A NEW PERSPECTIVE ON INVENTORY SYSTEMS

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ABSTRACT. In this article a single item inventory model with backlogging is analyzed, which is a generalization of the most well-known simple models. This formulation enables us to separate the analysis of the system to the analysis of the control rule (reduced to the analysis of a Markov chain) and of the time stationary distribution for the arrival process of customers. This facilitates a much better understanding of such systems. A simple sample path argument enables a straightforward derivation of average holding costs, ordering costs, services measures. A recently developed algorithm of Laplace transform inversion technique provides us with an efficient tool for the computation of these cost expressions.

1. Introduction

Until a few years ago, single item inventory models with backlogging were among the most frequently discussed models in the literature. The most well-known models are the (s, S) model (cf. [8, 11, 3]), the (s, Q) model (cf. [2, 1]), the simple $(S-1, S)$ model (cf. [4]) and the periodic review version of these models, that is, the (R, s, S), the (R, s, Q) and respectively the (R, S) models. These papers analyzed the structure of the models, their behaviour and tried to develop heuristical algorithms for calculating optimal costs. As we already know the most important characteristics of the models above, the general opinion seems to support that this area is exhausted, such that major breakthroughs are not to be expected. In our opinion, however, the state of the art in single item inventory models with backlogging suffers from a serious drawback: there is an undoubted lack of insight which is the discovery that the analysis of all these models reduces to the analysis of the embedded Markov chain of the inventory position process.

In order to demonstrate this new result in a general way, without restricting ourselves to the conditions of one of the specific control rules mentioned above, a general inventory model with backlogging is defined in such a way that the control rule of the system depends solely on the inventory position process. Hence, the analysis of the general model yields the most prominent characteristics of these cases, without restricting to separate heuristical treatments.

Sahin's formula (cf. [8]) gives us a relation among the three most significant stochastic processes describing a single item inventory control system with backlogging,

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that is, the inventory position process, the net inventory or netstock process and the demand process. Considering a general stochastic compound demand process, we find that this process is asymptotically independent of the inventory position process and this is one of the key issues in further analysis, since we are only interested in long run behaviour of such a system. Sahin's relation enables us to separate our analysis to that of the analysis of the control rule, thus the asymptotic behaviour of the embedded Markov chain of the inventory position process and to that of the analysis of the time stationary distribution (cf. [9]) for the arrival process. This brings us to a much better understanding of the problem. Obviously this method is much easier than the often cumbersome task of dealing with the joint process of the inventory position and the compound demand processes. Furthermore, the structure of the problem is much clearer and it also enables us to analyze and compare different demand processes and/or control rules simultaneously. It should be emphasized that with this framework we can analyze general compound renewal demand processes and also non-homogeneous compound processes. The other innovative feature of the paper is that we exploit the fact that the sample paths of the netstock and inventory position processes yield a step function, imposing a cost structure to it, and thus we obtain long run average costs and service measures in a straightforward manner, with very simple algebraic functions. Since it is easy to derive closed form expressions of the Laplace transforms of these costs and measures, we can make use of a recently developed Laplace transform inversion technique (cf. [6]). This facilitates us to compute these costs and measures in any point. The obtained results are exact almost up to machine precision.

The discussion of the periodic review "version" of this article is the topic a future paper.

2. Preliminaries and notations

Throughout this paper we deal with single item inventory systems with backlogging. The demand process $\mathbf{D}$, is a general, truly stochastic compound process, where $\mathbf{D}(t)$ represents the aggregate demand up to time t

$$\mathbf{D}(t) := \sum_{n=0}^{\mathbf{N}(t)} \mathbf{Y}_n. \tag{2.1}$$

The individual demands $\mathbf{Y}_n,\ n \in I\!N$ ($\mathbf{Y}_0 := 0$) are independent and identically distributed random variables, and independent of the arrival process of customers, $\mathbf{N}$. Customers' interarrival times are described by the process $\mathbf{X}_n,\ n \in I\!N$. Note that the arrival times of the customers are given by $\mathbf{t}_n := \mathbf{X}_1 + \ldots + \mathbf{X}_n,\ n \in I\!N$ ($\mathbf{t}_0 := 0$) and the related stochastic counting process $\{\mathbf{N}(t) : t \geq 0\}$ is given by

$$\mathbf{N}(t) := \sum_{n=1}^{\infty} 1_{\{\mathbf{t}_n \leq t\}}.$$

Further, there are two important stochastic processes which describe such an inventory control model. The netstock or net inventory process $\mathbf{IN} := \{\mathbf{IN}(t) : t \geq 0\}$, where $\mathbf{IN}(t)$ is the netstock level, i.e. the stock on hand minus the backordered amount at time t. The other stochastic process is the inventory position process $\mathbf{IP} := \{\mathbf{IP}(t) : t \geq 0\}$, where $\mathbf{IP}(t)$ is the inventory position, i.e. the net stock plus outstanding orders at time t. The control rule associated with the system is such that it only depends on the inventory position. That is, there is a predetermined threshold s, which is called the reorder level, such that if the inventory position reaches or goes below s at the arrival moment $\mathbf{t}_n$ ($n \in \mathbb{N}$) of a customer, then the inventory manager places a replenishment order. The size of the replenishment order only depends on the inventory position and it is given by $\mathbf{Z}_n := \mathbf{IP}(\mathbf{t}_n+) - \mathbf{IP}(\mathbf{t}_n-) + \mathbf{Y}_n$ (obviously, $\mathbf{IP}(\mathbf{t}_n+)$ is determined by the precise ordering policy). It is also natural to assume that the inventory position has a maximum value $0 < S < \infty$, otherwise our problem is degenerate. After placing a replenishment order it takes $L > 0$ time units for the outstanding order to reach the facility. In our analysis L is fixed and we refer to it as the *lead time*. In 1990 Sahin derived the following expression, which is a relation among the netstock, inventory position and demand processes (cf. [8]). This relation is a key tool for this paper. If the stochastic demand process $\mathbf{D}$ is càdlàg (that is right continuous with left limits) then

$$\phi_L \mathbf{IN}(t) = \mathbf{IP}(t) - \phi_t \mathbf{D}(0, L], \ \mathbb{P} \text{ - almost surely} \tag{2.2}$$

for every $t \geq 0$ where $\phi_s, \ s \geq 0$ is a shift operator such that $\phi_s(\mathbf{X})(t) := \mathbf{X}(t+s)$ for every $t \geq 0$ and $\mathbf{X}$ a stochastic process. Also for a general stochastic process $\mathbf{X}$ the notation $\mathbf{X}(a, b]$ means $\mathbf{X}(b) - \mathbf{X}(a)$.

Another important tool in our analysis is the Laplace Stieltjes transform.[1] For $h : [0, \infty) \to \mathbb{R}$, a function of bounded variation, this transform is defined by

$$LS_h(\alpha) = \int_{0-}^{\infty} \exp(-\alpha x) h(dx),$$

while the Laplace transform of h is given by

$$L_h(\alpha) = \int_{0-}^{\infty} \exp(-\alpha x) h(x) dx.$$

Clearly the parameter α is chosen in such a way that the above integrals are well defined. Also, for any function $q : \mathbb{R} \to \mathbb{R}$, vanishing on $(-\infty, 0]$, and a cumulative distribution function F on $[0, \infty)$ we introduce the convolution $q \star F : [0, \infty) \to \mathbb{R}$ given by

$$(q \star F)(x) := \int_{0-}^{x} q(x-y) F(dy); \tag{2.3}$$

and inductively we can define $F^{k*} := F \star F^{(k-1)*}$. It follows now from the previous definitions that

$$L_{q \,\star\, F}(\alpha) = L_q(\alpha) LS_F(\alpha), \tag{2.4}$$

and this important relation will be used throughout the rest of this paper. With the Laplace inversion algorithm as described in Den Iseger (cf. [6]), we obtain a piece-wise polynomial approximation in fractions of time. The calculations are numerically stable, while the approximation is precise almost up to machine precision. With the help of this algorithm we can calculate numerically all the important cost and service measures of the inventory system.

[1] All the arguments and results remain valid in case of Fourier transforms.

3. The netstock and inventory position processes

It is of great importance to realize that the sample paths of the net inventory process yield a step function, that is, there are jumps occurring in the sample paths but it is constant between two jumps. At customers' arrival moments $\mathbf{t}_n$, downwards jumps occur in the sample paths of the netstock process due to customers' individual demand. Hence the size of such a jump equals $\mathbf{Y}_n, n \in I\!N$. It is important to observe that the same downwards jumps of size $\mathbf{Y}_n$, at time point $\mathbf{t}_n$, occur in the sample paths of the inventory position process. These type of jumps will be referred to as *type I jumps*. If a certain individual demand (say, occurring at time point $\mathbf{t}_m$) causes the inventory position to drop (jump!) below the reorder level s, then, according to the policy, there will be a replenishment order of size $\mathbf{Z}_m$ placed at time point $\mathbf{t}_m$, which arrives at time point $\mathbf{t}_m + L$ to the facility. The arrival of the replenishment order causes at time point $\mathbf{t}_m + L$ an upwards jump of size $\mathbf{Z}_m$ in the sample paths of the netstock process, which we define as a *type II jump*. By definition, the type II jump occurs in the sample paths of the inventory position process at time point $\mathbf{t}_m$ and its size equals $\mathbf{Z}_m - \mathbf{Y}_m$. The moments of type II jumps in the sample path of the inventory position form a subset of customers' arrival moments. In what follows we will show that the inventory position process at customers' arrival moments, $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N \cup \{0\}\}$, forms a Markov chain. That is what it makes more convenient to relate the type II jumps to the inventory position process.

The definitions of this general control system imply for the inventory position process that $\mathbf{IP}(\mathbf{t}_n)$ only depends on the previous state $\mathbf{IP}(\mathbf{t}_{n-1})$, the individual demand of the nth customer $\mathbf{Y}_n$, and the magnitude of the replenishment order $\mathbf{Z}_n$, if there was any order placed at $\mathbf{t}_n$. Since the individual demands $\mathbf{Y}_n$ are independent of the arrival process of customers (the same holds for the size of the replenishment order!) it follows that $\{\mathbf{IP}(\mathbf{t}_n) : n \in I\!N \cup \{0\}\}$ is a Markov chain. By the definition of the general inventory control system we know that the inventory position has a maximum S, and a minimum s; it also clearly reaches every state between s and S with a positive probability. This implies straightforwardly that this Markov chain is irreducible and aperiodic with all states being positive recurrent, that is, the Markov chain has a unique limiting distribution (cf. [7]) given by

$$\lim_{n\uparrow\infty} I\!P\{\mathbf{IP}_n \leq x\} = I\!P\{\mathbf{IP}_\infty \leq x\} \tag{3.1}$$

where $\mathbf{IP}_\infty$ is a random variable distributed with the limiting distribution of the Markov chain $\{\mathbf{IP}_n, n \in I\!N \cup \{0\}\}$. The step function structure of the sample paths of the inventory position also implies that

$$\mathbf{IP}(t) = \mathbf{IP}_{\mathbf{N}(t)}, \text{ for all } t \geq 0. \tag{3.2}$$

Since $\mathbf{N}(t) \to \infty$ a.s. as $t \to \infty$ we obtain by relation (3.2) that $\mathbf{IP}(t)$ is asymptotically independent of $\mathbf{N}(t)$, thus also independent of $\mathbf{D}(t)$, and its limiting distribution is given by

$$\lim_{t\uparrow\infty} I\!P\{\mathbf{IP}(t) \leq x\} = I\!P\{\mathbf{IP}_\infty \leq x\}, \tag{3.3}$$

where $\mathbf{IP}_\infty$ is defined by relation (3.1). We will now give two examples which are related to two of the most well-known policies in the literature.

3.1. **The (s, S) policy.** Under this rule an order is triggered at the moment the level of the inventory position drops below the reorder level s $(0 < s < S)$. The size of the order is such that the level of the inventory position process is raised to order-up-to level S. That is, this control policy only depends on the inventory position process; hence, as derived at the beginning of section 3 the inventory position in the moments of customer arrivals $\{\mathbf{IP}_n, \, n \in \mathbb{N}\}$ is a Markov chain which possesses a unique limiting distribution. For notational convenience, define the sequence of random variables $\{\mathbf{V}_n : n \in \mathbb{N} \cup \{0\}\}$ as the difference between the order-up-to level S and the inventory position at moment $\mathbf{t}_n, n \in \mathbb{N} \cup \{0\}$:

$$\mathbf{V}_n := S - \mathbf{IP}(\mathbf{t}_n), \; n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Since $\{\mathbf{IP}(\mathbf{t}_n) : n \in \mathbb{N} \cup \{0\}\}$ is a Markov chain, obviously $\{\mathbf{V}_n : n \in \mathbb{N} \cup \{0\}\}$ is also a Markov chain equipped with unique limiting distribution. By the definition of the policy it immediately follows that

$$\mathbf{V}_{n+1} = (\mathbf{V}_n + \mathbf{Y}_{n+1}) \, \mathbf{1}_{\{\mathbf{V}_n + \mathbf{Y}_{n+1} \leq S-s\}}, \;\; n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

We aim to show now that the unique limiting distribution of the Markov chain $\{\mathbf{V}_n : n \in \mathbb{N}\}$ is of the form

$$\lim_{n \uparrow \infty} \mathbb{P}\{\mathbf{V}_n \leq x\} = \mathbb{P}\{\mathbf{V}_\infty \leq x\} = \frac{U_0(x)}{U_0(S-s)}, \tag{3.6}$$

where U_0 denotes the renewal function related to the renewal sequence $\{\mathbf{Y}_0, \mathbf{Y}_0 + \mathbf{Y}_1, \ldots\}$ given by

$$U_0(x) := \sum_{k=0}^{\infty} F_Y^{k\star}(x).$$

In a future paper we will exploit relation (3.6) to prove optimality. Relation (3.5) implies straightforwardly that for every $0 \leq x \leq S - s$

$$F_V(x) = C + (F_V \star F_Y)(x), \tag{3.7}$$

where $C := 1 - (F_V \star F_Y)(S-s)$ is a normalization constant. Since relation (3.7) is a renewal type equation, it follows (cf. [7]) that its uniquely determined solution is given by

$$F_V(x) = C U_0(x). \tag{3.8}$$

The constant C can be easily determined by the condition $F_V(S-s) = 1$, therefore we obtain that the unique invariant distribution of the Markov chain $\mathbf{V}_n$ is given by relation (3.6). As a standard result from renewal theory (cf. [7]), if x is big enough, that is, $S - s$ is large, than the renewal function $U(x)/x \longrightarrow 1/\mathbb{E}\mathbf{X}_1$. This implies that (3.6) converges to $x/(S-s)$, that is, the limiting distribution converges to a uniform distribution. In the next subsection it is proved that the limiting distribution of the Markovian inventory position process related to an (s, Q) model is given by the uniform distribution. This result suggests that for large Q and $S - s$ these models are very similar.

3.2. **The (s, nQ) policy.** According to this inventory rule an order is triggered at the moment the inventory position drops below or equals the reorder level s. The order size is chosen to be an integer multiple of Q, such that after ordering the inventory position process will be between s and $s + Q$. Hadley and Whitin (cf. [5]) proved that the transition matrix of the Markov chain $\{\mathbf{IP}(\mathbf{t}_n) : n \in \mathbb{N} \cup \{0\}\}$ is double stochastic, hence it follows straightforwardly that its limiting distribution is given by the uniform distribution on $(s, s+Q]$, that is

$$\lim_{t\uparrow\infty} \mathbf{IP}(\mathbf{t}_n) = s + Q\mathbf{U}, \ \ n \in \mathbb{N} \cup \{0\},$$

with $\mathbf{U}$ a uniformly distributed random variable on $(0,1]$. Together with the average holding and ordering cost expressions this result was also found by Chen and Zheng (cf. [1]), for a compound Poisson demand process. Having this result it is now possible to derive the important costs and measures. In the following section the costs imposed to the general system will be introduced. Having the results of the present section, it turns out that these expressions are not difficult to derive.

4. Costs

As it will be explained in detail later the general cost (thus also including service measures) consists of two parts: the cost of the inventory system and the cost of the control rule. The cost of the control rule is associated with the inventory position process while the cost of the system is associated with the net inventory process. There are three types of events with respect to this process, namely type I jumps, type II jumps and the sample paths of the netstock process being constant between two jumps. It is natural to define three types of costs related to the three types of events. Therefore, when $\mathbf{IN}(t) = \mathbf{IN}(\mathbf{t}_n) = x$ a.s. for $\mathbf{J}_n \leq t < \mathbf{J}_{n+1}$, where $x \in \mathbb{R}$ is a constant and $\mathbf{J}_n, n \in \mathbb{N}$ are the points of time when a jump occurs, then it is natural and trivial to introduce a cost rate function $f(x)$ related to this event. This cost will give us a very important characteristic, the average holding cost (and penalty cost), therefore we refer to this type of cost in the remainder of the paper as the average holding cost. Similarly, we introduce a cost function g_1 related to the type I jumps of the sample paths of the netstock process, that is, the cost of the jump in time point $\mathbf{t}_n$ is given by $g_1(\mathbf{IN}(\mathbf{t}_n), \mathbf{Y}_n)$. This type of "cost" usually provides us with service measures, since it is related to the arrival of customers. Therefore we refer to the cost of the type I jumps as service measures. Introduce also a function G_2, related to the type II jumps, that is, the cost of the control policy: for a replenishment order placed at time point $\mathbf{t}_n$ it is given by $G_2(\mathbf{Z}_n)$. By the definition of $\mathbf{Z}_n$, $\mathbf{Z}_n = h(\mathbf{IP}_n - \mathbf{Y}_n)$, where h is a function dependent on the control rule, the cost of the control rule is given by $g_2(\mathbf{IP}(\mathbf{t}_n) - \mathbf{Y}_n)$, with $g_2 = G_2 \circ h$. Before starting with the actual computation of these costs we discuss some properties related to the expected long run average cost associated with a stochastic process. The average cost associated with a function l and a stochastic process $\mathbf{X}$ is given by

$$\lim_{t\uparrow\infty} \mathbb{E}\left(\frac{1}{t}\int_0^t l(\mathbf{X}(s))ds\right).$$

Using Fubini's theorem the previous relation equals to

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{E}\left(l(\mathbf{X}(s))\right)ds = \lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \int_{-\infty}^{\infty} l(x)dF_{X(s)}(x)ds.$$

Using again Fubini's theorem for the previous relation, we obtain that the average cost equals

$$\mathbb{E}\left(l(\mathbf{X}^c_\infty)\right),$$

where $\mathbf{X}_\infty^c$ is a random variable distributed with the distribution given by

$$F_\infty^c(x) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t F_{X(s)}(x)ds. \tag{4.1}$$

Observe that if the limiting distribution of the stochastic process $\mathbf{X}$ exists then it coincides with the distribution defined by relation (4.1). Throughout this paper we will call the distribution defined by relation (4.1) the *time stationary distribution* for $\mathbf{X}$ (cf. [9], p.24-25). Obviously, the requirement that for a stochastic process its time stationary distribution would exist is much weaker than that of a "normal" limiting distribution.

4.1. **Average holding cost.** Since we are interested in long run average costs we aim to compute the expression

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \mathbb{E} f(\phi_L \mathbf{IN}(s))ds. \tag{4.2}$$

Relation (2.2) of Sahin gives us a powerful tool to compute the average cost. By the definition of the demand process (2.1) the average cost equals

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \mathbb{E} f\left(\mathbf{IP}(s) - \sum_{k=0}^{\phi_t \mathbf{N}(0,L]} \mathbf{Y}_k\right) dt. \tag{4.3}$$

We assume that the time stationary distribution for the stochastic counting process $\mathbf{N}$ exists, which most of the time is not a strong condition. As deduced in section 3 the limiting distribution of the inventory position process $\mathbf{IP}(t)$ exists and is given by relation (3.3). Further, it also follows that $\mathbf{IP}(t)$ and $\mathbf{N}(t)$ are asymptotically independent. Since $\mathbf{IP}(t)$ has a pointwise limit distributionally, in relation (4.3) it is possible to consider the pointwise limit of $\mathbf{IP}(t)$ and the time stationary distribution for $\phi_t \mathbf{N}(0, L]$ simultaneously, obtaining

$$\mathbb{E} f\left(\mathbf{IP}_\infty - \sum_{k=0}^{\mathbf{N}_\infty^c(0,L]} \mathbf{Y}_k\right), \tag{4.4}$$

where $\mathbf{N}_\infty^c(0, L]$ is a random variable distributed with the time stationary distribution for $\phi_t \mathbf{N}(0, L]$. Since $\mathbf{IP}_\infty, \mathbf{N}_\infty^c(0, L]$ and $\mathbf{Y}_k$ are pair by pair independent, this implies that equation 4.4 equals

$$\mathbb{E}_{\mathbf{IP}_\infty} \left(\left(f * F_{D_\infty(0,L]} \right) (\mathbf{IP}_\infty) \right), \tag{4.5}$$

where $\mathbf{D}_\infty(0, L] := \sum_{k=1}^{\mathbf{N}_\infty^c(0,L]} \mathbf{Y}_k$. Observe that

$$\mathbb{P}\{ \sum_{k=0}^{\mathbf{N}_\infty^c(0,L]} \mathbf{Y}_k \le x\} = \sum_{k=0}^{\infty} \mathbb{P}\{\mathbf{N}_\infty^c(0, L] = k\} F_Y^{k*}(x),$$

and taking the Laplace Stieltjes transform of this we obtain

$$LS_{F_{D_\infty}}(\alpha) = \sum_{k=0}^{\infty} \mathbb{P}\{\mathbf{N}_\infty^c(0, L] = k\} LS_{F_Y}^k(\alpha) = P_{\mathbf{N}_\infty^c(0,\, L]}(LS_{F_Y}(\alpha)),$$

where $P_{\mathbf{N}_\infty^c(0,\, L]}(\cdot)$ denotes the z-transform of $\mathbf{N}_\infty^c(0, L]$. In conclusion, if we can determine $P_{\mathbf{N}_\infty^c(0,\, L]}$ then with the previously mentioned Laplace transform inversion

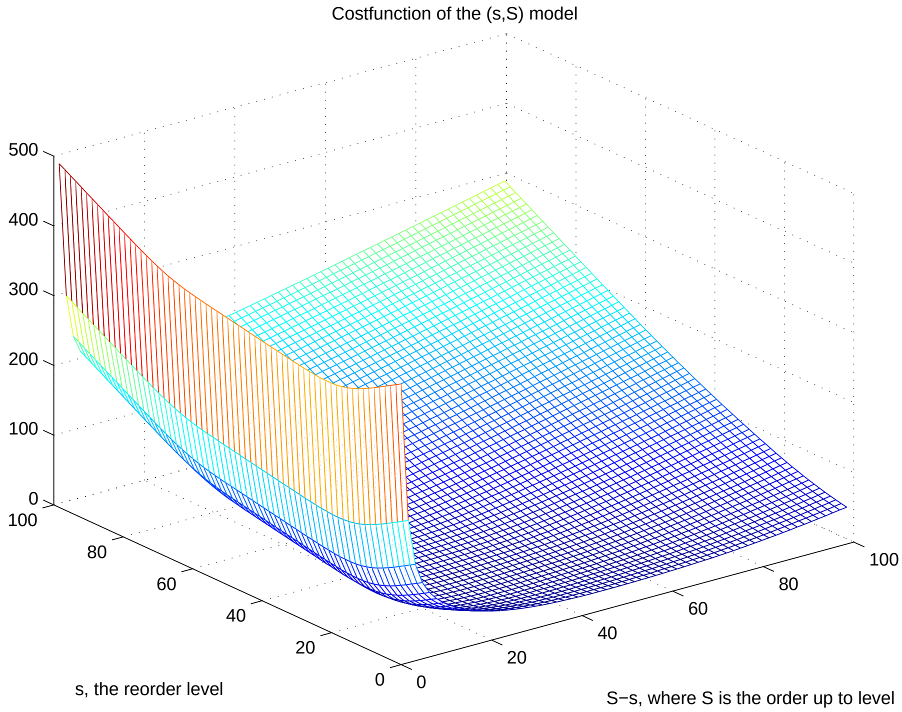

FIGURE 1. Average holding cost in case of an (s, S) policy with non-homogeneous compound Poisson demand; parameters are $K = 20$, $L = 1$, $\lambda_1 = 25/2$, $\lambda_2 = 45/2$, $q = 50$, $p = 3$, $h_1 = 1$, $h_2 = 3$)

algorithm we obtain a piece-wise polynomial approximation for $f * F_{D_\infty(0,L]}$, say $P_{f*F_{D_\infty(0,L]}}$. We are now able to approximate equation (4.5) by

$$\mathbb{E}_{\mathbf{IP}_\infty}\left(P_{f*F_{D_\infty(0,L]}}(\mathbf{IP}_\infty)\right) \tag{4.6}$$

Furthermore, in case of **compound renewal demand**, we obtain for the stochastic counting process that

$$\lim_{t\uparrow\infty} \phi_t \mathbf{N}(0, L] = \lim_{t\uparrow\infty} (\mathbf{N}(t + L) - \mathbf{N}(t)) \stackrel{d}{=} \mathbf{N}_0(L - \mathbf{A}), \tag{4.7}$$

where $\mathbf{A}$ is a random variable distributed with the limiting distribution of the residual life process (cf. [10]) and $\mathbf{N}_0$ denotes the arrival process with a renewal in time point 0. Let us use the notation

$$\Psi_k(t) := \mathbb{P}\{\mathbf{N}_0(t) = k\},$$

then the probability distribution of (4.7) equals $(\Psi_k \star F_A)(L)$. Since

$$\Psi_k = F_X^{k*} - F_X^{(k+1)*},$$

and the Laplace-Stieltjes transform of F_A is given by

$$LS_{F_A}(\beta) = \frac{1 - LS_{F_X}(\beta)}{\beta \mathbb{E} X_1},$$

it follows that the two dimensional Laplace transform of $\mathbf{D}(0, L]$ is given by

$$\frac{(1 - LS_{F_X}(\beta))^2}{\alpha\beta^2 \mathbb{E}\mathbf{X}(1 - LS_{F_X}(\beta) LS_{F_Y}(\alpha))}.$$

Hence we are able to calculate the long run average cost with the help of the two dimensional inversion algorithm (cf. [6]).

In case of **non-homogeneous compound Poisson demand** with arrival rate given by $\Lambda(t)$, $t \geq 0$, we obtain that the z-transform of the time stationary distribution for the stochastic counting process is given by

$$P_{\mathbf{N}^c_\infty(0,\,L]}(z) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-z)\int_{s-L}^s \Lambda(z)dz\right) ds. \tag{4.8}$$

Therefore the average cost can again easily be computed as it was described earlier. It should be mentioned that for a non-homogeneous demand process a static policy is not optimal. The analysis of a dynamic policy related to non stationary demand is the subject of a future paper.

Example: In Figure 1. we plotted the values of the average cost of an (s, S) policy with variable s and $S - s$ values in case when demand is given by a non-homogeneous compound Poisson process. The demand rate function varies every (unit) interval, such that if $t \in [2k, 2k+1)$ then $\Lambda(t) = \lambda_1$ and if $t \in [2k+1, 2k+2)$ then $\Lambda(t) = \lambda_2$. The individual demands follow a Gamma distribution with shape parameter 2.5 and scale parameter 2.5 (cf. [10]). Furthermore we considered a piecewise linear cost rate function given by

$$f(x) = \begin{cases} -px & \text{if } x < 0 \\ h_1 x & \text{if } 0 \leq x \leq q \\ h_1 q + h_2(x - q) & \text{if } x \geq q \end{cases} \tag{4.9}$$

where q denotes a critical level of inventory, from which the inventory holding cost increases to h_2 per unit ($h_2 > h_1 > 0$). Observe that we also included a fixed ordering cost $K > 0$ (see section 4.3) for every placement of a replenishment order.

4.2. **Service measures.** The long run average cost of the (type I) jumps associated with the function g_1 is given by

$$\lim_{t\uparrow\infty} \mathbb{E}\left(\frac{1}{t}\sum_{k=1}^{\mathbf{N}(t)} g_1(\mathbf{IN}(\mathbf{t}_k-), \mathbf{Y}_k)\right). \tag{4.10}$$

Define the measure

$$d\lambda_t(s) := \frac{1}{t} 1_{\{s \leq t\}} d\lambda(s)$$

where

$$\lambda(s) := \sum_{k=1}^{\infty} 1_{\{\mathbf{t}_k \leq s\}}.$$

Observe that by denoting $\bar{\mathbf{X}}(s)$ a stochastic process having a distribution defined by

$$\mathbb{P}\{\bar{\mathbf{X}}(s) \leq x\} := \mathbb{P}\{\mathbf{X}(s-) \leq x |\text{ there is a type I jump at } s\}, \tag{4.11}$$

one obtains by the definition of $d\lambda_t$ that the average cost up to time t equals

$$\mathbb{E}\left(\int_0^\infty g_1(\mathbf{IN}(s-), \mathbf{Y}_{\mathbf{N}(s)})d\lambda_t(s)\right) = \mathbb{E}\left(\int_0^\infty g_1(\mathbf{I\bar{N}}(s), \mathbf{Y}_{\mathbf{N}(s)})d\lambda_t(s)\right).$$

Furthermore, by the previous definitions it is obvious that $\mathbf{I\bar{N}}(s), \mathbf{Y}_{\mathbf{N}(s)}$ and $d\lambda_t(s)$ are independent, and using Fubini's theorem in the previous relation it follows that

$$\lim_{t\uparrow\infty} \mathbb{E}\left(\frac{1}{t}\int_0^t g_1(\mathbf{IN}(s-), \mathbf{Y}_{\mathbf{N}(s)})d\lambda(s)\right) = \mathbb{E} g_1(\mathbf{I\bar{N}}^c_\infty, \mathbf{Y}_\infty),$$

where $\mathbf{I\bar{N}}^c_\infty$ is a random variable distributed with the time stationary distribution for $\mathbf{I\bar{N}}$, given by

$$\lim_{t\uparrow\infty} \int_0^\infty \mathbb{P}\{\mathbf{I\bar{N}}(s) \le x\} d\mathbb{E}\lambda_t(s)$$

and

$$\mathbb{E}\lambda_t(s) = \frac{1}{t} 1_{\{s\le t\}} \sum_{k=1}^\infty \mathbb{P}\{\mathbf{t}_k \le s\},$$

and obviously $\mathbf{Y}_\infty \stackrel{d}{=} \mathbf{Y}_1$. By relation (2.2) of Sahin and by the same argument as

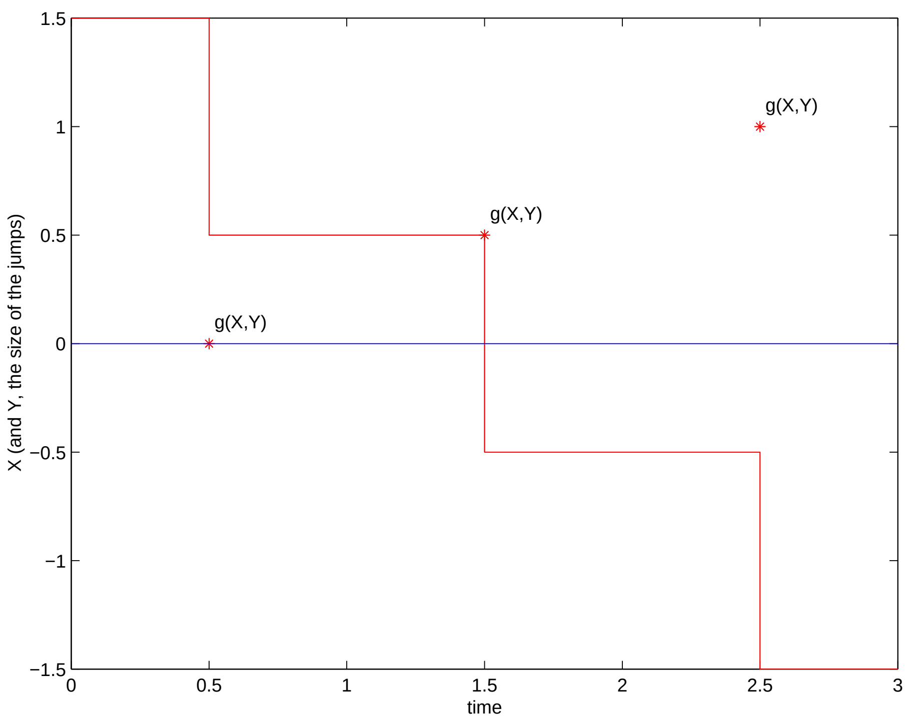

FIGURE 2. The g_1 costrate function, related to the type I jumps

in section 4.1 before relation (4.4), the average cost of the jumps (4.10) equals

$$\mathbb{E} g_1(\mathbf{IP}_\infty - \sum_{k=1}^{\bar{\mathbf{N}}_\infty^c(0,L]} \mathbf{Y}_k, \mathbf{Y}_\infty),$$

where $\bar{\mathbf{N}}_\infty^c(0,L]$ is a random variable distributed with the time stationary distribution for $\bar{\mathbf{N}}$, given by

$$\lim_{t\uparrow\infty} \int_0^\infty \mathbb{P}\{\phi_{s-L}\mathbf{N}(0,L] \leq x | \text{ there is a type I jump at } s\} d\mathbb{E}\lambda_t(s). \tag{4.12}$$

Intuitively, this limiting distribution is the probability that the number of jumps in the interval $(s-L, s]$, given there is a jump at s, is less or equal x, times the probability that there is a jump at s (that is, $d\mathbb{E}\lambda_t(s) = \mathbb{P}\{\text{there is a jump in } [s, s+ds]\}\frac{1}{t}$).

In case of **compound renewal demand**, $d\mathbb{E}\lambda_t(s)$ equals $\frac{1}{t}1_{\{s\leq t\}}dM(s)$, where M represents the renewal function associated with the renewal sequence defined by the arrival moments of customers. By a reversed time argument we obtain that

$$\lim_{t\uparrow\infty} \mathbb{P}\{\mathbf{N}(t-) - \mathbf{N}((t-) - L) = k | \text{ there is a type I jump at } t\} = \mathbb{P}\{\mathbf{N}(L) = k\}, \tag{4.13}$$

that is $\bar{\mathbf{N}}_\infty^c \stackrel{d}{=} \mathbf{N}(L)$, where $\bar{\mathbf{N}}_\infty^c$ is defined by relation (4.12). It is well known that

$$\lim_{t\uparrow\infty} dM(t) = \frac{dt}{\mathbb{E}\mathbf{t}_1},$$

therefore relation (4.10) equals

$$\frac{1}{\mathbb{E}\mathbf{t}_1} \mathbb{E} g_1(\mathbf{IP}_\infty - \sum_{k=1}^{\mathbf{N}(L)} \mathbf{Y}_k, \mathbf{Y}_\infty). \tag{4.14}$$

In case of **non-homogeneous compound Poisson** demand with rate $\Lambda(t)$ we obtain by the PASTA property that $\phi_s\bar{\mathbf{N}}(0,L] \stackrel{d}{=} \phi_s\mathbf{N}(0,L]$ ($\bar{\mathbf{N}}$ defined by (4.11)) and $d\mathbb{E}\lambda_t(s) = \Lambda(s)ds$, hence the z-transform of $\bar{\mathbf{N}}_\infty^c(0,L]$ is given by the relation

$$P_{\bar{\mathbf{N}}_\infty^c(0,\,L]}(z) = \lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \exp\left(-(1-z)\int_{s-L}^s \Lambda(z)dz\right)\Lambda(s)ds. \tag{4.15}$$

For the average cost of the jumps we obtain

$$\mathbb{E} g_1(\mathbf{IP}_\infty - \sum_{k=1}^{\bar{\mathbf{N}}_\infty^c(0,L]} \mathbf{Y}_k, \mathbf{Y}_1).$$

Both of the cases can be solved with the algorithm described in section 4.1.

Example One of the examples for the cost of the type I jumps would be the expected number of items short up to time t, which is one of the most frequently used service measures in the literature. In this case the function g_1 related to the jumps is given by

$$g_1(X, Y) := (Y - X)^+ - (-X)^+, \tag{4.16}$$

where X is the level from where the jump occurs and Y is the size if the jump. Obviously, $X := \mathbf{IN}(\mathbf{t}_k-)$ and $Y := \mathbf{Y}_k$. Figure 2. provides some intuition for the definition of the function g_1 in this case. A special case of a general compound renewal demand process with Gamma distributed arrival process (shape=5/2,scale= 1/14) and i.i.d. Gamma distributed individual demands with shape resp. scale

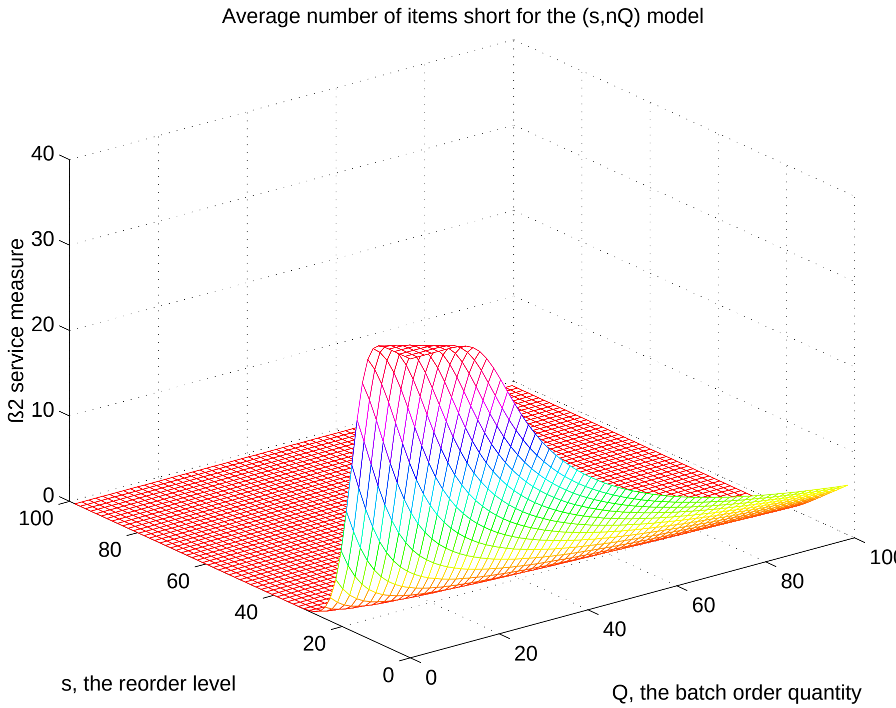

FIGURE 3. Average number β_2 of items short in case of an (s, Q) policy $(L = 0.5)$

parameters $\alpha = \beta = 2.5$ are considered in case of an (s, Q) control rule. The fill rate, given by relation (4.14) with g_1 given by (4.16), is plotted in Figure 3., with respect to the decision variables s and Q.

4.3. **The cost of the control rule.** As we discussed at the beginning of section 4, the type II jumps are related to the inventory position process. These jumps in the sample paths of the inventory position process occur due to placement of replenishment orders. This implies a suggestive name for this type of cost: *the cost of the control rule*. Thus, with the same definitions of measures and costs as in section 4.2 we obtain for the cost of the control rule that

$$\lim_{t\uparrow\infty}\int_0^\infty \mathbb{E} g_2(\mathbf{IP}(s-) - \mathbf{Y}_{\mathbf{N}(s)}) d\lambda_t(s). \tag{4.17}$$

Using the results of section 3 and section 4.2 we obtain that this equals

$$\mathbb{E} g_2(\mathbf{IP}_\infty - \mathbf{Y}_1) = \lim_{t\uparrow\infty}\int_0^\infty d\mathbb{E}\lambda_t(s) = \mathbb{E} g_2(\mathbf{IP}_\infty - \mathbf{Y}_1)\lim_{t\uparrow\infty}\frac{\mathbb{E}\mathbf{N}(t)}{t}.$$

The most obvious example of cost of type II jumps is the setup cost. In this case the cost rate function is given by

$$g_2(A) = K1_{\{A\le s\}},$$

where K and s are given parameters. A more general case would be setup cost aggregated with a variable cost, dependent on the amount of ordered items. In case of an (s, S) policy

$$g_2(A) = (K + c(S - A))1_{\{A \le s\}}.$$

In case of an (s, Q) policy we have to take into account that we order a multiple of Q, that is

$$\text{if } \ (k-1)Q \le \mathbb{E}(s - (\mathbf{IP}_\infty - \mathbf{Y}_1)) < kQ \ \text{ then order } \ kQ,$$

for every $k = 1, 2, \dots$ Since $\mathbf{IP}_\infty = s + Q\mathbf{U}$ as derived earlier one obtains that the variable cost equals to

$$cQ \sum_{k=1}^{\infty} k\mathbb{P}\{kQ \le Q\mathbf{U} + \mathbf{Y}_1 < (k+1)Q\} = cQ \sum_{k=1}^{\infty} (1 - \Phi(kQ)),$$

where $\Phi(x) := (F_U \star F_Y)(x)$. With the help of the Laplace transform inversion algorithms we can calculate easily the ordering and variable costs.

5. Conclusions

Having proved that the inventory position related to our general model is a Markov chain in the points of customers' arrival equipped with a unique limiting distribution, together with Sahin's formula it enables us to separate our analysis to that of the control policy and of the demand process. In this way the analysis of our model and the structure of the single item inventory models is clear and easily perspicuous. Making use of the fact that the sample paths of the netstock and inventory position processes yield a step function, we impose a cost structure to it. Due to the perspicuous structure of the model, it is now easy to derive the cost expressions for general cost functions, moreover, one can obtain all the specific cost and service measures by merely substituting the appropriate cost function into these expressions. All the costs can be computed numerically almost up to machine precision with the help of a recently developed Laplace transform inversion algorithm.

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