New bounce solutions and vacuum tunneling in de Sitter spacetime

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Abstract. I describe a class of oscillating bounce solutions to the Euclidean field equations for gravity coupled to a scalar field theory with multiple vacua. I discuss their implications for vacuum tunneling transitions and for elucidating the thermal nature of de Sitter spacetime.

INTRODUCTION

The problem of vacuum tunneling in de Sitter spacetime has recently acquired renewed relevance. In part, this is due to developments in string theory, which suggest that vacuum tunneling may be of relevance for understanding transitions between the various potential vacua that populate the string theory landscape. But, it is also of interest for the light that it can shed on the nature of de Sitter spacetime. In this talk I will describe some recent work [1] with Jim Hackworth in which we explored some aspects of the subject that have received relatively little attention.

De Sitter spacetime is the solution to Einstein's equations when there is a constant positive vacuum energy density V_{vac}, but no other source. Globally, it can be represented as the hyperboloid $x^2 + y^2 + z^2 + w^2 - v^2 = H^{-2}$ in a flat five-dimensional space with metric $ds^2 = dx^2 + dy^2 + dz^2 + dw^2 - dv^2$, where

$$H^2 = \frac{8\pi}{3} \frac{V_{\text{vac}}}{M_{\text{Pl}}^2}. \tag{1}$$

The surfaces of constant v are three-spheres, with the sphere of minimum radius, H^{-1}, occurring at $v = 0$. However, the special role played by this surface is illusory. De Sitter spacetime is homogeneous, and a spacelike three-sphere of minimum radius can be drawn through any point.

An important property of de Sitter spacetime is the existence of horizons. Just as for the case of a black hole, the existence of a horizon gives rise to thermal radiation, characterized by a temperature

$$T_{\text{dS}} = H/2\pi. \tag{2}$$

However, there are important differences from the black hole case. A black hole horizon has a definite location, independent of the observer. Further, although an observer's motion affects how the thermal radiation is perceived, the radiation has an unambiguous, observer-independent consequence — after a finite time, the black hole evaporates. By contrast, the location of the de Sitter horizon varies from observer to observer. Although comoving observers detect thermal radiation with a temperature T_{dS}, this radiation does not in any sense cause the de Sitter spacetime to evaporate. As we will see, tunneling between different de Sitter vacua provides further insight into the thermal nature of de Sitter spacetime and the meaning of T_{dS}.

Of course, the relevance of de Sitter spacetime to our Universe comes from the fact that in the far past, during the inflationary era, and in the far future, if the dark energy truly corresponds to a cosmological constant, the Universe approximates a portion of de Sitter spacetime. The underlying assumption is that results derived in the context of the full de Sitter spacetime are applicable to a region that is approximately de Sitter over a spacetime volume large compared to H^{-4}.
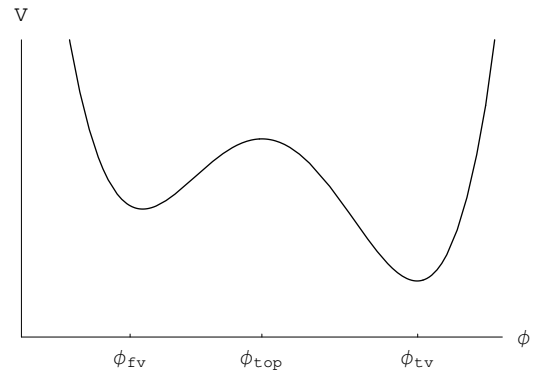
FIGURE 1. The potential for a typical theory with a false vacuum

VACUUM DECAY IN FLAT SPACETIME

The prototypical example for studying vacuum decay is a scalar field theory with a potential, such as that shown in Fig. 1, that has both an absolute minimum (the "true vacuum") and a higher local minimum (the "false vacuum"). For the purposes of this talk I will assume that $V(\phi) > 0$, so that both vacua correspond to de Sitter spacetimes. The false vacuum is a metastable state that decays by quantum mechanical tunneling. It must be kept in mind, however, that the tunneling is not from a homogeneous false vacuum to a homogeneous true vacuum, as might be suggested by the plot of $V(\phi)$. Rather, the decay proceeds via bubble nucleation, with tunneling being from the homogeneous false vacuum to a configuration containing a bubble of (approximate) true vacuum embedded in a false vacuum background. After nucleation, the bubble expands, a classically allowed process.

I will begin the discussion of this process by recalling the simplest case of quantum tunneling, that of a point particle of mass m tunneling through a one-dimensional potential energy barrier $U(q)$ from a initial point q_{init} to a point q_{fin} on the other side of the barrier. The WKB approximation gives a tunneling rate proportional to e^{-B}, where

$$B = 2 \int_{q_{\text{init}}}^{q_{\text{fin}}} dq \sqrt{2m[U(q) - E]} . \qquad (3)$$

This result can be generalized to the case of a multi-dimensional system with coordinates $q_1, q_2, \ldots, q_N$. Given an initial point q_j^{init}, one considers paths $q_j(s)$ that start at q_j^{init} and end at some point q_j^{fin} on the opposite side of the barrier. Each such path defines a one-dimensional tunneling integral B. The WKB tunneling exponent is obtained from the path that minimizes this integral [2] As a bonus, this minimization process also determines the optimal exit point from the barrier.

By manipulations analogous to those used in classical mechanics (but with some signs changed), this minimization problem can be recast as the problem of finding a stationary point of the Euclidean action

$$S_{\text{E}} = \int_{\tau_{\text{init}}}^{\tau_{\text{fin}}} d\tau \left[\frac{m}{2} \left(\frac{dq^j}{d\tau} \right)^2 + U(q) \right] . \qquad (4)$$

One is thus led to solve the Euclidean equations of motion

$$0 = m \frac{d^2 q_j}{d\tau^2} + \frac{\partial U}{\partial q_j} . \qquad (5)$$

The boundary conditions are that $q_j(\tau_{\text{init}}) = q_j^{\text{init}}$ and (because the kinetic energy vanishes at the point where the particle emerges from the barrier) that $dq_j/d\tau = 0$ at τ_{fin}. The vanishing of $dq_j/d\tau$ at the endpoint implies that the solution can be extended back, in a "τ-reversed"

fashion, to give a solution that runs from q_j^{init} to q_j^{fin} and back again to q_j^{init}. This solution is known as a "bounce", and the tunneling exponent is given by

$$
\begin{aligned}
B &= \int d\tau \left[\frac{m}{2} \left(\frac{dq^j}{d\tau} \right)^2 + U(q) - U(q^{\text{init}}) \right] \\
&= S_{\text{E}}(\text{bounce}) - S_{\text{E}}(\text{false vacuum}) , \qquad (6)
\end{aligned}
$$

with the factor of 2 in Eq. (3) being absorbed by the doubling of the path. It is essential to remember that τ is not in any sense a time, but merely one of many possible parameterizations of the optimal tunneling path.

The translation of this to field theory [3] is straightforward: The coordinates q_j become the field variables $\phi(\mathbf{x})$, and the path $q_j(\tau)$ becomes a series of three-dimensional field configurations $\phi(\mathbf{x}, \tau)$. The Euclidean action is

$$S_{\text{E}} = \int d\tau \, d^3 \mathbf{x} \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] \qquad (7)$$

and so one must solve

$$\frac{d^2 \phi}{d\tau^2} + (\nabla \phi)^2 = \frac{dV}{d\phi} . \qquad (8)$$

The boundary conditions are that the path must start at the homogeneous false vacuum configuration, with $\phi(\mathbf{x}, \tau_{\text{init}}) = \phi_{\text{fv}}$, and that $d\phi/d\tau = 0$ at τ_{fin} for all $\mathbf{x}$. (Because ϕ_{fv} is a minimum of the potential, it turns out that $\tau_{\text{init}} = -\infty$.) A three-dimensional slice through the solution at τ_{fin} gives the most likely field configuration for the nucleated true vacuum bubble. This configuration, $\phi(\mathbf{x}, \tau_{\text{fin}})$, gives the initial condition for the subsequent real-time evolution of the bubble. As with the single-particle case, a τ-reflected solution is conventionally added to give a full bounce.

Despite the fact that the spatial coordinates $\mathbf{x}$ and the path parameter τ have very different physical meanings, there is a remarkable mathematical symmetry in how they enter. This suggests looking for solutions that have an SO(4) symmetry; i.e., solutions for which ϕ is a function of only $s = \sqrt{\mathbf{x}^2 + \tau^2}$. For such solutions, the field equation reduces to

$$\frac{d^2 \phi}{ds^2} + \frac{3}{s} \frac{d\phi}{ds} = \frac{dV}{d\phi} . \qquad (9)$$

The boundary conditions are

$$\frac{d\phi}{ds} \bigg|_{s=0} = 0 , \qquad \phi(\infty) = \phi_{\text{fv}} , \qquad (10)$$

where the first follows from the requirement that the solution be nonsingular at the origin, and the second

ensures both that a spatial slice at $\tau = -\infty$ corresponds to the initial state, and that the slices at finite τ have finite energy relative to the initial state. Note that while $\phi(0)$ is required to be on the true vacuum side of the barrier, it is not equal to (although it may be close to) ϕ_{tv}.

Although the tunneling exponent is readily obtained from the WKB approach, the prefactor, including (in principle) higher order corrections, is most easily calculated from a path integral approach [4]. The basic idea is to view the false vacuum as a metastable state with a complex energy, with the imaginary part of the energy density yielding the decay rate per unit volume. The false vacuum energy is obtained by noting that for large $\mathscr{T}$

$$I(\mathscr{T}) = \int [d\phi]\, e^{-S_E(\phi)} \sim e^{-E_{\text{fv}}\mathscr{T}}, \qquad (11)$$

where the path integral is over configurations with $\phi(\mathbf{x}, \tau = \pm \mathscr{T}/2) = \phi_{\text{fv}}$.

This path integral can be calculated by summing the contributions from the various stationary points, each of which gives a factor of $(\det S'')^{-1/2}\, e^{-S}$. Here S'' is the functional second derivative of the action, evaluated at the stationary point; i.e., the product of the frequencies of the normal modes. The first stationary point, a homogeneous false vacuum configuration with $\phi(\mathbf{x}, \tau) = \phi_{\text{fv}}$ everywhere, gives a contribution $Ae^{-S_{\text{fv}}}$, where the real prefactor A includes the (properly renormalized) determinant factor and $S_{\text{fv}} = V(\phi_{\text{fv}})\mathscr{T}\mathscr{V}$. Here $\mathscr{V}$ denotes the volume of space and is understood to be taken to infinity at the end of the calculation.

The next stationary point is the bounce solution to Eq. (8). The calculation of the determinant factor here is complicated by the fact that $S''(\phi_{\text{bounce}})$ has one negative and four zero eigenvalues. The former implies a factor of i, which I will display explicitly. The latter require the introduction of collective coordinates; integrating over these gives a factor of $\mathscr{T}\mathscr{V}$, corresponding to the fact that the bounce can be centered anywhere in the four-dimensional Euclidean space.

Finally, the approximate stationary points corresponding to multibounce solutions also contribute, with the n-bounce contribution including a factor of $(\mathscr{T}\mathscr{V})^n/n!$ from integrating over the positions of n identical bounces. Putting all this together gives a result that can be written as

$$
\begin{aligned}
I(\mathscr{T}) &= Ae^{-S_{\text{fv}}} + i\mathscr{V}\mathscr{T}Je^{-S_{\text{bounce}}} + \cdots \\
&= Ae^{-S_{\text{fv}}}\left[1 + i\mathscr{V}\mathscr{T}Je^{-B} + \frac{1}{2}\left(i\mathscr{V}\mathscr{T}Je^{-B}\right)^2 + \cdots\right] \\
&= Ae^{-\mathscr{T}\mathscr{V}V(\phi_{\text{fv}})}\exp\left[i\mathscr{V}\mathscr{T}Je^{-B}\right]. \qquad (12)
\end{aligned}
$$

Here J includes both determinant and Jacobean factors, with the latter arising from the introduction of the collective coordinates; for present purposes, the important

point is that it is real. Extracting the energy density from the exponent in Eq. (12) gives an imaginary part that is proportional to $\mathscr{V}$, corresponding to the fact that a bubble can nucleate anywhere. The quantity we actually want is the nucleation rate per unit volume,

$$\Gamma = -\frac{2\,\text{Im}\,E_{\text{fv}}}{\mathscr{V}} = 2Je^{-B}. \qquad (13)$$

The path integral approach provides the vehicle for extending [5, 6] the calculation to finite temperature T, with the path integral over configurations extending from $\tau = -\infty$ to $\tau = \infty$ replaced by one over over configurations that are periodic in τ with periodicity $1/T$. At low temperature, where $1/T$ is larger than the characteristic radius of the four-dimensional bounce, there is little change from the zero-temperature nucleation rate. However, in the high-temperature regime where $1/T$ is much smaller than this characteristic radius the path integral is dominated by configurations that are constant in τ. A spatial slice at fixed τ gives a configuration, with total energy E_{crit}, that contains a single critical bubble. The exponent in the nucleation rate takes the thermal form

$$B = \frac{E_{\text{crit}}}{T} - \frac{E_{\text{fv}}}{T}. \qquad (14)$$

Note that, in contrast to the zero temperature case, there is no spatial slice corresponding to the initial state. Only through the boundary conditions at spatial infinity does the bounce solution give an indication of the initial conditions.

ADDING GRAVITY

Coleman and De Luccia [7] argued that the effects of gravity on vacuum decay could be obtained by adding an Einstein-Hilbert term to the Euclidean action and then seeking bounce solutions of the resulting field equations; as before, the tunneling exponent would be obtained from the difference between the actions of the bounce and the homogeneous initial state. Their treatment did not include the calculation of the prefactor, an issue that remains poorly understood.

If one assumes O(4) symmetry, as in the flat spacetime case, the metric can be written as

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2, \qquad (15)$$

where $d\Omega_3^2$ is the metric on the three-sphere, and the scalar field depends only on ξ. The Euclidean action becomes

$$S_E = 2\pi^2 \int d\xi \left[\rho^3\left(\frac{1}{2}\dot{\phi}^2 + V\right)\right.$$

$$+\frac{3M_{\mathrm{Pl}}^2}{8\pi}\left(\rho^2\ddot\rho+\rho\dot\rho^2-\rho\right)\Bigg]\,,\quad(16)$$

with dots denoting derivatives with respect to ξ. The Euclidean field equations are

$$\ddot\phi+\frac{3\dot\rho}{\rho}\,\dot\phi=\frac{dV}{d\phi}\qquad(17)$$

and

$$\dot\rho^2=1+\frac{8\pi}{3M_{Pl}^2}\rho^2\left(\frac{1}{2}\dot\phi^2-V\right)\,.\qquad(18)$$

One can show that if $V(\phi)$ is everywhere positive, as I am assuming here, then $\rho(\xi)$ has two zeros and the Euclidean space is topologically a four-sphere. One of the zeros of ρ can be chosen to lie at $\xi=0$, while the other is located at some value ξ_{max}. Requiring the scalar field to be nonsingular then imposes the boundary conditions

$$\left.\frac{d\phi}{d\xi}\right|_{\xi=0}=0\,,\qquad\left.\frac{d\phi}{d\xi}\right|_{\xi_{\mathrm{max}}}=0\,.\qquad(19)$$

The symmetry of these boundary conditions should be contrasted with the flat space boundary conditions of Eq. (10). Note that there is no requirement that scalar field ever achieve either of its vacuum values, although $|\phi(\xi_{\mathrm{max}})-\phi_{\mathrm{fv}}|$ is typically exponentially small in cases where gravitational effects are small.

Somewhat surprisingly, the Euclidean solution corresponding to a homogeneous false vacuum is not an infinite space, but rather a four-sphere of radius

$$H_{\mathrm{fv}}^{-1}=\sqrt{\frac{3M_{\mathrm{Pl}}^2}{8\pi V(\phi_{\mathrm{fv}})}}\,.\qquad(20)$$

Its Euclidean action is

$$S_{\mathrm{E}}=-\frac{3}{8}\frac{M_{\mathrm{Pl}}^4}{V(\phi_{\mathrm{fv}})}\,.\qquad(21)$$

If the parameters of the theory are such that the characteristic radius of the flat space bounce is much less than H_{fv}^{-1}, then the curved space bounce will be roughly as illustrated in Fig. 2a, with the small region near $\xi=0$ corresponding to the true vacuum region of the flat space bounce, the equatorial slice giving the optimal configuration for emerging from the potential barrier, and a slice such as that indicated by the lower dotted line roughly corresponding to the state of the system before the tunneling process. A bounce solution such as this yields a nucleation rate that only differs only slightly from the flat space result.

On the other hand, there are choices of parameters that give a bounce solution similar to that indicated in Fig. 2b, with a true vacuum region that occupies a significant fraction of the Euclidean space. In this case, there is no slice that even roughly approximates the initial state, suggesting that one should view this as more analogous to a thermal transition in flat space than to zero-temperature quantum mechanical tunneling.

Indeed, for a bounce such as this the true and false vacuum regions can perhaps be viewed as being on a similar footing, so that the bounce can describe either the nucleation of a true vacuum bubble in a region of false vacuum, or the nucleation of a false vacuum bubble in a true vacuum region [8]. The rate for the former case would be

$$\Gamma_{\mathrm{fv}\to\mathrm{tv}}\sim\exp\left\{-[S_{\mathrm{E}}(\mathrm{bounce})-S_{\mathrm{E}}(\mathrm{fv})]\right\}\,,\qquad(22)$$

while for the latter,

$$\Gamma_{\mathrm{tv}\to\mathrm{fv}}\sim\exp\left\{-[S_{\mathrm{E}}(\mathrm{bounce})-S_{\mathrm{E}}(\mathrm{tv})]\right\}\,.\qquad(23)$$

The ratio of these is

$$\begin{aligned}\frac{\Gamma_{\mathrm{tv}\to\mathrm{fv}}}{\Gamma_{\mathrm{fv}\to\mathrm{tv}}}&=\exp\left\{S_{\mathrm{E}}(\mathrm{tv})-S_{\mathrm{E}}(\mathrm{fv})\right\}\\&=\exp\left\{\frac{3}{8}\frac{M_{\mathrm{Pl}}^4}{V(\phi_{\mathrm{fv}})}-\frac{3}{8}\frac{M_{\mathrm{Pl}}^4}{V(\phi_{\mathrm{tv}})}\right\}\,.\qquad(24)\end{aligned}$$

If $V(\phi_{\mathrm{fv}})-V(\phi_{\mathrm{tv}})\ll V(\phi_{\mathrm{fv}})$, the geometry of space is roughly the same in the two vacua, and we can sensibly ask about the relative volumes of space occupied by the false and true vacua. In the steady state, this will be

$$\begin{aligned}\frac{\mathscr{V}_{\mathrm{fv}}}{\mathscr{V}_{\mathrm{tv}}}&=\frac{\Gamma_{\mathrm{tv}\to\mathrm{fv}}}{\Gamma_{\mathrm{fv}\to\mathrm{tv}}}\\&\approx\exp\left\{-\frac{4\pi}{3}H^{-3}[V(\phi_{\mathrm{fv}})-V(\phi_{\mathrm{tv}})]/T_{\mathrm{dS}}\right\}\,.(25)\end{aligned}$$

The last line of this equation, which gives the ratio as the exponential of an energy difference divided by the de Sitter temperature, is quite suggestive of a thermal interpretation of tunneling in this regime.

It is not hard to show that the flat space Euclidean field equations always have a bounce solution. This is no longer true when gravity is included, as we will see more explicitly below. However, Eqs. (17) and (18) always have a homogeneous Hawking-Moss [9] solution that is that is qualitatively quite different from the flat space bounce. Here ϕ is identically equal to its value ϕ_{top} at the top of the barrier, while Euclidean space is a four-sphere of radius $H_{\mathrm{top}}^{-1}\equiv\sqrt{3M_{\mathrm{Pl}}^2/8\pi V(\phi_{\mathrm{top}})}$. From this solution one infers a nucleation rate

$$\Gamma_{\mathrm{fv}}\sim\exp\left\{-\frac{3}{8}\frac{M_{\mathrm{Pl}}^4}{V(\phi_{\mathrm{fv}})}+\frac{3}{8}\frac{M_{\mathrm{Pl}}^4}{V(\phi_{\mathrm{top}})}\right\}\,.\qquad(26)$$

from the false vacuum
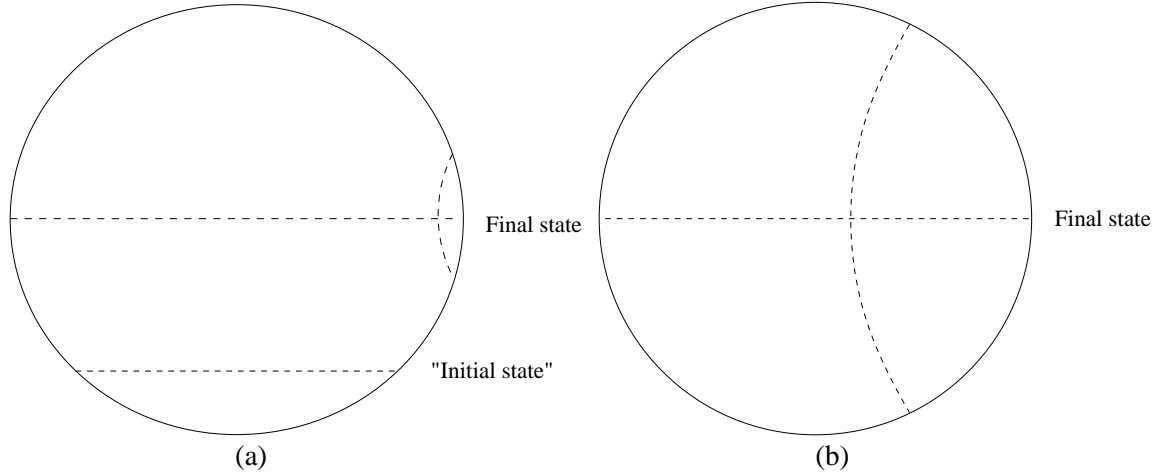
FIGURE 2. Schematic illustration of a Coleman-De Luccia bounce solution in two limiting regimes. In both, ϕ is near its true vacuum value in the region to the right of the dashed arc, while on the left side it is near the false vacuum. In both cases the equatorial slice denotes a three-sphere corresponding to the spatial hypersurface on which the bubble nucleates. The lower dashed line in (a) represents a three-sphere indicative of the initial false vacuum state; this has no analogue in the regime illustrated in (b).

OTHER TYPES OF BOUNCES?

Given the existence of the Hawking-Moss solution, it is natural to inquire whether the inclusion of gravity allows any other new classes of Euclidean solutions. In particular, might there be "oscillating bounce" solutions in which ϕ crosses the potential barrier not once, but rather $k > 1$ times, between $\xi = 0$ and $\xi = \xi_{\text{max}}$?

There can indeed be such solutions [10]. In examining their properties, we focussed on the case where $V(\phi_{\text{top}}) - V(\phi_{\text{tv}}) \ll V(\phi_{\text{tv}})$. This simplifies the calculations considerably, but does not seem to be essential for our final conclusions. With this assumption, the metric is, to a first approximation, that of a four-sphere of fixed radius H^{-1}, and $\xi_{\text{max}} = \pi/H$. We then only need to solve the scalar field Eq. (17). Defining $y = H\xi$, we can write this as

$$\frac{d^2\phi}{dy^2} + 3\cot y \frac{d\phi}{dy} = \frac{1}{H^2}\frac{dV}{d\phi}. \qquad (27)$$

It is convenient to start by first examining "small amplitude" solutions in which $\phi(0)$ and $\phi(\pi)$ are both close to ϕ_{top}. Let us assume that near the top of the barrier V can be expanded as[1]

$$\tilde{V}(\phi) = V(\phi_{\text{top}}) - \frac{H^2\beta}{2}(\phi - \phi_{\text{top}})^2 + \frac{H^2\lambda}{4}(\phi - \phi_{\text{top}})^4 + \cdots \qquad (28)$$

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[1] The omission of cubic terms here is only to simplify the algebra. There is no difficulty, and little qualitative change, in including such terms. The details are given in Ref. [1].

with

$$\beta = \frac{|V''(\phi_{\text{top}})|}{H^2}. \qquad (29)$$

Keeping only terms linear in $(\phi - \phi_{\text{top}})$ in Eq. (17) gives

$$0 = \frac{d^2\phi}{dy^2} + 3\cot y \frac{d\phi}{dy} + \beta(\phi - \phi_{\text{top}}), \qquad (30)$$

whose general solution is

$$\phi(y) - \phi_{\text{top}} = AC_\alpha^{3/2}(\cos y) + BD_\alpha^{3/2}(\cos y), \qquad (31)$$

where $C_\alpha^{3/2}$ and $D_\alpha^{3/2}$ are Gegenbauer functions of the first and second kind and $\alpha(\alpha + 3) = \beta$. The vanishing of $d\phi/dy$ at $y = 0$ implies that $B = 0$; the analogous condition at $y = \pi$ is satisfied only if α is an integer, in which case $C_\alpha^{3/2}$ is a polynomial.

While the linearized equation only has solutions for special values of β, this condition is relaxed when the nonlinear terms are included. Furthermore, the nonlinear terms fix the amplitude of the oscillations, which is completely undetermined at the linear level. The problem can be analyzed by an approach similar to that used to treat the anharmonic oscillator. Any function with $d\phi/dy$ vanishing at both $y = 0$ and $y = \pi$ can be expanded as

$$\phi(y) = \phi_{\text{top}} + \frac{1}{\sqrt{|\lambda|}}\sum_{M=0}^{\infty} A_M C_M^{3/2}(y). \qquad (32)$$

Substituting this into Eq. (17) and keeping terms up to cubic order in $(\phi - \phi_{\text{top}})$ gives

$$0 = \sum_{M=0}^{\infty} C_M^{3/2}(\cos y)\Big[[\beta - M(M+3)]A_M$$
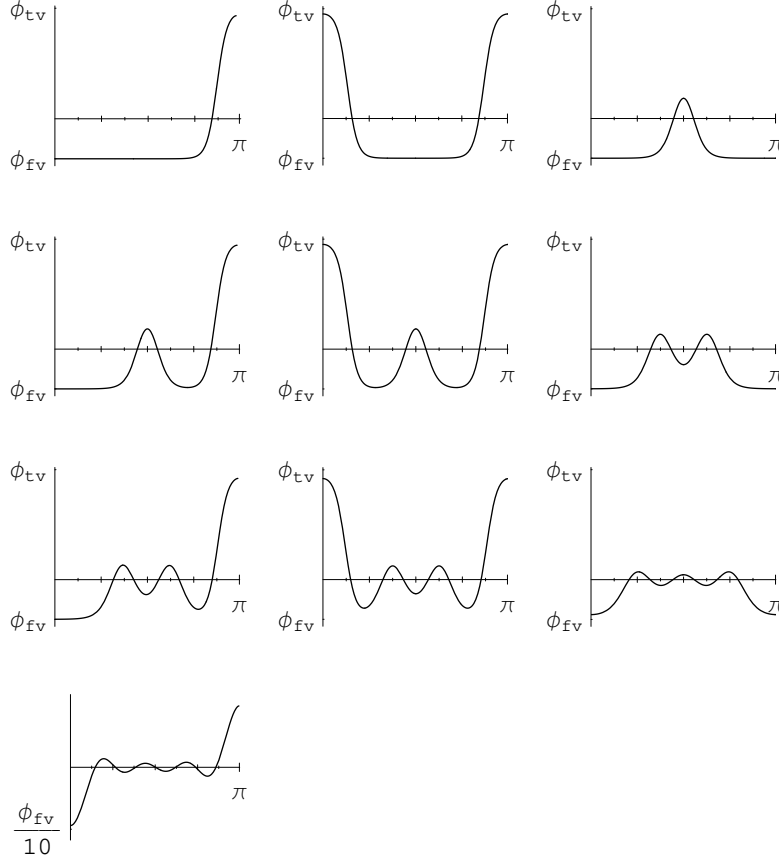
FIGURE 3. Bounce solutions for a scalar potential [1] with cubic and quartic interactions and $\beta = 70.03$.

$$-\text{sgn}(\lambda) \sum_{I,J,K} A_I A_J A_K q_{IJK;M} \Bigg], \qquad (33)$$

where the $q_{IJK;M}$ arise from expanding products of three Gegenbauer polynomials. Requiring that the quantities multiplying each of the $C_M^{3/2}$ separately vanish yields an infinite set of coupled equations. These simplify, however, if $|\Delta| \equiv |\beta - N(N+3)| \ll 1$ for some N. In this case, one coefficient, A_N, is much greater than all the others. The $M = N$ term in Eq. (33) then gives (to leading order)

$$A_N = \pm \sqrt{\frac{\Delta}{\text{sgn}(\lambda) q_{NNN;N}}}, \qquad (34)$$

where $q_{NNN;N} > 0$.

If $\lambda > 0$, Eq. (34) only gives a real value of A_N if $\beta > N(N+3)$. As β is increased through this critical value, two solutions appear. These are essentially small oscillations about the Hawking-Moss solution, with $\phi(0) \approx \phi_{\text{top}} \pm A_N C_N^{3/2}(1)$ and $\phi(\pi) \approx \phi_{\text{top}} \pm A_N C_N^{3/2}(-1)$. Between these endpoints, ϕ crosses the top of the barrier

N times. If N is even, the two solutions are physically distinct, with one having ϕ on the true vacuum side of the barrier at both endpoints, and the other having both endpoint values on the false vacuum side. If N is odd, the two solutions are just "y-reversed" images of each other.

As β is increased further, the endpoints move down the sides of the barrier, until eventually the small amplitude approximation breaks down. Nevertheless, we would expect the solutions to persist, with $\phi(0)$ and $\phi(\pi)$ each moving toward one of the vacua. When β reaches the next critical value, $(N+1)(N+4)$, two new solutions, with $N+1$ oscillations about ϕ_{top}, will appear, but the previous ones will remain. Thus, for $N(N+1) < \beta < (N+1)(N+4)$, we should expect to find solutions with $k = 0, 1, 2, \ldots, N$ oscillations. We have confirmed these expectations by numerically integrating the bounce equations for various values of the parameters; the solutions for a typical potential with $\beta = 70.03$ are shown in Fig. 3.

The fact that the number of solutions should increase with β is physically quite reasonable. One would expect

the minimum distance needed for an oscillation about ϕ_{top}, like the thickness of the bubble wall itself, to be roughly $|V''|^{-1/2}$. Hence, the number of oscillations that can fit on a sphere of radius H^{-1} should be of order $H^{-1}/|V''|^{-1/2} = \sqrt{\beta}$. In particular, this suggests that for $\beta < 4$ there should not even be a $k = 1$ Coleman-De Luccia bounce [11].

It is thus somewhat puzzling to note the implications of Eq. (34) for the case where λ is negative. Here, increasing β through a critical value causes two solutions to merge into the Hawking-Moss solution and disappear, suggesting that the number of solutions is a decreasing function of β. The resolution to this can be found by analytically and numerically examining various potentials that are unusually flat at the top. In all the cases we have examined, the number of solutions is governed by a parameter γ that measures an averaged value of $|V''|/H^2$ over the width of the potential barrier. When γ is sufficiently small, there are no bounce solutions (other than the Hawking-Moss, which is always present). As γ is increased, new solutions appear at critical values. These first appear as solutions with finite values of $\phi(0) - \phi_{\text{top}}$. They then bifurcate, with $\phi(0)$ for one solution moving toward a vacuum and $\phi(0)$ for the other moving toward ϕ_{top}, eventually reaching it and disappearing when β is at a critical value. The net effect is that the number of solutions generally increases with γ, although it is not strictly monotonic.

INTERPRETING THE OSCILLATING BOUNCES

How should these oscillating bounce solutions be interpreted? For the flat space bubble, a spacelike slice through the center of the bounce gives the initial conditions for the real-time evolution of the system after nucleation; these predict a bubble wall with a well-defined trajectory and a speed that soon approaches the speed of light. The interpretation of the Coleman-De Luccia bounce is similar. The main new feature here is the fact that the spacelike slice is finite. Formally, this corresponds to the fact that de Sitter spacetime is a closed universe, even though we expect the bubble nucleation process to proceed similarly in a spacetime that only approximates de Sitter locally.

The Hawking-Moss solution can be interpreted as corresponding to a thermal fluctuation of all of de Sitter space (or, more plausibly, of an entire horizon volume) to the top of the potential barrier. Strictly speaking, classical Lorentzian evolution would leave ϕ at the top of the barrier forever. However, this is an unstable configuration, and so would be expected to break up, in a stochastic fashion, into regions that evolve toward one vacuum

or the other.

The oscillating bounce solutions yield a hybrid of these two extremes. The endcap regions near $\xi = 0$ and $\xi = \xi_{\text{max}}$ clearly evolve into vacuum regions analogous to those from the Coleman-De Luccia bounce, while the intermediate, "oscillating", region is like that emerging from a Hawking-Moss mediated transition. As with the Hawking-Moss solution, the bounce carries no information about the initial state, and there is not even any correlation between the vacua in the endcaps and the initial vacuum state. Thus, like Hawking-Moss, it is reminiscent of finite temperature tunneling in the absence of gravity, and provides evidence of the thermal nature of de Sitter spacetime.

The relative importance of the various solutions depends on the values of their Euclidean actions. Although the details vary with the particular form of the potential, the various regimes are characterized by a parameter γ measuring an averaged value of $|V''|/H^2$. If $\gamma \gg 1$, there is a Coleman-De Luccia bounce, a Hawking-Moss solution, and many oscillating bounces. However, the Coleman-De Luccia bounce has a much smaller action than the others, and so dominates. This is a regime of quantum tunneling transitions followed by deterministic classical evolution. At the other extreme is the case where $\gamma \lesssim 1$, where the Hawking-Moss is the only solution to the bounce equations. This is a regime of thermal transitions followed by stochastic real-time evolution. In between is a transitional region, with thermal effects still important. It is here that the oscillating bounces are most likely to play a role.

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