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## Techniques for calculations with $n$ PI effective actions

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**Abstract.** We consider a symmetric scalar theory with quartic coupling in 2- and 3-dimensions and compare the self-consistent 4-point vertex obtained from the 4PI effective action with the Bethe-Salpeter 4-vertex from 2PI effective action. We show that when the coupling is large the contributions from the higher order effective action are large. We also show that one can solve the 2PI equations of motion in 4-dimensions, without introducing counter-terms, using a renormalization group method. This method provides a promising starting point to study the renormalization of higher order  $n$ PI theories.

### 1 Introduction

The resummation of certain classes of Feynman diagrams to infinite loop order is a useful method in quantum field theory when there is no small expansion parameter and standard perturbative methods do not apply. The  $n$ PI effective action formalism [1–7] is a non-perturbative approach in which the action is expressed as a functional of  $n$ -point functions which are determined through self-consistent stationary equations after the effective action is expanded to some order in the loop or  $1/N$  expansion. These self-consistent equations of motion resum certain classes of diagrams to infinite order. An important application of the  $n$ PI effective theory is the calculation of transport coefficients [8, 9]. In many cases, one needs to use a higher order ( $n > 2$ ) effective action to obtain even leading order transport coefficients [10–12]. A major problem with higher order  $n$ PI calculations is the issue of renormalization. The renormalization of the 2PI effective theory requires introducing a set of vertex counter-terms which obey different renormalization conditions and approach each other in the limit that the order of the approximation is taken to infinity [13–16]. It is not known how to renormalize an  $n$ PI theory for  $n > 2$ .

We calculate the self-consistent 4-vertex from a 4-Loop 4PI symmetric scalar theory with quartic interaction in 3-dimensions, where there are no vertex divergences and renormalization requires only a mass counter-term. We compare the result with the perturbative 4-vertex, and the Bethe-Salpeter 4-vertex obtained from the 2PI effective action. We find that all three vertices agree well when the coupling is small. In addition, the 2PI and 4PI vertices agree well for certain (diagonal) momentum configurations. For large coupling and non-diagonal momentum arguments, the three vertices differ significantly. These results indicate that higher order effective actions can play an important role in calculations of non-perturbative quantities.

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As a first step towards a 4-dimension 4PI calculation, we do a 4-dimension 2PI calculation using a renormalization group (RG) technique, which allows us to perform the renormalization without using counter-terms. We check our method by comparing the result with the traditional 2PI calculation. The method is straightforward to extend to higher order  $n$ PI theories.

## 2 Notation

In most equations in this paper we suppress the arguments that denote the space-time dependence of functions. As an example of this notation, the quadratic term in the action is written:

$$\frac{i}{2} \int d^4x d^4y \varphi(x) G_{\text{no-int}}^{-1}(x-y) \varphi(y) \longrightarrow \frac{i}{2} \varphi G_{\text{no-int}}^{-1} \varphi. \quad (1)$$

We use the notation  $G_{\text{no-int}}$  for the bare propagator because we reserve  $G_0$  for the propagator in the RG calculation in the limit that the regulator goes to zero. The classical action is

$$S[\varphi] = \frac{i}{2} \varphi G_{\text{no-int}}^{-1} \varphi - \frac{i}{4!} \lambda \varphi^4, \quad iG_{\text{no-int}}^{-1} = -(\square + m^2). \quad (2)$$

For notational convenience we use a scaled version of the physical coupling constant ( $\lambda_{\text{phys}} = i\lambda$ ). The extra factor of  $i$  will be removed when rotating to Euclidian space to do numerical calculations. The 4PI effective action is obtained from the Legendre transformation of the connected generating functional:

$$Z[R_1, R_2, R_3, R_4] = \int [d\varphi] \text{Exp}[i(S_{cl}[\varphi] + R_1\varphi + \frac{1}{2}R_2\varphi^2 + \frac{1}{3!}R_3\varphi^3 + \frac{1}{4!}R_4\varphi^4)], \quad (3)$$

$$W[R_1, R_2, R_3, R_4] = -i \text{Ln} Z[R_1, R_2, R_3, R_4],$$

$$\Gamma[\phi, G, V_3, V_4] = W - R_1 \frac{\delta W}{\delta R_1} - R_2 \frac{\delta W}{\delta R_2} - R_3 \frac{\delta W}{\delta R_3} - R_4 \frac{\delta W}{\delta R_4}.$$

The 2PI effective action can be obtained from (3) by setting  $R_3 = R_4 = 0$ . Higher order effective actions can be obtained by adding additional source terms. The Legendre transforms can be done using the method of subsequent transformations [4, 5] which involves starting from an expression for the 2PI effective action and exploiting the fact that the source terms  $R_3$  and  $R_4$  can be combined with the corresponding bare vertex by defining a modified interaction vertex. We consider only the symmetric theory for which odd  $n$ -point functions are zero. We write the result as a function of renormalized variables without introducing additional subscripts:

$$\Gamma[\phi, G, V] = \Gamma_{\text{no-int}}[\phi, G] + \Gamma_{\text{int}}[\phi, G, V], \quad (4)$$

$$\Gamma_{\text{no-int}}[\phi, G] = \frac{i}{2} \phi G_{\text{no-int}}^{-1} \phi + \frac{i}{2} \text{Tr} \ln G^{-1} + \frac{i}{2} \text{Tr} G_{\text{no-int}}^{-1} G,$$

Throughout this paper we use the notation  $i\Gamma = \Phi$  where both  $\Gamma$  and  $\Phi$  carry the same subscripts or superscripts. For example, for the 2PI effective action we write  $i\Gamma[\phi, G] = \Phi[\phi, G]$ , for the interacting part of the 2PI effective action we have  $i\Gamma_{\text{int}}[\phi, G] = \Phi_{\text{int}}[\phi, G]$ , etc. We show the result for  $\Phi_{\text{int}}$ , not including counter-terms, for the symmetric 2PI and 4PI theories in Fig. 1. 2.

## 3 4PI in 3-dimensions

The self-consistent 2- and 4-point functions in the symmetric phase of the 4PI theory are obtained by solving simultaneously the equations of motion:

$$\left. \frac{\delta \Gamma[G, V]}{\delta G} \right|_{G=\tilde{G}, V=\tilde{V}} = 0, \quad \left. \frac{\delta \Gamma[G, V]}{\delta V} \right|_{G=\tilde{G}, V=\tilde{V}} = 0. \quad (5)$$

For the 2PI theory the effective action depends only on the self-consistent 2-point function, and we have only the first equation in (5).

The equations of motion obtained from the 4-Loop 4PI theory are:

$$\begin{aligned} \Sigma(P) &= i(\delta Z P^2 - \delta m^2) + \frac{1}{2}(\lambda + \delta\lambda) \int dQ G(Q) \\ &+ \frac{1}{6}(\lambda + \delta\lambda_{bb}) \int dQ \int dK V(P, Q, K) G(Q) G(K) G(Q + K + P). \end{aligned} \quad (6)$$

$$\begin{aligned} V(P_a, P_b, P_c) &= \lambda + \delta\lambda + V_s(P_a, P_b, P_c) + V_t(P_a, P_b, P_c) + V_u(P_a, P_b, P_c), \\ V_s(P_a, P_b, P_c) &= \frac{1}{2} \int dQ V(P_a, P_c, Q) G(Q) G(Q + P_a + P_c) V(P_b, P_d, -Q), \\ V_t(P_a, P_b, P_c) &= V_s(P_a, P_c, P_b), \\ V_u(P_a, P_b, P_c) &= V_s(P_a, P_b, P_d), \quad P_d = -(P_a + P_b + P_c), \end{aligned} \quad (7)$$

where we have defined  $\Sigma = 2\delta\Phi/\delta G$ . The equation of motion for the 2-point function has been simplified using the eom for the 4-point function, which allows one to cancel the 3-loop diagram which would otherwise appear.

Although the 2PI effective action does not explicitly contain a 4-vertex, it is well known that it can be used to obtain a non-perturbative 4-point vertex called the Bethe-Salpeter (BS) vertex [13]. The equation that determines the 2PI BS vertex is obtained by calculating the functional derivatives of the 2PI effective action with respect to the 2-point function and the bilocal source. We give here only the result, for further details see [17]. We consider systems in thermal equilibrium for which the system is invariant under space-time translations. We define the 4-kernel  $\Lambda = 4\delta^2\Phi_{\text{int}}/\delta G^2$  and write the BS equation in momentum space as:

$$M(P, -P, Q, -Q) = \Lambda(P, -P, Q, -Q) + \frac{1}{2} \int dK \Lambda(P, -P, K, -K) G^2(K) M(K, -K, Q, -Q). \quad (8)$$

Equation (8) is shown diagrammatically in figure 2.

We solve the self-consistent equations of motion for the 2- and 4-point functions in Euclidean space using a numerical lattice method, for details see [18, 19]. There are no ultra-violet divergences in the 4-point vertices in less than four dimensions, and therefore we set the coupling strength counter-term to zero, i.e.,  $\delta\lambda = 0$ . For the 2-point function, the only fundamental divergences are the tadpole and sunset diagrams. The tadpole diagram has a momentum independent divergence in 2D and is finite in 3D. The sunset diagram is finite in 2D and has a momentum independent divergence in 3D. Since we have no momentum dependent divergences, we can also set  $\delta Z = 0$  and renormalize the propagator with the counter-term  $\delta m^2$ . To determine this counter-term we use the renormalization condition  $\Sigma(0) = 0$ , which means we can drop the tadpole diagram. Note that if one expands the equation of motion for the 4-vertex  $V$  obtained from the 4PI effective action, or the equation for the BS vertex from the 2PI effective action, each self-energy insertion is accompanied by the mass counter-term that makes it finite, and therefore there are no sub-divergences. In Fig. 3 we show some of our results. At large coupling, the 4PI vertex differs significantly from the 2PI and perturbative ones.

## 4 2PI in 4-dimensions

In 4-dimensions, we do not know how to renormalize the 4PI theory, but we can work in 4-dimensions at the 2PI level. To perform the renormalization, one starts by adding counter-terms to each local, mass

dimension 4 operator in the effective action

$$\Phi_{\Delta} = \frac{1}{4!}\Delta\lambda_4\phi^4 + \frac{1}{4}\Delta\lambda_{\text{tp}}\phi^2G + \frac{1}{8}\Delta\lambda_{\text{et}}G^2. \quad (9)$$

In addition, one includes the usual counter-terms in the skeleton expansion of the effective action, to the approximation order. For example, to order  $\lambda^3$  we have:

$$\begin{aligned} \Phi_{\text{ct}} &= -\frac{i}{2}(\delta Z_2\Box + \delta m_2^2)\phi^2 - \frac{i}{2}(\delta Z_0\Box + \delta m_0^2)\text{Tr}G + \frac{1}{4!}\delta\lambda'_4\phi^4 + \frac{1}{4}\delta\lambda'_{\text{tp}}\phi^2G \\ &+ \frac{1}{3}\lambda\delta\lambda_{\text{egg}}\phi^2G^3 + \frac{1}{8}\delta\lambda'_{\text{et}}G^2 + \frac{1}{24}\delta\lambda_{\text{bb}}\lambda G^3 + \mathcal{O}(\lambda^4). \end{aligned} \quad (10)$$

Primes are used for counter-terms which have partners in (9). We define

$$\delta\lambda_4 = \delta\lambda'_4 + \Delta\lambda_4, \quad \delta\lambda_{\text{tp}} = \delta\lambda'_{\text{tp}} + \Delta\lambda_{\text{tp}}, \quad \delta\lambda_{\text{et}} = \delta\lambda'_{\text{et}} + \Delta\lambda_{\text{et}}. \quad (11)$$

The coupling counter-terms in  $\Phi_{\text{ct}}$  are chosen to cancel divergences in the integrals in the 4-kernels, and the coupling counter-terms in  $\Phi_{\Delta}$  cancel the remaining divergences in the resummed 4-point vertices. The 4-kernels are divergent (for example  $\Lambda = \Delta\lambda_{\text{et}} + \Lambda_f$  where  $\Lambda_f$  is the quantity that is made finite by  $\lambda'_{\text{et}} + \lambda_{\text{bb}} + \dots$ ), but this is not a problem since the 4-kernels are not directly related to physical quantities. The 2-point functions contain coupling constant counter-terms that must be obtained self-consistently from the BS equation.

In the truncated theory, the different  $n$ -point functions are not the same, and similarly the counter-terms which are differentiated by subscripts are not the same. Renormalizability requires only that the untruncated (exact) theory contains one mass counter-term, one wave-function renormalization counter-term and one coupling constant counter-term, which produce one renormalized 2-point function and one renormalized 4-point function. The counter-terms introduced in (9 - 11) must therefore satisfy  $\{\delta m_2^2, \delta m_0^2\} \rightarrow \delta m^2$ ,  $\{\delta Z_2, \delta Z_0\} \rightarrow \delta Z$  and  $\{\delta\lambda_4, \delta\lambda_{\text{tp}}, \delta\lambda_{\text{egg}}, \delta\lambda_{\text{et}}, \delta\lambda_{\text{bb}} \dots\} \rightarrow \delta\lambda$  when the order of the approximation is taken to infinity. In this paper we consider only the symmetric theory at the 4-Loop level, and therefore we only need the counter-terms  $\delta m_0^2$ ,  $\delta Z_0$  and  $\delta\lambda_{\text{et}}$ . We use the renormalization conditions:

$$-iG^{-1}(0) = m^2, \quad -i\frac{d}{dP^2}G^{-1}\Big|_{P=0} = -1, \quad M(0,0) = \lambda,$$

where the notation  $(0)$ ,  $(0,0)$ , etc., indicates that all momentum components of each leg are set to zero.

## 5 Renormalization Group method

Using the functional renormalization group method, we do not use counter-terms but instead we add to the action in (2) a non-local regulator term

$$S_{\kappa}[\varphi] = S[\varphi] + \Delta S_{\kappa}[\varphi], \quad \Delta S_{\kappa}[\varphi] = -\frac{1}{2}\hat{R}_{\kappa}\varphi^2. \quad (12)$$

The bare mass and coupling are defined at an ultra-violet scale  $\mu$  which must be specified (we use  $\mu$  instead of the traditional  $\Lambda$  because that letter has already been used for the 4-point kernel). The parameter  $\kappa$  has dimensions of momentum and the regulator  $\hat{R}_{\kappa}(Q)$  is chosen to have the following properties: when  $Q \ll \kappa$ ,  $\hat{R}_{\kappa}(Q) \sim \kappa^2$ , and when  $Q \geq \kappa$ ,  $\hat{R}_{\kappa}(Q) \rightarrow 0$ . The effect is therefore that (1) for  $Q \ll \kappa$  the regulator is a large mass term which suppresses quantum fluctuations with wavelengths  $1/Q \gg 1/\kappa$  and; (2) fluctuations with  $Q \gg \kappa$  and wavelengths  $1/Q \ll 1/\kappa$  are not affected by the presence of the regulator.

The  $n$ -point functions of the theory depend on the parameter  $\kappa$  and the goal is to calculate them in the limit  $\kappa \rightarrow 0$ , where the full quantum theory is restored. One obtains a hierarchy of coupled differential ‘flow’ equations for the derivatives of the  $n$ -point functions with respect to  $\kappa$ . The formal relationship between the RG method and 2PI theories has been studied in [23–26], and the connection with higher  $n$ PI theories was developed in [27]. We will show that when the 2PI effective action is used, the hierarchy of flow equations is truncated when the effective action is. There are several advantages to truncating at the level of the effective action. One is that it is straightforward to systematically extend the order of the approximation. Another is that the truncation respects the symmetries of the original theory, to the order of the approximation [21, 22]. To solve the flow equations, one chooses  $\mu$  large enough that when  $\kappa = \mu$  the theory is classical and the 2- and 4-point functions are known functions of the bare parameters. The flow equations can then be integrated from the scale  $\kappa = \mu$ , using the known classical solutions as boundary conditions, to the scale  $\kappa = 0$ , at which the desired quantum solutions are obtained.

A fundamental technical difficulty with the RG formalism is created by the fact that the renormalization conditions are defined in terms of the quantum ( $\kappa \rightarrow 0$ )  $n$ -point functions, which are obtained only after the calculation is finished. We want to specify chosen values for the renormalized mass and coupling, but the required calculational input is the bare parameters, not the renormalized ones. An arbitrary choice of the bare parameters will not produce the chosen renormalized parameters, and we do not know in advance which choice of bare parameters will. We must “tune” the bare parameters, so that the renormalized mass and coupling that are produced by the calculation are the ones that were originally specified.

The 2PI effective action in the FRG formalism is obtained from equations (3) using the regulated action (12):

$$Z_\kappa[R_1, R_2] = \int [d\varphi] \exp \left\{ i \left( S[\varphi] + R_1 \varphi + \frac{1}{2} R_2 \varphi^2 - \frac{1}{2} \hat{R}_\kappa \varphi^2 \right) \right\}, \quad (13)$$

$$W_\kappa[R_1, R_2] = -i \ln Z_\kappa[R_1, R_2], \quad (14)$$

$$\frac{\delta W_\kappa[R_1, R_2]}{\delta R_1} = \langle \varphi \rangle \equiv \phi, \quad \frac{\delta W_\kappa[R_1, R_2]}{\delta R_2} = \frac{1}{2} \langle \varphi^2 \rangle = \frac{1}{2} (\phi^2 + G). \quad (15)$$

The expectation values are calculated in the presence of the regulator and therefore depend on the parameter  $\kappa$ , which means that the relations between  $(\phi, G)$  and  $(R_1, R_2)$  are  $\kappa$ -dependent. The 2PI effective action is obtained by taking the double Legendre transform of the generating functional  $W_\kappa[R_1, R_2]$  with respect to the sources  $R_1$  and  $R_2$  and taking  $\phi$  and  $G$  as the independent variables (see equation (3)):

$$\hat{\Gamma}_\kappa[\phi, G] = W_\kappa - R_1 \frac{\delta W_\kappa}{\delta R_1} - R_2 \frac{\delta W_\kappa}{\delta R_2} = W_\kappa - R_1 \phi - \frac{1}{2} R_2 (\phi^2 + G). \quad (16)$$

After performing the Legendre transform, the functional arguments of the effective action  $\phi$  and  $G$  are independent of the regulator function and the parameter  $\kappa$ , but the non-interacting propagator does depend on  $\kappa$ . We define

$$iG_{\text{no-int}\kappa}^{-1} = iG_{\text{no-int}}^{-1} - \hat{R}_\kappa = -\square - (m^2 + \hat{R}_\kappa). \quad (17)$$

It is useful to define an effective action that corresponds to the original classical action at the scale  $\mu$ :

$$\Gamma_\kappa = \hat{\Gamma}_\kappa - \Delta S_\kappa(\phi). \quad (18)$$

To make the equations look nicer we define an imaginary regulator function  $R_\kappa = -i\hat{R}_\kappa$  (the extra factor  $i$  will be removed when we rotate to Euclidian space to do the numerical calculation). The regulated 2PI effective action satisfies the flow equation [28]:

$$\partial_\kappa \Phi_\kappa = \frac{1}{2} \partial_\kappa R_\kappa (\langle \varphi^2 \rangle - \phi^2) = \frac{1}{2} \partial_\kappa R_\kappa G. \quad (19)$$

This result has the same form for any  $n$ PI effective action. The difference in the flow equations for different effective actions is contained in the definition of the expectation values.

A hierarchy of flow equations can be obtained by taking functional derivatives of (19). Using the 1PI effective action (19) becomes

$$\partial_\kappa \Phi_{1\text{PI}\cdot\kappa} = -\frac{1}{2} \partial_\kappa R_\kappa \left[ \frac{\delta^2 \Phi_{1\text{PI}\cdot\kappa}}{\delta \phi^2} + R_\kappa \right]^{-1}. \quad (20)$$

and taking derivatives with respect to  $\phi$  produces the well known infinite hierarchy of functional renormalization group equations. Practical calculations require a truncation of this hierarchy and there is a priori no clear way to decide how to perform this truncation.

Using a similar method we can obtain flow equations for the kernels obtained from the regulated 2PI effective action. We define ( $\kappa$  dependent) kernels

$$\Phi_{\text{int}\cdot\kappa}^{(n,m)} = 2^m \frac{\delta^n}{\delta \phi^n} \frac{\delta^m}{\delta G^m} \Phi_{\text{int}} \Big|_{\substack{G=G_\kappa \\ \phi=0}}. \quad (21)$$

Functionally differentiating (19) produces an infinite hierarchy of coupled equations for the flow of these kernels:

$$\partial_\kappa \Phi_{\text{int}\kappa}^{(n,m)} \Big|_{\substack{G=G_\kappa \\ \phi=0}} = \frac{1}{2} \partial_\kappa (R_\kappa + \Sigma_\kappa) G_\kappa^2 \partial_\kappa \Phi_{\text{int}\kappa}^{(n,m+1)} \Big|_{\substack{G=G_\kappa \\ \phi=0}}. \quad (22)$$

The self-consistent 2-point function which solves the equation of motion depends on  $\kappa$ , and therefore we do not need to use a tilde to denote the self-consistent solution (as we did in the 2PI calculation), but write it instead as  $G_\kappa$ .

Equation (22) gives a series of infinite hierarchies of coupled equations for the 2PI kernels in which kernels with fixed  $n$  and different  $m$  are coupled together. However, unlike the hierarchy produced from the 1PI effective action, when the 2PI effective action is truncated at some finite loop order, the hierarchy in (22) is also truncated.

We consider the symmetric theory and give the first two flow equations including terms up to order  $\lambda^2$  in the skeleton expansion. In Euclidian space we obtain

$$\partial_\kappa \Sigma_\kappa(P) = \frac{1}{2} \int dQ \partial_\kappa (\Sigma_\kappa(Q) + R_\kappa(Q)) G_\kappa^2(Q) \Lambda_\kappa(Q, P), \quad (23)$$

$$\partial_\kappa \Lambda_\kappa(P, K) = \frac{1}{2} \int dQ \partial_\kappa [R_\kappa(Q) + \Sigma_\kappa(Q)] G_\kappa^2(Q) \Lambda_\kappa^{03}(Q, P, K). \quad (24)$$

Using  $(n, m) = (0, 3)$  can write an equation for the flow of the kernel  $\Lambda_\kappa^{03}$  of the form  $\partial_\kappa \Lambda_\kappa^{03} \sim \int dQ \partial_\kappa G_\kappa \Lambda_\kappa^{04}$ . At the level of our approximation however, the kernel  $\Lambda_\kappa^{04}$  is a constant, and therefore the right side of the equation for  $\partial_\kappa \Lambda_\kappa^{03}$  is an exact differential which can be integrated directly. The integration constant must be set to zero because there is no 6-vertex in the Lagrangian. Equivalently, one can simply obtain  $\Lambda_\kappa^{03}$  directly from the effective action using (21):

$$\begin{aligned} & \Lambda^{03}(Q, P, K) \\ & = -\lambda^2 (G_\kappa(Q + P + K) + G_\kappa(Q + P - K) + G_\kappa(Q - P + K) + G_\kappa(Q - P - K)). \end{aligned} \quad (25)$$

In order to solve these flow equations, one must specify the boundary conditions from which the flow starts at  $\kappa = \mu$ . These boundary conditions must be consistent with the renormalization conditions that we want to impose at the  $\kappa = 0$  end of the flow. One can show [20] that the boundary conditions

$$\mathbf{G}_\mu^{-1} = Z_\mu(P^2 + m_\mu^2), \quad \Lambda_\mu(P, Q) = -\lambda_\mu \quad (26)$$

are consistent with the renormalization conditions analogous to (12).

The equations (23, 24, 25) can be solved simultaneously using the boundary conditions (26). The momentum integral in (24) is completely finite except for a momentum independent piece which can be absorbed into the integration constant. Similarly, the momentum integral in (23) is finite except for a momentum independent divergence that can be absorbed into the definition of  $m_\mu$ . The result is therefore that all of the divergent contributions have been absorbed into the definitions of the parameters  $m_\mu$  and  $\lambda_\mu$ . For a given choice of the function  $R_\kappa$ , the theory is completely specified by the flow equations and the initial conditions, and one may “forget” their origins from a functional integral. In fact, since the integral on the right side of the flow equation (24) is finite and the integration constant is known, we do not need to solve the flow equation for  $\Lambda_\kappa$ , but can write directly:

$$\Lambda_\kappa(P, Q) = -\lambda_\mu + \frac{\lambda^2}{2} \int dQ G_\kappa(Q) [G_\kappa(Q + P - K) + G_\kappa(Q + P + K)]. \quad (27)$$

It is easy to see that this expression satisfies (24) together with (25). One can also verify that (27) satisfies the boundary condition (26), by showing that in the limit  $\mu \gg \{P, K\}$  the integral reduces to a constant, which can be absorbed into the definition of  $\lambda_\mu$ .

The expression in equation (27) is just the 2PI kernel with the tree term replaced by the vertex  $\lambda_\mu$ . However, we can *not* start from some kind of similarly modified 2PI expression for  $\Sigma_\kappa$ , because the flow equation for the 2-point function (23) contains an embedded sub-divergence which cannot be removed unless  $\Lambda_\kappa$  is calculated self-consistently from its flow equation.

## 6 Numerical Technique

In order to do the numerical calculation, we restrict to a box in co-ordinate space of finite volume  $L^3\beta$ . Fourier transforming to momentum space one obtains discrete frequencies and momenta. This can be written

$$\int \frac{dp_4}{2\pi} \prod_{i=1}^3 \int_{-\infty}^{\infty} \frac{dp_i}{2\pi} f(p_4, p_i) \rightarrow \frac{m_t m_s^3}{(2\pi)^4} \sum_{n_4 = -\frac{N_t}{2} + 1}^{\frac{N_t}{2}} \prod_{i=1}^3 \sum_{n_i = -\frac{N_s}{2} + 1}^{\frac{N_s}{2}} f(m_t n_4, m_s n_i), \quad (28)$$

$$m_t = 2\pi T = 2\pi/(N_t a_t), \quad m_s = 2\pi L^{-1} = 2\pi/(N_s a_s), \quad L = a_s N_s, \quad T = 1/a_t N_t. \quad (29)$$

The parameters  $a_t$  and  $a_s$  are the lattice spacing in the temporal and spatial directions. Indices which fall outside of the range  $\{-N/2 + 1, N/2\}$  are wrapped inside using periodic boundary conditions. The scalar  $\phi^4$  theory in 4-dimensions is non-interacting if it is considered as a fundamental theory valid for arbitrarily high momentum scales (quantum triviality), but the renormalized coupling is non-zero if the theory has an ultra-violet cutoff and an infra-red regulator. In our calculation the lattice spacing parameters  $a_t$  and  $a_s$  provide an ultra-violet cutoff for the  $p_4$  and  $p_i$  momentum integrals, and the mass  $m$  regulates the momentum integrals in the infra-red. The theory has a Landau pole at a scale which decreases as the coupling increases and, in order to avoid unphysical behaviour, the ultra-violet cut off must be less than the scale at which the Landau pole appears. This means that problems will occur when  $\lambda$  is very large, or  $a_s$  is very small.

We will do the numerical calculations using the renormalized parameters  $m = 1$  and  $\lambda = 1$ . We choose  $N_s = 32$ , and  $L = a_s N_s = 2$ . The renormalization is done with  $N_t = 128$ ,  $a_t = 1/16$ , and finite temperature calculations are done with  $126 \geq N_t \geq 8$ . All calculations are done using fast Fourier transforms, to improve performance. We use  $a_t = 1/16$ ,  $a_s = 1/16$ ,  $N_s = 32$  and renormalize the theory at  $T_0 = (N_t^{\max} a_t)^{-1}$ . The maximum value of  $N_t$  is limited by computation time and memory. We use  $N_t^{\max} = 128$  which gives  $T_0 = 0.125$ . In the RG calculation we use  $\mu = 100$  and solve the flow equations in  $N_\kappa = 50$  steps down to  $\kappa_{\min} = 0.01$ . We have checked that the RG results are unchanged if  $\kappa_{\min}$  is reduced or  $\mu$  is increased. To study finite temperatures we use a range of values for  $N_t$  such that

$$T_0 = [a_t N_t^{\max}]^{-1} \ll m < T_{\max} = [a_t N_t^{\min}]^{-1}. \quad (30)$$

We use  $N_t^{\min} = 8$  which gives  $T_{\max} = 2m$ .

Figure 4 shows the inverse 2-point function and Bethe-Salpeter vertex as functions of temperature. The 2PI and RG calculations agree well, which shows that the 2PI calculation can be done without counter-terms by using a RG regulator and solving the flow equations. From the plot of  $G^{-1}(\mathbf{0})$  versus  $T$  we see that

$$T_0 \ll T^* \equiv m(T^*) \approx 1, \quad (31)$$

which verifies that  $N_t = 128$  can be taken as the zero temperature limit.

We test the renormalization by reducing the lattice spacing in the spatial direction ( $a_s$ ) while holding the spatial length of the box ( $L = a_s N_s$ ) fixed. In Fig. 5 we plot  $M(0, x, 0, 0)$  versus  $\log 1/a_s$  for  $x = 0$  and  $x = N/4$ . For comparison, we repeat the 2PI calculation using an incorrect renormalization procedure, by adding vertex counter-terms ( $\lambda \rightarrow \lambda + \delta\lambda$ ) to the basketball diagram (see Fig. 1). The graph shows that in the incorrect calculation,  $M(0, x, 0, 0)$  changes with  $a_s$ . The 2PI and RG calculations are almost flat, which shows that the renormalization is done correctly. When  $a_s$  becomes very small, the curves in the figure bend, because of the influence of the Landau pole.

## 7 Conclusions

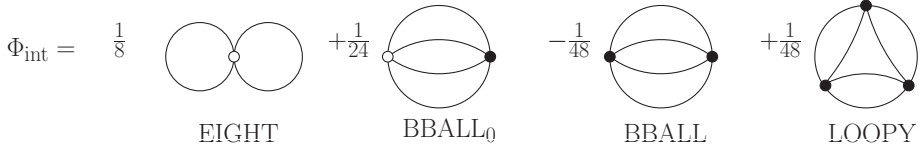
The method that has been used to renormalize the 2PI effective action cannot be used with higher order effective actions. On the other hand, it is reasonable to hope that the RG method we have developed in this paper might be applicable to higher order  $n$ PI theories. The number of bare parameters is fixed by the structure of the Lagrangian, and a complicated interdependent set of counter-terms is replaced by a hierarchy of flow equations that are straightforward to derive using the technique developed in this paper. Renormalization conditions can be enforced on higher order Bethe-Salpeter equations [17]. The RG formalism therefore appears to be a promising approach to the renormalization of higher order  $n$ PI theories.

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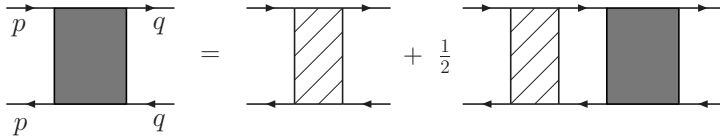
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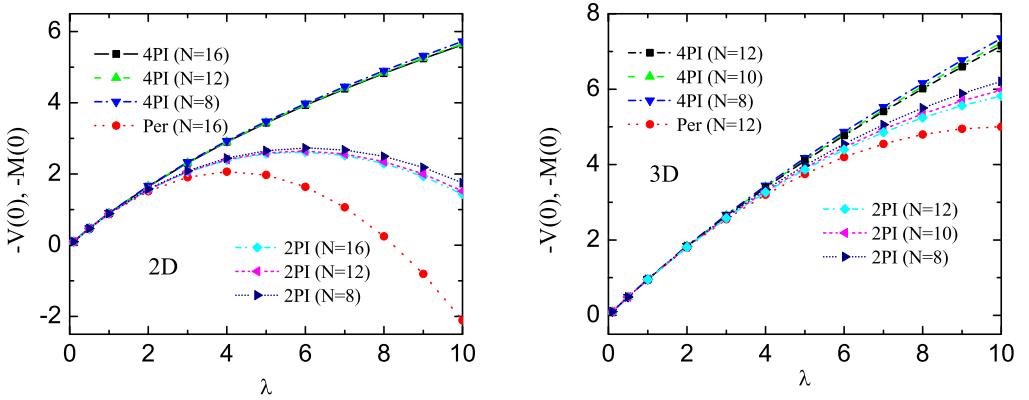
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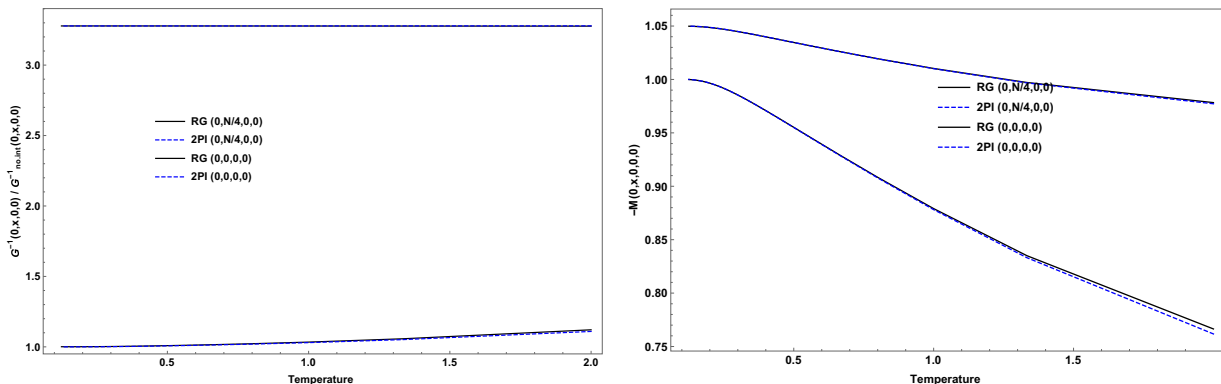
**Figure 1.** The interacting part of the 4-Loop 4PI effective action, not including counter-terms. Solid dots represent self-consistent vertices and open circles are bare vertices. The corresponding expression for the 2PI effective action can be obtained by replacing the self-consistent vertices with bare ones.



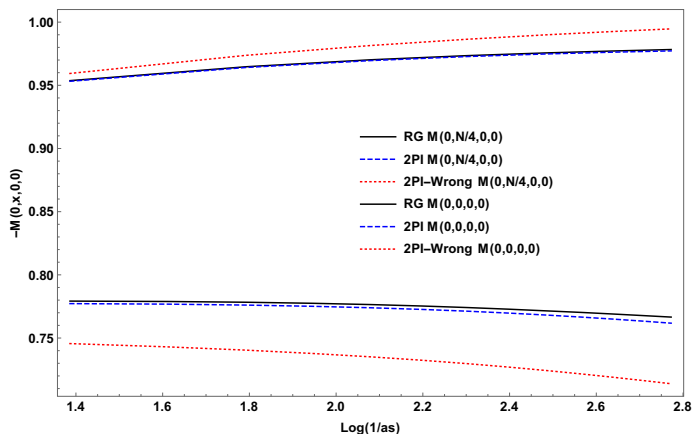
**Figure 2.** Diagrammatic representation of the BS equation in equation (8). Grey boxes and boxes with oblique lines in them represent the vertex  $M$  and kernel  $\Lambda$ , respectively.



**Figure 3.** (color online). Comparison of  $V$  and  $M$  as functions of the coupling strength  $\lambda$ . All the momentum arguments are chosen to be vanishing. The calculations are done in 2D with  $N = 16, 12,$  and  $8$  (left panel), and in 3D with  $N = 12, 10,$  and  $8$  (right panel). For comparison we also show the perturbative result in each graph, which is the dotted line joining round markers (red).



**Figure 4.** The inverse propagator  $G^{-1}(0, x, 0, 0)/G^{-1}_{\text{no-int}}(0, x, 0, 0)$  and Bethe-Salpeter vertex  $M(0, x, 0, 0)$  as functions of the temperature, for two different choices of momentum  $x$ , from the RG and 2PI calculations. In both panels the solid (black) curve is the RG calculation and the dashed (blue) curve is the 2PI calculation. The top two lines are  $x = N/4$ , and the bottom two lines are  $x = 0$ . In all cases, the 2PI curves lie almost exactly on top of the corresponding RG ones.



**Figure 5.** The dependence of  $M(0, x, 0, 0)$  on the lattice separation in the spatial directions, for the RG calculation, the 2PI calculation, and an incorrect version of the 2PI calculation which is included for comparison. The temperature is 2.0 in mass units. The RG, 2PI and incorrect 2PI calculations are respectively the solid (black), dashed (blue) and dotted (red) lines. The top three curves are  $x = N/4$  and the bottom three are  $x = 0$ .