LQ Control without Riccati Equations: Deterministic Systems

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Abstract

We study a deterministic linear-quadratic (LQ) control problem over an infinite horizon, and develop a general approach to the problem based on semi-definite programming (SDP) and related duality analysis. This approach allows the control cost matrix R to be non-negative (semi-definite), a case that is beyond the scope of the classical approach based on Riccati equations. We show that the complementary duality condition of the SDP is necessary and sufficient for the existence of an optimal LQ control. Moreover, when the complementary duality does hold, an optimal state feedback control is constructed explicitly in terms of the solution to the semidefinite program. On the other hand, when the complementary duality fails, the LQ problem has no attainable optimal solution, and we develop an ϵ-approximation scheme that achieves asymptotic optimality.

Key words: LQ control, semidefinite programming, complementary duality, generalized Riccati equation.

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1 Introduction

Consider the following deterministic linear-quadratic (LQ) control problem:

$$\text{(LQ)} \quad \min \quad J(x_0, u(\cdot)) := \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \tag{1}$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in \Re^n. \end{cases} \tag{2}$$

Here and throughout the paper, A, B, and Q, R are constant matrices, and Q and R are both symmetric matrices; and ^{T} denotes the transpose of matrices and vectors; the control $u(\cdot)$ is an element of $L^2(\Re^m)$, the set of all $\Re^m$-valued, measurable functions satisfying

$$\int_0^{+\infty} \|u(t)\|^2 dt < +\infty,$$

where $\|u(t)\| := [\sum_i u_i(t)^2]^{1/2}$.

The LQ control problem, initiated by Kalman [5], is one of the most important classes of optimal control problems, in both theory and applications. In the deterministic case, it is well known that when $R \succ 0$ (positive definite) and $Q \succeq 0$ (non-negative definite), the problem (LQ) can be solved elegantly via the (algebraic) Riccati equation:

$$Q + A^T P + PA - PBR^{-1}B^T P = 0. \tag{3}$$

Furthermore, the optimal control can be obtained explicitly in a feedback form:

$$u^*(t) = -R^{-1}B^T P x^*(t).$$

The Riccati equation has been a primary, if not predominant, tool in studying LQ problems in the literature. However, an immediate drawback of this approach is the requirement that $R \succ 0$, which may exclude many meaningful problems. Consider the following two examples.

Example 1.1 The Riccati approach provides no information at all on the following simple LQ problem, in which $R \equiv 0$:

$$\min \quad \int_0^\infty x^2(t) dt$$

$$\text{s.t.} \quad \begin{cases} \dot{x}(t) = -x(t) + u(t), \\ x(0) = x_0 \in \Re^1. \end{cases}$$

Example 1.2 Consider the following two-dimensional problem:

$$\min \quad \int_0^\infty [12x_1^2(t) + 12x_1(t)x_2(t) + 3x_2^2(t) + u_1^2(t)] dt$$

$$\text{s.t.} \quad \begin{cases} \dot{x}_1(t) = x_1(t) + x_2(t) + u_1(t) + u_2(t), \\ \dot{x}_2(t) = 4x_1(t) + x_2(t) + u_1(t) - 2u_2(t), \\ x(0) = x_0 \in \Re^2. \end{cases}$$

Here,

$$Q = \begin{bmatrix} 12, & 6 \\ 6, & 3 \end{bmatrix} \succeq 0, \qquad R = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \succeq 0,$$

and both matrices are singular. The Riccati approach again fails.

Another development in recent years is the rise of semidefinite programming (SDP) as a computational tool to solve Riccati equations; refer to [16]. Specifically, based on Schur's lemma, the Riccati equation in (3) is reformulated as an SDP, which, in turn, can be efficiently solved by interior point techniques (e.g., [6, 8, 14, 16]). However, the non-singularity of R is also a starting point of this treatment.

In this paper, we study the LQ control problem allowing R to be possibly singular. The central idea of our approach is based on a very simple observation: SDP, in contrast to the Riccati equation, does *not* involve any matrix inverse; hence, in particular, it is not restricted to a non-singular R. Therefore, SDP should apply to a broader class of LQ control problems, much beyond the scope of the Riccati equation in (3). Indeed, this simple idea calls for solving (LQ) directly through SDP — a more general approach of *LQ without Riccati*.

Below, in §2, we present the generalized Riccati equation (i.e., the counterpart of (3) allowing $R \succeq 0$), the SDP primal-dual problems that correspond to (LQ), and related preliminary materials. In §3, we demonstrate that the so-called *complementary duality* — a condition between the primal and dual SDP problems — is the key linkage between the SDP and (LQ). First, when $Q \succ 0$, we show that complementary duality is *equivalent* to the existence of an optimal control that solves (LQ). In addition, we also establish another equivalent condition, in terms of the optimal solution to the primal SDP and the generalized Riccati equation. Next, we show that when Q is possibly singular, these equivalent conditions still hold, provided an additional stability condition on the feedback control is in force.

Furthermore, when the complementary duality holds and when $Q \succ 0$, we show in §4 that it is possible to "regularize" the (LQ) problem so as to remove the singularity of R through an orthonormal transformation. In the same section we also propose an ϵ-approximation scheme that achieves asymptotic optimality when complementary duality fails. In §5, we present a set of examples to illustrate various aspects of the SDP approach.

We close this section by summarizing some of the regularity conditions concerning the problem (LQ).

(i) An open-loop control $u(\cdot)$ is called *admissible* (w.r.t. x_0), if it is (asymptotically) *stabilizing* (w.r.t. x_0), namely, if the state process under the control, $x(\cdot)$ of (2), with initial state x_0, satisfies $\lim_{t\to+\infty} x(t) = 0$. The set of all admissible controls w.r.t. x_0 is denoted as U^{x_0}.

(ii) A feedback control $u(t) = Kx(t)$, where K is a constant matrix, is called (asymptotically) *stabilizing*, if for every initial state x_0, we have $\lim_{t\to+\infty} x(t) = 0$, where $x(\cdot)$ is the solution to (2), with $u(t) = Kx(t)$.

(iii) Accordingly, the system in (2) is called (asymptotically) *stabilizable*, if there exists a stabilizing feedback control of the form $u(t) = Kx(t)$.

(iv) (LQ) is called *well-posed* (w.r.t. x_0), if its cost objective has a finite infimum:

$$-\infty < \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) < +\infty.$$

(v) (LQ) is called *attainable* (w.r.t. x_0), if it is well-posed and if there exists a control that attains the infimum $\inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot))$, in which case the control is called *optimal*.

2 Generalized Riccati Equation and SDP

Throughout this paper we assume that

$$R \succeq 0. \tag{4}$$

Note that this condition is *necessary* for the LQ problem (LQ) to be well-posed (cf. [17, Chapter 6, Proposition 2.4]).

Since we allow the matrix R to be singular, the classical Riccati equation is no longer defined. A natural extension is to consider the following *generalized Riccati equation*:

$$F(P) := A^T P + PA + Q - PBR^+B^T P = 0 \tag{5}$$

where R^+ stands for the pseudo-inverse of R. Note that R^+ satisfies the following properties (refer to [9], and note the symmetry of R):

$$R^+ \succeq 0, \quad (R^+)^T = R^+, \quad R^+R = RR^+;$$

$$RR^+R = R, \qquad R^+RR^+ = R^+.$$

Next, we introduce an affine transformation of the matrix P,

$$\mathcal{L}(P) := \begin{bmatrix} R, & B^T P \\ PB, & Q + A^T P + PA \end{bmatrix}. \tag{6}$$

The following lemma ([1]) shows that $F(P)$ and $\mathcal{L}(P)$ are closely related.

Lemma 2.1 *$\mathcal{L}(P) \succeq 0$ if and only if $F(P) \succeq 0$ and $(I - RR^+)B^T P = 0$.*

Consider the following SDP:

$$\begin{array}{lll}(\mathrm{P}) & \max & \langle I, P\rangle \\ & \text{s.t.} & \mathcal{L}(P) \succeq 0 \\ & & P \in \mathcal{S}^{n\times n}.\end{array}$$

Here and below, $\mathcal{S}^{n\times n}$ denotes the set of $n \times n$ symmetric matrices, and $\langle X, Y\rangle := \sum_{i,j} X_{ij} Y_{ij}$ denotes the matrix inner-product. In particular, $\langle I, P\rangle$ (with I being the identity matrix) is equal to the trace of the matrix P. Note that Lemma 2.1 implies that any feasible solution of (P) is necessarily constrained by the equality $(I - RR^{+})B^T P = 0$.

In the classical setting when $R \succ 0$, it is known that the solution to the Riccati equation $F(P) = 0$ can be obtained by solving the SDP problem (P); refer to [3, 16]. In the more general setting here (allowing $R \succeq 0$), the SDP is still a well defined problem; in particular, it does not impose any restrictions on the definiteness of R. Hence, a viable approach to (LQ) is to solve the SDP first, and then study the relationship between the SDP solution and the solution to (LQ).

To this end, consider the dual of (P), which is also an SDP. Let

$$Z := \begin{bmatrix} Z_b, & Z_u \\ Z_u^T, & Z_n \end{bmatrix}$$

denote the dual variable associated with the primal constraint $\mathcal{L}(P) \succeq 0$, with Z_b, Z_u and Z_n being a block partitioning of Z with appropriate dimensions. (Here we borrowed a similar notion from linear programming: The index set b denotes the "basis" part of the partition, and n denotes the "non-basis" part of the partition; see e.g. [12]. Note that in linear programming, the so-called strict complementarity holds, and hence the non-basis part of optimal dual variables are strictly positive.)

The dual of (P) is

$$\begin{array}{lll}(\mathrm{D}) & \min & \langle R, Z_b\rangle + \langle Q, Z_n\rangle \\ & \text{s.t.} & I + Z_u^T B^T + B Z_u + Z_n A^T + A Z_n = 0 \\ & & Z \succeq 0.\end{array}$$

The semidefinite programs are known to be special forms of conic optimization problems, for which there exists a well-developed duality theory; see, e.g. [8, 16, 6] and the references therein. Key points of the theory can be highlighted as follows:

- The weak duality always holds, i.e. any feasible solution to the primal (maximization) problem always possesses an objective value that is no greater than the (dual) objective value of any dual feasible solution (the dual being a minimization problem).

- In contrast, the strong duality — that the optimal values of the primal and dual problems coincide — needs not always hold (unlike the case of linear programming).

- A sufficient condition for the strong duality is that there exists a pair of *complementary* optimal solutions, i.e., both the primal and dual SDP problems have attainable optimal solutions, and that these solutions are complementary to each other (i.e., no duality gap). For (P) and (D) above, this means that the primal optimal solution P^* and the dual optimal solution Z^* both exist and satisfy $\mathcal{L}(P^*)Z^* = 0$.

- If both (P) and (D) satisfy the *strict feasibility* (also known as *Slater's condition*), namely, there exist primal and dual feasible solutions P^0 and Z^0 such that $\mathcal{L}(P^0) \succ 0$ and $Z^0 \succ 0$, then the complementary solutions exist.

A mild regularity condition, which is assumed throughout the paper, is that the system in (2) is stabilizable as defined at the end of §1. In terms of SDP, this is equivalent to (D) satisfying Slater's condition. Refer to the lemma below.

Lemma 2.2 The following conditions are equivalent.

(i) The system in (2) is stabilizable.

(ii) Problem (D) satisfies Slater's condition.

Proof. First assume that the system in (2) is stabilizable by some feedback control $u(t) = Kx(t)$. Then all the eigenvalues of the matrix $A - BK$ have negative real parts. Consequently, the following Lyapunov equation has a positive solution $Z_n \succ 0$:

$$(A + BK)Z_n + Z_n(A + BK)^T = -I.$$

Set $Z_u = KZ_n$. Then the above relation can be rewritten as

$$I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n = 0.$$

Now choose $Z_b = \epsilon I + Z_u(Z_n)^{-1}Z_u^T$. Then by Schur's lemma, $Z = \begin{bmatrix} Z_b, & Z_u \\ Z_u^T, & Z_n \end{bmatrix}$ is strictly feasible to (D), namely, Problem (D) satisfies Slater's condition.

Conversely, if Z is strictly feasible to (D), then $Z_n \succ 0$ by Schur's lemma. Putting $K = Z_u(Z_n)^{-1}$, then Z satisfying the equality constraint of (D) yields

$$(A+BK)Z_n + Z_n(A+BK)^T = -I.$$

By constructing a quadratic Lyapunov function $x^T Z_n x$, it is easily verified that the system in (2) is stabilizable. □

The above lemma suggests that, in view of the duality of the semidefinite programming, the stability can be regarded as a *dual* concept of the optimality.

As to the primal problem (P), in the classical non-singular setting (i.e., when $Q \succ 0, R \succ 0$), we have

Lemma 2.3 Suppose $Q \succ 0, R \succ 0$. Then, there exists a *maximal* solution $P^* \succ 0$ to the Riccati equation $F(P^*) = 0$ (i.e., $P^* - P \succeq 0$ for any symmetric P that satisfies $F(P) \succeq 0$). And, P^* is also the unique optimal solution to the SDP problem (P). In this case, (LQ) has an optimal feedback control $u(t) = -R^{-1}B^T P^* x(t)$, with an optimal value $x_0^T P^* x_0$ for any initial state x_0.

Proof. It is known that the classical Riccati equation $F(P) = 0$ has a maximal solution $P^* \succ 0$ when $Q \succ 0$ and $R \succ 0$ (cf. [11]). From Schur's lemma and the maximality of P^*, it is clear that P^* is an optimal solution to (P).

To show that it is the unique optimal solution, let $\bar{P}$ be any optimal solution to (P). Then $\langle I, P^* - \bar{P} \rangle = 0$. However, $P^* - \bar{P} \succeq 0$ due to the maximality of P^*. Hence $P^* = \bar{P}$. The rest of the lemma is well known. □

Underlying the elegant results summarized in Lemma 2.3 is the fact that in the non-singular setting both primal and dual SDP's satisfy Slater's condition. To see this, note that $P^0 = 0$ is strictly feasible for the primal problem (as evident from $\mathcal{L}(P) \succ 0$ following (6), taking into account $Q \succ 0$ and $R \succ 0$), while the dual is strictly feasible by virtue of the system in (2) being stabilizable as discussed earlier. Hence complementary duality holds automatically in the non-singular setting. This leads to a constructive way of solving the Riccati equation through solving the SDP, for which efficient interior point codes are available (e.g., [14]). However, with the possible singularity of R, the situation becomes more complicated, as the primal problem may no longer satisfy Slater's condition, and consequently, complementary duality may fail. (Refer to Example 5.1 below.)

In contrast to Lemma 2.3, we have the following result, which will also be used later.

Proposition 2.4 Suppose $Q \succeq 0, R \succeq 0$. Then (P) has a unique optimal solution $P^* \succeq 0$.

Proof. Consider the following perturbed problem of (P) along with its dual, where $\epsilon > 0$:

$$\begin{array}{lll} (P_\epsilon) & \max & \langle I, P\rangle \\ & \text{s.t.} & \left[\begin{array}{cc} R+\epsilon I, & B^T P \\ PB, & Q+\epsilon I + A^T P + PA \end{array}\right] \succeq 0 \end{array}$$

and

$$\begin{array}{lll} (D_\epsilon) & \min & \langle R+\epsilon I, Z_b\rangle + \langle Q+\epsilon I, Z_n\rangle \\ & \text{s.t.} & I + Z_u^T B^T + BZ_u + Z_n A^T + AZ_n = 0 \\ & & Z := \left[\begin{array}{cc} Z_b, & Z_u \\ Z_u^T, & Z_n \end{array}\right] \succeq 0. \end{array}$$

Both problems satisfy Slater's condition, and therefore complementary optimal solutions exist. Observe that the feasible set of (D_ϵ) is independent of ϵ. Take any dual feasible solution Z^0. By weak duality, we have

$$\langle I, P\rangle \leq \langle R+\epsilon I, Z_b^0\rangle + \langle Q+\epsilon I, Z_n^0\rangle. \tag{7}$$

Further, by Lemma 2.3, the unique optimal solution for (P_ϵ), denoted P_ϵ^*, is positive definite: $P_\epsilon^* \succ 0$. This, together with (7), implies in particular that P_ϵ^* is contained in a compact set, where $0 \leq \epsilon \leq \epsilon_0$, and $\epsilon_0 > 0$ is a pre-determined constant. Now, take a convergent subsequence such that

$$\lim_{i\to\infty} P_{\epsilon_i}^* = P_0^* \succeq 0,$$

with $\epsilon_i \to 0$ as $i \to \infty$. Clearly, P_0^* is a feasible solution of (P) since the feasible region of (P_ϵ) monotonically shrinks as $\epsilon \downarrow 0$. Now it suffices to show that $P_0^* = P^*$. Indeed, since P^* is feasible for (P_ϵ), and by Lemma 2.3 P_ϵ^* is the maximal solution to the corresponding Riccati equation, we have $P_\epsilon^* \succeq P^*$, resulting in $P_0^* \succeq P^*$. But P^* is optimal; hence, $\langle I, P^*\rangle \geq \langle I, P_0^*\rangle$. Therefore, we have $P^* = P_0^* \succeq 0$. The uniqueness is evident from the above argument. □

3 Main Results

For the most part of this section we assume $Q \succ 0$. The following Theorem 3.1 summarizes our main results. Towards the end of the section, in Theorem 3.6, we point out the necessary modifications when allowing $Q \succeq 0$.

Theorem 3.1 Suppose $Q \succ 0$. The following three statements are equivalent:

(A) (P) and (D) have complementary optimal solutions.

(B) (P) has an optimal solution P^* which satisfies the generalized Riccati equation $F(P) = 0$.

(C) (LQ) has an attainable optimal feedback control,

$$u^*(t) = -R^+ B^T P^* x^*(t), \tag{8}$$

where P^* is an optimal solution to (P).

As discussed earlier, in the non-singular setting when $Q \succ 0$ and $R \succ 0$, complementary duality, and hence **(A)** is automatically satisfied. Therefore, Theorem 3.1 reduces to Lemma 2.3.

Among the three equivalent conditions in the above theorem, **(C)** concerns the original (LQ) problem, whereas **(A)** and **(B)** are easy to verify numerically — via SDP. (Notice that **(B)** does not require solving the generalized Riccati equation, which could be a difficult task; rather, it only verifies if an optimal solution to (P) satisfies the Riccati equation.)

From another angle, Theorem 3.1 puts verifying complementary duality of the SDP on the same footing as solving the original (LQ) problem; whereas **(B)** is a kind of intermediary between the two, with the generalized Riccati equation substituting for the dual SDP. Note that computationally **(B)** is not needed, as most SDP codes are primal-dual based, which directly solves (verifies) **(A)**.

To prove Theorem 3.1, we first show **(A)**$\Rightarrow$**(B)**, which asserts that complementary duality is the key for the primal SDP solution to satisfy the generalized Riccati equation.

Theorem 3.2 If (P) and (D) have complementary optimal solutions P^* and Z^*, respectively, then P^* must satisfy the generalized Riccati equation: $F(P^*) = 0$.

Proof. By Lemma 2.1, we have $(I - RR^+)B^T P^* = 0$. Thus, the following Schur decomposition holds true:

$$\mathcal{L}(P^*) = \begin{bmatrix} I, & 0 \\ P^* B R^+, & I \end{bmatrix} \begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & R^+ B^T P^* \\ 0, & I \end{bmatrix}. \tag{9}$$

From the relation $\mathcal{L}(P^*) Z^* = 0$, it follows that

$$\begin{aligned}
& \begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & R^+ B^T P^* \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} \\
= & \begin{bmatrix} R(Z_b^* + R^+ B^T P^* (Z_u^*)^T), & R(Z_u^* + R^+ B^T P^* Z_n^*) \\ F(P^*)(Z_u^*)^T, & F(P^*) Z_n^* \end{bmatrix} \\
= & \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{aligned}$$

Therefore

$$F(P^*)(Z_u^*)^T = 0 \quad \text{and} \quad F(P^*)Z_n^* = 0,$$

and hence,

$$Z_u^* F(P^*) = 0 \quad \text{and} \quad Z_n^* F(P^*) = 0.$$

Since Z^* is dual feasible, we have

$$I + (Z_u^*)^T B^T + B Z_u^* + Z_n^* A^T + A Z_n^* = 0.$$

Multiplying $F(P^*)$ on both sides above yields

$$\begin{aligned} 0 &= F(P^*)(I + (Z_u^*)^T B^T + B Z_u^* + Z_n^* A^T + A Z_n^*) F(P^*) \\ &= F(P^*)^2, \end{aligned}$$

which implies $F(P^*) = 0$. $\square$

Next, we establish **(B)$\Rightarrow$(C)**, which relates the SDP to the original (LQ) problem.

Theorem 3.3 If (P) has an optimal solution P^* satisfying $F(P^*) = 0$, then (LQ) has an attainable optimal feedback control as determined by (8).

Proof. To start with, consider any primal feasible solution P, and any admissible (therefore stabilizing) control $u(\cdot) \in U^{x_0}$. We have,

$$\begin{aligned} \frac{d}{dt}(x(t)^T P x(t)) &= (Ax(t) + Bu(t))^T P x(t) + x(t)^T P (Ax(t) + Bu(t)) \\ &= x(t)^T (A^T P + PA) x(t) + 2u(t)^T B^T P x(t). \end{aligned} \tag{10}$$

Integrating (10) over $[0, \infty)$ and making use of the fact that $x(t)^T P x(t) \to 0$ as $t \to \infty$, we have

$$0 = x_0^T P x_0 + \int_0^\infty \Big[x(t)^T (A^T P + PA) x(t) + 2u(t)^T B^T P x(t) \Big] dt.$$

Therefore,

$$\begin{aligned} & J(x_0, u(\cdot)) \\ = & \int_0^\infty \Big[x(t)^T Q x(t) + u(t)^T R u(t) \Big] dt \\ = & \ x_0^T P x_0 \\ & + \int_0^\infty \Big[x(t)^T (A^T P + PA + Q) x(t) + 2u(t)^T B^T P x(t) + u(t)^T R u(t) \Big] dt \\ = & \ x_0^T P x_0 \\ & + \int_0^\infty \Big[(u(t) + R^+ B^T P x(t))^T R (u(t) + R^+ B^T P x(t)) + x(t)^T F(P) x(t) \Big] dt. \end{aligned} \tag{11}$$

Since P is feasible, we have $F(P) \succeq 0$. This means

$$J(x_0, u(\cdot)) \geq x_0^T P x_0, \tag{12}$$

for any P feasible to (P) and for any admissible control $u(\cdot) \in U^{x_0}$. On the other hand, under the feedback control $u^*(t) = -R^+ B^T P^* x(t)$, taking into account $P^* \succeq 0$ (Proposition 2.4), we have

$$\begin{aligned}
0 &\leq J(x_0, u^*(\cdot)) \\
&= \int_0^\infty \Big[x(t)^T Q x(t) + u^*(t)^T R u^*(t)\Big] dt \\
&= \lim_{t\to\infty} \int_0^t \Big[x(\tau)^T Q x(\tau) + u^*(\tau)^T R u^*(\tau)\Big] d\tau \\
&= \lim_{t\to\infty} \{x_0^T P^* x_0 - x(t)^T P^* x(t) \\
&\quad + \int_0^t \Big[x(\tau)^T (A^T P^* + P^* A + Q) x(\tau) + 2u(\tau)^T B^T P^* x(\tau) + u^*(\tau)^T R u^*(\tau)\Big] d\tau\} \\
&\leq x_0^T P^* x_0 \\
&\quad + \lim_{t\to\infty} \int_0^t \Big[(u^*(\tau) + R^+ B^T P^* x(\tau))^T R (u^*(\tau) + R^+ B^T P^* x(\tau)) + x(\tau)^T F(P^*) x(\tau)\Big] d\tau \\
&= x_0^T P^* x_0.
\end{aligned} \tag{13}$$

First of all the above shows that the feedback control $u^*(\cdot)$ incurs a finite cost (w.r.t. any initial state x_0), then it must be stabilizing (and hence admissible). This is because a finite cost in (1) implies $\lim_{t\to+\infty} x^*(t)^T Q x^*(t) = 0$, where $x^*(\cdot)$ is the corresponding state trajectory; and since $Q \succ 0$, we must have $\lim_{t\to+\infty} x^*(t) = 0$. On the other hand, (13) yields $J(x_0, u^*(\cdot)) \leq x_0^T P^* x_0$. Thus, in view of (12) we conclude that $u^*(\cdot)$ is an optimal control. □

The last piece in establishing the equivalence relations in Theorem 3.1 is to show **(C)⇒(A)**. To do so, we need to first establish another result, which is useful in its own right. We want to show that **(A)** is, in fact, implied by a weaker version of **(B)**. That is, complementary duality is actually necessary for any non-negative and *feasible* (as opposed to optimal) solution of (P) to satisfy the generalized Riccati equation.

Theorem 3.4 If (P) has a feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, then there exist complementary optimal solutions to (P) and (D); and in particular, P^* is optimal to (P).

Proof. Denote $K := -R^+ B^T P^*$. First we show that the feedback control given by $u(t) = Kx(t)$ must be stabilizing. Indeed, going through the same calculation as (13) and noting the assumption that $P^* \succeq 0$, we conclude that $u(\cdot)$ incurs a finite cost with respect to any initial state. Hence it must be stabilizing (as shown in the proof of Theorem 3.3). It follows that the following Lyapunov equation

$$(A + BK)Y + Y(A + BK)^T + I = 0$$

has a positive solution; let it be $Y^* \succ 0$. Let

$$Z_n^* = Y^*, \qquad Z_u^* = KY^*, \qquad Z_b^* = KY^*K^T.$$

By this construction, we can easily verify the following:

$$\begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} = \begin{bmatrix} I, & K \\ 0, & I \end{bmatrix} \begin{bmatrix} 0, & 0 \\ 0, & Z_n^* \end{bmatrix} \begin{bmatrix} I, & 0 \\ K^T, & I \end{bmatrix} \succeq 0,$$

and

$$I + (Z_u^*)^T B^T + BZ_u^* + Z_n^* A^T + AZ_n^* = 0.$$

Therefore, Z^* is a feasible solution of (D). Moreover,

$$\begin{aligned}
&\mathcal{L}(P^*) \begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} R, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_b^*, & Z_u^* \\ (Z_u^*)^T, & Z_n^* \end{bmatrix} \\
= &\begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} R(Z_b^* - K(Z_u^*)^T), & R(Z_u^* - KZ_n^*) \\ F(P^*)(Z_u^*)^T, & F(P^*)Z_n^* \end{bmatrix} \\
= &\begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}.
\end{aligned}$$

This means that P^* and Z^* are complementary solutions. In particular, P^* is optimal to (P). □

We are now ready to close the loop of equivalence, to show **(C)⇒(A)**, which indicates that complementary duality is not only sufficient but also necessary for solving (LQ).

Theorem 3.5 If (LQ) has an attainable optimal control w.r.t. any initial condition x_0, then (P) and (D) must have complementary optimal solutions.

Proof. Since (LQ) has an attainable optimal control w.r.t. any initial condition x_0, it is known ([2, p.21]) that there exists $M \succeq 0$ such that

$$\inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0.$$

For the time being, suppose the matrix M is a feasible solution to (P). Fix an initial x_0 and let $u^*(\cdot)$ be the optimal control w.r.t. x_0. Since M is feasible to (P), using (11), we obtain the following identity:

$$\begin{aligned} J(x_0, u^*(\cdot)) &= x_0^T M x_0 \\ &+ \int_0^\infty \Big[(u^*(t) + R^+ B^T M x(t))^T R (u^*(t) + R^+ B^T M x(t)) + x(t)^T F(M) x(t)\Big] dt. \end{aligned}$$

Since $J(x_0, u^*(\cdot)) = x_0^T M x_0$, we have

$$\int_0^\infty \Big[(u^*(t) + R^+ B^T M x(t))^T R (u^*(t) + R^+ B^T M x(t)) + x(t)^T F(M) x(t)\Big] dt = 0.$$

Thus, $x(t)^T F(M) x(t) = 0$ for all $t \in [0, \infty)$. Since x_0 can be chosen arbitrarily, we conclude that $F(M) = 0$. The desired result then follows from Theorem 3.4. Furthermore, we know M must be optimal to (P).

What remains is to show the primal feasibility of M. To this end we consider the perturbed problem (P_ϵ) and its dual (D_ϵ) introduced in the proof of Proposition 2.4. For the optimal solution of (P_ϵ), denoted P_ϵ^*, there is a convergent subsequence such that

$$\lim_{i \to \infty} P_{\epsilon_i}^* = P_0^*,$$

with $\epsilon_i \to 0$ as $i \to \infty$. Clearly, P_0^* is a feasible solution of (P), since the feasible region of (P_ϵ) shrinks as $\epsilon \to 0$. We now show that $P_0^* = M$.

First, it follows from Lemma 2.3 that

$$\inf_{u(\cdot) \in U^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) = x_0^T P_{\epsilon_i}^* x_0$$

for all i, where

$$J_\epsilon(x_0, u(\cdot)) = \int_0^\infty [x(t)^T (Q + \epsilon I) x(t) + u(t)^T (R + \epsilon I) u(t)] dt.$$

Let $\bar{u}(\cdot)$ be the optimal control of (LQ) w.r.t. x_0 and $\bar{x}(\cdot)$ be the corresponding state. Then, we have

$$\begin{aligned} 0 &\leq \inf_{u(\cdot) \in U^{x_0}} J_\epsilon(x_0, u(\cdot)) - \inf_{u(\cdot) \in U^{x_0}} J(x_0, u(\cdot)) \\ &= \inf_{u(\cdot) \in U^{x_0}} J_\epsilon(x_0, u(\cdot)) - J(x_0, \bar{u}(\cdot)) \\ &\leq J_\epsilon(x_0, \bar{u}(\cdot)) - J(x_0, \bar{u}(\cdot)) \\ &= \epsilon \int_0^\infty (\|\bar{x}(t)\|^2 + \|\bar{u}(t)\|^2) dt. \end{aligned}$$

It then follows that

$$\lim_{i \to \infty} x_0^T P_{\epsilon_i}^* x_0 \equiv \lim_{i \to \infty} \inf_{u(\cdot) \in U^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) = \inf_{u(\cdot) \in U^{x_0}} J(x_0, u(\cdot)) \equiv x_0^T M x_0.$$

The above yields

$$x_0^T P_0^* x_0 = x_0^T M x_0$$

for all x_0, implying $M = P_0^*$. This shows that M is indeed a primal feasible solution, and consequently (P) and (D) have complementary optimal solutions as we discussed before. $\square$

To conclude this section we discuss the more general case that $Q \succeq 0$. The key advantage with a nonsingular Q, as we observed above, is that for any primal feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, the control in (8) is automatically stabilizing (see the proof of Theorem 3.3). This stability is no longer guaranteed when Q is possibly singular (see Example 5.3 below).

Theorem 3.6 Suppose $Q \succeq 0$.

(i) If (P) and (D) have complementary optimal solutions P^* and Z^*, respectively, then P^* must satisfy the generalized Riccati equation, $F(P) = 0$, and $u(t) = -R^+ B^T P^* x(t)$ is the optimal control that solves (LQ).

(ii) If (P) has a feasible solution P^* satisfying $P^* \succeq 0$ and $F(P^*) = 0$, and if the control $u(t) = -R^+ B^T P^* x(t)$ is stabilizing, then it must be the optimal solution to (LQ).

(iii) If (LQ) has an attainable optimal control, w.r.t. any initial condition x_0, then (P) must have an optimal solution P^* satisfying $F(P^*) = 0$. Moreover, if the feedback control $u(t) = -R^+ B^T P^* x(t)$ is stabilizing, then (P) and (D) must have complementary optimal solutions.

Proof. (i) This follows from Theorem 3.2, as the proof there does not require Q to be non-singular.

(ii) In the proof of Theorem 3.3, we established that the control in (8) is stabilizing due to Q being non-singular. Here, the stability of the control is assumed. On the other hand, note that the proof of Theorem 3.3 does not require the optimality of P^*; it only utilizes the fact that $P^* \succeq 0$ (which is implied by the optimality of P^* via Proposition 2.4). Hence the proof of Theorem 3.3 applies to the present case.

(iii) The proof of Theorem 3.5 implies that there is a primal feasible solution M with

$$\inf_{u(\cdot) \in U^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0, \text{ and } F(M) = 0.$$

On the other hand, in view of (12), we have

$$x_0^T M x_0 \equiv \inf_{u(\cdot) \in U^{x_0}} J(x_0, u(\cdot)) \geq x_0^T P x_0,$$

for any P feasible to (P). Hence M is optimal for (P). Let $P^* = M$. However, *a priori*, we do not know whether the control $u(t) = -R^+ B^T P^* x(t)$ is necessarily the the optimal control for (LQ); hence, its assumed stabilizing property is needed to guarantee the complementary optimal solutions. (Note, in Theorem 3.4, this stabilizing property is automatic, since $Q \succ 0$.)
$\square$

4 Regularization and Asymptotic Optimality

Complementary duality, as discussed in the last section, plays a central role in linking the SDP to the original (LQ) problem. Here, we further examine cases in which complementary duality holds or fails.

First, consider the case when complementary duality holds. Further, suppose $Q \succ 0$. According to Theorem 3.1, in this case there exists an optimal control to (LQ). Here, we further demonstrate that the (LQ) problem can be transformed into one with a *non-singular* R matrix.

Notice that $R \in \mathcal{S}^{m\times m}$ can be diagonalized by an orthonormal transformation as follows:

$$R = W^T \Lambda_r W$$

where $W^T W = I$ and Λ_r denotes a diagonal matrix whose first r diagonal components are positive and whose last $m-r$ components are zeros, with r being the rank of R.

Theorem 4.1 Suppose $Q \succ 0$; and suppose (P) and (D) have complementary optimal solutions. Suppose $R \in \mathcal{S}^{m\times m}$, and rank($R$)=$r$; and write $R = W^T \Lambda_r W$ as an orthonormal transformation. Then, the last $m-r$ columns of the matrix BW must be zero vectors.

Proof. By Theorem 3.1, the existence of a pair of complementary optimal solutions for (P) and (D), P^* and Z^*, implies that

$$F(P^*) = Q + A^T P^* + P^* A - P^* B R^+ B^T P^* = 0.$$

Furthermore, we claim that P^* cannot be singular. To see this, consider any ξ satisfying $P^*\xi = 0$. Pre- and post-multiplying ξ^T and ξ on both sides of the above equation yields $\xi^T Q \xi = 0$, and consequently $\xi = 0$, due to $Q \succ 0$. This shows that P^* is non-singular.

Once again observe $F(P^*) = 0$ where P^* is primal optimal. It follows from (9) that $\mathrm{rank}(\mathcal{L}(P^*)) = \mathrm{rank}(R) = r$. Observe that

$$\begin{bmatrix} W, & 0 \\ 0, & I \end{bmatrix} \mathcal{L}(P^*) \begin{bmatrix} W^T, & 0 \\ 0, & I \end{bmatrix} = \begin{bmatrix} \Lambda_r, & W B^T P^* \\ P^* B W^T, & Q + A^T P^* + P^* A \end{bmatrix}.$$

This implies that

$$\text{rank}([\Lambda_r, WB^T P^*]) = r,$$

and hence the last $m-r$ rows of $WB^T P^*$ must be zero vectors. Because P^* is non-singular as argued above, we conclude that the last $m-r$ rows of WB^T must all be zero vectors too. $\Box$

Theorem 4.1 implies that in the case when R is singular and when the SDP identifies complementary dual solutions, by a transformation on the control variables, $\bar{u}(t) := W^T u(t)$, the last $m-r$ control variables, $\bar{u}_{r+1}(t), \cdots, \bar{u}_m(t)$, will vanish from both the cost objective and the system dynamics. Hence, they can be removed from the problem. The new LQ problem becomes one with a nonsingular cost matrix that is the non-zero diagonal block of Λ_r.

Whereas the orthonormal transformation can always be applied to R no matter what, the key to the above analysis lies in identifying the complementary dual solutions. Without the existence of these solutions, (LQ) does not possess an optimal contorl, as stipulated in Theorem 3.1, and the orthonormal transformation becomes quite meaningless.

Next, we consider the case when complementary duality fails. Following Theorem 3.1, we know there is no optimal solution to the original LQ problem. In this case, we propose to consider the perturbed problem, (LQ_ϵ), obtained by keeping all the data A and B unchanged, and letting $Q_\epsilon = Q + \epsilon I$ (this transformation is not necessary when Q is non-singular) and $R_\epsilon = R + \epsilon I$ with $\epsilon > 0$. The corresponding perturbed SDP's have already appeared in the proof of Theorem 3.5.

By Lemma 2.3, (LQ$_\epsilon$) has an optimal feedback control for each $\epsilon > 0$; and both (P_ϵ) and (D_ϵ) satisfy Slater's condition.

Theorem 4.2 Suppose $Q \succ 0$. Let $J_\epsilon^*(x_0)$ and $J^*(x_0)$ be the optimal values of (LQ_ϵ) and (LQ), respectively. Then,

$$\lim_{\epsilon \downarrow 0} J_\epsilon^*(x_0) = J^*(x_0).$$

Proof. Let the optimal solution of (P_ϵ) be P_ϵ^*. Following Lemma 2.3, we know that

$$u^\epsilon(t) = -(R + \epsilon I)^{-1} B^T P_\epsilon^* x^\epsilon(t) \tag{14}$$

is optimal for (LQ$_\epsilon$), with the corresponding optimal objective value equal to $J_\epsilon^*(x_0) = x_0^T P_\epsilon^* x_0$.

Following the same argument as in the proof of Proposition 2.4, we know that P_ϵ^* is contained in a compact set, with $0 < \epsilon \leq 1$. Moreover, since by definition $J_\epsilon^*(x_0)$ decreases monotonically as $\epsilon \downarrow 0$, so does P_ϵ^*. Therefore, P_ϵ^* itself also converges as $\epsilon \downarrow 0$.

What remains is to show that $x_0^T P_0^* x_0$ is equal to the true infimum of (LQ), now denoted as $J^*(x_0)$. To this end, first note that

$$x_0^T P_\epsilon^* x_0 = J_\epsilon^*(x_0) \geq J^*(x_0),$$

where the inequality is due to the positive perturbation in (P$_\epsilon$). Letting $\epsilon \to 0$, we obtain

$$x_0^T P_0^* x_0 \geq J^*(x_0).$$

On the other hand, since P_0^* is feasible to (P) (see the proof of Proposition 2.4), it follows from (12) that

$$J^*(x_0) = \inf_{u(\cdot)\in U^{x_0}} J(x_0, u(\cdot)) \geq x_0^T P_0^* x_0.$$

This completes the proof. $\Box$

The above theorem says that the optimal values of the original and the perturbed problems are very close to each other or, more precisely, they are the same asymptotically. The next result is concerned with the asymptotically optimal feedback control of the original LQ problem.

Theorem 4.3 The feedback control $u^\epsilon(\cdot)$ constructed by (14) is asymptotically optimal for (LQ), namely,

$$\lim_{\epsilon\downarrow 0} J(x_0, u^\epsilon(\cdot)) = J^*(x_0).$$

Proof. Denote by $J_\epsilon(x_0, u(\cdot))$ the cost of the perturbed problem (LQ$_\epsilon$) under an admissible control $u(\cdot) \in U^{x_0}$ w.r.t. the initial state x_0. Then for any $\eta > 0$, there is an ϵ_0 such that when $0 < \epsilon < \epsilon_0$:

$$\begin{aligned} J^*(x_0) &\leq J(x_0, u^\epsilon(\cdot)) \\ &\leq J_\epsilon(x_0, u^\epsilon(\cdot)) \\ &= J_\epsilon^*(x_0) \\ &\leq J^*(x_0) + \eta, \end{aligned}$$

where the last inequality is due to Theorem 4.2. This proves our claim. $\Box$

5 Examples

In this section we present several examples including the two in §1 to illustrate our results. The first two examples demonstrate that in the absence of complementary duality, the LQ control has no solution. The third example illustrates that when Q is singular, the stability of the feedback control is not guaranteed. The fourth example is a positive one that illustrates how the SDP approach identifies the optimal feedback control when R is singular. All these examples are beyond the scope of the classical Riccati approach; whereas the SDP approach developed here completely solves the problems: it either identifies the optimal control or declares that no solution exists. The last two examples illustrate the use of regularization and the ϵ-approximation scheme presented in §4.

Example 5.1 Consider the problem in Example 1.1, where $A = -1$, $B = 1$, $Q = 1$ and $R = 0$ (hence $R^+ = 0$). The corresponding SDP reads:

$$\begin{array}{lll} (P_1) & \max & p \\ & \text{s.t.} & \begin{bmatrix} 0, & p \\ p, & 1-2p \end{bmatrix} \succeq 0. \end{array}$$

This problem has a unique feasible solution $p = 0$, which is hence optimal. The generalized Riccati equation in this case is $F(p) = 1 - 2p = 0$. So the optimal solution does not satisfy the generalized Riccati equation.

In view of Theorem 3.2, we expect complementary duality to fail. Consider the dual of (P_1),

$$\begin{array}{lll} (D_1) & \inf & z_n \\ & \text{s.t.} & 1 + 2z_u - 2z_n = 0 \\ & & z := \begin{bmatrix} z_b, & z_u \\ z_u, & z_n \end{bmatrix} \succeq 0. \end{array}$$

This problem is feasible and satisfies Slater's condition. Moreover, it has a finite optimal value equal to 0, which, however, is not attainable. Because whenever $z_n = 0$ we must have $z_u = -1/2$, violating $z \succeq 0$.

Example 5.2 Consider a two-dimensional problem with

$$A = \begin{bmatrix} 1, & 0 \\ 1, & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 1, & -\frac{1}{2} \\ -\frac{1}{2}, & 1 \end{bmatrix} \succ 0, \qquad R = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \succeq 0.$$

Note that $R^+ = R$. It is easy to see that the system is stabilizable (for example, $(u_1(t), u_2(t) = (-2x_1(t), 0)$ is a stabilizing control). The generalized Riccati equation is:

$$Q + A^T P + PA - PRP = 0. \tag{15}$$

If $P = \begin{bmatrix} p_1, & p_3 \\ p_3, & p_2 \end{bmatrix} \in \mathcal{S}^{2\times 2}$ is a feasible solution to the primal SDP (P), then by Lemma 2.1, it is necessary that $(I - RR^+)B^T P = 0$. This leads to $p_2 = p_3 = 0$. Thus P must be singular. Because Q is non-singular, similar to the argument in the proof of Theorem 4.1 we know that the generalized Riccati equation cannot be satisfied, and therefore by Theorem 3.1 the LQ problem does not have any attainable optimal control. In addition, by Theorem 3.2, there is no complementary dual solutions to (P) and (D). (One can also verify this directly; details are left to the reader.).

Example 5.3 Consider a 2-dimensional problem with

$$A = \begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1, & 1 \\ 1, & 1 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 2, & -2 \\ -2, & 2 \end{bmatrix} \succeq 0, \qquad R = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \succeq 0.$$

This system is easily seen to be stabilizable as $A - B = -I$. On the other hand, one can verify that

$$P^* = \begin{bmatrix} 1, & -1 \\ -1, & 1 \end{bmatrix} \succeq 0$$

is feasible to the primal SDP and satisfies the generalized Riccati equation. However,

$$A - BR^+B^TP^* \equiv A,$$

which has an eigenvalue 1. Hence the feedback control $u^*(t) = -R^+B^TP^*x^*(t)$ is *not* stabilizing.

Example 5.4 Consider Example 1.2. Here

$$A = \begin{bmatrix} 1, & 1 \\ 4, & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1, & 1 \\ 1, & -2 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 12, & 6 \\ 6, & 3 \end{bmatrix} \succeq 0, \qquad R = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix} \succeq 0.$$

Again this system is easily seen to be stabilizable. To identify a non-negative feasible solution P^* to (P) with $F(P^*) = 0$, first consider the constraint $(I - RR^+)B^TP^* = 0$ as stipulated by Lemma 2.1. This gives rise to

$$P^* = \begin{bmatrix} 2p, & p \\ p, & \frac{p}{2} \end{bmatrix}$$

for some p. Substituting the above into the generalized Riccati equation yields

$$A^TP^* + P^*A + Q - P^*BR^+B^TP^* \equiv \begin{bmatrix} -9p^2 + 12p + 12, & -\frac{9}{2}p^2 + 6p + 6 \\ -\frac{9}{2}p^2 + 6p + 6 & -\frac{9}{4}p^2 + 3p + 3 \end{bmatrix} = 0.$$

Solving for p leads to $p = 2$. Thus,

$$P^* = \begin{bmatrix} 4, & 2 \\ 2, & 1 \end{bmatrix} \succeq 0$$

is a primal feasible solution that satisfies $F(P) = 0$. On the other hand, in this case

$$A - BR^+B^TP^* = \begin{bmatrix} -5, & -2 \\ -2, & -2 \end{bmatrix},$$

which has eigenvalues -1 and -6. Hence $u^*(t) = -R^+B^TP^*x^*(t)$ is stabilizing. By Theorem 3.6-(ii), this control must be optimal.

Example 5.5 Consider a 2-dimensional problem with

$$A = \begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2, & 2 \\ 2, & 2 \end{bmatrix},$$

and

$$Q = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix} \succ 0, \qquad R = \begin{bmatrix} 1, & 1 \\ 1, & 1 \end{bmatrix} \succeq 0.$$

Similar to Example 5.3, this problem is stabilizable. Solving the primal SDP problem (P) yields

$$P^* = \begin{bmatrix} 1/2, & 0 \\ 0, & 1/2 \end{bmatrix}$$

which satisfies the generalized Riccati equation $F(P) = 0$. By Theorem 3.3, the following is the optimal feedback control:

$$u^*(t) = -R^+ B^T P^* x^*(t) = -\frac{1}{2} \begin{bmatrix} 1, & 1 \\ 1, & 1 \end{bmatrix} x^*(t),$$

or,

$$u_1^*(t) = -\frac{1}{2} x_1^*(t) - \frac{1}{2} x_2^*(t), \qquad u_2^*(t) = -\frac{1}{2} x_1^*(t) - \frac{1}{2} x_2^*(t).$$

Clearly, by a variable transformation $\bar{u}(t) := \frac{\sqrt{2}}{2}(u_1(t) + u_2(t))$ the original problem will be regularized as Theorem 4.1 asserts.

Example 5.6 Continue with Example 5.1. With perturbation, the primal SDP becomes

$$\begin{array}{lll} (P_\epsilon) & \max & p \\ & \text{s.t.} & \begin{bmatrix} \epsilon, & p \\ p, & 1-2p \end{bmatrix} \succeq 0. \end{array}$$

The optimal solution is:

$$p_\epsilon^* = \sqrt{\epsilon^2 + \epsilon} - \epsilon.$$

Clearly, $p_\epsilon^* = O(\sqrt{\epsilon})$, and hence the optimal value, $p_\epsilon^* x_0^2$, converges to 0 as $\epsilon \downarrow 0$. Further, to obtain an asymptotic optimal feedback control for the original problem, we apply the *same* feedback law

$$u^\epsilon(t) = -(\sqrt{1 + \frac{1}{\epsilon}} - 1) x^\epsilon(t),$$

which is optimal to the perturbed problem, to the original problem. It is straightforward to check that the corresponding cost under this control is

$$J(x_0, u^\epsilon(\cdot)) = \frac{\sqrt{\epsilon}}{2\sqrt{1+\epsilon}} x_0^2,$$

which converges to 0 as $\epsilon \downarrow 0$.

6 Concluding remarks

We have developed an SDP-based approach to the deterministic LQ control problem where the control cost matrix is possibly singular. Whereas the classical Riccati approach does not apply, our approach gives a complete solution to the problem: it either derives the optimal feedback control or determines that there is no optimal control for the problem. Specifically, we solve a pair of primal-dual SDP problems; if complementary dual solutions exist, then an optimal feedback control is explicitly constructed (based on the primal solution); otherwise, if complementary duality fails, we are guaranteed that there is no optimal control for the LQ problem. In the latter case, we have developed a convergent approximation scheme that achieves asymptotic optimality.

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