos n\theta W(\theta)\,d\theta\right|}{n(1-F(n))m^{-2}(n)}=O(B^{\frac{1}{2}(\gamma-1)}).$$

The proof of (c′) is now completed and we obtain by combination of (a), (b), (c)

$$\lim_{n\to\infty}\frac{u_n-u_{[np]}}{n(1-F(n))m^{-2}(n)}=\log p\quad\forall p>0.$$

This implies by Lemma 2, [3], Proposition 2, and [2], p. 41,

$$\lim_{n\to\infty}\frac{\sum_{p=0}^{n}u_p-nu_n}{n^2(1-F(n))m^{-2}(n)}=1.$$

On the other hand we proved in [7], Theorem 3,

$$\lim_{n\to\infty} \frac{\sum_{p=0}^{n} u_p - \dfrac{n}{m(n)}}{n^2(1-F(n))m^{-2}(n)} = 1.$$

Combination of both relations yields

$$\lim_{n\to\infty} \frac{u_n - \dfrac{1}{m(n)}}{n(1-F(n))m^{-2}(n)} = 0.$$

We now prove the converse statement of the theorem. This statement is obvious since $-u_{[t]} \in \Pi^\infty$ implies $-1/[t] \sum_{p=0}^{[t]} u_p \in \Pi^\infty$ and [7], Theorem 1, then yields $1-F(n) \in \mathrm{RVS}_{-1}^\infty$.

As an application of the foregoing we can sharpen the result concerning the limit distribution of the residual waiting time. Following Example (b) of [6], p. 332, and the above results it is easy to prove

(i) $1-F(n) \in \mathrm{RVS}_{-\alpha}^\infty(\alpha > 1) \Rightarrow \lim_{n\to\infty} \dfrac{W_n(r) - \dfrac{1}{\mu}(1-F(r-1))}{n(1-F(n))} = \dfrac{1-F(r-1)}{\mu^2(\alpha-1)}$

(ii) $1-F(n) \in \mathrm{RVS}_{-1}^\infty \Rightarrow \lim_{n\to\infty} \dfrac{W_n(r) - \dfrac{1-F(r-1)}{m(n)}}{n(1-F(n))m^{-2}(n)} = 0$

with $W_n(r) \triangleq \Pr\{$first occurrence of ε after the nth trial takes place at the $(n+r)$th trial$\}$.

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