

*Probability in the Engineering and Informational Sciences*, **30**, 2016, 326–344.  
doi:10.1017/S0269964816000036

# A MARKOVIAN ANALYSIS OF IEEE 802.11 BROADCAST TRANSMISSION NETWORKS WITH BUFFERING

GUY FAYOLLE

*INRIA Paris-Rocquencourt, Domaine de Voluceau  
BP 105, 78153 Le Chesnay Cedex  
France  
Email: [Guy.Fayolle@inria.fr](mailto:Guy.Fayolle@inria.fr)*

PAUL MUHLETHALER

*INRIA Paris-Rocquencourt, Domaine de Voluceau  
BP 105, 78153 Le Chesnay Cedex  
France  
E.mail: [paul.muhlethaler@inria.fr](mailto:paul.muhlethaler@inria.fr)*

The purpose of this paper is to analyze the so-called back-off technique of the IEEE 802.11 protocol in broadcast mode with waiting queues. In contrast to existing models, packets arriving when a station (or node) is in back-off state are not discarded, but are stored in a buffer of infinite capacity. As in previous studies, the key point of our analysis hinges on the assumption that the time on the channel is viewed as a random succession of transmission slots (whose duration corresponds to the length of a packet) and mini-slots during which the back-off of the station is decremented. These events occur independently, with given probabilities. The state of a node is represented by a two-dimensional Markov chain in discrete-time, formed by the back-off counter and the number of packets at the station. Two models are proposed both of which are shown to cope reasonably well with the physical principles of the protocol. The stability (ergodicity) conditions are obtained and interpreted in terms of maximum throughput. Several approximations related to these models are also discussed.

## 1. INTRODUCTION

Several studies [1,4] have recently been devoted to the analysis of the IEEE 802.11 protocol, both in the unicast and broadcast modes. Nonetheless, to the best of our knowledge, no model has taken into account the possibility of having buffers to store arriving packets which, due to channel occupancy, cannot be transmitted. When using the framework developed in [1,4], it is impossible to rigorously evaluate the stability of the protocol. In this paper, we use the same key assumption concerning the slots as in [1,4], but we couple it with a Markovian analysis of nodes using the IEEE 802.11 back-off with an infinite buffer for

the packets. We discuss how the way of sensing slots on the channel can give rise to two closely related models. The first one is analyzed in Section 3 and considers the so-called *greedy mode*, while the second one in Section 4 deals with a *fair load* situation. In both cases, ergodicity conditions and the value of the maximum throughput are obtained. The waiting time distribution of a packet is tackled in Section 5.

## 2. SYSTEM PARAMETERS

In contrast to the famous ALOHA protocol where the back-off does not take into account the activity of the channel, the back-off scheme of IEEE 802.11 monitors the channel in order to schedule the transmission of its pending packets. When a node receives a packet to send, it first selects a random back-off number, which indicates the number of mini-slots (MSs) the node has to wait before transmitting. A node with a pending packet senses the channel, and decreases its back-off counter by one each time an idle MS is detected. When its back-off counter reaches the value zero, the node transmits the packet.

The following parameters are ubiquitous in the analytic studies of IEEE 802.11 broadcast:

- $\sigma$ , the duration of an MS, which is the time needed for a station to sense whether the channel is busy.
- The back-off window size  $W$ , expressed as a multiple of  $\sigma$ .
- $T$ , the packet duration, a priori larger than  $\sigma$ .

The three above parameters are really essential in the IEEE 802.11 back-off scheme. They lead to the main assumption that is usually made in the analysis of this scheme (see [1,4]), stating that the channel consists in a random succession of *full slots* (dedicated to packet transmission) and *MSs* (short slots during which the channel is idle). To complete the description of the model, we also need to introduce the following three additional quantities.

- $\lambda$  is the Poisson arrival rate at a station.
- $M$ , the number of stations in the network.
- $\tau$ , the probability that a station will transmit a packet, stressing that this can take place only when its back-off counter has reached zero. We will return to this parameter later.

## 3. MODEL OF AN ISOLATED STATION IN A GREEDY MODE

This section is devoted to the analysis of the basic component of the network, namely a single station operating in compliance with some of the principles of the IEEE 801.11 protocol.

The channel is supposed to be sampled at discrete time instants  $Z_i$ , so that it is sensed during consecutive time intervals  $Z_i - Z_{i-1}, i \geq 1$ , which are equal either to  $T$  (a *normal slot*) or to  $\sigma$  (an *MS* (see Section 3.1)). In some sense, we have thus introduced two embedded time-scales.

In general, the selection between these two types of slots is random: a slot is a normal slot with probability  $r$ , and an MS with probability  $1 - r$ . Nonetheless the node is said to be *greedy*, since, when the back-off reaches 0, a transmission takes place in a slot which is then necessarily a normal slot of length  $T$ . In particular, this implies that a sequence of slots is not exactly obtained as a pure outcome of repeated Bernoulli trials.

Now, starting from the basic one-dimensional models proposed in [1,4], we construct a two-dimensional Markov chain, taking into account the possibility of waiting queues at the stations.

An arbitrary station (or node) will be represented by a stochastic process, which is a two-dimensional random walk  $(K_{Z_i}, N_{Z_i})_{i \geq 1}$ , where  $Z_i$  is an embedded increasing sequence of discrete random times at which the chain is observed. Indeed, at time  $Z_i$ ,  $K_{Z_i}$  stands for the value of the back-off counter and  $N_{Z_i}$  is the number of packets in the waiting queue at a station. The Markovian evolution of  $(K_{Z_i}, N_{Z_i})$  is described in detail in the next section.

### 3.1. Dynamics

Taking into account the definitions given in Section 2, the sequence  $Z_i, i \geq 1$  in the so-called *greedy mode* satisfies the recursive stochastic relationship

$$Z_{i+1} = Z_i + \Delta_i \left[ \mathbb{1}_{\{K_{Z_i} > 0\}} + \mathbb{1}_{\{K_{Z_i} = N_{Z_i} = 0\}} \right] + T \mathbb{1}_{\{K_{Z_i} = 0, N_{Z_i} > 0\}} \quad (3.1)$$

with

$$\Delta_i = T \mathbb{1}_{\{B_i\}} + \sigma(1 - \mathbb{1}_{\{B_i\}}), \quad (3.2)$$

where  $\mathbb{1}_{\{\cdot\}}$  denotes the indicator function and  $B_i$  is the event  $\{\text{channel busy}\}$  at time  $Z_i$ . From the assumptions made above,  $\{\mathbb{1}_{\{B_i\}}, i \geq 1\}$  form a sequence of independent identically distributed random variables with  $\mathbb{P}(B_i) = r$ .

Without entering a quite standard formalism, we just state that the underlying probability space for the random walk  $(K_{Z_i}, N_{Z_i})_{i \geq 1}$  is obtained by combining the external Poisson arrival process and the sequence of independent Bernoulli trials related to the  $B_i$ 's.

Then we are in a position to write Kolmogorov's forward equations for the Markov chain  $(K_{Z_i}, N_{Z_i})$ . The reader will see that the *greediness* appearing in the title of the section stems from the last term in the right-hand side member of (3.1).

Let us define the conditional probability  $p(k, n; Z_i) \stackrel{\text{def}}{=} P(K_{Z_i} = k, N_{Z_i} = n; Z_i)$ . Then, for all  $k \geq 0, n \geq 1$ ,

$$\begin{aligned} p(k, n; Z_i + T) &= \mathbb{1}_{\{k > 0\}} \mathbb{1}_{\{B_i\}} \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j; Z_i) \\ &\quad + \frac{1}{W+1} \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n+1-j; Z_i) \\ &\quad + \mathbb{1}_{\{B_i\}} \frac{1}{W+1} \frac{e^{-\lambda T} (\lambda T)^n}{n!} p(0, 0; Z_i), \end{aligned} \quad (3.3)$$

$$\begin{aligned} p(k, n; Z_i + \sigma) &= \mathbb{1}_{\{k < W\}} (1 - \mathbb{1}_{\{B_i\}}) \sum_{j=0}^n \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j; Z_i) \\ &\quad + (1 - \mathbb{1}_{\{B_i\}}) \frac{1}{W+1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^n}{n!} p(0, 0; Z_i). \end{aligned} \quad (3.4)$$

These equations deserve some explanation.

- The first term of the right-hand member of (3.3) corresponds to the transmission of a packet by another station when the current station is in back-off, noting that arrivals may occur during this slot. The second term corresponds to a packet transmission by

the current station and the choice of a back-off number for the next pending packet; new arrivals may still occur. The last term concerns arrivals at an empty current station during the transmission of a packet by the network; a random back-off is selected for the first writing packet (if any).

- It is worth remarking that for  $k = 0$  the first term in the right-hand member of (3.3) does not exist, since in this case it is impossible to reach the state  $(0, n, Z_i)$  from any state  $(0, j, Z_{i-1}), j \leq n$ . Also, the second term in (3.3) indicates that a transmission takes place when the back-off counter equals 0.
- In (3.4), the first term of the right-hand member corresponds to the current station decrementing its back-off counter; new arrivals may occur during the MS, while the second term takes into account the new arrivals during an MS in the current station which is idle; if at least one packet has arrived, then a random back-off number is selected for this packet.

On the other hand, the following boundary equations hold for  $k = 0$  and for all  $Z_i, i \geq 1$ .

$$\begin{cases} p(k, 0; Z_i) = 0, & \forall k \geq 1, \\ p(0, 0; Z_i + T) = e^{-\lambda T} [p(0, 1; Z_i) + \mathbb{1}_{\{B_i\}} p(0, 0; Z_i)], \\ p(0, 0; Z_i + \sigma) = e^{-\lambda \sigma} (1 - \mathbb{1}_{\{B_i\}}) p(0, 0; Z_i), \end{cases} \quad (3.5)$$

where  $p(0, 0, Z_i)$  is the empty (or idle) state, through which the system exhibits some regeneration properties.

In the following, we focus on the steady-state behavior of system (3.3)–(3.5), which is simply obtained by letting  $i \rightarrow \infty$ , since in this case  $Z_i \rightarrow \infty$  almost surely (and even surely!).

Setting

$$p(k, n) \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} \mathbb{E}[p(k, n; Z_i)],$$

we obtain, for all  $n \geq 1, 0 \leq k \leq W$ ,

$$\begin{aligned} p(k, n) &= \mathbb{1}_{\{k < W\}} (1 - r) \sum_{j=0}^n \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k + 1, n - j) \\ &\quad + \mathbb{1}_{\{k > 0\}} r \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n - j) \\ &\quad + \frac{1}{W + 1} \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n + 1 - j) \\ &\quad + \frac{1}{W + 1} \left[ \frac{(1 - r) e^{-\lambda \sigma} (\lambda \sigma)^n}{n!} + \frac{r e^{-\lambda T} (\lambda T)^n}{n!} \right] p(0, 0), \end{aligned} \quad (3.6)$$

and

$$\begin{cases} p(k, 0) = 0, & \forall 1 \leq k \leq W, \\ e^{-\lambda T} p(0, 1) = p(0, 0) [1 - r e^{-\lambda T} - (1 - r) e^{-\lambda \sigma}]. \end{cases} \quad (3.7)$$

Let us observe that here  $r$  is an exogenous parameter, but later in Sections 3.4 and 4.1, it will depend on the state of the system.

### 3.2. Formal Solution of the Steady-State Equations

Let us introduce the generating functions

$$F_k(x) = \sum_{n \geq 0} p(k, n) x^n, \quad \forall 0 \leq k \leq W,$$

which will be sought to be *analytic* in the unit disc  $\mathcal{D} = \{x : |x| < 1\}$  and continuous on the boundary  $|x| = 1$ .

LEMMA 3.1: *The generating functions  $F_k(x)$ ,  $k \geq 0$ , satisfy the linear system*

$$\begin{cases} a(x)F_1(x) = \widetilde{F}_0(x) + C(x), \\ a(x)F_{k+1}(x) = b(x)F_k(x) + C(x), \quad \forall 1 \leq k \leq W-1, \\ b(x)F_W(x) + C(x) = 0, \end{cases} \quad (3.8)$$

where

$$\begin{cases} \widetilde{F}_0(x) = F_0(x) - p(0, 0) \\ a(x) = (1-r)e^{-\lambda\sigma(1-x)}, \\ b(x) = 1 - re^{-\lambda T(1-x)}, \end{cases} \quad (3.9)$$

and

$$C(x) = \frac{-e^{-\lambda T(1-x)}}{(W+1)x} F_0(x) + \frac{p(0, 0)}{W+1} \left[ 1 + \left( \frac{1}{x} - r \right) e^{-\lambda T(1-x)} - (1-r)e^{-\lambda\sigma(1-x)} \right]. \quad (3.10)$$

PROOF: System (3.8) is easily obtained after multiplying (3.6) by  $x^n$ , summing from  $n \geq 1$  onwards, and taking into account the first boundary condition in (3.7). The final form of  $C(x)$  in (3.10) is derived from the second boundary condition in (3.7). More detail is given in Appendix A. ■

PROPOSITION 3.2: *Letting*

$$u(x) \stackrel{\text{def}}{=} \frac{b(x)}{a(x)},$$

*the functions  $F_k(x)$ ,  $1 \leq k \leq W$ , are given by the relations*

$$F_k(x) = \frac{\widetilde{F}_0(x)}{a(x)} \left[ \frac{u^{k-1}(x) - u^W(x)}{1 - u^{W+1}(x)} \right], \quad (3.11)$$

*and  $F_0(x)$  has the explicit form*

$$F_0(x) = p(0, 0) \frac{Q(x)}{R(x)}, \quad (3.12)$$

where

$$R(x) = \frac{e^{-\lambda T(1-x)}}{x} - \frac{(W+1)u^W(x)(1-u(x))}{1 - u^{W+1}(x)}, \quad (3.13)$$

$$Q(x) = \left[ 1 + \left( \frac{1}{x} - r \right) e^{-\lambda T(1-x)} - (1-r)e^{-\lambda\sigma(1-x)} \right] - \frac{(W+1)u^W(x)(1-u(x))}{1 - u^{W+1}(x)}. \quad (3.14)$$

PROOF: The result is obtained in two steps. First a straightforward calculus exploiting system (3.8) yields (3.11). Then, instantiating  $k = W$  in Eq. (3.11) and comparing it with the third relation in (3.8), we get

$$u^W(x)\widetilde{F}_0(x) + C(x) \frac{1 - u^{W+1}(x)}{1 - u(x)} = 0,$$

which, by using (3.9), leads to the final Eqs. (3.12)–(3.14). ■

The only remaining unknown  $p(0, 0)$  will now be obtained from the normalization condition

$$\sum_{k \geq 0, n \geq 0} p(k, n) = 1.$$

PROPOSITION 3.3: *Under ergodicity conditions, the probability  $p(0, 0)$  that the system is idle is given by the formula*

$$p(0, 0) = \frac{1 - \lambda B}{1 - \lambda A + \frac{\lambda W(B - A)}{2(1 - r)}} \quad (3.15)$$

where

$$\begin{cases} A &= (1 - r)(T - \sigma) + \frac{W[rT + (1 - r)\sigma]}{2(1 - r)}, \\ B &= T + \frac{W[rT + (1 - r)\sigma]}{2(1 - r)}. \end{cases} \quad (3.16)$$

PROOF: By (3.11), since  $u(1) = 1$  and  $a(1) = 1 - r$ , we get

$$F_0(1) + \frac{\widetilde{F}_0(1)}{1 - r} \sum_{k=1}^W \frac{W - k + 1}{W + 1} = 1,$$

that is

$$F_0(1) + \frac{W\widetilde{F}_0(1)}{2(1 - r)} = 1. \quad (3.17)$$

Hence, instantiating  $x = 1$  in Eqs. (3.12)–(3.14), using the relation

$$\lim_{x \rightarrow 1} \frac{Q(x)}{R(x)} = \frac{Q'(1)}{R'(1)},$$

since  $Q(1) = R(1) = 0$ , where  $Q'(1)$  and  $R'(1)$  are the respective derivatives of  $Q(x)$  and  $R(x)$  at  $x = 1$ , from Eq. (3.17) we obtain

$$p(0, 0) \left[ \left( 1 + \frac{W}{2(1 - r)} \right) \frac{Q'(1)}{R'(1)} - \frac{W}{2(1 - r)} \right] = 1.$$

Setting for the present calculation

$$f(x) = \frac{(W + 1)u^W(x)(1 - u(x))}{1 - u^{W+1}(x)} = \frac{(W + 1)u^W(x)}{\sum_{0 \leq i \leq W} u^i(x)},$$

we have

$$f'(1) = \frac{Wu'(1)}{2} = -\frac{\lambda W[rT + (1-r)\sigma]}{2(1-r)}.$$

Hence, by (3.13) and (3.14),

$$R'(1) = -1 + \lambda T - f'(1) = -1 + \lambda T + \frac{\lambda W[rT + (1-r)\sigma]}{2(1-r)},$$

$$Q'(1) = -1 + (1-r)\lambda(T - \sigma) - f'(1) = -1 + (1-r)\lambda(T - \sigma) + \frac{\lambda W[rT + (1-r)\sigma]}{2(1-r)},$$

which by an elementary computation yields Eq. (3.15). ■

*Remark 3.4:* As we shall see in Theorem 3.5, the system is ergodic if and only if

$$R'(1) < 0,$$

in which case the following inequalities hold:

$$\begin{cases} Q'(1) \leq R'(1) < 0, \\ \frac{Q'(1)}{R'(1)} \geq 1. \end{cases}$$

Two limit cases can also be checked.

- $\lambda \rightarrow 0 \Rightarrow p(0,0) \rightarrow 1$ .
- It should be observed that  $Q'(1) = R'(1)$  if and only if  $r = \sigma = 0$ , which corresponds to the trivial situation  $p(0,0) = 1$ .

We bear in mind that functions  $F_k(x)$ ,  $k \geq 0$ , are sought to be analytic in the unit disc. Hence, ergodicity conditions will be obtained from (3.12) by studying the possible zeros of  $R(x)$  in the closed unit disc  $\mathcal{D}$ , noting *en passant* that  $Q(x)$  and  $R(x)$  have no common root in  $\mathcal{D}$ , but at  $x = 1$ . This is the objective of the next theorem.

### 3.3. Stability Condition of the Greedy Model

With the notation of Proposition 3.3, we have the following:

**THEOREM 3.5:** *The system is ergodic if and only if  $R'(1) = \lambda B - 1 < 0$ , that is*

$$\lambda < \frac{1}{T \left[ 1 + \frac{rW}{2(1-r)} \right] + \frac{W\sigma}{2}}. \quad (3.18)$$

*In other words, the ergodicity region in the parameter space  $\{\lambda, \sigma, T, r\}$  is delimited by the surface  $S(\lambda, \sigma, T)$  expressed by the equation*

$$\lambda[(Wr + 2(1-r))T + W(1-r)\sigma] = 2(1-r).$$

PROOF: Let

$$g(x) = \frac{e^{-\lambda T(1-x)}[1 - u^{W+1}(x)]}{1 - u(x)}, \quad h(x) = (W + 1)xu^W(x),$$

one can see at once that, for all  $x \neq 1$ , the zeros of  $R(x)$  in  $\mathcal{D}$  coincide with the roots of

$$g(x) - h(x) = 0.$$

From (3.9)

$$u(x) = \frac{e^{\lambda\sigma(1-x)}[1 - re^{-\lambda T(1-x)}]}{(1 - r)},$$

so that,  $\forall x \in \mathcal{D}$ ,

$$|u(x)| = \frac{|e^{\lambda\sigma(1-x)}|}{1 - r} |1 - re^{-\lambda T(1-x)}| \geq |e^{\lambda\sigma(1-x)}| \geq 1,$$

which yields

$$\left| \frac{1 - u^{W+1}(x)}{1 - u(x)} \right| = |1 + u(x) + \cdots + u^W(x)| \leq (W + 1)|u^W(x)|.$$

Therefore, on the unit circle  $|x| = 1, x \neq 1$ , we have  $|g(x)| < |h(x)|$ . Since the functions  $g(x)$  and  $h(x)$  are analytic in  $\mathcal{D}$ , we will be in a position to apply the well-known Rouché's theorem (see e.g., [3]), after carrying out an adequate modification of the unit circle  $\mathcal{C}$  around the point  $x = 1$ . Since

$$g(x) - h(x) = \frac{x(1 - u^{W+1}(x))}{1 - u(x)} R(x),$$

we have

$$g(x) - h(x) = (W + 1)R'(1)(x - 1) + o((x - 1)^2).$$

Let us assume  $R'(1) < 0$ . We construct a contour  $\mathcal{C}_\varepsilon$  consisting essentially of the unit circle  $\mathcal{C}$  continuously distorted around  $x = 1$  by a small notch (an arc of a circle of radius  $\varepsilon$ ), keeping the point 1 inside the domain  $\mathcal{D}_\varepsilon$  bounded by  $\mathcal{C}_\varepsilon$ . Then, choosing  $\varepsilon$  sufficiently small, we have

$$|g(x)| < |h(x)|, \quad \forall x \in \mathcal{C}_\varepsilon.$$

Consequently, by Rouché's theorem, functions  $h(x)$  and  $h(x) - g(x)$  have the same number of zeros inside  $\mathcal{D}_\varepsilon$ , that is, exactly one located at  $x = 1$ , since  $h(x)$  vanishes only at  $x = 0$ . We conclude that  $F_0(x)$  given by (3.12) is analytic in  $\mathcal{D}$  and the system is ergodic.

Conversely, if  $R'(1) > 0$ , we can construct a contour  $\mathcal{D}_\varepsilon$  that does include the point  $x = 1$ , and the same argument shows that  $R(x)$  has a root different from 1 inside the unit disc  $\mathcal{D}$ , and in this case  $F_0(x)$  is not analytic in  $\mathcal{D}$ . ■

We observe that, on the one hand, for  $r$  tending toward 1,  $\lambda$  tends toward 0. The activity outside the current station is prominent, which means that no bandwidth is left for the current station. On the other hand, if  $r \rightarrow 0$ , then  $\lambda$  can reach the value  $\lambda_{\max} = \frac{1}{T + W\sigma/2}$ . In this case, the current station can consume all the bandwidth, the term  $\frac{W\sigma}{2}$  being the overhead due to the back-off scheme.

### 3.4. Dynamics of a Network of Greedy Broadcasting Stations

Consider a network consisting of  $M$  identical stations operating in a broadcast mode together with a tagged station. The key feature of the protocol IEEE 802.11 – namely a transmission is permitted only when the back-off counter is at zero – can be reflected in the simple equality

$$\tau = \sum_{n \geq 1} p(0, n). \quad (3.19)$$

Hence, from the point of view of the tagged station, the  $M$  stations represent the outside world and the event  $\{\text{Channel busy}\}$  takes place with probability

$$r = 1 - (1 - \tau)^M, \quad M \geq 0. \quad (3.20)$$

The problem is now to recast the ergodicity condition (3.18) given in Theorem 3.5 in the light of Eqs. (3.19), (3.20). According to Proposition 3.3 and Remark 3.4,  $\tau$  defined by (3.19) satisfies

$$\tau = F_0(1) - p(0, 0) = p(0, 0) \left( \frac{Q'(1)}{R'(1)} - 1 \right) = p(0, 0) \frac{\lambda(B - A)}{1 - \lambda B},$$

that is, by (3.15) and (3.16),

$$\tau = \frac{\lambda(B - A)}{1 - \lambda A + \frac{\lambda W(B - A)}{2(1 - r)}} = \frac{\lambda[rT + (1 - r)\sigma]}{1 - \lambda A + \frac{\lambda W(B - A)}{2(1 - r)}} = \frac{\lambda[rT + (1 - r)\sigma]}{1 - \lambda T + \lambda[rT + (1 - r)\sigma]}. \quad (3.21)$$

Observe at once that the denominator in the last equality of (3.21) necessarily imposes

$$\lambda T < 1, \quad (3.22)$$

with an obvious interpretation. Henceforth, condition (3.22) will be assumed in the rest of this section.

So, (3.20) and (3.21) yield the following fixed point equation in  $r$ , written as

$$(1 - r)^{1/M} = \frac{1 - \lambda T}{1 - \lambda(1 - r)(T - \sigma)}. \quad (3.23)$$

On the other hand, setting

$$z = (1 - r)^{1/M}, \quad (3.24)$$

one can also view (3.23) as a polynomial equation in  $z$ , namely

$$P(z) \stackrel{\text{def}}{=} \lambda(T - \sigma)z^{M+1} - z + (1 - \lambda T) = 0. \quad (3.25)$$

LEMMA 3.6: Under condition (3.22), Eq. (3.25) viewed as an equation in  $z$  has only one root in the real interval  $[0, 1]$ .

PROOF: Under the assumption  $\lambda T < 1$ , we have

$$P(0) > 0, \quad P(1) < 0,$$

and two cases must be considered.

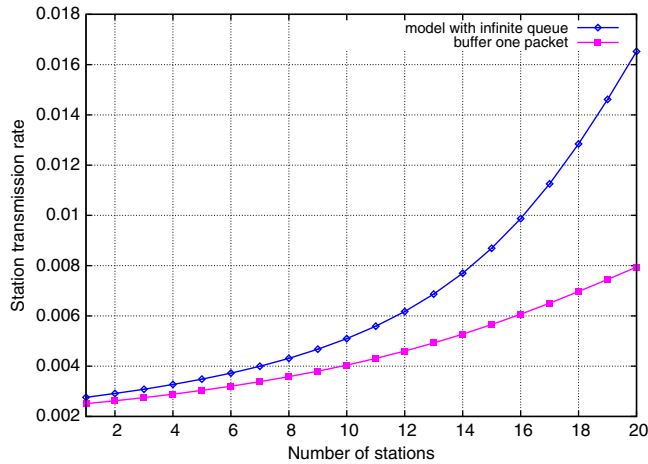


FIGURE 1. Computation of  $\tau(M)$  with and without buffer.

- (i)  $T < \sigma$  (purely academic situation). Then, on the interval  $[0, 1]$ , the derivative  $P'(z)$  in (3.24) is negative, and  $P(z)$  vanishes exactly once.
- (ii)  $T > \sigma$  (the real world!). On the unit circle  $|z| = 1$ , we have

$$|z + 1 - \lambda T| \geq ||z| - (1 - \lambda T)| \geq \lambda T \geq |\lambda(T - \sigma)z^{M+1}| = \lambda(T - \sigma),$$

and the result follows by an immediate application of Rouché's theorem.

The proof of the lemma is concluded. ■

We adopt the following figures  $\lambda = 0.05$ ,  $T = 1$ ,  $\sigma = 0.05$ ,  $W = 31$ , and we vary  $M$ . We compute  $\tau$  in our model and we compare it with the value of  $\tau$  obtained in the model without any buffer of [4]. In Figure 1, we observe that  $\tau$  is larger in our model compared to the model without any buffer for the same input load. This is because the model without buffer drops packets when one packet arrives during the back-off period of a preceding packet. In Figure 2, we compare the throughput without collision of our model with the throughput without collision of the model without buffer. For the same reason, the throughput without collision of our model is higher.

We are now in a position to compute the maximum throughput of the system under the stability condition (3.18). This is the subject of the following and final theorem.

**THEOREM 3.7:**

1 The greedy broadcast network is ergodic if and only if

$$2z^{M+1} > W(1 - z), \quad (3.26)$$

where  $z$  defined by (3.24) is the unique real root of Eq. (3.25) in the interval  $[0, 1]$ .

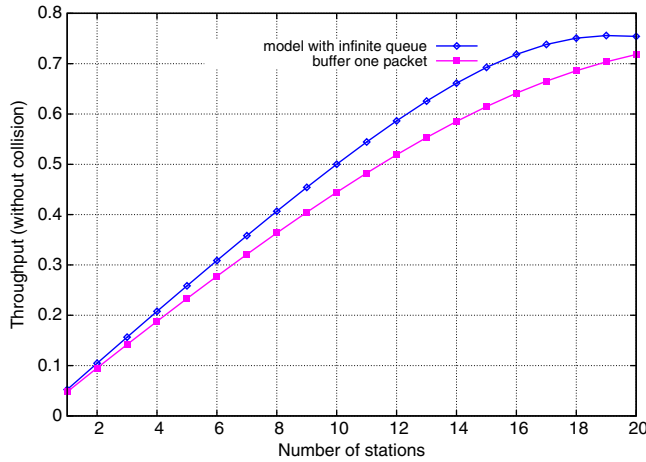


FIGURE 2. Network throughput *without collision* versus the number of stations.

2 When the system is ergodic, the maximum achievable throughput of the system takes one of the two equivalent forms

$$\lambda_{\max} = \frac{1-u}{T(1-u^{M+1}) + \sigma u^{M+1}} = \frac{1-u}{T + \frac{W(\sigma-T)(1-u)}{2}}, \quad (3.27)$$

where  $u$  is the unique root on  $[0, 1]$  of the equation

$$2u^{M+1} = W(1-u). \quad (3.28)$$

PROOF: Rewrite  $B$  given by (3.16) as

$$B = T + \frac{W[(1-r)(\sigma-T) + T]}{2(1-r)},$$

which, using (3.25), yields the pleasant factorized form

$$1 - \lambda B = (1 - \lambda T) \left[ 1 - \frac{W(1-z)}{2z^{M+1}} \right].$$

Hence, the necessary and sufficient ergodicity condition  $\lambda B - 1 < 0$  comes down to

$$1 - \frac{W(1-z)}{2z^{M+1}} > 0,$$

which is exactly (3.26). Finally, the quantity  $\lambda_{\max}$  is simply obtained by saturating inequality (3.26), which gives (3.28) and then (3.27). ■

We exploit formulas (3.27) and (3.28). We still have  $T = 1$ ,  $\sigma = 0.05$ ,  $W = 31$ , but  $\lambda = 0.05$ , and we vary  $M$  between 1 and 100. We plot the maximum throughput of one station and the cumulated throughput of all the stations which we call the network offered load. This is shown in Figure 3 where we vary the number  $M$  of stations in the network. One can observe that the network offered load can be larger than 1. This can be explained by the fact that two stations (or more) can send a packet during the same slot. The network offered load is the maximum total load the network can submit while remaining stable. For a given number of stations, it is also possible to study the maximum network throughput (without collision) when we vary  $W$ . This study is presented in Figure 4.

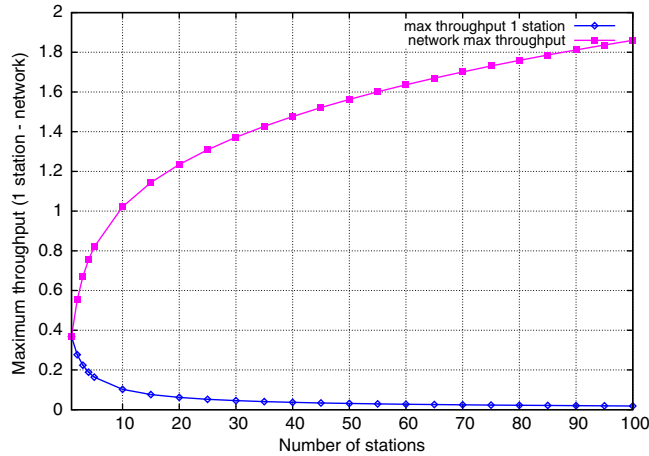


FIGURE 3. Network throughput versus the number of stations.

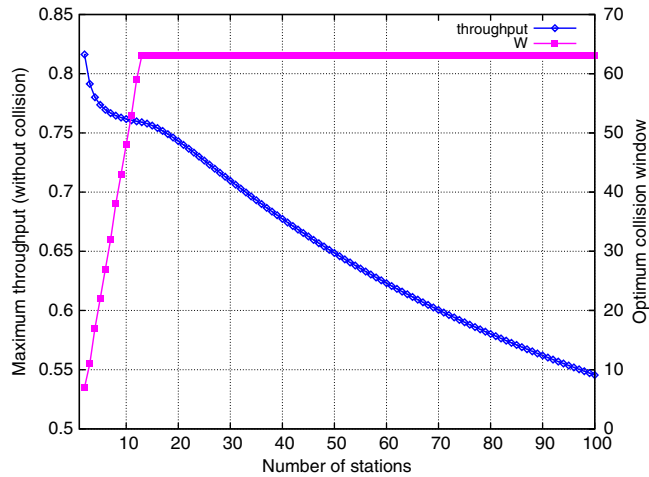


FIGURE 4. Optimized network throughput (without collision) and corresponding value of  $W$  versus the number of stations  $M$ .

#### 4. A FAIR LOAD MODEL

Here a station does not impose an extra load, as this was the case in the greedy model, and the sequence of times  $Z_i, i \geq 0$  is given by recursive scheme

$$Z_i = Z_{i-1} + \Delta_i,$$

where  $\Delta_i$  is still given by (3.2). A station with a pending packet and a back-off counter at 0 may send its packet with probability  $r$  or return to back-off with probability  $1 - r$ . Thus, in contrast to Section 3, slots on the channel are selected at random and form a renewal process: a full slot (length  $T$ ) with probability  $r$ , or an MS (length  $\sigma$ ) with probability  $1 - r$ , independently of the state of the station. So,  $r$  is considered as an exogenous parameter. But if the station is able to sense the channel, one could think of  $r$  being a function of the state, possibly estimated via some adaptive scheme like for example in [2] for Aloha-type systems.

Mutatis mutandis with respect to Section 3.1, one sees that the stationary distribution of a *fair loaded* station satisfies the following Kolmogorov's equations, for all  $n \geq 0, 0 \leq k \leq W$ ,

$$\begin{aligned} q(k, n) = & \mathbb{1}_{\{k < W\}}(1-r) \sum_{i=0}^n \frac{e^{-\lambda\sigma}(\lambda\sigma)^i}{i!} q(k+1, n-i) + \mathbb{1}_{\{k > 0\}} r \sum_{i=0}^n \frac{e^{-\lambda T}(\lambda T)^i}{i!} q(k, n-i) \\ & + \frac{1}{W+1} \left[ r \sum_{i=0}^n \frac{e^{-\lambda T}(\lambda T)^i}{i!} q(0, n+1-i) + (1-r) \sum_{i=0}^n \frac{e^{-\lambda\sigma}(\lambda\sigma)^i}{i!} q(0, n-i) \right] \\ & + \frac{re^{-\lambda T}(\lambda T)^n}{(W+1)n!} q(0, 0), \end{aligned} \quad (4.1)$$

and

$$\begin{cases} q(k, 0) = 0, & \forall 1 \leq k \leq W, \\ re^{-\lambda T} q(0, 1) = q(0, 0) [1 - re^{-\lambda T} - (1-r)e^{-\lambda\sigma}]. \end{cases} \quad (4.2)$$

The first line of Eq. (4.1) corresponds to the transition when the back-off counter is greater than or equal to 1. If the current slot is an MS (with probability  $1-r$ ) the back-off counter is decremented, otherwise if it is a normal slot (with probability  $r$ ) and the back-off counter remains unchanged. In both the cases, new arrivals may occur and increment the number of pending packets. The second line of Eq. (4.1) deals with the transition when the back-off counter is at 0. With probability  $r$ , there is a transmission with a new draw of the back-off (if any pending packets) and new arrivals may occur. With probability  $1-r$  the current slot is an MS and the packet returns to back-off, and there still may be new arrivals. The third line of Eq. (4.1) corresponds to the transition from the idle state of the station to a state where a back-off value is selected with probability  $1/(W+1)$ .

Let us introduce as before the generating functions

$$G_k(x) = \sum_{n \geq 0} q(k, n)x^n, \quad 0 \leq k \leq W,$$

sought to be *analytic* in the unit disc  $\mathcal{D} = \{x : |x| < 1\}$  and continuous on the boundary  $|x| = 1$ . Then the following results hold, the explicit proofs of which are omitted, as they mimic the arguments of Section 3.2. Some hints are given in Appendix B.

LEMMA 4.1: *The generating functions  $G_k(x), k \geq 0$ , satisfy the linear system*

$$\begin{cases} a(x)G_1(x) = \widetilde{G}_0(x) + D(x), \\ a(x)G_{k+1}(x) = b(x)G_k(x) + D(x), & \forall 1 \leq k \leq W-1, \\ b(x)G_W(x) + D(x) = 0, \end{cases} \quad (4.3)$$

where

$$\widetilde{G}_0(x) = G_0(x) - q(0, 0),$$

$a(x), b(x)$  being given by (3.9), and

$$D(x) = \frac{-G_0(x)}{(W+1)} \left[ \frac{re^{-\lambda T(1-x)}}{x} + (1-r)e^{-\lambda\sigma(1-x)} \right] + \frac{q(0, 0)}{W+1} \left[ 1 + r \left( \frac{1}{x} - 1 \right) e^{-\lambda T(1-x)} \right].$$

PROPOSITION 4.2: *Letting*

$$u(x) \stackrel{\text{def}}{=} \frac{b(x)}{a(x)},$$

*the functions  $G_k(x)$ ,  $1 \leq k \leq W$ , are given by the relations*

$$G_k(x) = \frac{\widetilde{G}_0(x)}{a(x)} \left[ \frac{u^{k-1}(x) - u^W(x)}{1 - u^{W+1}(x)} \right], \quad (4.4)$$

*and  $G_0(x)$  has the explicit form*

$$G_0(x) = q(0, 0) \frac{\overline{Q}(x)}{\overline{R}(x)}, \quad (4.5)$$

*where*

$$\overline{R}(x) = \frac{re^{-\lambda T(1-x)}}{x} + (1-r)e^{-\lambda \sigma(1-x)} - \frac{(W+1)u^W(x)(1-u(x))}{1-u^{W+1}(x)}, \quad (4.6)$$

$$\overline{Q}(x) = \left[ 1 + r \left( \frac{1}{x} - 1 \right) e^{-\lambda T(1-x)} \right] - \frac{(W+1)u^W(x)(1-u(x))}{1-u^{W+1}(x)}. \quad (4.7)$$

PROOF: Along arguments quite similar to those used in Section 3.2, one can show

$$u^W(x) \widetilde{G}_0(x) + D(x) \frac{1 - u^{W+1}(x)}{1 - u(x)} = 0,$$

which directly yields (4.5)–(4.7). ■

THEOREM 4.3: *The fair load station is ergodic if and only if  $\overline{R}'(1) < 0$ , that is*

$$\lambda < \frac{r(1-r)}{[1-r+W/2][rT+(1-r)\sigma]}. \quad (4.8)$$

*In this case,*

$$q(0, 0) = 1 - \frac{\lambda[rT+(1-r)\sigma] \left[ 1 + \frac{W}{2(1-r)} \right]}{r}. \quad (4.9)$$

PROOF: As in Proposition 3.3, by using (4.4) and (4.5), the normalization condition becomes

$$G_0(1) + \frac{W\widetilde{G}_0(1)}{2(1-r)} = 1,$$

whence, after some algebra

$$q(0, 0) = \frac{1}{\frac{W}{2(1-r)} \left( \frac{\overline{Q}'(1)}{\overline{R}'(1)} - 1 \right) + \frac{\overline{Q}'(1)}{\overline{R}'(1)}} = \frac{-\overline{R}'(1)}{r}, \quad (4.10)$$

which is exactly equivalent to (4.9). One can remark that when  $r \searrow 0$ , the system remains ergodic if and only if

$$\frac{\lambda}{r} \nearrow \frac{1}{\sigma(1+W/2)},$$

in which case  $q(0, 0) \searrow 0$ . ■

#### 4.1. Dynamics of a Network of Fair Broadcasting Stations

We proceed as in Section 3.4, setting

$$r = 1 - (1 - \bar{\tau})^M, \quad (4.11)$$

with

$$\bar{\tau} = \sum_{n \geq 1} q(0, n). \quad (4.12)$$

Then the following result similar to Theorem 3.7 holds.

**THEOREM 4.4:** *The fair broadcast network is ergodic if and only if*

$$\lambda < \bar{\lambda}_{\max},$$

where the maximum achievable throughput  $\bar{\lambda}_{\max}$  takes the form

$$\bar{\lambda}_{\max} = \frac{1 - u}{T + \frac{W\sigma(1-u)}{u(2+W)-W}}, \quad (4.13)$$

where  $u$  is the unique root in  $[0, 1]$  of (3.28), that is

$$2u^{M+1} = W(1 - u).$$

**PROOF:** We outline the arguments, as they mimic those of Theorem 3.7. The algebra is even simpler. Indeed, by (4.12) and (4.10) one can write

$$\bar{\tau} = q(0, 0) \left( \frac{\bar{Q}'(1)}{\bar{R}'(1)} - 1 \right) = \frac{\bar{R}'(1) - \bar{Q}'(1)}{r} = \frac{\lambda[rT + (1-r)\sigma]}{r},$$

so that, using (4.8)

$$\lim_{\lambda \rightarrow \bar{\lambda}_{\max}} \bar{\tau} = \frac{1}{1 + \frac{W}{2(1-r)}}. \quad (4.14)$$

Then, putting  $1 - r = u^M$ , it appears that  $u$  satisfies exactly (3.28), and we obtain

$$\bar{\lambda}_{\max} = \frac{1}{\left[1 + \frac{W}{2(1-r)}\right] \left[T + \frac{(1-r)\sigma}{r}\right]} = \frac{1 - u}{T + \frac{(1-r)\sigma}{r}},$$

which with (4.11) and (4.14) yield (4.13). ■

It is clearly worth comparing the two maxima  $\lambda_{\max}$  and  $\bar{\lambda}_{\max}$  given respectively by (3.27) and (4.13). To this end, it suffices to check that the quantity

$$\delta = \frac{\sigma - T}{2} - \frac{\sigma}{u(2+W)-W} = \frac{1}{2} \left[ \sigma - T - \frac{\sigma}{u(1-u)^M} \right],$$

is clearly negative as  $u < 1$ . Hence

$$\bar{\lambda}_{\max} < \lambda_{\max}.$$

We exploit the formula giving the maximum station throughput. We still have  $\lambda = 0.01$ ,  $T = 1$ ,  $\sigma = 0.05$ ,  $W = 31$  and we vary  $M$  between 1 and 100. We plot the maximum throughput

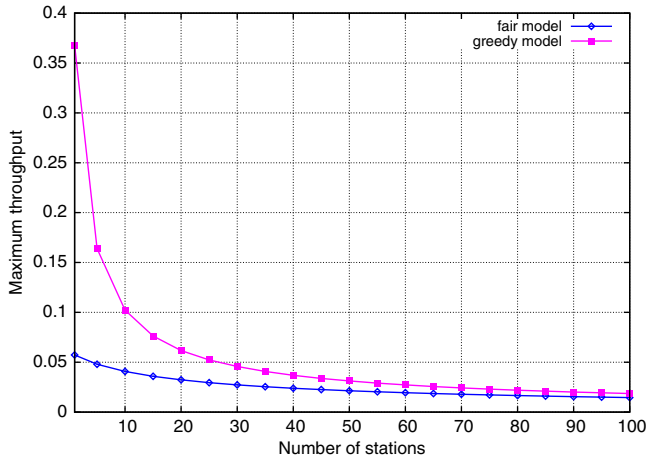


FIGURE 5. Maximum station throughput versus the number of stations  $M$  in the fair and greedy models.

of one station for the fair and greedy models. This is shown in Figure 5 where we vary the number  $M$  of stations in the network. As foreseen, the maximum achievable throughput is lower in the fair model compared to the greedy model. However, in both models  $u$  is given by the unique Eq. (3.28), so that the station transmission rates coincide, as well as the network throughput without collision.

## 5. ABOUT THE WAITING TIME DISTRIBUTION IN THE GREEDY MODEL

From now on, the system will be assumed to evolve at *steady state*. We intend to compute the stationary distribution of the virtual waiting time of a packet arriving at an arbitrary epoch  $Z_i$ .

For an arbitrary probability distribution  $F(\cdot)$ , denote by  $F^*$  its Laplace–Stieltjes transform (LST)

$$F^*(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} dF(t) dt, \quad \Re(s) \geq 0.$$

Setting

$$f(s) = \frac{(1-r)e^{-s\sigma}}{1-re^{-sT}}, \quad v(s) = \frac{e^{-sT}[f^{W+1}(s) - 1]}{(W+1)(f(s) - 1)}, \quad (5.1)$$

we have the following result.

**THEOREM 5.1:** *The Laplace transform of the stationary virtual waiting time distribution of a packet arriving at time  $Z_i$  has the form*

$$\psi^*(s) = \sum_{k=0}^W f^k(s) F_k(v(s)). \quad (5.2)$$

In particular, the expectation  $-\psi^*(0)'$  can be computed from formulas (3.11), (3.12).

PROOF: Let  $A(k, n; Z_i)$  be the conditional waiting time of a packet which at time  $Z_i$  sees the system in the state  $(K_{Z_i}, N_{Z_i}) = (k, n)$ . With the notation of Section 3.1, we can write

$$A(k, n; Z_{i-1}) = (1 - \mathbb{1}_{\{B_{i-1}\}})[\sigma + A(k-1, n; Z_i)] + \mathbb{1}_{\{B_{i-1}\}}[T + A(k, n; Z_i)], \quad \forall k \geq 1, \quad (5.3)$$

remembering the transmission discipline at the buffer is first-in-first-out (FIFO), without forgetting the boundary equation for  $k = 0$ , namely

$$A(0, n; Z_{i-1}) = T + \sum_{\ell=0}^W \mathbb{1}_{\{U=\ell\}} A(\ell, n; Z_i), \quad \forall k \geq 1, \quad (5.4)$$

where  $U$  is random variable uniformly distributed on the integers  $0, 1, \dots, W$ .

Let  $\varphi_i^*(s; k, n)$  be the LST corresponding to the distribution of  $A(k, n; Z_i)$ . Then (5.3) implies

$$\varphi_{i-1}^*(s; k, n) = (1-r)e^{-s\sigma}\varphi_i^*(s; k-1, n) + re^{-sT}\varphi_i^*(s; k, n), \quad \forall k \geq 1,$$

whence, letting  $i \rightarrow \infty$  and  $\varphi^* \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} \varphi_i^*$  and using the definition (5.1),

$$\varphi^*(s; k, n) = f^k(s)\varphi^*(s; 0, n). \quad (5.5)$$

Similarly, from (5.4), the LST  $\varphi^*(s; 0, n)$  satisfies the following stationary recursive relationship holds.

$$\varphi^*(s; 0, n) = \frac{e^{-sT}}{W+1} \sum_{\ell=0}^W \varphi^*(s; \ell, n-1) = \left[ \frac{f^{W+1}(s) - 1}{f(s) - 1} \right] \frac{e^{-sT}\varphi^*(s; 0, n-1)}{W+1}. \quad (5.6)$$

Setting  $\varphi^*(s; 0, 0) \equiv 1$ , we get from (5.1), (5.5), (5.6),

$$\varphi^*(s; k, n) = f^k(s)v(s)^n,$$

and the sought waiting time distribution is given by

$$\psi^*(s) = \sum_{k=0}^W \sum_{n=0}^{\infty} p(k, n)\varphi^*(s; k, n),$$

which reduces immediately to (5.2). ■

*Remark 5.2:* To compute the waiting-time distribution of a packet at arrival epoch, one has to switch from a discrete-time model to a continuous-time one. We leave this question as an exercise, just briefly sketching how it can be solved in two steps.

1. First, find the distribution of the stationary Markov chain  $(K_t, N_t)$  from  $(K_{Z_i}, N_{Z_i})$ , by applying for instance a Palm probability inversion formula to the marked point process  $Z_i$  with marks  $(K_{Z_i}, N_{Z_i})$ . But a rather direct way (however slightly tedious) is to proceed as in the classical derivation of residual life times for renewal processes: fix the unique interval  $[Z_{i(\tau)}, Z_{i(\tau)+1}]$  containing the arrival instant  $\tau$ , which allows to derive the number of new packets arrived between  $Z_{i(\tau)}$  and  $Z_{i(\tau)} + \tau$ . Here the stationary sequence  $Z_i$  satisfies

$$\lim_{i \rightarrow \infty} \mathbb{E}(Z_{i+1} - Z_i) = [r + \alpha(1-r)]T + (1-r)(1-\alpha)\sigma,$$

where  $\alpha = F_0(1) - p(0, 0)$ .

2. Then proceed as in the proof of Theorem 5.1.

## 6. PARTIAL CONCLUSION

We have proposed two models for a CSMA protocol using the back-off of IEEE 802.11, one being very close to the real protocol for broadcast packets. In contrast to existing works, we assumed that nodes can store arriving packets. With a classical formalism of normal slot and MSs, it was possible to compute several important characteristic parameters: station transmission rate, stability conditions, delay for a random packet.

It is worth to emphasize that these protocols can stabilize *without external control*, in contrast to many famous pioneers such as ALOHA, CSMA, etc. In this respect, the window-back-off mechanism exhibits some common features with the so-called tree (or stack) algorithms, which are stable provided that the external load remains below some critical value.

The numerical results obtained so far in this paper seem to match the performance of the IEEE 802.11 broadcast protocol. More simulations (left as future work) should very certainly confirm this matching. At last, studying the IEEE 802.11 protocol for point-to-point traffic under the same hypothesis of infinite buffer, via the same mathematical approach, might be an interesting follow-up of this paper.

### Acknowledgement

This paper is dedicated to Erol Gelenbe for his 70th Birthday. In particular, the first author remembers the old days (1975) of a cooperation on a model of the ALOHA protocol, a topic which stimulated his longstanding interest in stochastic modeling.

### References

1. Bianchi, G. (2000). Performance Analysis of the IEEE 802.11 distributed coordination function. *IEEE Journal of Selected Areas in Communications* 18: pp. 535–547.
2. Fayolle, G., Gelenbe, E., & Labetoulle, J. (1977). Stability and optimal control of the packet switching broadcast channel. *Journal of Association for Computing Machinery* 24: pp. 375–386.
3. Titchmarsh, E.C. (1939). *The theory of functions*. Amen House, London: Oxford University Press, 2nd ed.
4. Wang, J.-P., Abolhasan, M., Franklin, D.R., & Safaei, F. (2009). Characterising the behaviour of IEEE 802.11 Broadcast Transmissions in Ad Hoc Wireless LANs. In *IEEE International Conference on Communications, 2009 (ICC'09)*, pp. 1–5.

## APPENDIX A: EQUATION (3.8) OF THE GREEDY MODEL

Next is the intermediate step leading to system (3.8). Indeed, multiplying (3.6) by  $x^n$ , summing from  $n \geq 1$  onwards, and taking into account the first boundary condition in (3.7), namely  $p(k, 0) \equiv 0, \forall k \geq 1$ , we get

$$\begin{aligned} F_k(x) &= \mathbb{1}_{\{k < W\}}(1-r)e^{-\lambda\sigma(1-x)}F_{k+1}(x) + \mathbb{1}_{\{k > 0\}}re^{-\lambda T(1-x)}F_k(x) \\ &\quad + \frac{1}{(W+1)x} \left[ e^{-\lambda T(1-x)}(F_0(x) - p(0, 0)) - p(0, 1)e^{-\lambda T}x \right] \\ &\quad + \frac{p(0, 0)}{(W+1)} \left[ r(e^{-\lambda T(1-x)} - e^{-\lambda T}) + (1-r)(e^{-\lambda\sigma(1-x)} - e^{-\lambda\sigma}) \right], \end{aligned}$$

which yields in particular

$$a(x)F_{k+1}(x) = b(y)F_k(x) + C(x), \quad \forall 1 \leq k \leq W-1,$$

whence (3.8) follows easily.

**APPENDIX B: EQUATION (4.3) OF THE FAIR LOAD MODEL MODEL**

Just as in the case of the greedy model, (4.3) is obtained by means of the following system, derived from (4.1), (4.2),

$$\begin{aligned}
 G_k(x) = & \mathbb{1}_{\{k=0\}}q(0,0) + \mathbb{1}_{\{k < W\}}(1-r)e^{-\lambda\sigma(1-x)}G_{k+1}(x) + \mathbb{1}_{\{k > 0\}}re^{-\lambda T(1-x)}G_k(x) \\
 & + \frac{r}{(W+1)x} \left[ e^{-\lambda T(1-x)}(G_0(x) - q(0,0)) - p(0,1)e^{-\lambda T}x \right] \\
 & + \frac{1-r}{(W+1)} \left[ e^{-\lambda\sigma(1-x)}G_0(x) - q(0,0)e^{-\lambda\sigma} \right] + \frac{rq(0,0)(e^{-\lambda T(1-x)} - e^{-\lambda T})}{W+1}.
 \end{aligned}$$