

# Energy Levels of “Hydrogen Atom” in Discrete Time Dynamics

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## Abstract

We analyze dynamical consequences of a conjecture that there exists a fundamental (indivisible) quant of time. In particular we study the problem of discrete energy levels of hydrogen atom. We are able to reconstruct potential which in discrete time formalism leads to energy levels of unperturbed hydrogen atom. We also consider linear energy levels of quantum harmonic oscillator and show how they are produced in the discrete time formalism. More generally, we show that in discrete time formalism finite motion in central potential leads to discrete energy spectrum, the property which is common for quantum mechanical theory. Thus deterministic (but discrete time!) dynamics is compatible with discrete energy levels.

## 1 Introduction

Discovery of discrete energy levels for atoms demonstrated that the classical Newtonian model could not be used to describe this phenomenon. Now days this phenomenon is described in the framework of quantum mechanics. The advantages of the computational methods of quantum mechanics are

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well known. They are widely used for computation of energy levels not only of atoms, but also in essentially more complicated situations. However, despite this computational power, quantum mechanics induced many still unsolvable phenomenological problems, see [1]-[37] for extended discussions. One of distinguishing features of quantum mechanics (at least in Copenhagen interpretation) is the impossibility to provide realist deterministic description of quantum reality. In particular, here particles do not have well defined trajectories.

The impossibility to use the deterministic evolutionary model looks rather counterintuitive. Moreover, it contradicts to many quantum experiments where "trajectories" of particles are well observed, e.g. in Wilson's camera. Of course, in modern quantum phenomenology this problem is solved via the principle of complementarity. Nevertheless, this strong deviation from our intuitive picture of physical reality induces a rather general opinion that quantum mechanics has a lot of mysteries [27].

This unsatisfactory status of quantum mechanics induces new and new attempts to create new models that would be closer to our physical intuition. We just mention one model, Bohmian mechanics [20, 21].

Last years there were performed intensive investigations to reconsider probabilistic foundations of quantum mechanics, see e.g. [36, 37]. These investigations are of the great importance. It is well known that "quantum probability" differs strongly from "classical probability". Typically, see e.g. [27], it is pointed out that quantum randomness is irreducible and fundamental (in the opposite to classical randomness that could be reduced to e.g. randomness of initial conditions). This irreducible quantum randomness is deeply connected with the impossibility of realist deterministic models that would reproduce quantum probabilities. It seems to be impossible to imagine that quantum particles have deterministic trajectories, since the existence of trajectories should imply classical probabilistic rules. But these rules are violated (for example, in the two-slit experiment [27], see also investigations on EPR-Bohm-Bell consideration and chameleon effect [17, 18, 19]).

In papers [38, 39, 40, 41] there was developed the contextual approach to quantum probabilities that might be used to explain the origin of quantum randomness in the classical (but contextual!) probabilistic framework. Therefore the probabilistic constraint to create realist deterministic models need not be taken into account<sup>1</sup>.

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<sup>1</sup>Of course, there are also Bell's inequality arguments. But there arguments are strongly

What kind of realist deterministic models could be created for quantum phenomena?

As we have already mentioned, we could not directly use Newtonian mechanics even for the simplest quantum system – the hydrogen atom. Thus Newton equations should be changed. The most straightforward idea is to change classical forces (in the case of atom Coulomb’s law) and find new forces, quantum forces. Such forces should ”drive” particles along trajectories that reproduce quantum data. There is a rather common view point that one of successful realizations of such a program is given by Bohmian mechanics. However, this problem is not simple, since Bohmian mechanics is, in fact, not mechanics, but a field model. There exists an additional field equation for quantum potential. We do not know any model that would give the realization of the discussed program of ”modernization” of Newtonian mechanics.

In the present paper we consider discrete time Newtonian model. This is a kind of classical physical model (in particular, realist deterministic). The only difference is discreteness of time. Thus we use the classical force – interaction picture, but the discrete time version of Newton’s second law. One of the main advantages of this model is its simplicity. We do not change phenomenology of classical physics (position, velocity, force-interaction, trajectories). We only change the mathematical representation of time.

On the other hand, discreteness is the main distinguishing feature of quantum physics. In fact, M. Planck and A. Einstein obtained Wien’s law simply by assuming discreteness of energy. However, this discreteness approach was not developed further to get a formalism based only on the discreteness postulate. Discreteness of quantum observables was reproduced by using an advanced mathematical formalism based on the representation of the observables in the complex Hilbert space. As we have already mentioned, the use of this formalism (despite its computational advantages) induced many phenomenological problems. We are trying to modify classical physics by starting with one natural postulate:

**TD: “Time is discrete”.**

Here, in particular, continuous Newton equations (differential equations) are just approximations of real equations of motions, namely, Newton’s differential equations coupled to nonlocality. And this problem is far from problems considered in the present paper.

ference equations<sup>2</sup>. In the present paper we use the discrete time model for the hydrogen atom. The discrete-time postulate implies discrete orbits and energy levels. At the same time we have deterministic motion along orbits. In the limit we get continuous motion along circular orbit (a kind of Bohr's correspondence principle).

In the present paper we do not try to develop some statistical theory (a kind of Born approach). Our investigation has some similarities with the original paper of W. Heisenberg (that was not statistical one), [1, 2]. We hope that starting with only discrete time postulate we would be able to develop simpler formalism to calculate spectra of quantum observables - without using noncommutative calculus and without a cardinal change of the phenomenology of classical physics.

It is interesting to point that there is some similarity between our approach and G. 't Hooft's approach [7, 8, 9] where a general scheme was proposed that maps states of quantum field system to the states of a completely deterministic field model. Although in this note we do not consider the field-particle duality it would be interesting to study this problem.

Some statistical consequences of the postulate (TD) were investigated in our previous papers [42, 43, 44].

## 2 Discrete Time Dynamics

In classical mechanics a dynamical function  $A = A(p, q)$  (here  $p$  and  $q$  are momenta and coordinates of the system) evolves according to the following well known equation [4]

$$D_t A = \{A, H\} \quad (1)$$

where  $H = H(p, q)$  is a Hamiltonian of the system and in the right hand side is a Poisson bracket, which could be presented as

$$\{A, B\} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q} \quad (2)$$

The left hand side of (1) contains a continuous time derivative

$$D_t A = \frac{dA}{dt}$$

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<sup>2</sup>We reverse the modern viewpoint on the description of physical reality. Not (discrete) difference equations are used to approximate (continuous) differential equations, but the inverse!

As it was mentioned earlier we are interested in construction dynamics with discrete time. This is done with the help of *discrete derivative* which is postulated to be

$$D_t^{(\tau)} A = \frac{1}{\tau} [A(t + \tau) - A(t)],$$

where  $\tau$  is the discreteness parameter. This parameter is finite and is treated in the same way as Plank constant in quantum mechanical formalism. In particular if  $\tau$  is small relative to dimensions of the system then classical approximation with continuous derivative might work well (although this could not be the case all the time in the same sense as there are examples when quantum formalism is reasonable even for macroscopic systems, for example in superfluidity).

Summarizing, the discrete time dynamical equation is postulated to be

$$D_t^{(\tau)} A = \{A, H\}, \quad (3)$$

where  $A(p, q)$  is a real-valued function of real-valued dynamical variables and in the right hand side there is classical Poisson bracket (2). The equation (3) could be solved in the sense that we can write

$$A(t + \tau) = A(t) + \tau \{A, H\} \quad (4)$$

thus providing the evolution of any dynamical function  $A = A(p, q)$ .

Note that in our model the coordinate space is continuous.

### 3 Motion in Central Potential

Here we will study the properties of motion in central potential  $U = U(r)$  in discrete time formalism. As we will see discreteness of time enriches mechanics with some new properties which are usually thought as having quantum nature. In particular as it will be shown below in discrete time mechanics stationary orbits (i.e. finite motion) have discrete energy spectrum. We point that the phase space is assumed here to be a continuous real manifold.

Following the general approach described in the previous section we start from the classical Hamiltonian and then write the dynamical equations. In polar coordinates  $(r, \varphi)$  the Hamiltonian of the system with mass  $m$  in central potential  $U(r)$  is given by

$$H(r, p_r, \varphi, p_\varphi) = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + U(r), \quad (5)$$

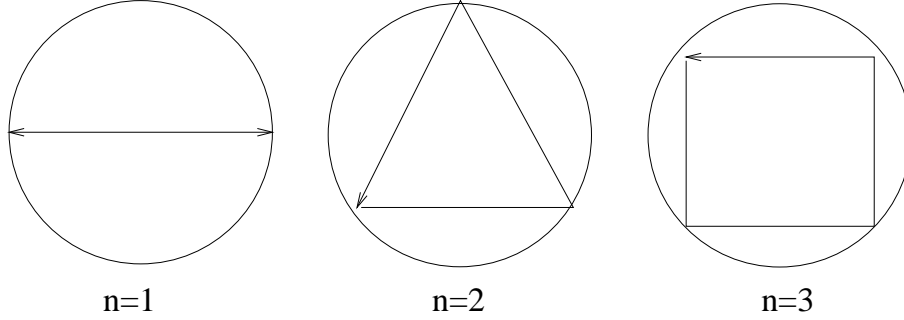


Figure 1: First three trajectories.

where  $p_r$  and  $p_\varphi$  denote momenta corresponding to  $r$  and  $\varphi$  – radial and angular coordinates respectively. Using (4) let us write the dynamical equations. We obtain

$$r(t + \tau) = r(t) + \tau \frac{p_r}{m} \quad (6)$$

$$p_r(t + \tau) = p_r(t) + \tau \left( \frac{p_\varphi^2}{mr^3} - \frac{\partial U}{\partial r} \right) \quad (7)$$

$$\varphi(t + \tau) = \varphi(t) + \tau \frac{p_\varphi}{mr^2} \quad (8)$$

$$p_\varphi(t + \tau) = p_\varphi(t) \quad (9)$$

The equation (9) corresponds to angular momentum conservation in central field – this is a direct analog of the angular momentum conservation law in classical (continuous time) dynamics and is a consequence of the fact that our Hamiltonian does not depend on  $\varphi$ , (it is a so called *cyclic variable*).

Let us limit ourself to circular stationary periodic orbits. In this case we should have  $r(t + \tau) = r(t)$  and thus using (6) we see that the radial momentum should be zero,

$$p_r(t) = 0 \quad (10)$$

From (7) and (10) we obtain the following condition

$$\frac{p_\varphi^2}{mr^3} = \frac{\partial U}{\partial r} \quad (11)$$

Let us finally come to the angular coordinate  $\varphi$ . For stable motion the following *periodicity* condition should be satisfied (Fig. 1)

$$\varphi(n\tau) = \varphi(0) + 2\pi, \quad (12)$$

where  $n = 1, 2, \dots$  (note that discreteness of  $n$  is a consequence of discreteness of time). Using (8) and (12) we get

$$n\tau \frac{p_\varphi}{mr^2} = 2\pi,$$

or

$$p_\varphi = \frac{2\pi mr^2}{n\tau} \quad (13)$$

From (11) and (13) we get the following equation for the radius of the  $n$ -th orbit

$$\frac{4\pi^2 mr}{n^2 \tau^2} = \frac{\partial U}{\partial r} \quad (14)$$

If potential  $U(r)$  is known then from equation (14) we can find  $r = r_n$ . If we consider physical potentials, i.e. potentials for which the force,  $-U'(r)$ , is smooth, negative, and is strictly monotonically decreasing in absolute value as  $r$  grows, vanishing on infinity, then solution of (14) always exists and unique. Now upon substituting  $r_n$  to the Hamiltonian (5) we obtain energy levels  $E_n$ .

For circular periodic orbits the original Hamiltonian (5) due to (10) and (11) simplifies to the following form

$$H = \frac{1}{2} r \frac{\partial U}{\partial r} + U(r)$$

Thus the expression for energy  $E_n$  of the  $n$ -th stable periodic orbit in terms of its radius  $r_n$  is given by

$$E_n = \frac{1}{2} r_n \left. \frac{\partial U}{\partial r} \right|_{r=r_n} + U(r_n), \quad n = 1, 2, \dots \quad (15)$$

As we see if potential  $U(r)$  allows stationary periodic motion and equation (14) has unique positive solutions  $r_n$  then the energy spectrum is discrete. This situation is directly analogous to quantum mechanics where for finite motion we might expect discrete energy levels.

## 4 Energy Levels of Hydrogen Atom

Our task in this section is to study whether the discrete time dynamics in central field described can lead to energy levels of hydrogen atom. We

treat energy levels as given (measured) quantities and our task is to find the corresponding potential. We restrict ourself to the simplest case when the atom is unperturbed by external electric or magnetic fields and thus currently we do not study splitting of the energy levels (note that in order to observe “degenerate” levels in quantum mechanical treatment one needs to somehow perturb the system in such a way that the levels split).

We start from the following energy spectrum for hydrogen atom [13]

$$E_n = -\frac{\gamma}{n^2}, \quad n = 1, 2, \dots, \quad (16)$$

where  $\gamma \approx 13.6 \text{ eV}$  is ionization energy of the hydrogen atom. The task is to find such  $U = U(r)$  that leads to the spectrum (16). Using (14), (15), and (16) we get

$$\frac{1}{2}\xi \frac{r_n^2}{n^2} + U(r_n) = -\frac{\gamma}{n^2}, \quad (17)$$

where the constant  $\xi$  is given by (note that  $\xi$  depends on discreteness parameter  $\tau$ )

$$\xi = \frac{4\pi^2 m}{\tau^2} \quad (18)$$

and  $n = 1, 2, \dots$ . Let us rewrite equation (17) in the following form

$$U(r_n) = -\frac{1}{n^2} \left( \frac{1}{2}\xi r_n^2 + \gamma \right) \quad (19)$$

We want to find  $U(r)$ . The idea is to obtain dependence of  $r_n = f(n)$  on  $n$  and then inverting it substitute  $n = f^{-1}(r_n)$  in (19), here  $f^{-1}$  denotes function inverse to  $f$ . As a result of this procedure we will get rid of explicit dependence of the right hand side of (19) on  $n$ , it will depend only on  $r_n$ . Then we interpolate the result for any  $r \geq 0$ . The resulting  $U(r)$  we check by substitution to (14)-(15).

Let us proceeding as described above. First, we have to find dependence of  $r_n$  on  $n$ . To do this let us assume that  $n$  is continuous and take the derivative in  $n$  of both sides of (19). This gives

$$\left. \frac{\partial U}{\partial r} \right|_{r=r_n} \frac{dr_n}{dn} = \frac{\xi r_n^2}{n^3} - \frac{\xi r_n r'_n}{n^2} + \frac{2\gamma}{n^3} \quad (20)$$

Now, from (14) we find

$$\left. \frac{\partial U}{\partial r} \right|_{r=r_n} = \xi \frac{r_n}{n^2} \quad (21)$$



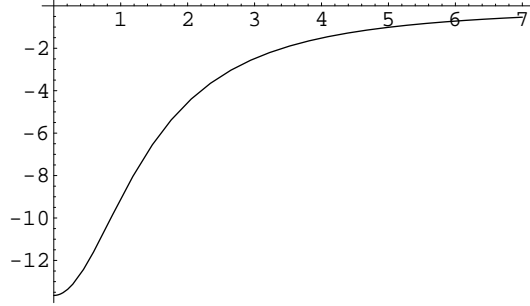


Figure 2: The shape of potential (25) which leads to energy levels of the hydrogen atom (we put  $\beta = \xi = \gamma = 1$ ).

Substituting (21) to (20) we obtain the following differential equation for  $r_n$

$$2\xi n r_n r'_n - \xi r_n^2 - 2\gamma = 0 \quad (22)$$

Solving (22) and taking into consideration only positive solution we obtain

$$r_n = \sqrt{\frac{-2\gamma + e^{2\beta\xi}n}{\xi}}, \quad (23)$$

where the constant  $\beta$  is due to integration. Inverting (23) we get

$$n = e^{-2\beta\xi}(2\gamma + r_n^2\xi) \quad (24)$$

Substituting (24) to (19) we obtain

$$U(r_n) = -\frac{e^{4\beta\xi}}{4\gamma + 2r_n^2\xi}$$

or performing interpolation, i.e. putting  $r$  in place of  $r_n$  we finally get

$$U(r) = -\frac{e^{4\beta\xi}}{4\gamma + 2r^2\xi} \quad (25)$$

Now, substituting (25) to (14)-(15) we come to the expected energy levels (16) of hydrogen atom.

The form of the potential (25) is presented on Fig.2. It is interesting to note that unlike Coulomb's potential it is nonsingular at  $r = 0$ .

## 5 Spectrum of Harmonic Oscillator

Procedure described in the previous section could be used to obtain potentials corresponding to arbitrary energy spectrum. Here we deduce the potential which results in linear energy levels of the homogeneous two dimensional quantum harmonic oscillator.

In quantum mechanics homogeneous two dimensional harmonic oscillator is a system described by the potential

$$U(r) = \frac{1}{2}m\omega^2 r^2, \quad (26)$$

Solving the Schrödinger equation in potential (26) results in the following energy spectrum

$$E_\Lambda = \hbar\omega(\Lambda + 1), \quad \Lambda = 0, 1, \dots \quad (27)$$

The simplest way to get this relation is to note that we deal with two uncoupled oscillators (indeed, if  $x$  and  $y$  are Cartesian coordinates then equation (26) takes the form  $U = \frac{1}{2}m\omega^2(x^2 + y^2)$ ), each having energy  $E_k = \hbar\omega(k + \frac{1}{2})$ , then the total energy is just the sum of two such terms and we get (27).

Let us first rewrite (27) in terms of  $n = 1, 2, \dots$ , we have  $n = \Lambda + 1$  and thus

$$E_n = \alpha n, \quad n = 1, 2, \dots, \quad (28)$$

where  $\alpha = \hbar\omega$ . Our task is to find potential  $U = U(r)$  which result in energy levels (28). Proceeding as in the previous section we assume that  $n$  is continuous and write the differential equation

$$\frac{dE_n}{dn} = \alpha \quad (29)$$

Now, using (14) and (15) equation (29) reduces to

$$2\xi \frac{r_n r'_n}{n^2} - \xi \frac{r_n^2}{n^3} = \alpha, \quad (30)$$

where  $\xi$  is given by (18). The positive solution of (30) is given by

$$r_n = \sqrt{\beta n + \frac{\alpha n^3}{2\xi}}, \quad (31)$$

where  $\beta$  is a constant due to integration. This leads us to a cubic equation in terms of  $n$ , which has one real solution

$$n = \frac{\mathcal{V}(r_n)}{3\alpha} - \frac{2\beta\xi}{\mathcal{V}(r_n)},$$

where  $\mathcal{V}(r)$  is given by

$$\mathcal{V}(r) = 3^{1/3} \left( 9r^2\alpha^2\xi + \sqrt{3}\sqrt{27r^4\alpha^4\xi^2 + 8\alpha^3\beta^3\xi^3} \right)^{1/3}$$

We finally get the expression for potential

$$U(r) = \frac{\mathcal{V}(r)}{3} - \frac{2\alpha\beta\xi}{\mathcal{V}(r)} - \frac{9r^2\mathcal{V}(r)^2\alpha^2\xi}{2(\mathcal{V}(r)^2 - 6\alpha\beta\xi)^2}$$

If we put  $\beta = 0$  in (31) the potential takes simple form

$$U(r) = \frac{3}{2} \left( \frac{r^2\alpha^2\xi}{4} \right)^{1/3} \quad (32)$$

Substituting (32) to (14)-(15) we get correct energy levels (28). It is interesting to provide the expression (32) when all constants are substituted, we have

$$U(r) = \frac{3}{2} \left( r \frac{\hbar\omega\pi\sqrt{m}}{\tau} \right)^{2/3}$$

## 6 General Case of Arbitrary Spectrum

One can find explicit relations for the radii of the orbits for arbitrary spectrum  $E_n$ . Indeed, From (14) and (15) we want to find  $U$  as a function of  $r$  in terms of a given energy spectrum  $E_n$ . As we did above we assume the continuity of the parameter  $n$  and take derivative of both parts of (15) in  $n$ , we have

$$E'_n = r'_n \frac{\xi r_n}{n^2} - \frac{\xi r_n^2}{n^3} + r'_n \frac{\partial U}{\partial r} \Big|_{r=r_n},$$

where prime denotes the derivative in  $n$ . The use of (14) allows us to get a differential equation for  $r_n$  in terms of only known quantities. We have

$$2r_n r'_n - \frac{1}{n} r_n^2 = \frac{n^2}{\xi} E'_n$$

Introducing a new variable  $\rho = r^2$  we obtain a linear differential equation which could be rewritten as

$$\frac{d}{dn} \left( \frac{\rho_n}{n} \right) = \frac{1}{\xi} n E_n'$$

which can be integrated to obtain

$$r_n = \sqrt{\frac{1}{\xi} n \left( n E_n - E_1 - \int_1^n E_k dk + \epsilon \right)} \quad (33)$$

Equation (33) expresses  $n$ -th radius in terms of  $n$ , i.e. it has the form  $r = f(n)$  now if we invert it we relate  $n$  in terms of  $r$ ,  $n = f^{-1}(r)$ , which if substituted to (15) gives an equation for  $U$  in terms of  $r$  only (actually in terms of  $r_n$ , but we perform interpolation effectively ignoring the fact that the relation strictly holds only for orbit radii).

Note that in (33) we have a constant  $\epsilon$  (having units of energy) arising due to the integration, this means that we have a set of potentials resulting in the same energy spectrum. As we see from (33) the constant  $\epsilon$  could be determined if for example the smallest radius  $r_1$  is known. In sections 4 and 5 the situation was the same resulting in the constant  $\beta$  (see (23) and (31)). Since  $\epsilon$  is more interesting from the point of view of physical interpretation we provide the expressions for  $\beta$  in terms of  $\epsilon$ . For the case of energy levels of hydrogen atom we have (see (23))

$$\beta_{hydr} = \frac{1}{2\xi} \ln(\epsilon + \gamma)$$

and for the case of harmonic oscillator we have (see (31))

$$\beta_{osc} = \frac{\epsilon - \frac{1}{2}\alpha}{\xi}$$

## 7 Energy Spectrum in Various Potentials

As we already seen, for a given potential it is straightforward to compute corresponding energy levels. Indeed, from (14) we find  $r = r_n$  and upon substitution to (15) we get  $E_n$ . Like in quantum mechanics, potential  $U = U(r)$  should be attractive and strong enough to result in finite motion. Below

we consider several common central potentials which result in rather simple expressions for energy spectrum. In what follows the constant  $\xi$  is given by (18), note that it depends on time discreteness parameter as  $1/\tau^2$ .

a). Coulomb potential

$$U(r) = -\frac{\alpha}{r}, \quad r_n = n^{2/3} \left( \frac{\alpha}{\xi} \right)^{1/3}, \quad E_n = -\frac{1}{2n^{2/3}} (\alpha^2 \xi)^{1/3}$$

Note that energy spectrum is different from the  $-\gamma/n^2$  spectrum obtained in quantum mechanics for this potential (see section on energy levels of hydrogen atom for detailed discussion).

b). Linear potential

$$U(r) = \alpha r, \quad r_n = \frac{n^2 \alpha}{\xi}, \quad E_n = \frac{3n^2 \alpha^2}{2\xi}$$

c). Logarithmic potential

$$U(r) = \alpha \ln r, \quad r_n = n \sqrt{\frac{\alpha}{\xi}}, \quad E_n = \alpha \left[ \frac{1}{2} + \ln \left( n \sqrt{\frac{\alpha}{\xi}} \right) \right]$$

For potentials (a), (b), and (c)  $r_n > 0$  if  $\alpha > 0$  in all three cases it corresponds to the attraction field.

d). Polynomial potential

$$U(r) = \alpha r^\sigma, \quad r_n = \left( \frac{n^2 \alpha \sigma}{\xi} \right)^{\frac{1}{2-\sigma}}, \quad E_n = \frac{1}{2} \alpha (2 + \sigma) \left( \frac{n^2 \alpha \sigma}{\xi} \right)^{\frac{\sigma}{2-\sigma}} \quad (34)$$

This case generalizes cases (a) and (b) described above, although because of importance of these potentials we provided corresponding expressions explicitly. Note that if we make  $\sigma$  in (34) satisfy the following equation

$$\frac{2\sigma}{2-\sigma} = 1,$$

i.e.  $\sigma = 2/3$  then we get the linear energy spectrum  $E_n \sim n$  for the 2D quantum harmonic oscillator (see previous section for detailed discussion).

## 8 Discussion and Conclusion

In this paper we have shown that discrete time formalism leads to some distinguishable properties of micro-observables that are used to be described with quantum mechanics. In particular it was shown that finite motion results in discrete energy spectrum. Of the main interest in this paper is discrete energy levels of hydrogen atom. We have shown that for unperturbed hydrogen atom the discrete time formalism is able to give correct energy spectrum, more precisely we have reconstructed the corresponding “micro”-potential. Here we did not consider Stark or Zeeman effects for hydrogen atom, it would be interesting to study them from the point of view of discrete time formalism.

As we have seen in above the discrete time model requires potentials which are different from QM potentials. One may argue this as a disadvantage of the model. We pay attention that there are no reasons to expect to reproduce QM by using the standard classical potentials; Bohr, Zommerfeld, Heisenberg and many others tried to do this, but they did not succeed. D. Bohm developed a new model [20] in which quantum mechanics can be reproduced on the classical basis, but, of course, the classical potential could not be preserved – it is perturbed by the quantum potential. And the latter looks not so natural from the classcial viewpoint, see e.g. the quantum potential for the two slit experiment in [20]. The discrete time model has an analogy with Bohmian mechanics – it tries to reproduce QM by changing potentials. But there is of course the fundamental difference: the only postulate that is used in the proposed approach is that there exists a quant of time  $\tau$ .

Another interesting point is that one might expect that our dynamical equations are essentially difference equations which might produce discrete spectrum. This is not correct – we recall that in our model only time is discrete, but space is still continuous.

There is still an open problem of the quantative value of the discreteness parameter  $\tau$ . One might speculate its relation with Plank time constant [34, 35] – the smallest measurable time interval in ordinary QM and gravity – which is quantatively given by

$$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \approx 5.3910^{-44} \text{ (sec.)}$$

The detailed analysis of this issue is out of the scope of the present article.

There might be an interesting interconnection on how the discrete time is used in information dynamics theory [10, 12] and the discrete time dynamics as it appears in presented study. In particular, it would be interesting to consider the equations for information dynamics with discrete time.

Finally, we would like to comment also that there might be a deep interrelation between the energy-time uncertainty relations [28, 29, 30] and Bohr-Somerfeld quantization rules [22] in quantum mechanics and our discrete time model. In particular one can try to write the Bohr-Somerfeld semi-classical quantization rules for energy and time as canonical variables. For a system with conserved energy one might get  $E_n T_n \sim n\hbar$ , this relation holds for example for energy levels  $E_n$  and classical periods  $T_n$  of Hydrogen atom. On the other hand the relation (12) in discrete mechanics could be treated as the condition for the period to take only discrete values  $T_n \sim n\tau$ . We can see that although relations are similar there is an extra factor  $E_n$  in the quantum-mechanical relation. In fact one may argue that if we make the  $\tau$  in equations of motion depend on the energy of the system as

$$\tau = \tau_0 \frac{\varepsilon}{E},$$

where  $\tau_0$  is the “fundamental” time quantum and  $\varepsilon$  a “fundamental” energy quantum, we get precisely the semiclassical quantization rules. The question arise how to treat the energy  $E$  here and what will happen with the dynamics. Further investigation of this interrelation will be discussed elsewhere.

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