# The Induced Path Function, Monotonicity and Betweenness

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**Abstract**

The induced path function $J(u,v)$ of a graph consists of the set of all vertices lying on the induced paths between vertices $u$ and $v$. This function is a special instance of a transit function. The function $J$ satisfies betweenness if $w \in J(u,v)$ implies $u \notin J(w,v)$ and $x \in J(u,x)$ implies $J(u,x) \subseteq J(u,v)$, and it is monotone if $x, y \in J(u,v)$ implies $J(x,y) \subseteq J(u,v)$. The induced path function of a connected graph satisfying the betweenness and monotone axioms are characterized by transit axioms.

Key words : transit function, induced path, betweenness, monotone, long cycle, house, domino, $P$-graph.

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# 1 Introduction

In [18] the notion of transit function is introduced as a means to study how to move around in discrete structures. Basically, it is a function satisfying three simple axioms on a set $V$, which is provided with a structure $\sigma$. Prime examples of such a structure are: a set of edges $E$, so that we are considering a graph $G = (V, E)$, or a partial ordering $\leq$, so that we are considering a partially ordered set $(V, \leq)$. The idea is to study transit functions that have additional properties defined in terms of the structure $\sigma$. For instance, the transit function may be defined in terms of paths in the graph $G = (V, E)$. Such transit functions are called *path transit functions* on $G$ in [18]. A prime example is the interval function (geodesic interval function) $I : V \times V \to 2^V$ of a connected graph $G$, where $I(u, v)$ is the set of vertices lying on shortest paths between $u$ and $v$. This function has been widely studied from many different perspectives, to name a few: convexity, see e.g. [9], [17], [28], medians, see e.g. [17], [14], monotonicity, see e.g. [17]. For the induced path function $J : V \times V \to 2^V$ of a connected graph $G$, where $J(u, v)$ is the set of vertices lying on induced paths between $u$ and $v$, similar questions and problems have been studied: convexity, see e.g. [4], [8], [16], [10], median-type properties, see [16], monotonicity, see e.g. [3], [4]. This exemplifies the basic idea for introducing the concept of transit function in [18]: transfer ideas, questions and problems from one transit function to another and see whether interesting problems arise. This was the motivation to study the analogues of these questions for the all-paths function $A$ on a graph: now $A(u, v)$ consists of the vertices on all $u, v$-paths, see [2]. The convexity related to the all-paths function was already studied much earlier, see e.g. [25], [7]. Note that any transit function has an associated convexity. Such convexities are called interval convexities in [28]. Those related to path transit functions are discussed in more detail in [5].

In [19, 20, 21] Nebeský obtained some quite interesting results. He characterized the functions that are the geodesic interval function of some graph without any reference to the notion of distance. That is, a function $I : V \times V \to 2^V$ is the geodesic interval function of some graph if and only if $I$ satisfies a set of axioms that are phrased in terms of $I$ only. This immediately poses the problem for other transit functions: can they be characterized in terms of such transit axioms only? For the all-paths function $A$ this was done in [2]. Surprisingly, such a characterization of the induced path function $J$ is not possible, as was shown by Nebeský in [22] using first order logic.

The aim of this paper is to study special cases in which $J$ can still be characterized by such transit axioms only. These cases are where $J$ has the properties of a betweenness, and where $J$ is *monotone*, that is, all sets $J(u, v)$ are $J$-convex. As one might expect, the characterizations we seek for $J$ in this paper involve forbidden (induced) subgraphs. The most important ones are the house, the domino and the holes, see Fig. 1. Another one is the $P$-graph, see Fig. 2. The so-called $HHD$-free graphs and $HHP$-free graphs that appear over and over below also have other interesting aspects. These classes of graphs have important applications as far as elimination orderings in graphs are concerned. $HHD$-free and $HHP$-free graphs are natural generalizations of the class of chordal graphs in connection with the lexicographic breadth first search (LexBFS) and maximum cardinality search (MCS) orderings in graphs ([24, 27]. In [6], using a relaxation of the induced path convexity known as $m^3$-convexity, it is proved that graphs for which LexBFS (MCS) is a semi-simplicial ordering is precisely the class of $HHD$-free ($HHP$-free) graphs. See also

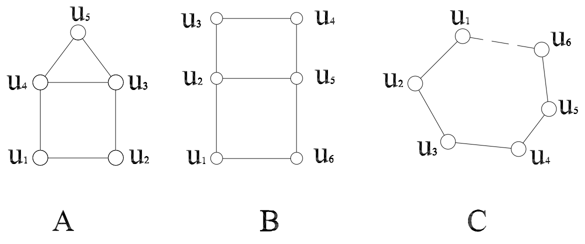


**Figure 1:** A: house, B: domino,C: hole

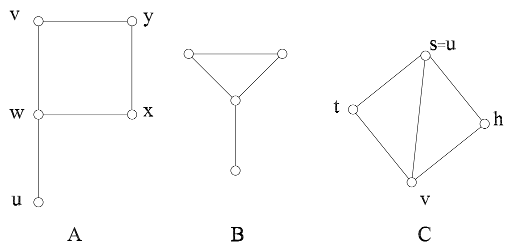


**Figure 2:** A: $P-graph$, B: $K_{1,3}+e$, C: $K_4-e$

[11].

In section 2 we give the definition of transit function, betweenness and monotonicity, and introduce the axioms which are needed for the characterization of the *induced path function* $J$ in terms of these transit axioms. In addition to the basic transit axioms and natural betweenness axioms, we present six more axioms in which the last is the monotone axiom while the others are special types of betweenness axioms. We characterize the graphs for which the induced path function $J$ satisfies these axioms. In section 3 we prove our main theorems, which characterize the induced path function satisfying the betweenness and monotone axioms. In this paper, using the above characterization, we also characterize the classes of $HHD$-free and $HHP$-free graphs.

All graphs in this paper are connected, nontrivial, finite, simple and loopless. In the sequel a *long cycle* or *hole* is a cycle of length at least five and the *P-graph* is the graph formed by a cycle of length four together with a pendant edge at one of its vertices. A *house* is a five cycle with an extra edge. A *domino* is a six cycle with an extra edge between antipodal vertices. See Figures 1 and 2 for these graphs. By an $HHP$-free graph, or an $HHD$-free graph we mean the graph for which the *house*, the *holes*, and the $P$-graph, respectively, the *house*, the *holes* and the *domino* are forbidden induced subgraphs.

# 2 The Induced Path Function

Let $V$ be a (finite) set. A *transit function* on $V$ is a function $R : V \times V :\longrightarrow 2^V$ satisfying the following three axioms:

$(t1)$ $\quad u \in R(u,v)$, for any $u$ and $v$ in $V$,

$(t2)$ $\quad R(u, v) = R(v, u)$, for all $u$ and $v$ in $V$.
$(t3)$ $\quad R(u, u) = \{u\}$, for all $u$ in $V$.

If, moreover, $G = (V, E)$ is a graph with vertex set $V$, then we say that $R$ is a transit function on $G$. The underlying graph $G_R$ of a transit function $R$ is the graph with vertex-set $V$, where two distinct vertices $u$ and $v$ are joined by an edge if and only if $R(u, v) = \{u, v\}$. Note that, in general, $G$ and $G_R$ will not be isomorphic graphs. Transit functions were introduced in [18]. Prime examples of transit functions on a graph $G$ are the (geodesic) interval function $I$, the all-paths function $A$, and the *induced path function* $J$, which is defined by

$$J(u, v) = \{w \in V | w \text{ lies on some induced } u, v\text{-path in } G\}.$$

These three functions are so-called path transit functions because they are defined in terms of paths of $G$, see [18] and [5] for more information on path transit functions. The geodesic intervals $I(u, v)$ in $G$ also have the structure of a betweenness, but the other two do not. Hence the following *betweenness axioms* were introduced in [18] to model the idea of betweenness. The first tells us that, if $x$ is between $u$ and $v$ but distinct from $v$, then $v$ is *not* between $u$ and $x$. The second tells us that, if $x$ is between $u$ and $v$ and $y$ is between $u$ and $x$, then $y$ is between $u$ and $x$. A transit function $R$ on $V$ is called a *betweenness*, if it satisfies

$(b1)$ $\quad x \in R(u, v),\, x \neq v \Longrightarrow v \notin R(u, x)$,
$(b2)$ $\quad x \in R(u, v) \Longrightarrow R(u, x) \subseteq R(u, v)$.

It is easy to see that $A$ is a betweenness on $G$ if and only if $G$ is a tree. In [16] it was shown that $J$ is a betweenness on $G$ if and only if $G$ is $HHD$-free. Note that only few aspects of the betweenness properties of $I$ are reflected in these two axioms. To capture all aspects would require a long and complicated list of axioms. Moreover, we would not get anything that could be "transferred" to other transit functions, the whole idea behind this approach. Therefore, this notion of betweenness is weaker than existing ones in the literature, see e.g. [26].

If $R$ is a betweenness on $V$, then we have the following lemma.

**Lemma 1** *If the transit function $R$ on a nonempty set $V$ is a betweenness, then the underlying graph $G_R$ of $R$ is connected.*

**Proof.** Let $u, v$ be any two distinct vertices of $G_R$. We prove the existence of a $u, v$-path in $G_R$ using induction on $|R(u, v)|$. If $|R(u, v)| = 2$, then $R(u, v) = \{u, v\}$, by transit axiom $(t1)$. Therefore, by the definition of $G_R$, we have $uv \in E(G_R)$, which is a $u, v$-path in $G_R$. So the lemma holds for $|R(u, v)| = 2$. Assume that there is a $u, v$-path in $G_R$ for any two distinct vertices $u, v$ with $|R(u, v)| < n$ $(n > 2)$. Since $n > 2$, there is a vertex $w \neq u, v$ with $w \in R(u, v)$. Hence by $(b1)$ we have $u \notin R(w, v)$ and $v \notin R(u, w)$. Also by $(b2)$ we have $R(u, w) \subseteq R(u, v)$ and $R(w, v) \subseteq R(u, v)$. Therefore $|R(u, w)| < |R(u, v)|$ and $|R(w, v)| < |R(u, v)|$. Hence, by the induction hypothesis, the existence of a $u, w$-path and a $w, v$-path follows. Concatenating the two paths we obtain a $u, v$-walk which proves the lemma. ■

**Remark 1** *The two betweenness axioms* $(b1)$, $(b2)$ *are necessary for the connectedness of* $G_R$.

For example, on $V = \{a, b, c, d\}$, the function $R$, defined by $R(u, u) = \{u\}$ for every $u \in V$, $R(a, b) = \{a, b, c\}, R(a, c) = \{a, c, d\}, R(a, d) = \{a, b, d\}, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(c, d) = \{c, d\}$, is a transit function satisfying $(b1)$, but not $(b2)$ and it can be easily verified that $G_R$ is disconnected. On $V = \{a, b, c\}$, the function $R$ defined by $R(a, b) = R(b, c) = R(c, a) = V$ and $R(u, u) = \{u\}$ for every $u \in V$ is a transit function satisfying $(b2)$, but not $(b1)$. Here also $G_R$ is disconnected.

In Lemma 1, only the connectivity of the underlying graph is established, but nothing pertinent can be said yet about the question whether $G$ and $G_R$ are isomorphic or not. Moreover, a betweenness in general will not be the induced path function of some graph. Hence, we need some more transit axioms for our purposes.

An axiom that played an important role in the study of median graphs and median structures is that of *monotonicity*, see [17]. There it was introduced for the interval function $I$, but in [18] it is introduced as a transit axiom:

$(m)$ $\quad x, y \in R(u, v) \Longrightarrow R(x, y) \subseteq R(u, v).$

Note that in the terminology of convexity this axiom can be read as follows: the $R$-intervals $R(u, v)$ are $R$-convex. For references on convexity, and monotonicity of $I$, $J$, and $A$, see the Introduction.

The following five new transit axioms all reflect some aspect that the betweenness of the function $I$ possesses. Let $R$ be a transit function on a connected graph $G = (V, E)$. For any $u, v \in V$, we define the following axioms.

$(J1)$ $\quad w \in R(u, v), w \neq u, v, \Longrightarrow$ there exists $u_1 \in R(u, w) \setminus R(v, w), v_1 \in R(v, w) \setminus R(u, w)$, such that $R(u_1, w) = \{u_1, w\}, R(v_1, w) = \{v_1, w\}$ and $w \in R(u_1, v_1)$,

$(J2)$ $\quad R(u, x) = \{u, x\}, R(x, v) = \{x, v\}, u \neq v, R(u, v) \neq \{u, v\} \Longrightarrow x \in R(u, v)$.

$(J3)$ $\quad x \in R(u, y), y \in R(x, v), x \neq y, u \neq v, R(u, v) \neq \{u, v\} \Longrightarrow x \in R(u, v)$.

$(J2')$ $\quad x \in R(u, y), y \in R(x, v), x \neq y, |R(u, x)| = |R(x, y)| = |R(y, v)| = 2, u \neq v, R(u, v) \neq \{u, v\} \Longrightarrow x \in R(u, v)$.

$(J3')$ $\quad x \in R(u, y), y \in R(x, v), R(x, y) \neq \{x, y\}, \ x \neq y, u \neq v, R(u, v) \neq \{u, v\} \Longrightarrow x \in R(u, v)$.

Note that, although we use the letter $J$ to name these axioms, only the axioms $(J2)$ and $(J2')$ are satisfied by the induced path function of any graph.

First we examine the graphs for which $J$ satisfies the other axioms. We can easily verify that if $G$ is not $HHD$-free, then $(J1)$ is not satisfied. We will prove the converse.

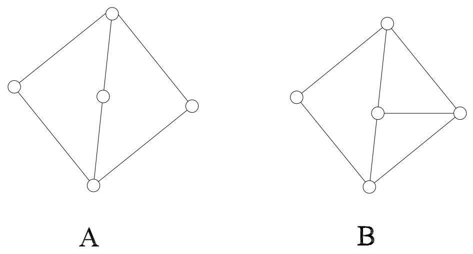


**Figure 3:** A: $K_{2,3}$, B: $W_4 - e$

In the proof we use the following notation. Let $P$ be a path in a graph $G$, and let $x, y$ be two vertices on $P$. The $x \to \ldots P \ldots \to y$ denotes the subpath of $P$ between $x$ and $y$, that is, we walk from $x$ to $y$ *along* $P$.

**Theorem 1** *The induced path function $J$ on a graph $G$ satisfies $(J1)$ if and only if $G$ is $HHD$-free.*

**Proof.** First assume that $G$ is not $HHD$-free. Then $G$ contains a *house*, a *hole* or a *domino*. In each case we can find three vertices $u$,$v$ and $w$ with $u$ and $w$ adjacent and $v$ not adjacent to $u$ or $w$ such that $w \in J(u, v)$ and $J(u, w) = \{u, w\} \subset J(v, w)$. Hence we cannot find a $u_1$ as required by the axiom $(J1)$. So $(J1)$ is not satisfied.

Now assume that $(J1)$ is not satisfied. Then there exist vertices $u, v, w, u_1, v_1$ and an induced $u, v$-path $P$ with $u \to \ldots P \ldots \to u_1 \to w \to v_1 \to \ldots P \ldots \to v$ such that either $u_1 \in J(v, w)$ or $v_1 \in J(u, w)$. Suppose $u_1 \in J(v, w)$. Then there exists an induced $w, v$-path $Q$ containing $u_1$. Evidently $Q$ starts with the edge $wu_1$. Let $v_2$ be the first vertex on $Q$ which is also a vertex on the $w \to \ldots P \ldots \to v$. Then $v_2 \neq v_1$, otherwise $wv_1$ will act as a chord. Since $P$ is an induced path, $u_1v_1 \notin E(G)$. Hence $Q' = u_1 \to \ldots Q \to \ldots v_2$ is an induced $u_1, v_2$-path of length greater than or equal to two and $P' = u_1 \to w \to v_1 \to \ldots P \to \ldots v_2$ is another induced $u_1, v_2$-path of length at least three. They together form a cycle of length at least five. To avoid a long cycle, there must exist chord between an internal vertex of $P'$ and $Q'$. Let $v_3$ be the vertex on $P'$ closest to $u_1$ having a chord to $Q'$, and let $v_4$ be the vertex on $Q'$ closest to $u_1$ having a chord to $v_3$. Then $w \to v_1 \to \ldots P \ldots \to v_3 \to v_4 \to \ldots Q \ldots \to u_1 \to w$ is an induced cycle (say) $\mathcal{C}$. Since $\mathcal{C}$ cannot be a long cycle we have $v_3 = v_1$ and $v_4$ adjacent to $u_1$. Hence $\mathcal{C}$ is an induced cycle of length four. Consider the cycle $v_1 \to \ldots P \ldots \to v_2 \to \ldots Q \ldots \to v_4 \to v_1$. If it is of length three or four, then together with $\mathcal{C}$ we get a house or a domino. So it is a cycle of length at least five. Again, to avoid a hole, there must be chords. As above, we choose a chord "closest" to $v_1$ and $v_4$, which yields a 3-cycle or 4-cycle. But now this cycle together with $\mathcal{C}$ is a house or a domino. Thus we have a contradiction, which concludes the proof. ■

**Corollary 1** *Let $J$ be the induced path function of a connected graph $G$. The $J$ is a betweenness if and only if $J$ satisfies $(J1)$.*

Note that the equivalence of $(b1), (b2)$ on the one hand and $(J1)$ on the other hand in this corollary is a special case that only holds for the induced path function of a graph. For arbitrary transit functions this equivalence need not hold.

Let $x \to \ldots P \ldots \to y$ be path. If we choose any vertex on this path such that it may not be $x$, then we say that we choose it from $(x) \to \ldots P \ldots \to y$. If it must be distinct from $x$ as well as $y$, then we say that we choose it from $(x) \to \ldots P \ldots \to (y)$, etcetera.

**Lemma 2** *Let $G$ be connected graph and $J$ be the induced path function of $G$ satisfying $(b1)$. Then $J$ is monotone if and only if $G$ is HHD-free and has no $K_{2,3}$ or $W_4 - e$ as induced subgraph.*

**Proof.** If $G$ has a long cycle, house, domino, $K_{2,3}$ or $W_4 - e$ as induced subgraph, then it can be easily verified that the induced path function $J$ of $G$ does not satisfy both $(b1)$ and $(m)$. Conversely, assume that $G$ does not contain any of these graphs as induced subgraphs. Let $u, v, x, y$ be any four vertices in $G$ such that $x, y \in J(u, v)$. If $x \in J(u, y)$ or $x \in J(y, v)$, then, by $(b1)$, we have $J(x, y) \subseteq J(u, v)$. So assume that $x \notin J(u, y) \cup J(v, y)$. Therefore, there exist induced $u, v$-paths $P_x$ and $P_y$ such that $P_x$ contains $x$, but not $y$ and $P_y$ contains $y$, but not $x$. We show that $J(x, y) \subseteq J(u, v)$. Assume the contrary. So there exists an interior vertex $z$ on an induced $x, y$-path $Q$ such that $z \notin J(u, v)$. We may choose $x$ and $y$ such that $x$ is the common vertex of $Q$ and $P_x$, and $y$ is the common vertex of $Q$ and $P_y$ and no vertex on $(x) \to \ldots Q \ldots \to (y)$ lies in $J(u, v)$. Since $x \notin J(u, y)$, it follows that $u \to \ldots P_x \ldots \to x \to \ldots Q \ldots \to y$ is not an induced $u, y$-path. Hence, there must be a chord $x_1 y_1$ from $u \to \ldots P_x \ldots \to (x)$ to $(x) \to \ldots P \ldots \to y$. Amongst such chords we choose one with $x_1$ closest to $x$ and then $y_1$ closest to $x$. Then $u \to \ldots P_x \ldots \to x_1 \to y_1 \to \ldots Q \ldots \to x \to \ldots P_x \ldots \to v$ is a $u, v$-path containing at least one vertex on $(x) \to \ldots P \ldots \to y$. Hence by the choice of $Q$, it can not be an induced $u, v$-path. Now, if there would be a chord between $(y_1) \to \ldots Q \ldots \to (x)$ and $(x) \to \ldots P_x \ldots \to v$, then we would find an induced $u, v$-path containing an internal vertex of $Q$, contradicting the choice of $Q$. Therefore the only possible chord is between $x_1$ and $x$. Then $x_1 \to x \to \ldots P \ldots \to y_1 \to x_1$ is an induced cycle. To avoid long cycles, the length of the subpath $x \to \ldots P \ldots \to y_1$ must be at most two. Similarly, since $x \notin J(y, v)$, we can find a vertex $x_2$ on $x \to \ldots P_x \ldots \to v$ adjacent to $x$ and a vertex $y_2$ on $P$ adjacent to $x_2$ so that $x_2 \to y_2 \to \ldots Q \ldots \to x \to x_2$ is an induced cycle of length at most four. Now $u \to \ldots P_x \ldots \to x_1 \to y_1 \to \ldots Q \ldots \to y_2 \to x_2 \to \ldots P_x \ldots \to v$ is a $u, v$-path containing the internal vertex $y_1$ of $Q$. To avoid this path being induced, the only possibility is that either $y_1$ or $y_2$ coincides with $y$. Let us assume that $y_2 = y$. We consider two cases.

**Case 1.** $y_1 = y$.

In this case to avoid an induced long cycle, the subgraph induced by $x \to \ldots Q \ldots \to y$ together with the vertices $x_1$ and $x_2$ must be isomorphic to $K_{2,3}$.

**Case 2.** $y_1 \neq y$.

Here also, to avoid a long cycle the subgraph induced by $x \to \ldots Q \ldots \to y$ together with the vertices $x_1$ and $x_2$ must be isomorphic to $W_4 - x$ or house according as $x_1$ is adjacent to $y_2$ or not. ■

# 3 A characterization of the $J$ function satisfying betweenness

Let $J$ be a transit function, and let $G_J$ be its underlying graph. In general, the induced path function $J_{G_J}$ of $G_J$ may be quite different from the original transit function $J$, even if $J$ satisfies some axioms reflecting properties of the induced path function of a graph.

In this section we consider a set of axioms on a transit function $J$ such that we have the very nice property $J = J_{G_J}$. Thus we obtain a partial analogue of Nebeský's very nice characterization of the interval function $I$ in terms of transit axioms only.

Let $J$ be a transit function on a non-empty finite set $V$ satisfying some or all of the axioms $(b1), (b2), (m), (J1), (J2), (J2'), (J3), and (J3')$. Using this set of axioms we give two characterizations of the induced path function $J$ on the underlying graph $G_J$. For proving our main theorems we need the following lemmas. Note that the tricky part in the proofs is that we do not know yet whether $J$ is the induced path function of $G_J$.

**Lemma 3** *Let $J$ be a transit function on a non-empty finite set $V$ satisfying the axioms $(b1), (J2)$ and $(J3)$ with underlying graph $G_J$. Then $G_J$ is $HHP$-free.*

**Proof.** Suppose $G_J$ contains a house as an induced subgraph with vertices shown in Figure 1(A). Then by $(J2)$ we have $u_1 \in J(u_2, u_4)$, $u_4 \in J(u_1, u_5)$ and $u_1 \neq u_4$. Hence by $(J3)$we have $u_1 \in J(u_2, u_5)$. Similarly we have $u_2 \in J(u_1, u_5)$, which violates $(b1)$. If $G_J$ contains a long cycle, say $C = u_1 u_2, \ldots, u_n$ with $n \geq 5$ as an induced subgraph, then by applying $(J2)$ and $(J3)$ successively we get that $u_2 \in J(u_1, u_{n-1})$ and $u_1 \in J(u_2, u_{n-1})$, which violates $(b1)$. Similarly, if $G_J$ has a $P$ as an induced subgraph, then we can also derive a contradiction. For, let the vertices of the induced $P$-graph be $u, w, x, y, v$ with $w, x, y, v$ forming the four cycle $C_4$ and $w$ adjacent to $u, v$ and $x$. By $(J(2)$ we have $w \in J(u, x)$, $x \in J(w, y)$ and $y \in J(x, v)$. Moreover we have $u \neq v, x \neq y$ and $J(u, v) \neq \{u, v\}$. Therefore by $(J3)$ we have $x \in J(u, v)$. By the same argument it follows that $v \in J(u, x)$, which contradicts $(b1)$. ■

**Lemma 4** *Let $J$ be a transit function on a non-empty finite set $V$ satisfying the axioms $(b1), (J2), (J2')$ and $(J3')$ with underlying graph $G_J$. Then $G_J$ is $HHD$-free.*

**Proof.** Suppose $G_J$ contains a house as an induced subgraph with vertices shown in Figure 1(A). Then by $(J2)$ we have $u_1 \in J(u_2, u_4)$, $u_4 \in J(u_1, u_5)$, and $|J(u_2, u_1)| = |J(u_1, u_4)| = |J(u_4, u_5)| = 2$ with $u_1 \neq u_4, u_2 \neq u_5$. Hence by $(J2')$ we have $u_1 \in J(u_2, u_5)$. Similarly, $u_2 \in J(u_1, u_5)$, which violates $(b1)$. If $G_J$ contains a long cycle, say $C = u_1, u_2, \ldots, u_n$ with $(n \geq 5)$ as an induced subgraph, then by applying $(J2), (J2')$ and $(J3')$ successively we get that $u_2 \in J(u_1, u_{n-1})$ and $u_1 \in J(u_2, u_{n-1})$ which violates $(b1)$. Assume that $G_J$ contains a domino as an induced subgraph, (say) with vertices $u_1, u_2, u_3, u_4, u_5, u_6$ as shown in Figure 1(B). Here also using $(J2), (J2')$ and $(J3')$ we get that $u_3 \in J(u_1, u_4)$ and $u_4 \in J(u_1, u_3)$, which violates $(b1)$. ■

**Lemma 5** *Let $J$ be a transit function on a non-empty finite set $V$ satisfying the axioms $(b1), (b2),$ and $(J1)$ with underlying graph $G_J$. If $w \in J(u, v), w \neq u, v$, then there exists a sequence $u_1, u_2, \ldots, u_k \in V$ satisfying the conditions* *(i)* $\quad J(u_{i+1}, u) \subset J(u_i, u), \quad i = 0, 1, 2, \ldots,$ *where* $u_0 = w, u_{k+1} = u$,

*(ii)* $\quad u_i \in J(u_{i-1}, u_{i+1}), i = 1, 2, 3, \ldots, k$
*such that* $w, u_1, u_2, \ldots, u_k, u$ *is a path in* $G_J$.

**Proof.** Since $w \in J(u, v)$ and $w \neq u, v$, by $(J1)$ there exists $u_1 \in J(u, w) \setminus J(v, w)$ and $v_1 \in J(v, w) \setminus J(u, w)$ such that $J(u_1, w) = \{u_1, w\}$, $J(w, v_1) = \{w, v_1\}$ and $w \in J(u_1, v_1)$. Since $u_1 \in J(u, w)$ by $(J1)$, there exists $u_2 \in J(u, u_1) \setminus J(u_1, w)$ such that $u_1 \in J(u_2, w)$. Now applying $(J1)$ successively to $J(u, u_2)$ and so on, we get a sequence of vertices $w, u_1, u_2, u_3, \ldots, u_k$ such that

$(i)$ $\quad J(u_{i+1}, u) \subset J(u_i, u)$ using $(b1)$ and $(b2), i = 0, 1, 2, \ldots, k$ where $u_0 = w, u_{k+1} = u$,

$(ii)$ $\quad u_i \in J(u_{i-1}, u_{i+1}), i = 1, 2, 3, \ldots, k$

$(iii)$ $\quad u_i u_{i+1} \in E(G_J), i = 0, 1, 2, 3, \ldots, k$

By $(i)$, there exists a vertex $u_k$ in the sequence $u_1, u_2, \ldots$ such that $|J(u_k, u)| = 2$. Hence, by $(t1)$ and $(t2)$, it follows that $u_k u \in E(G_J)$. So $w, u_1, u_2, \ldots, u_k u$ is a $w, u$-walk in $G_j$. Take $u_i, u_j$ with $i < j$, then $u_j \in J(u_i, u)$, by $(i)$ and $u_i \notin J(u_j, u)$ by $(b1)$. So $u_i \neq u_j$ and hence the lemma. ■

Now we are ready for the main results of our paper: the characterization of transit functions in terms of transit axioms only that are precisely the induced path function of some graph. Because of Nebeský's impossibility result in [22], we have to restrict ourselves to special instances. In our case this means that we restrict ourselves to transit functions that are a betweenness.

**Theorem 2** *Let $V$ be a finite non-empty set and $J$ be a transit function on $V$ satisfying the axioms $(b1), (b2), (J1), (J2)$ and $(J3)$. Let $G_J$ be the underlying graph of the transit function $J$. Then $J$ is precisely the induced path function of $G_J$.*

**Proof.** Let $u$ and $v$ be two distinct vertices of $G_J$, and let $w$ be a vertex in $J(u, v)$. Since $G_J$ is connected, there is an induced $u, v$-path. Hence the lemma holds when $w = u$ or $v$. So let us assume that $w \neq u, v$. Then by Lemma 5, there exists a $u, w$-path $P_u : w = u_0 u_1 u_2 \ldots u_k u_{k+1} = u$ satisfying

$(i)$ $\quad u_i \in J(u_{i-1}, u_{i+1}), i = 1, 2, 3, \ldots, k$

$(ii)$ $\quad u_i u_{i+1} \in E(G_J), i = 1, 2, 3, \ldots, k$

$(iii)$ $\quad J(u_{i+1}, u) \subset J(u_i, u)$, $i = 1, 2, 3, \ldots, k$

and a $v,w$-path $P_v : w = v_0v_1v_2...v_{k'}v_{k'+1} = v$ satisfying conditions similar to (i), (ii) and (iii) such that $w \in J(u_1, v_1)$.

*Claim 1*: $P_u$ is an induced $u,w$-path.

We need to prove that $u_iu_{i+l} \notin E(G_J)$, for $i = 0,1,2,\ldots,k-l$ with $l \geq 2$. When $l = 2$, the result follows by $(i)$. In the case $l = 3$, assume the contrary, that is $u_iu_{i+3} \in E(G_J)$. Then, by $(J2)$ we have $u_i \in J(u_{i+1}, u_{i+3})$. By $(ii)$, $J(u_{i+3}, u) \subset J(u_{i+2}, u) \subset J(u_{i+1}, u)$. Hence $u_i \in J(u_{i+1}, u)$. But by $(iii)$ we have $u_{i+1} \in J(u_i, u)$ which contradicts $(b2)$, hence $u_iu_{i+3} \notin E(G_J)$. Since induced long cycles are forbidden by Lemma 3, Claim 1 follows.

*Claim 2*: No vertex $u_i$, with $i = 1,2,3,\ldots,k$, is adjacent to a vertex in $v_1, v_2, \ldots, v_{k'+1}$.

Now, $w \in J(u_1, v_1)$. Therefore $u_1v_1 \notin E(G_J)$. If $u_1v_2 \in E(G_J)$, then by $(J2)$ we have $u_1 \in J(w, v_2)$, since $wv_2 \notin E(G_J)$. Also we have $J(w, v_2) \subset J(w, v) \implies u_1 \in J(w, v)$, which violates $(J1)$. Therefore $u_1v_2 \notin E(G_J)$. Similarly $u_2v_1 \notin E(G_J)$. Now we prove that no vertex in $u_1, u_2, \ldots, u_{k+1}$ is adjacent to a vertex in $v_3, v_4, \ldots, v_{k'+1}$. Suppose not and let $u_r$ be the first vertex in the $u_i$'s, $i = 1,2,\ldots,k+1$ which is adjacent to a vertex in the $v_j$'s, $j = 1,2,\ldots,k'+1$. Let $v_s$ be the first vertex in the $v_j$'s adjacent to $u_r$. Then $u_r, v_s, v_{s-1}, \ldots, w, u_1, u_2, \ldots, u_{r-1}, u_r$ is an induced long cycle, which is a contradiction by Lemma 3. By producing a similar contradiction we can prove that $v_1$ and $v_2$ are not adjacent to any vertex in $u_1, u_2, \ldots, u_{k+1}$. This settles Claim 2.

Now we prove that no vertex in $u_1, u_2, \ldots, u_{k+1}$ coincides with vertex in $v_1, v_2, \ldots, v_{k'+1}$. Evidently $u_1 \neq v_1$. Suppose $u_i = v_j$, for some $i$ and $j$ except $i = j = 1$. Without loss of generality we may assume that $i \geq j$. Then $u_{i-1}$ is adjacent to $v_j$, which is a contradiction by Claim 2. Hence $P_u \cup P_v$ is an induced $u,v$-path and $w$ lies on it.

For any vertex $w$ on some induced $u,v$-path $P$, we prove that $w \in J(u,v)$, by induction on the length $l(P)$ of $P$. If $w = u$ or $v$, then evidently $w \in J(u,v)$. So assume that $w \neq u, v$, so that $l(P) \geq 2$. When $l(P) = 2$, the result follows by $(J2)$. Assume that the result is true for $l(P) < m$. Suppose now that $l(P) = m$ with $m > 2$. Then, either $u$ or $v$ has a neighbor on $P$ different from $w$. Let $u_1 \neq w$ be the neighbor of $u$ on $P$. So $u_1$ lies on the induced $w,u$-subpath of $P$ and $w$ lies on the induced $v,u_1$-subpath of $P$. By the induction hypothesis we have $w \in J(v, u_1)$ and $u_1 \in J(w, u)$, hence by $(J3)$ we have $w \in J(v, u)$. Since $J$ is a transit function it follows that $w \in J(u, v)$. ■

**Theorem 3** *Let $V$ be a finite non-empty set and $J$ be a transit function on $V$ satisfying the axioms $(b1), (b2), (J1), (J2), (J2'), (J3')$. Let $G_J$ be the underlying graph of the transit function $J$. Then $J$ is precisely the induced path function of $G_J$.*

**Proof.** The first part of the proof is essentially the same as that of the previous Lemma. For the second part, that is when $w$ is a vertex on some induced $u,v$-path $P$, we will prove that $w \in J(u,v)$, by induction on the length $l(P)$ of $P$. The cases when $w = u$ or $v$ and when $l(P) = 2$ are also the same as that in the previous Lemma. So assume that $l(P) \geq 3$. If $l(P) = 3$, the result follows by $(J2')$. Suppose $l(P) = 4$. If $w$ is adjacent to $u$ or $v$, say $u$, then $w \in J(u, v_1)$ and $v_1 \in J(w, v)$, since the result holds when $l(P) = 3$, where $v_1$ is a vertex adjacent to $v$ on $P$. Also $w$ is not adjacent to $v_1$ and hence $J(w, v_1) \neq \{w, v_1\}$. Therefore by $J(3')$ we have $w \in J(u, v)$. If $w$ is not adjacent to both $u$ and $v$, then $u_1 \in J(u, v)$ by the previous argument, where $u_1$ is a vertex adjacent to $u$ on $P$. Therefore $J(u_1, v) \subseteq J(u, v)$ by $(b2)$. Also $w \in J(u_1, v)$, since the result is true for $l(P) = 3$. Hence $w \in J(u, v)$. Assume that the result is true when $l(P) < m$. Let $l(P) = m$ with $m > 4$. Consider the case when $w$ is adjacent to either $u$ or $v$. Let us assume that $w$ is adjacent to $u$. Let $v_1$ be the neighbor of $v$ on $P$. Since $m > 4$, $J(w, v_1) \neq \{w, v_1\}$. Also by the induction hypothesis, $w \in J(u, v_1)$ and $v_1 \in J(w, v)$. Hence by $J3'$ we have $w \in J(u, v)$. Now consider the case when $w$ is not adjacent to $u$ or $v$. In this case we can find a vertex $u_1$ on $P$ adjacent to $u$ or $v_1$. Assume that $u_1$ is adjacent to $u$ so that $u_1$ is not adjacent to $w$. Then by the induction hypothesis and $(J3')$ we get $w \in J(v, u)$. Since $J$ is a transit function this implies $w \in J(u, v)$. ■

In the above instances we have results that can be written in mathematical shorthand as $J = J_{G_J}$, so we start with $J$, then construct the underlying graph $G_J$, and then consider the induced path function of this graph. In a similar way, we could start with a connected graph $G$, then consider its induced path function $J_G$, and then construct the underlying graph of this transit function $G_{J_G}$. Then the question is, under what conditions will we have that these two graphs are isomorphic, or in mathematical shorthand: $G = G_{J_G}$? It turns out that this is not an easy question. As first steps in this direction we present the following results. For these we need an extra axiom that relates the transit function $J$ and the graph $G = (V, E)$ on which this transit function is defined.

$(e)$ $uv \in E$ if and only if $|J(u, v)| = 2$.

**Theorem 4** *Let $G = (V, E)$ be a connected graph. Let $J : V \times V \to 2^V$ be a transit function satisfying the axioms $(b1), (b2), (J1), (J2), (J3)$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $G$ is $HHP$-free.*

**Proof.** Let $G$ be an $HHP$-free connected graph. If a transit function $J$ on the $V(G)$ satisfies the axioms $(b1), (b2), (J1), (J2), (J3)$, then $G_J$ is $HHP$-free by Lemma 3. By axiom $(e)$ we have $G \cong G_J$. Hence $J$ is the induced path function of $G_J$ by Lemma 2. Conversely, let $J$ be the induced path function of a connected graph $G$. Then $J$ satisfies the axioms $(b1), (b2), (b3), (J1), (J2), (J3)$ if and only if $G$ is $HHP$-free by Theorem 1 and Lemma 3. ■

The analogue of Theorem 3 is as easily obtained. We omit the proof.

**Theorem 5** *Let $G = (V, E)$ be a connected graph. Let $J : V \times V \to 2^V$ be a transit function satisfying the axioms $(b1), (b2), (J1), (J2), (J2'), (J3')$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $G$ is $HHD$-free.*

From the above results, we easily deduce various similar results. We list these as observations. The proofs are straightforward, hence omitted.

**Observation 1** *Let $G = (V, E)$ be a connected graph and $J : V \times V :\to 2^V$ be a transit function on $G$ satisfying $(b1), (m), (J1), (J2), (J3)$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $G$ has no long cycle, house,$P$, $K_{2,3}$ and $W_4 - e$ as induced subgraphs.*

**Observation 2** *Let $G = (V, E)$ be a connected graph and $J : V \times V \to 2^V$ be a transit function on $G$ satisfying $(b1), (m), (J1), (J2), (J2'), (J3')$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $G$ has no long cycle, house, domino, $K_{2,3}$ and $W_4 - e$ as induced subgraphs.*

**Observation 3** *Let $G = (V, E)$ be a connected $HHP$-free graph. Let $J$ be a be a transit function on $G$ satisfying the axioms $(J1), (J2), (J3)$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $J$ is a betweenness.*

**Observation 4** *Let $G = (V, E)$ be a connected $HHD$-free graph. Let $J$ be a transit function on $G$ satisfying the axioms $(J1), (J2), (J2'), (J3')$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $J$ is a betweenness.*

**Observation 5** *Let $G = (V, E)$ be a connected $HHP$, $K_{2,3}$ and $W_4 - e$-free graph. and $J$ be a transit function on $G$ satisfying $(J1), (J2), (J3)$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $J$ satisfies the betweenness axiom $(b1)$ and the monotonicity axiom $(m)$.*

**Observation 6** *Let $G = (V, E)$ be a connected $HHD$, $K_{2,3}$ and $W_4 - e$-free graph and $J$be a transit function on $G$ satisfying $(J1), (J2), (J2'), (J3')$, and $(e)$. Then $J$ is the induced path function of $G$ if and only if $J$ satisfies the betweenness axiom $(b1)$ and the monotonicity axiom $(m)$.*

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