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Endogenous Technological Change under Uncertainty

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Abstract

How does risk or uncertainty in the productivity of research affect the growth rate of the economy? To answer this question, a model of endogenous technological change is used where sustained growth stems from intentional investments in R&D from profit-maximizing firms. The uncertainty arises from the productivity of these investments in R&D. The main result of this analysis is that the relationship between long-run growth and uncertainty (on the productivity of knowledge creation) depends on two main factors - the returns to scale in knowledge creation (increasing or non-increasing) and the value of the elasticity of intertemporal substitution (higher or lower than some critical value). Based on empirical studies on the returns to scale in knowledge creation ("non-increasing") and the value of the elasticity of intertemporal substitution ("higher than the critical value"), we expect a negative relationship between long-run growth and uncertainty regarding the productivity of knowledge creation.

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1 Introduction

Investments in research and development (R&D) or, more generally, investments in the creation of knowledge are the driving force behind the advance-

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ment of the technology. More investments will generally lead to a higher rate of technological change, and, consequently, to higher economic growth. However, the return to these investments is not known in advance, that is, the productivity of knowledge creation is uncertain. Hence, there is a link between uncertainty and (long-run) growth. The objective of this paper is to find out the nature (positive or negative) of this link and to identify the main factors that determine this nature.

In the present paper, uncertainty derives from randomness in the productivity of R&D. In general, one part of uncertainty is due to individual, firm-specific (idiosyncratic) uncertainty, while the other part arises from economy-wide (common) shocks, which have the same impact on all firms. Here, the analysis will focus on common shocks[1], such as technology and policy shocks. Hence, the question is how uncertainty in the productivity of knowledge creation, which is common across firms, affects the growth rate of the economy.

Many researchers have studied the link between investment and uncertainty, mainly theoretically. According to this literature[2], the sign of the effect of uncertainty on (capital) investment is ambiguous; the effect depends on the degree of competition in product markets, the degree of irreversibility of investment, the production technology (constant or decreasing returns to scale), the incompleteness of (financial) markets and the attitude of firms toward risk. The few empirical studies on the link between investment and uncertainty (measured by various volatility measures) seem to indicate a negative effect of volatility on *private* investment (as opposed to aggregate investment, that is, private and public investment, where no such correlation is found).[3]

The present paper is concerned with the link between *growth* and uncertainty and, therefore, in the spirit of the new or endogenous growth literature, with the link between uncertainty and human capital or knowledge instead of physical capital. Although there are many studies on stochastic endogenous growth models[4], so far there have been few studies that analyze the effect of uncertainty, i.e., the volatility of the shocks, on (the distribution of) the

[1] Schankerman (2001) finds that idiosyncratic shocks do not account for much (approximately 25%) of the variation in investment decisions. Nearly 75% of the microvariance is due to heterogeneity in micro level responses to aggregate (common) shocks.

[2] E.g. Hartman (1972), Abel(1983), McDonald and Siegel (1986), Pindyck (1988), Caballero(1991), Dixit and Pindyck (1994).

[3] E.g. Ramey and Ramey (1995), Guiso and Parigi (1999) and Aizenman and Marion (1999).

[4] See e.g. King and Rebelo (1988), King, Plosser and Rebelo (1988), Obstfeld (1994), Hopenhayn and Muniagurria (1996).

long-run growth rate. For example, Jones, Manuelli and Stacchetti (1999) show that the relationship between volatility in technology shocks and mean growth can be either positive or negative, depending on the curvature of the utility function. However, they expect a positive relationship between uncertainty and growth on the basis of a quantitative analysis. De Hek (1999) analyses the link between volatility in the productivity of knowledge creation and growth within two different models. Under the assumption of a logarithmic utility function, in the human capital accumulation model of Lucas (1988) volatility has a negative effect on the expected growth rate; in Romer's (1986) model of learning-by-doing and knowledge spillovers the sign of the growth-uncertainty relationship can be either positive or negative, where the key parameter is again the curvature of the utility function.[5] In a recent paper Blackburn and Pelloni (2001) investigate the relationship between growth and volatility in learning-by-doing economies. They find that the correlation between long-term growth and short-term volatility depends on the source of stochastic fluctuations and the functioning of the labor market. As regards the former, long-run growth is negatively related to the volatility of (non-neutral) nominal shocks, but positively related to the volatility of real shocks.

In an influential empirical study Ramey and Ramey (1995) find evidence of a negative relationship between economic growth and the volatility of economic fluctuations. This negative relationship is mainly due to the volatility of the *innovations* to growth. This measure corresponds more closely to the notion of *uncertainty*. At face value this result seems to contradict those of Kormendi and Meguire (1985) who find that the standard deviation of output growth has a significant positive effect on growth. However, Ramey and Ramey (1995, p.1145) argue that in the regressions of Kormendi and Meguire, the positive effect of the standard deviation may be capturing the effect of predictable movements in growth. In that way, both results are consistent: volatility of the innovations seems to have a negative effect, while volatility in the *predicted* variable has a positive effect on growth.

The analysis in this paper differs from the analyses in Jones, Manuelli and Stacchetti (1999) and De Hek (1999) by choice of endogenous growth model and source of uncertainty. Here, a model of endogenous technological change is used. This choice of model is motivated by the fact that in this model long-run growth in the economy is driven by improvements in the technology that arise from intentional investments in R&D from profit-maximizing firms.

[5] See also Roche (1999), who finds a negative relationship between uncertainty and the expected consumption growth rate in a continuous time framework, and De Hek and Roy (2001), who analysed conditions for sustained growth in the presence of stochastic shocks.

Uncertainty derives from the productivity of the investments in R&D. Uncertainty is modeled by assuming a probability distribution over the relevant variable. This random variable is independently and identically distributed over time. Hence producers who invest in R&D face the same uncertainty in every period and cannot learn from past experiences.[6]

The main result of this analysis is that the relationship between long-run growth and uncertainty (on the productivity of knowledge creation) depends on two main factors - the returns to scale in knowledge creation (increasing or non-increasing) and the value of the elasticity of intertemporal substitution (higher or lower than some critical value). Under the assumption of a logarithmic utility function, more uncertainty leads to less time spent on research and a smaller average rate of technological change if there are non-increasing returns to scale. If there are increasing returns to scale, there are two equilibria and more uncertainty leads to more (less) time spent on research and a higher (lower) average rate of technological change if the economy is in the low (high) research level equilibrium. Under the restriction that the economy is in its unique long-run equilibrium (where the growth rate of output is governed by a unique invariant probability measure), the same results apply in the case of a CES utility function if the elasticity of intertemporal substitution (EIS) is higher than some critical value. If the EIS is lower than the critical value, the results go in the opposite direction. Based on empirical studies on the returns to scale in knowledge creation ("non-increasing") and the value of the EIS ("higher than the critical value"), we expect a negative relationship between long-run growth and uncertainty regarding the productivity of knowledge creation.

2 The Model

The model that will be developed in this section is based on the models of endogenous technological change of Romer (1990) and Aghion and Howitt (1998, Chapter 3).

[6] This kind of uncertainty precludes the use of real options theory, which assumes that the uncertainty will be at least partly revealed over time.

2.1 Technology

The consumption-capital good in the economy, final output Y, is produced according to

$$Y_t = L_t^{1-\beta} \int_0^1 A_{it} x_{it}^{\beta} di, \tag{1}$$

where x_{it} is the quantity of intermediate (or capital[7]) good i, L_t is the quantity of labor employed to produce final output and A_{it} is an index for the technology or knowledge in firm (or sector) i. At each date, the representative final-output firm decides on the quantity of each intermediate good by maximization of its profits. This implies that the inverse demand function, relating the price p_{it} of intermediate good i to its supply, is given by

$$p_{it} = \beta L_t^{1-\beta} A_{it} x_{it}^{\beta-1}, \forall i \in [0,1]. \tag{2}$$

The wage rate $w_{L,t}$ of (skilled) labor used in the final-output sector is equal to its marginal product,

$$w_{L,t} = (1-\beta) L_t^{-\beta} \int_0^1 A_{it} x_{it}^{\beta} di. \tag{3}$$

Each intermediate good is produced by a firm that has an infinitely-lived patent on that design (or can in some other way effectively prevent other competitors from entering the market, without affecting the profit maximization). Due to this monopoly power an intermediate firm can devote resources, i.e., labor, to research and development (R&D), which enhances the state of the technology of that firm. A higher state of the technology might be seen as an improvement of the quality of the firm's product and implies higher profits. The intermediate sector uses labor to conduct research. Labor or human capital[8] in sector i is denoted by h_{it}. Average or total human capital used to conduct research is then given by $H_t = \int_0^1 h_{jt} dj$. The total labor force in the economy is fixed and set to 1, i.e., $L_t + H_t = 1$ for all t.

Suppose that technology or knowledge evolves according to

$$A_{i,t+1} = \left(1 + \eta_{t+1} h_{it}^{\gamma} H_t^{\theta}\right) A_{it}, \tag{4}$$

[7] Intermediate goods and capital (goods) are used interchangeably throughout the paper.

[8] In this paper, the amount of human capital used in sector i, h_{it}, is defined to be the amount of labor used in sector i, l_{Ait}, times the (constant) skill level, say $\overline{u}$. Normalizing $\overline{u}$ to 1 implies that $h_{it} = l_{Ait}$.

where $\gamma > 0$ is a returns-to-scale parameter, $\theta > 0$ a parameter controlling the spill-over effect of average (or total) human capital, $H_t = \int_0^1 h_{jt} dj$, and η a random variable representing the productivity of human capital in the accumulation of knowledge. In every period, η may take any value on some interval I. As a result, the return to research is uncertain. The *probability distribution* of the return is, however, known and fixed. More formally, assume that the sequence of shocks $\{\eta_t\}$ satisfies:

> *$\{\eta_t\}$ is a sequence of independently and identically distributed (i.i.d.) random variables with probability distribution μ and support $I = [\eta^{low}, \eta^{high}]$, $\eta^{high} > \eta^{low} > 0$.*

Clearly, more (less) uncertainty is associated with higher (lower) variability. To determine the effect of changing the variability on the expectation of a function of the random variable, the following result is very useful.

Lemma 1 *Given that Y is more variable than X, $Ef(X) > (\geq)\ Ef(Y)$ if f is strictly (weakly) concave, while $Eg(X) < (\leq)\ Eg(Y)$ if g is strictly (weakly) convex.*

Proof. See Rothschild and Stiglitz (1971) for a derivation of this result. (For a formal definition of variability see Rothschild and Stiglitz, 1970). ■

Therefore, to determine the effect of increasing (or decreasing) variability on $Ef(X)$, it is sufficient to find out whether $f(.)$ is strictly concave or strictly convex. E.g. if $f(X)$ is strictly concave, increasing the variability of X leads to a decrease in the expectation of $f(X)$. If $f(.)$ is neither concave, nor convex, the theory does not predict the effect of increasing variability.

Assume that each intermediate-good producer rents its capital from households in a perfectly competitive market, where the rental rate at each date t is ζ_t. The intertemporal expected profit maximization problem of an intermediate-good producer is given by:

$$\max E \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \delta_s \right) [p_{it} x_{it} - \zeta_t A_{it} x_{it} - w_{H,t} h_{it}] \tag{5}$$

$$s.t.\ A_{i,t+1} = \left(1 + \eta_{t+1} h_{it}^{\gamma} H_t^{\theta}\right) A_{it},$$

where E is the expectation operator, $\delta_s \equiv \frac{1}{1+r_s}$ the discount factor, with r_s representing the interest rate (with $r_0 = 0$). The transversality condition, which is given in the appendix, is assumed to be satisfied.

Returns on investment in R&D are uncertain. In each period the impact of research on each firm's stock of knowledge is randomly determined. Since η is assumed to be independent from i, this specification of the uncertainty implies that the shocks are economy wide, i.e., the same for each firm. Therefore, the riskiness of the investments in R&D is the result of changes in the economic climate, e.g., induced by technology or policy shocks.

With respect to the timing of the shocks, it is assumed that the shock η_t realizes at the beginning of period t. This implies that in period t, η_t is known but η_{t+1} is not known.

In maximizing the expected discounted stream of profits, the firm knows the demand for its product as given by equation (2). Therefore, replacing p_{it} in the maximization problem with the right-hand side of equation (2) and differentiating with respect to the two choice variables x_{it} and h_{it} leads to the first-order conditions. These two conditions can be written as

$$\zeta_t = \beta^2 \left(\frac{x_{it}}{L_t} \right)^{\beta - 1}, \tag{6}$$

$$w_{H,t} = E\left[\frac{\eta_{t+1} \gamma h_{it}^{\gamma-1} H_t^{\theta} A_{it} \beta (1-\beta) L_{t+1}^{1-\beta} x_{i,t+1}^{\beta}}{1 + r_{t+1}} \right]. \tag{7}$$

Let A_t denote the average productivity parameter across all firms at date t: $A_t \equiv \int_0^1 A_{it} di$. Because each sector i uses $A_{it} x_{it}$ units of capital, the total capital stock (measured in forgone consumption) is equal to $K_t \equiv \int_0^1 A_{it} x_{it} di$. According to equation (6), all firms produce the same amount at any given time: $x_{it} = x_t = \frac{K_t}{A_t}$ for all i. Next, suppose that initially at $t = 0$ every firm has the same productivity, that is, $A_{i0} = A_0$ for all i, which implies that $A_{it} = A_t$ for all i. Then equation (7) allows us to have $h_{it} = h_t$ for all i, which, in turn, implies that $H_t = h_t$. As a result, the aggregate technology (1) can now be expressed in the simpler form

$$Y_t = A_t x_t^{\beta} L_t^{1-\beta}. \tag{8}$$

For reasons of tractability, it is assumed that the intermediate good depreciates fully each period. This implies that the rental rate that would just compensate the owners of capital (the households) for interest and depreciation costs will be $\zeta_t = r_t + 1$. Combined with the fact that, in equilibrium, the wage rate in the intermediate sector is equal to the wage rate in the final-output sector, i.e., $w_{H,t} = (1-\beta) L_t^{-\beta} A_t x_t^{\beta}$, this implies that equation (7) can be written as

$$\beta L_t^{-\beta} x_t^{\beta} = E\left[x_{t+1} \gamma \eta_{t+1} h_t^{\gamma+\theta-1} \right]. \tag{9}$$

2.2 Preferences

On the consumer side, the objective of the representative agent (the number of agents is normalized to one) is to select consumption and savings to maximize the expected value of lifetime utility:

$$\max E \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho} \right)^{t} u(C_t) \tag{10}$$

$$s.t.\ b_{t+1} = (1+r_t)b_t + w_t + \pi_t - C_t, \tag{11}$$

where ρ represents the time preference of the consumers, C_t is consumption and b_t represents assets. The agent's sources of income are return on their stock of assets $(1+r_t)b_t$, wage income $w_t L_t + w_t h_t = w_t$ and profits $\pi_t = \beta Y_t - (1+r_t)K_t - w_t h_t$. Maximization with respect to consumption and savings implies that the optimal path of consumption follows the Euler equation,

$$u\prime(C_t) = E\left[u\prime(C_{t+1}) \frac{1+r_{t+1}}{1+\rho} \right]. \tag{12}$$

The associated transversality condition is given in the appendix.

2.3 Equilibrium

In equilibrium, the total amount of assets owned by the consumers should equal the capital stock: $b_t = K_t \equiv A_t x_t$. By imposing this (capital) market clearing condition, the budget constraint of the consumers implies that the market for goods clears: $C_t + K_{t+1} = Y_t$. Since the total amount of labor present in the economy is normalized to 1, equilibrium on the labor market requires that $L + h = 1$.

To be able to solve the consumer maximization problem along the optimal path, the utility function is restricted to the logarithmic one[9]:

$$u(C_t) = \ln(C_t). \tag{13}$$

By equations (6) and (8), the fact that $K_t = A_t x_t$ and the market clearing condition for goods, $1 + r_{t+1}$ can be written as $1 + r_{t+1} = \beta^2 \frac{Y_{t+1}}{K_{t+1}} = \beta^2 \frac{C_{t+1}+K_{t+2}}{K_{t+1}}$. This implies that the Euler equation turns into the following stochastic expectations difference equation:

$$K_{t+1} C_t^{-1} = \frac{\beta^2}{1+\rho} E\left[1 + K_{t+2} C_{t+1}^{-1} \right]. \tag{14}$$

[9]Notice that this setting with a logarithmic utility function, a Cobb-Douglas production function and complete depreciation is strongly related to Brock and Mirman (1972).

Notice that because the shock in period t realizes "sufficiently early" in period t, output Y_t and hence investment K_{t+1} is known at the moment a consumption choice is made.

This equation is solved (see appendix) by imposing the associated transversality condition. It follows that investment and consumption are a constant fraction of income:

$$K_{t+1} = \frac{\beta^2}{1+\rho} Y_t, \tag{15}$$

$$C_t = \left(1 - \frac{\beta^2}{1+\rho}\right) Y_t. \tag{16}$$

2.3.1 The optimal level of research

To solve for the optimal level of research activity, notice that equation (14) implies that $x_{t+1} = \frac{\beta^2}{1+\rho} \frac{Y_t}{A_{t+1}}$. Inserting this expression into equation (8) yields the following equilibrium research condition,

$$\beta\gamma(1-h_t) h_t^{\gamma+\theta-1} E\left(\frac{\eta_{t+1}}{1+\eta_{t+1} h_t^{\gamma+\theta}}\right) = 1+\rho. \tag{17}$$

Given the parameter values and probability measure of η, the intermediate producers choose the optimal amount of time spent on research, $h_t = \overline{h} \equiv \overline{h}(\beta, \gamma, \theta, \rho, \mu(\eta))$. For example, an increase in β, capital's share of output, raises the left-hand side of the above equation, inducing more time spent on research in equilibrium, at least in the case of nonincreasing returns to research $(\gamma + \theta \leq 1)$[10]. The reason behind this effect is that a higher capital's share shifts the inverse demand function (2) upwards, increasing the profitability of the intermediate producers and, hence, raises the productivity of research.

The optimal level of research determines the (gross) rate of technological change according to

$$g_{A,t} \equiv \frac{A_{t+1}}{A_t} = 1 + \eta_{t+1} \overline{h}^{\gamma+\theta}. \tag{18}$$

[10] The effect on the equilibrium research level depends on the returns to research. Section 2.4 discusses this in more detail in the case of increasing variability.

2.3.2 The equilibrium path of output

Given the rate of technological change, the growth rate of output can be determined. First, observe that along the optimal path $h_t = \overline{h}$, and therefore $L_t = \overline{L}$, are constant over time (not depending on the shock). Second, since investment is a constant fraction of output, the relative change in the level of the intermediate goods can be written as $\frac{x_{t+1}}{x_t} = \frac{Y_t}{Y_{t-1}} \frac{A_t}{A_{t+1}}$. As a result, using equation (8), the (gross) growth rate of output follows the stochastic difference equation

$$\frac{Y_{t+1}}{Y_t} = \left(\frac{Y_t}{Y_{t-1}}\right)^{\beta} \left(\frac{A_{t+1}}{A_t}\right)^{1-\beta} = \left(\frac{Y_t}{Y_{t-1}}\right)^{\beta} \left(1 + \eta_{t+1}\overline{h}^{\gamma+\theta}\right)^{1-\beta}, t = 1, 2, ... \tag{19}$$

This Markov process (in growth rates) describes the equilibrium path of output. Associated with each growth rate is a probability measure, say λ_t, and one way to determine the long-run behavior of the stochastic process is to look for an invariant probability measure and see whether the sequence of probability measures converges to it. Since this stochastic process is similar to the stochastic process obtained in De Hek (1999)[11], we can immediately conclude that there exists a unique invariant probability measure to which the growth rate (weakly) converges. This implies that we may indeed talk about *the* long-run behavior of the growth rate. Another result that immediately follows from the analysis in De Hek (1999) is on the relationship between the growth rate of output and the rate of technological change, given in the next proposition.

Proposition 2 *(A) The log of the growth rate converges in expectation to the log of the rate of technological change, that is,*

$$\lim_{t \longrightarrow \infty} E\left\{\left|\ln \frac{Y_{t+1}}{Y_t} - \ln\left[1 + \eta\overline{h}^{\gamma+\theta}\right]\right|\right\} = 0, \tag{20}$$

with the expectation taken with respect to the information available at time $t = 0$. (B) The expectation of the growth rate of output is strictly smaller than the expectation of the rate of technological change, that is,

$$E\frac{Y_{t+1}}{Y_t} < E\left[1 + \eta\overline{h}^{\gamma+\theta}\right]. \tag{21}$$

Proof. See the proof of Proposition 3 in De Hek (1999). ■

[11] See equation (8) and Proposition 1 in De Hek (1999).

2.4 The effect of uncertainty

This section studies the effect of uncertainty on the equilibrium research effort, $\overline{h}$, the average rate of technological change, Eg_A, and the average (asymptotic) growth rate of output, Eg_Y.

2.4.1 The effect of uncertainty on $\overline{h}$ and Eg_A

The effect of a higher volatility of the shock η on the optimal choice of h depends on the functional form of the equilibrium research condition (17) regarding the shock η and the variable h. Ignoring - for this moment - the time subscripts, the equilibrium research condition can be rewritten as

$$E\left(\frac{\beta\gamma(1-h)h^{\gamma+\theta-1}\eta}{1+\eta h^{\gamma+\theta}}\right)=1+\rho. \tag{22}$$

The first step in finding the effect of more uncertainty on research activity is to determine the effect of a higher volatility of η on the left-hand-side of this equation, which will be denoted by $E(\Phi)$. Since Φ is a concave function of η, a higher volatility of η has a negative effect on the expectation of Φ.

Second, the effect of a smaller $E(\Phi)$ on the equilibrium value of h depends on the functional form of $E(\Phi)$ as a function of h. If $\gamma+\theta\leq 1$, it is easy to see that $E(\Phi)$ is a decreasing function of h, as depicted in figure 1. A higher volatility, which decreases $E(\Phi)$ as a function of h, then leads to a smaller level of research. On the other hand, if $\gamma+\theta>1$, $E(\Phi)$ as a function of h is hump-shaped. This implies that there are two equilibrium values of h, a "low research level equilibrium" and a "high research level equilibrium" (that is, if the maximum of $E(\Phi)$ is higher than 1). An example of this situation is given in figure 2. There will actually be more time spent on research due to more uncertainty if the economy is in the low level equilibrium, as opposed to less research time in the high level equilibrium.

What is the effect of a change in the time spent on research on the rate of technological change? A reduction in the time spent on research, for example, implies that the expectation of g_A decreases, that is, the rate of technological change will be smaller on average. More formally, consider the two probability measures μ and μ^+, where μ^+ is more uncertain than μ, that is, it has the same mean but a higher volatility. Then the average rate of technological change under μ^+ is smaller than the average rate of technological change under μ for almost any sequence of realizations of η; i.e., it occurs almost surely. The effect of uncertainty on the time spent on research and the average rate of technological change is summarized in the next proposition.

Proposition 3 *(A) If $\gamma + \theta \leq 1$, then more uncertainty leads to (i) less time spent on research and (ii) a smaller rate of technological change on average. (B) If $\gamma + \theta > 1$, then more uncertainty leads to (i) more (respectively less) time spent on research and (ii) a higher (respectively smaller) rate of technological change on average if the economy is in the low (respectively high) research level equilibrium.*

Proof. See Appendix. ■

2.4.2 The effect of uncertainty on Eg_Y

To determine the effect of uncertainty on the path of final output, suppose that the volatility of the shocks increases from period $t+1$ on (that is, the shocks $\eta_{t+1}, \eta_{t+2}, ...$ are more variable than the shocks $\eta_t, \eta_{t-1}, ...$). Consider the following equation (which is an immediate consequence of equation 18):

$$g_{Y,t+T} = (g_{Y,t})^{\beta^T} \left(1 + \eta_{t+1}\overline{h}^{\gamma+\theta}\right)^{(1-\beta)\beta^{T-1}} ... \left(1 + \eta_{t+T}\overline{h}^{\gamma+\theta}\right)^{1-\beta}, \quad (23)$$

with $g_{Y,t} = Y_t/Y_{t-1}$. Taking first expectations and then logs on both sides gives

$$\ln Eg_{Y,t+T} = E\left[(g_{Y,t})^{\beta^T}\right] + \sum_{i=0}^{T-1} \ln E\left[\left(1 + \eta_{t+T-i}\overline{h}^{\gamma+\theta}\right)^{(1-\beta)\beta^i}\right]. \quad (24)$$

Notice that, e.g. in the case of $\gamma + \theta \leq 1$, more variability in the shocks lead to a fall in $\overline{h}_t$, the fraction of total labor used in research activities, which implies a rise in $\overline{L}_t$, the fraction of labor used in production. The quantities of the intermediate goods in period t do not change, because $x_t = \frac{\beta^2}{1+\rho}\frac{Y_{t-1}}{A_t}$ does not depend on the distribution of the future shocks. Therefore, more uncertainty leads to a higher growth rate of output in period t. However, if T goes to infinity[12],

$$\ln Eg_Y = \sum_{i=0}^{\infty} \ln E\left[\left(1 + \eta\overline{h}^{\gamma+\theta}\right)^{(1-\beta)\beta^i}\right], \quad (25)$$

where g_Y is the asymptotic growth rate of output. Because all expectations on the right-hand side of equation (23) are obtained from strictly concave functions of η, the higher variability of the shocks leads to a fall in the expected asymptotic growth rate of output.

[12] Since there is a unique invariant probability measure to which the growth rate of output is converging and the $g_{Y,t}$ are uniformly bounded and therefore uniformly integrable, $Eg_{Y,t} \longrightarrow Eg_Y$ (see Theorem 25.12 in Billingsley, 1986).

3 Precautionary saving

From earlier studies (e.g. De Hek, 1999 and Jones et al., 1999) we know that the positive effect of precautionary saving on growth may offset the negative effect if the elasticity of intertemporal substitution is low enough. Therefore, in order to incorporate this effect, we need to generalize the logarithmic utility function to the CES utility function,

$$U(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma}, \tag{26}$$

where $1/\sigma$ is the elasticity of intertemporal substitution (and σ the parameter measuring risk aversion). In the previous case with the logarithmic utility function, it was possible to solve for the optimal path, which gave rise to a Markov process in growth rates describing the equilibrium path of output. From that, we could conclude that there exists a unique invariant probability measure to which the growth rate converges. That is, the equilibrium path of output converges to the long-run equilibrium in which the growth rate of output is governed by a fixed probability measure. Hence, the long-run or asymptotic equilibrium may also be characterised by the term "balanced *expected* growth path" (BEGP). On the BEGP, the *expected* growth rate is constant. In the current situation of a CES utility function, we *cannot* in general solve for the optimal or equilibrium path, which implies that both existence and stability of a unique invariant probability measure cannot be established. However, suppose that the economy has a unique long-run equilibrium. Then, if the economy is in its long-run equilibrium, we *can* solve for the balanced expected growth path. That is,we can solve the stochastic expectations difference equation that follows from the Euler equation with the CES utility function. This is the approach that we will take in this section.

Let the economy be in its long-run equilibrium. Then the growth rate of output is governed by a unique invariant probability measure. In particular, $Eg_{Y,t} \equiv Eg_Y$ for all t. Making use of this fact that the expectation of the growth rate is independent from time, the solution to the stochastic expectations difference equation is again that investment and consumption are a constant fraction of income (see appendix). However, this fraction now involves an expectation term:

$$K_{t+1} = \frac{\beta^2}{1+\rho} E[g_Y^{1-\sigma}] Y_t, \tag{27}$$

$$C_t = \left(1 - \frac{\beta^2}{1+\rho} E[g_Y^{1-\sigma}]\right) Y_t. \tag{28}$$

This expectation term $E[g_Y^{1-\sigma}]$ contains the precautionary saving effect. To see this, consider an increase in the variability of g_Y. By Lemma 1, this decreases $E[g_Y^{1-\sigma}]$ if $\sigma < 1$ and increases $E[g_Y^{1-\sigma}]$ if $\sigma > 1$. In other words, if $\sigma < 1$, a more variable growth rate of output leads to more consumption and less savings, while if $\sigma > 1$, it leads to less consumption and more savings. Therefore, in the case of $\sigma > 1$, the presence of uncertainty implies extra, precautionary, savings, and *more* uncertainty leads to *more* savings.

Progressing in the same way as in section 2.3 yields the new equilibrium research condition,

$$\beta\gamma(1-h_t)h_t^{-1}E[g_Y^{1-\sigma}]E\left(\frac{\eta_{t+1}h_t^{\gamma+\theta}}{1+\eta_{t+1}h_t^{\gamma+\theta}}\right) = 1+\rho. \tag{29}$$

The difference with the former equilibrium research condition in the logarithmic utility case is the presence of the expectation term $E[g_Y^{1-\sigma}]$. To determine this expectation term, notice that, since investment is again a constant fraction of output, equation (19) describes the optimal path of output.[13] Therefore, equation (23) applies also in this case. By raising both sides of equation (23) to the power $1-\sigma$, it is easy to show that

$$\ln E[g_Y^{1-\sigma}] = \sum_{i=0}^{\infty} \ln E\left[\left(1+\eta\overline{h}^{\gamma+\theta}\right)^{(1-\sigma)(1-\beta)\beta^i}\right]. \tag{30}$$

3.1 The effect of uncertainty

As in section 2.4, the new equilibrium research condition (29) can be written as

$$E[g_Y^{1-\sigma}]E\left(\frac{\beta\gamma(1-h)h^{\gamma+\theta-1}\eta}{1+\eta h^{\gamma+\theta}}\right) = 1+\rho. \tag{31}$$

Let us denote the left-hand side of this equation by Ψ. Hence, Ψ is the product of two expectations. The effect of higher variability on the first expectation, $E[g_Y^{1-\sigma}]$, can be inferred from equation (30) using Lemma 1. In the case of $\sigma < 1$, a higher variability lowers the expectation, while in the case of $\sigma > 1$ the effect of increasing variability on $E[g_Y^{1-\sigma}]$ is positive. The second expectation is actually identical to the expectation in the "old" equilibrium research condition (22) which was denoted by $E(\Phi)$. Hence, the effect of higher variability on the second expectation is negative, irrespective

[13] The difference with the model in section 2 is that in the present model equation (19) only holds in the long-run equilibrium.

of the value of σ. Taking these two effects of increasing variability on the two expectations together implies that Ψ decreases if $\sigma < 1$. However, if $\sigma > 1$, the effect of uncertainty on Ψ and, hence, on the level of research and on the growth rate is ambiguous. To assess the relative strenghts of the two opposite effects in this case, we will approximate both expectations by use of a second-order Taylor series expansion. It turns out that there exists a critical value of σ, which will be denoted by $\widehat{\sigma}$, such that: If σ lies below this critical value, the total effect on Ψ is negative, while if σ is higher than $\widehat{\sigma}$, the total effect is positive.

3.1.1 Estimation of $\widehat{\sigma}$

First, notice that $E[g_Y^{1-\sigma}]$ can be approximated by its second-order Taylor series expansion around $E[g_Y] \equiv \overline{g_Y}$, i.e.,

$$E[g_Y^{1-\sigma}] \approx \overline{g_Y}^{1-\sigma} - \frac{1}{2}(1-\sigma)\sigma\overline{g_Y}^{-\sigma-1} var(g_Y) \equiv Q, \tag{32}$$

where $var(g_Y)$ is the variance of g_Y. To determine the variance of g_Y, observe that equation (21) implies (taking first logs and then expectations) that $E[\ln g_Y] = E[\ln X]$, where $X \equiv 1 + \eta\overline{h}^{\gamma+\theta}$. Approximating both expectations with a second-order Taylor series expansion around $\overline{g_Y}$, respectively $\overline{X} \equiv E[X]$, implies that

$$var(g_Y) \approx 2\overline{g_Y}^2 \ln\left(\frac{\overline{g_Y}}{\overline{X}}\right) + \left(\frac{\overline{g_Y}}{\overline{X}}\right)^2 var(X), \tag{33}$$

with $var(X) = h^{2(\gamma+\theta)}\sigma_\eta^2$.

Similarly,

$$E\left(\frac{\eta_{t+1}h_t^{\gamma+\theta}}{1+\eta_{t+1}h_t^{\gamma+\theta}}\right) \approx \frac{\overline{\eta}\overline{h}^{\gamma+\theta}}{1+\overline{\eta}\overline{h}^{\gamma+\theta}} - \frac{\overline{h}^{2(\gamma+\theta)}\sigma_\eta^2}{\left(1+\overline{\eta}\overline{h}^{\gamma+\theta}\right)^3} \equiv W. \tag{34}$$

To avoid complications, in the remainder of this section we assume that the variance of the shock is "small enough" such that both Q and W are positive:

> **Assumption A.** Let σ_η^2 be restricted such that $Q > 0$ and $W > 0$.

To approximate the effect of an increase in the variance of the shock η on the equilibrium research condition, we will determine its effect on $Q.W$. The result is given in the next proposition.

Proposition 4 *Let Q and W be given by equations (29) and (31), with $\overline{h} > 0$. Furthermore, let Assumption A hold. Then $\partial(Q.W)/\partial\sigma_\eta^2 = 0$ if and only if*

$$\frac{1}{\frac{1}{2}(\widehat{\sigma}-1)\widehat{\sigma}} = \overline{\eta}\overline{h}^{\gamma+\theta} - 2\frac{\overline{h}^{2(\gamma+\theta)}\sigma_\eta^2}{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2} - \ln\left(\frac{\overline{g_Y}}{1+\overline{\eta}\overline{h}^{\gamma+\theta}}\right) \qquad (35)$$

If the right-hand side of this equation is positive, then there exists $\widehat{\sigma}$ such that $\partial(Q.W)/\partial\sigma_\eta^2 < 0$ if $\sigma < \widehat{\sigma}$, and $\partial(Q.W)/\partial\sigma_\eta^2 > 0$ if $\sigma > \widehat{\sigma}$.

Proof. See appendix ■

To get some idea about the critical level of (the reciprocal of) the elasticity of intertemporal substitution, $\widehat{\sigma}$, we have to obtain values for the parameters together with the equilibrium value $\overline{h}$ in above equation. First, we assume a value of 1/3 for the capital coefficient, β, motivated by the usual data on capital's share of income. Next, consistent with the estimations in a number of empirical studies (Dinopoulos and Thompson, 1996, 2000, Thompson, 1996), we let $\gamma+\theta$ vary between 0.2 and 1. Furthermore, let σ_η^2 take a value of ca. 0.01. In comparison, in the Summers and Heston data (PWT 5.6), the average (across countries) variance of the per capita growth rate is ca. 0.0036. Jones (2002) provides a rough empirical measure of $\overline{h}$, the number of scientists and engineers engaged in research and development as a part of overall employment, based on data from the G-5 countries (France, West Germany, Japan, the United Kingdom and the United States). In 1975, for example, the fraction $\overline{h}$ was about 1/2 of one percent. By 1993, this fraction had risen to about 3/4 of a percent. Here, we take $\overline{h} = 0.01$. The average value of the shock, $\overline{\eta}$, varies according to the value of $\gamma+\theta$ to ensure a plausible long-run rate of technological change.

To simplify equation (32), we will argue that the last two terms on the right-hand side (RHS) can be ignored. Notice that, by Proposition 2(B), the last term - with the minus sign - contributes positively to the RHS. However, a numerical analysis shows that, with the given parameter values, the value of this term is in the order of magnitude of a few hundredth of a percent or even less.[14] Hence, in the remainder of this paper, the long-run rate of growth, $\overline{g_Y}$, is being treated as equal to the long-run rate of technological

[14] A value of $\overline{g_Y}$ is obtained by iteration (50,000 times) of $g_{Y,t+1} = (g_{Y,t})^\beta \left(1+\eta_{t+1}\overline{h}^{\gamma+\theta}\right)^{1-\beta}$, where the shocks $\{\eta_t\}$ are drawn from a given probability distribution, and taking the average of the growth rates $\{g_{Y,t}\}$ after removing the first 1000 values (to ensure convergence). Concerning the probability distribution, we used both uniform distributions and normal distributions with similar results.

change, $1+\overline{\eta}\overline{h}^{\gamma+\theta}$. Furthermore, inserting the range of values mentioned in the previous paragraph into the second term of the RHS of equation (32) shows that this term is not larger than a few tenth of a percent. Therefore, as long as the long-run rate of technological change is not too small (in the sense that it is significantly higher than the sum of both terms), we may safely ignore both terms. This implies that $\widehat{\sigma}$ can be approximated by

$$(\widehat{\sigma}-1)\widehat{\sigma}=\frac{2}{\overline{\eta}\overline{h}^{\gamma+\theta}}. \tag{36}$$

Consequently, if the average rate of technological change is 2.5%, the critical value of σ is as high as 10.

3.1.2 The effect of uncertainty on $\overline{h}$ and Eg_Y

The implication of a smaller or higher Ψ for the optimal amount of research, $\overline{h}$, depends on the functional form of Ψ as a function of h. As in the previous section, this depends on the nature of the returns to scale in knowledge creation. First, consider the case of decreasing returns to scale. Then, for $\sigma>1$ it is easy to show that Ψ is decreasing in h. For $\sigma<1$ the situation is more complex. However, numerical simulations indicate that also in this case Ψ is decreasing in h.[15] Second, if there are increasing returns to scale in knowledge creation, numerical simulations show that there are two equilibria (as in figure 2), provided that the return to research is large enough.

The analysis above implies that, if there are no increasing returns to scale in knowledge creation, i.e. $\gamma+\theta\leq 1$, more uncertainty regarding the productivity of research leads to a smaller average long-run growth rate if $\sigma<\widehat{\sigma}$, and to a larger average long-run growth rate if $\sigma>\widehat{\sigma}$. If there are increasing returns to scale in knowledge creation, there may exist two equilibria. The effect of uncertainty, then, depends on in which equilibrium the economy is located. If the economy is in the *high* research level equilibrium, uncertainty affects the long-run growth rate in the same way as above. However, if the economy is in the *low* research level equilibrium, the effect of uncertainty is exactly the opposite: More uncertainty regarding the productivity of research leads to a larger average long-run growth rate if $\sigma<\widehat{\sigma}$, and to a smaller average long-run growth rate if $\sigma>\widehat{\sigma}$.

[15] Notice that, based on the above considerations regarding $\overline{g_Y}$, it follows that $\overline{g_Y}$ is practically equal to $1+\overline{\eta}\overline{h}^{\gamma+\theta}$, which implies, by equations (29) and (30), that Q is practically equal to $\overline{X}^{1-\sigma}-\frac{1}{2}(1-\sigma)\sigma\overline{X}^{-\sigma-1}\overline{h}^{2(\gamma+\theta)}\sigma_\eta^2$. Then, without uncertainty, it is immediately clear that Ψ is decreasing in h. Therefore, Ψ is decreasing in h as long as σ_η^2 is 'small enough'. Numerical simulations indicate that Ψ is decreasing in h as long as Assumption A holds.

3.2 Discussion

If there are no increasing returns to scale in knowledge creation, and the elasticity of intertemporal substitution is not too small, that is $\sigma < \widehat{\sigma}$, more uncertainty regarding the productivity of research leads to a smaller average long-run growth rate. Hence, more uncertainty implies a *lower* average long-run growth rate if these two restrictions are satisfied.

The first restriction is that there are no increasing returns to R&D; i.e., $\gamma + \theta \leq 1$. The presence of constant or decreasing returns seems a fairly realistic assumption, which is confirmed by recent empirical evidence. For example, Dinopoulos and Thompson (1996, 2000) estimate versions of Romer's model of endogenous technological change and find positive, but decreasing, returns to R&D. Similar results are found in Hall, Griliches and Hausman (1986), Kortum (1993) and Thompson (1996).

The second restriction is on the elasticity of intertemporal substitution (EIS). The approximation of the critical value $\widehat{\sigma}$ in equation (33) shows that an average rate of technological change of 2.5% implies a critical value of 10. Even if the average rate of technological change is as high as 10%, this critical value is equal to 5. Since the EIS is equal to $1/\sigma$, this would restrict the EIS to be larger than $0.1 - 0.2$. Concerning the existing literature on this subject, on the one hand, empirical studies using aggregate consumption data typically find that the EIS is close to zero (Hall, 1988). On the other hand, calibrated macroeconomic models designed to match growth and business cycle facts typically require that the EIS be close to one (Weil, 1989, Lucas, 1990, among others). In a recent paper, Vissing-Jørgensen (2002) argues that accounting for limited asset market participation is important for estimating the EIS. She finds estimates of the EIS of around $0.3 - 0.4$ for stockholders and around $0.8 - 1$ for bondholders (the estimates are larger for households with larger asset holdings within these two groups). Similarly, Guvenen (2002) studies a dynamic macroeconomic model which incorporates limited asset market participation together with an EIS that increases with wealth to reconcile the conflicting evidence. He finds that the properties of aggregate variables directly linked to wealth, such as investment and output (growth), are almost entirely determined by the (high-elasticity) assetholders. (At the same time, since consumption is much more evenly distributed across households than is wealth, estimation using aggregate consumption uncovers the low EIS of the majority of households, i.e., the non-assetholders.) Hence, these studies indicate that the EIS, at least the EIS that is relevant for the issue of output growth in the present paper, is larger than 0.2.

Hence, given the available evidence, it seems that both restrictions are satisfied, implying that more uncertainty depresses the long-run growth rate.

The above analysis implies that increasing σ increases the positive effect of uncertainty on growth. Beyond some critical level of σ the positive effect even dominates the negative effect of uncertainty on growth. The intuition for this effect is the following. Due to risk aversion, more uncertainty reduces (the certainty equivalent of) the return on savings. Whether this leads to more or less savings depends on the relative strenghts of the income and substitution effects. The parameter σ controls the EIS. The higher σ, the lower is the elasticity and the less willing agents are to intertemporally substitute consumption (or, the more anxious agents are to smooth their consumption over time). As a result, the income effect (or precautionary saving effect) becomes stronger while the substitution effect becomes weaker with higher σ. Beyond the critical level of σ, the income effect dominates the substitution effect: the lower return on savings implies more savings. This rise in savings leads to an increase in investment and a lower (average) interest rate. The latter implies an increase in the return to research and, therefore, in the case of nonincreasing returns to research, both an increase in the amount of human capital used in research activities and a higher average growth rate.

4 Conclusion

The main question in this paper is how risk or uncertainty in the return to investment in R&D affects the growth rate of the economy. The analysis in this paper establishes a negative link between uncertainty on the return to investment in R&D and the long-run growth rate under two restrictions. The first restriction puts a lowerbound on the EIS. However, for realistic values of the rate of technological change, which provides the lowerbound on the EIS, recent studies on the estimation of the EIS indicate that this restriction is likely to be satisfied. Another variable which is restricted in order to get a negative link between growth and uncertainty is the return to R&D, which is assumed to be nonincreasing. A number of empirical studies have provided evidence that there are in fact decreasing returns to scale, so that this restriction also seems to be satisfied.

5 Appendix

5.1 Transversality conditions

The expected one-period profit of an intermediate-good producer on the BEGP, using equations (2) and (6), can be written as

$$\pi_i = \beta Y - (1+r)K - w\overline{h} = (1-\beta)(\beta - \frac{\overline{h}}{1-\overline{h}})Y, \tag{37}$$

since, by equation (6), $(1+r)K = \beta^2 Y$ and, by equation (3), $w\overline{h} = (1-\beta)Y\frac{\overline{h}}{L} = (1-\beta)Y\frac{\overline{h}}{1-\overline{h}}$.

Given the initial value Y_0, output Y_t can be written as

$$Y_t = \left(\prod_{s=1}^{t} g_{Y,s}\right) Y_0.$$

Let $z = (1-\beta)(\beta - \frac{\overline{h}}{1-\overline{h}})$. Then the optimization problem (5) can be written as

$$\max E\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\delta_s\right) z \prod_{s=1}^{t} g_{Y,s}Y_0 =$$

$$= \max E\sum_{t=0}^{\infty}(\delta_0 Y_0)(\delta_1 g_{Y,1})...(\delta_t g_{Y,t})z =$$

$$= \max \sum_{t=0}^{\infty}\left(E\left[\delta g_Y\right]\right)^t z.$$

The transversality condition ensures that the above summation exists and is, therefore, given by

$$E\left[\delta g_Y\right] < 1. \tag{38}$$

The well-known transversality condition of the consumer's optimization problem is given by

$$\lim_{t\to\infty} E\left(\frac{1}{1+\rho}\right)^t c_t^{-\sigma} b_{t+1} = 0. \tag{39}$$

5.2 Restriction for "$\pi > 0$"

Equation (34) implies that $\pi > 0$ iff $\beta > \frac{\overline{h}}{1-\overline{h}}$. Hence, expected profit π is positive if and only if $\overline{h} < \frac{\beta}{1+\beta}$. If $\beta = 1/3$, this implies that $\overline{h} < 1/4$.

5.3 Proof of proposition 3

This proof consists of proving the two steps taken in the text prior to the proposition. First, we have to prove that $G(\eta) \equiv \frac{\eta}{1+\eta h^{\gamma+\theta}}$ is a concave function of η. Let us write $G(\eta) = \frac{\eta}{1+b\eta}$, with $b = h^{\gamma+\theta}$. Differentiating $G(.)$ with respect to η shows that $\frac{\partial G}{\partial \eta} = \frac{1}{(1+b\eta)^2}$. Differentiating again with respect to η yields

$$\frac{\partial^2 G(\eta)}{\partial \eta^2} = \frac{-2b}{(1+b\eta)^3} < 0.$$

Hence, $G(\eta)$ is (strictly) concave for all $\eta \in [\underline{\eta}, \overline{\eta}]$. Hence, by Lemma 1, a higher volatility of η decreases $E(\Phi)$.

Second, it remains to be proved that $E(\Phi)$ is decreasing if $\gamma + \theta \leq 1$ and hump-shaped if $\gamma + \theta > 1$. To prove this, we define the function F as follows: $F(h) = \frac{m\eta(1-h)h^{\gamma+\theta-1}}{1+\eta h^{\gamma+\theta}}$, with $m = \beta\gamma/(1+\rho)$. Differentiating this function with respect to h yields

$$\frac{\partial F}{\partial h} = \frac{m\eta h^{\gamma+\theta-2}\left[(\gamma+\theta)(1-h) - \left(1+\eta h^{\gamma+\theta}\right)\right]}{(1+\eta h^{\gamma+\theta})^2}.$$

If $\gamma + \theta \leq 1$, it is evident that $F(h)$ is decreasing in h. In the case of $\gamma + \theta > 1$, it is clear that for small h (h close enough to zero such that $(\gamma+\theta)(1-h) > 1 + \eta h^{\gamma+\theta}$), $\frac{\partial F}{\partial h} > 0$. For large h (h close enough to one such that $(\gamma+\theta)(1-h) < 1 + \eta h^{\gamma+\theta}$), $\frac{\partial F}{\partial h} < 0$. Since $(\gamma+\theta)(1-h)$ is decreasing in h (from $\gamma+\theta$ to 0) and $1 + \eta h^{\gamma+\theta}$ is increasing in h (from 1 to $1+\eta$), there is a unique h for which $(\gamma+\theta)(1-h) = 1 + \eta h^{\gamma+\theta}$. Therefore, we can conclude that $F(h)$ is hump-shaped.

The first step implies that a higher volatility of η decreases $E(\Phi)$. The second step implies that depending on whether $\gamma + \theta \leq 1$ or $\gamma + \theta > 1$, $E(\Phi)$ is decreasing in h for all $h \in [0, 1]$ or hump-shaped. For example, in the first case, h has to fall in order to keep $E(\Phi)$ equal to 1.

5.4 Proof of Proposition 4

Since $Q > 0$ and $W > 0$, $\partial(Q.W)/\partial\sigma_\eta^2 = 0$ if and only if

$$\frac{\partial Q/\partial\sigma_\eta^2}{Q} = \frac{-\partial W/\partial\sigma_\eta^2}{W}. \tag{40}$$

From equations (29) and (30) we can derive that

$$\partial Q/\partial\sigma_\eta^2 = \frac{-\frac{1}{2}(1-\sigma)\sigma\overline{g_Y}^{1-\sigma}\overline{h}^{2(\gamma+\theta)}}{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2}.$$

Then we can write

$$\frac{Q}{\partial Q/\partial\sigma_\eta^2} = \frac{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2}{\frac{1}{2}(\sigma-1)\sigma\overline{h}^{2(\gamma+\theta)}} + \frac{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2 var(g_Y)}{\overline{g_Y}^2\overline{h}^{2(\gamma+\theta)}}$$

$$= \frac{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2}{\frac{1}{2}(\sigma-1)\sigma\overline{h}^{2(\gamma+\theta)}} + \sigma_\eta^2 + \frac{2(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2 \ln\left(\frac{\overline{g_Y}}{\overline{X}}\right)}{\overline{h}^{2(\gamma+\theta)}}.$$

According to equation (37), this should be equal to

$$\frac{W}{-\partial W/\partial\sigma_\eta^2} = \frac{\overline{\eta}\overline{h}^{\gamma+\theta}(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2}{\overline{h}^{2(\gamma+\theta)}} - \sigma_\eta^2.$$

Hence, $\partial(Q.W)/\partial\sigma_\eta^2 = 0$ if and only if

$$\frac{1}{\frac{1}{2}(\sigma-1)\sigma} = \overline{\eta}\overline{h}^{\gamma+\theta} - 2\frac{\overline{h}^{2(\gamma+\theta)}\sigma_\eta^2}{(1+\overline{\eta}\overline{h}^{\gamma+\theta})^2} - 2\ln\left(\frac{\overline{g_Y}}{1+\overline{\eta}\overline{h}^{\gamma+\theta}}\right).$$

5.5 Solution of the stochastic expectations difference equation

5.5.1 Logarithmic utility

The stochastic expectations difference equation is given by

$$K_{t+1}C_t^{-1} = aE\left[1 + K_{t+2}C_{t+1}^{-1}\right],$$

with $a = \frac{\beta^2}{1+\rho}$. Forward substitution implies that

$$K_{t+1}C_t^{-1} = \left(a + a^2 + a^3 + ...\right) + \lim_{n\longrightarrow\infty} a^n E[K_{t+n+1}C_{t+n}^{-1}].$$

The transversality condition, as given by equation (36) in the appendix (with $\sigma = 1$), implies that

$$\lim_{n\longrightarrow\infty} a^n E[K_{t+n+1}C_{t+n}^{-1}] < \lim_{n\longrightarrow\infty}\left(\frac{1}{1+\rho}\right)^n E[K_{t+n+1}C_{t+n}^{-1}] = 0.$$

Hence,

$$K_{t+1}C_t^{-1} = \frac{a}{1-a}.$$

Together with the budget constraint $K_{t+1}+C_t = Y_t$, this implies that $K_{t+1} = aY_t$ and $C_t = (1-a)Y_t$.

5.5.2 CES utility

In the case of the CES utility function, the stochastic expectations difference equation is given by

$$K_{t+1}C_t^{-\sigma} = aE_t\left[C_{t+1}^{1-\sigma} + K_{t+2}C_{t+1}^{-\sigma}\right],$$

which can be written as

$$K_{t+1}C_t^{-\sigma} = aE_t[C_{t+1}^{1-\sigma}] + a^2E_t[E_{t+1}[C_{t+2}^{1-\sigma}]] + ... + \lim_{n\longrightarrow\infty} a^n E[K_{t+n+1}C_{t+n}^{-\sigma}]. \tag{41}$$

To solve this equation, we assume that the economy is in its long-run (asymptotic) equilibrium, such that in every period the growth rate of output is drawn from *the same* probability measure. This allows us to write $E[g_Y^{1-\sigma}]$ instead of $E[g_{Y,t+1}^{1-\sigma}]$ in the third line below. Let $K_{t+1} = \lambda Y_t$ and $C_t = (1-\lambda)Y_t$, where λ is still to be determined. Then

$$\begin{aligned} a^2E_t[E_{t+1}[C_{t+2}^{1-\sigma}]] &= a^2E_t[E_{t+1}[(1-\lambda)^{1-\sigma}Y_{t+2}^{1-\sigma}]] \\ &= a^2(1-\lambda)^{1-\sigma}E_t[E_{t+1}[g_{Y,t+1}^{1-\sigma}Y_{t+1}^{1-\sigma}]] \\ &= a^2(1-\lambda)^{1-\sigma}E[g_Y^{1-\sigma}]E_t[Y_{t+1}^{1-\sigma}]]. \end{aligned}$$

Consequently, equation (41) transforms to

$$K_{t+1}C_t^{-\sigma} = a(1-\lambda)^{1-\sigma}E[Y_{t+1}^{1-\sigma}]\left[\frac{1}{1-aE[g_Y^{1-\sigma}]}\right],$$

by imposing the transversality condition and under the condition that $aE[g_Y^{1-\sigma}] < 1$. Inserting $K_{t+1} = \lambda Y_t$ and $C_t = (1-\lambda)Y_t$ into the above equation yields the solution for λ:

$$\lambda = aE[g_Y^{1-\sigma}].$$

5.6 Taylor series approximation

5.6.1 Determination of $E[g_Y^{1-\sigma}]$

Using the second-order Taylor series expansion around $E[g_Y^{1-\sigma}]$ implies

$$E[\ln(g_Y)] \approx \ln\overline{g_Y} - \frac{1}{2}\frac{Var(g_Y)}{\overline{g_Y}^2},$$

where $\overline{g_Y}$ and $Var(g_Y)$ represent respectively the mean and variance of g_Y. Let $X = 1 + \eta \overline{h}^{\gamma+\theta}$. From equation (21) it follows (taking first logs and then expectations) that

$$\begin{aligned} E[\ln(g_Y)] &\approx E[\ln X] \\ &\approx \ln \overline{X} - \frac{1}{2}\frac{var(X)}{\overline{X}^2}, \end{aligned}$$

where the second equality uses the second-order Taylor series expansion around $\overline{X}$, the mean of X. As a result,

$$var(g_Y) \approx 2\overline{g_Y}^2 \ln\left(\frac{\overline{g_Y}}{\overline{X}}\right) + \left(\frac{\overline{g_Y}}{\overline{X}}\right)^2 var(X),$$

with $var(X) = h^{2(\gamma+\theta)}\sigma_\eta^2$.

5.6.2 Determination of $E\left(\frac{\eta h^{\gamma+\theta}}{1+\eta h^{\gamma+\theta}}\right)$

First, we rewrite the expectation as

$$E\left(\frac{\eta h^{\gamma+\theta}}{1+\eta h^{\gamma+\theta}}\right) = E\left(\frac{1+\eta h^{\gamma+\theta}-1}{1+\eta h^{\gamma+\theta}}\right) = 1 - E\left(\frac{1}{1+\eta h^{\gamma+\theta}}\right).$$

The second-order Taylor series expansion around $\overline{\eta}$ yields

$$E\left(\frac{1}{1+\eta h^{\gamma+\theta}}\right) = \frac{1}{1+\overline{\eta} h^{\gamma+\theta}} + \frac{h^{2(\gamma+\theta)}}{(1+\overline{\eta} h^{\gamma+\theta})^3}\sigma_\eta^2.$$

This implies that

$$E\left(\frac{\eta h^{\gamma+\theta}}{1+\eta h^{\gamma+\theta}}\right) = 1 - \frac{1}{1+\overline{\eta} h^{\gamma+\theta}} - \frac{h^{2(\gamma+\theta)}}{(1+\overline{\eta} h^{\gamma+\theta})^3}\sigma_\eta^2.$$

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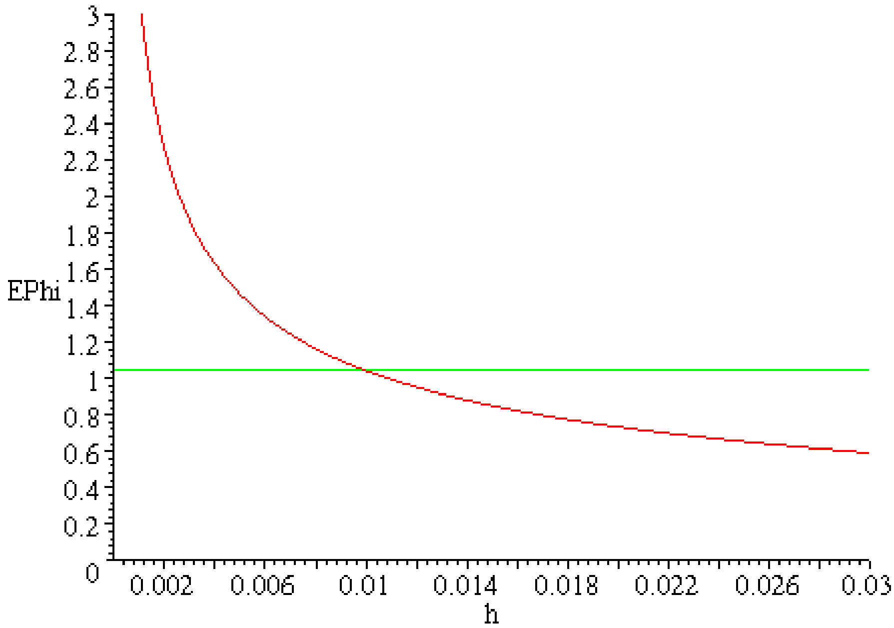

Figure 1: Equilibrium research condition, with $\gamma + \theta \leq 1$. The figure is based on equation (22) where the expectation is approximated with a second-order Taylor series expansion (see appendix). The parameter values used: $\beta = 1/3$, $\gamma = 0.5$, $\theta = 0.05$, $\rho = 0.05$, $\overline{\eta} = 0.85$, $\sigma_\eta^2 = 0.01$.

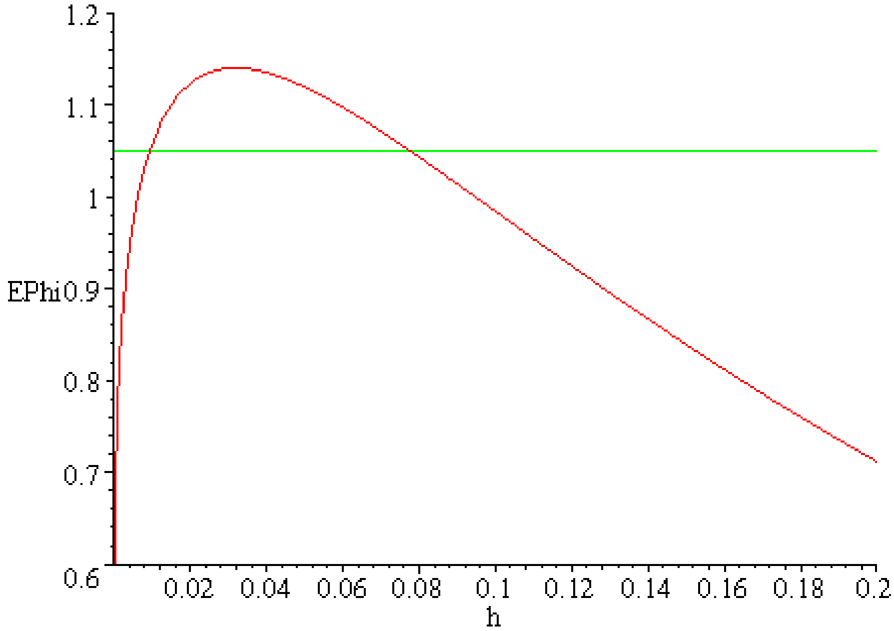

Figure 2: Equilibrium research condition, with $\gamma + \theta > 1$. The figure is based on equation (22) where the expectation is approximated with a second-order Taylor series expansion (see appendix). The parameter values used: $\beta = 1/3$, $\gamma = 1.1$, $\theta = 0.05$, $\rho = 0.05$, $\overline{\eta} = 6$, $\sigma_\eta^2 = 0.01$.