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The Elasticities of Complementarity and Substitution

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Abstract

This paper argues that the conventional definition of the elasticity of complementarity is not well suited to deal with the case of increasing returns. It proposes a slightly different formula, that uses a distance function formulation instead of a production function. The proposed definition coincides with the Hicksian measure in case the production function displays constant returns. It is more informative in case returns to scale are not constant, as it disentangles entry effects and substitution effects of factor supplies. The new definition is also preferable in that it is fully symmetric with the definition of the elasticity of substitution.

Keywords: distance function, elasticity of complementarity, returns to scale, tangency condition

JEL codes: D21, D43, L16

1 Introduction

The concepts of substitutes and complements play a central role in demand theory. They apply to the two sides of a demand system, the effect of prices on quantities and the effect of quantities on prices. In 1933, Hicks quantified both concepts through the definition of two elasticity measures, the elasticity of substitution and the elasticity of complementarity (see Hicks (1963)). The relation between these concepts has been a recurrent issue ever since. In Hicks's original definition the two measures were not clearly distinguished and, in fact, they are reciprocal in the two-factor, constant returns case. Hicks (1970) then extended the definition to the three-factor case and invented the name "elasticity of complementarity."He showed that the elasticity measures the degree to which factor prices change, following an increase in one of the factors, and keeping marginal cost constant. Such a measure is very convenient to provide an answer *e.g.* to questions of tax incidence, where one wants to know how a tax or subsidy on one factor affects the price of another factor. Seidman (1989) shows how in this context the elasticity of complementarity is more useful than the elasticity of substitution.

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The concepts were given a general formulation in the 1970's using duality theory (see e.g. Diewert (1971), Sato and Koizumi (1973)). This formulation clearly shows the intimate relationship that exists between both concepts. In this set-up, the elasticity of substitution is a unit-free measure of the local curvature of the cost function. This curvature measures to what extent the consumer or producer can avoid the increase in cost associated with a price increase. The elasticity of complementarity on the other hand is a local measure of the curvature of the production function. This curvature determines what change in factor prices suffices to restore equality of marginal product and cost, following a change in one of the factors.

The curvature of both functions, the cost function and the production function, can be linked via the duality relation that exists between the two. Sato and Koizumi (1973) first showed this for a general n-factor, single-output production function under constant returns to scale. Their result was extended by Syrquin and Hollender (1982) to the case of non-constant returns to scale. From this extension, it appears that there is an asymmetry between the definitions of the elasticities of complementarity and substitution. In particular, the elasticity of complementarity incorporates the returns to scale effects of the change in the production factors whereas the elasticity of substitution is invariant to the size of the scale effects.

In this paper I reconsider the definition of the elasticity of complementarity in the presence of increasing returns to scale. I argue that the asymmetry obtained by Syrquin and Hollender (1982) results from an improper generalisation of this elasticity to a setting of increasing returns. A useful definition of this elasticity should take into account the consequences of non-convexity of the production technology for output markets. Entry and exit of firms severs the direct link of aggregate supply of production factors and marginal costs of production, that is at the heart of the result obtained by Syrquin and Hollender. Using the concepts developed by Shephard (1953), I show that duality theory can perfectly match the increasing returns case without destroying symmetry. Moreover, the proposed definition is better suited to measure the responsiveness of factor prices to changes in factor supplies in the presence of non-competitive output markets. In addition, the measure generalises without problem to a multi-output setting.

The remaining part of this paper is organised as follows. Section 2 discusses the standard definition and Section 3 discusses the applicability of the elasticity of complementarity to a setting of imperfect competition. Section 3.1 then introduces the new definition and shows how this applies to the problem at hand. Section 4 shows how the new definition relates to the old one, and Section 5 applies the new definition to the problem of finding the effect of factor supplies on factor price responses when increasing returns are present. Section 6 offers some conclusions.

2 The Hicksian elasticities of substitution and complementarity

For the sake of comparison, I first present the standard definitions. Let $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$, $y \in \mathbb{R}$ and let the production function be given by

$$y = f(x) \tag{1}$$

It is assumed that f is strictly quasi-concave and twice differentiable. The dual cost function is defined as

$$C(y,p) = \min_x \{p'x;\ f(x) = y\} \tag{2}$$

The cost-minimizing inputs x are obtained by application of Shephard's lemma

$$x_i = \frac{\partial C}{\partial p_i} \Leftrightarrow$$
$$\frac{\partial \ln x_i}{\partial \ln p_j} = s_j \sigma_{ij} \tag{3}$$

where $s_i = p_i x_i / C$ denotes the cost share of factor i and where the partial elasticity of substitution σ_{ij} is defined as

$$\sigma_{ij} = \frac{C_{ij} C}{C_i C_j} \tag{4}$$

C_{ij} denotes the partial derivative of C with respect to p_i and p_j.

The elasticities of complementarity on the other hand are associated with the response of factor prices to changes in the factor supplies, *keeping marginal cost constant*. In fact, from (2) we find the first-order condition $p_i = \lambda f_i$, where the Lagrange multiplier λ equals marginal cost C_y and

$$\frac{\partial p_i}{\partial x_j} = \frac{p_i}{f_i} f_{ij} \quad \Rightarrow$$
$$\frac{\partial \ln p_i}{\partial \ln x_j} = \theta_j a_{ij} = \xi s_j a_{ij} \tag{5}$$

where $\theta_i = f_i x_i / f$ is the output elasticity of x_i, $\xi = \sum_i \theta_i$ is the scale elasticity (not necessarily constant), and $s_i = p_i x_i / C$ is the cost share. a_{ij} is the partial elasticity of complementarity, defined by Sato and Koizumi (1973) as

$$a_{ij} = \frac{f_{ij} f}{f_i f_j} \tag{6}$$

Comparing (5) with (3), two differences are noticeable: the p_i are a function of x and λ, not x and y, and (3) and (5) are not symmetric for $\xi \neq 1$.

This lack of symmetry also exists in the relation between σ_{ij} and a_{ij}. Define

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \tag{7}$$

then Syrquin and Hollender (1982) prove that

$$\sigma_{ij} = \frac{\xi}{\theta_i \theta_j} \frac{\operatorname{adj}(\mathbf{A})_{ij}}{|\mathbf{A}|} = \frac{1}{\xi} \frac{1}{s_i} \frac{1}{s_j} \frac{\operatorname{adj}(\mathbf{A})_{ij}}{|\mathbf{A}|} \tag{8}$$

where adj$(\mathbf{A})$ denotes the transpose of the matrix of cofactors of $\mathbf{A}$. They also prove a converse relation. Define

$$\Sigma_\delta = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} & \delta_1 \\ \vdots & & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} & \delta_n \\ \delta_1 & \cdots & \delta_n & 0 \end{pmatrix} \tag{9}$$

where $\delta_i = \frac{yC_{y_i}}{C_i} = \frac{\partial \ln x_i}{\partial \ln y}$. Syrquin and Hollender show that

$$a_{ij} = \frac{\text{adj}(\Sigma_\delta)_{ij}}{|\Sigma_\delta|} \frac{1}{s_i} \frac{1}{s_j} \frac{1}{\xi} - \frac{\partial \ln \lambda}{\partial \ln y} \tag{10}$$

Comparing (8) and (10), we see that these relations are not strictly dual if $\xi \neq 1$.

It is easy to check that whereas, for homothetic production functions, the elasticity of substitution is invariant to the scale elasticity, this does not hold for the elasticity of complementarity. In particular, for given elasticities of substitution and given budget shares, the elasticities of complementarity fall to zero for large scale elasticities, and vice versa for the elasticities of substitution in case of constant elasticities of complementarity. The two elasticity concepts therefore do not really provide a dual characterization of the curvature properties of the same production technology.

3 Increasing returns

The suitability of any definition of the elasticity of complementarity depends on its usefulness in answering a given set of questions. Intuitively, elasticities of complementarity provide a measure for the responsiveness of factor prices to changes in factor supplies. In competitive equilibrium marginal costs are given exogenously for the individual firm and marginal productivity conditions suffice to define equilibrium for the production side of the economy. With increasing returns however market equilibrium cannot be defined on the basis of the marginal productivity conditions only. An increase in factor supplies leads to an increase in production and a fall in average costs of firms. As a result, profits increase. The zero profit condition therefore becomes an independent equilibrium condition, maintained through entry and exit of firms. This implies that the production level of individual firms is not necessarily proportional with that of the industry as a whole. Since marginal costs are defined at the level of the individual firm, constancy of marginal costs at the firm level is no longer a defining characteristic of the effect on marginal productivity of a change of factor supply at the industry level.

To elaborate this point, consider an industry equilibrium, based on a cost function $C(y_i, p)$, where i is the firm index, $i = 1, \ldots, m$. We assume that all firms are identical and that the equilibrium is symmetric. In that case, the equilibrium can be written as

$$x_j = m \frac{\partial C(\bar{y}, p)}{\partial p_j}, \quad j = 1, \ldots, n \tag{11a}$$

$$p_y = M \frac{\partial C(\bar{y}, p)}{\partial \bar{y}} \tag{11b}$$

$$p_y \bar{y} = C(\bar{y}, p) \tag{11c}$$

where M denotes the mark-up, and $\bar{y}$ is the output per firm. Equilibrium on the factor markets is given by (11a), and the optimal supply of output is given by (11b). (11c) is the zero profit condition, which determines the number of firms. Because of homogeneity, the system (11a)-(11c) only determines relative prices. We can therefore rewrite the system by using *normalised prices*, $\pi = p/C(\bar{y}, p)$, $\pi_y = p_y/C(\bar{y}, p)$ (Samuelson (1947)):

$$m \frac{\partial C(\bar{y}, \pi)}{\partial \pi} = x \tag{12a}$$

$$C(\bar{y}, \pi) = 1 \tag{12b}$$

$$M \frac{\partial \ln C(\bar{y}, \pi)}{\partial \ln \bar{y}} = 1 \tag{12c}$$

(12b) defines the normalisation, and (12c) is the familiar tangency condition, that can be obtained from (11b) and (11c). This system determines $(\pi, \bar{y}, m)$, provided that $\frac{\partial^2 \ln C}{\partial y^2}$ is not zero.[1] Given $\bar{y}$, the normalized output price follows from the zero profit condition as $\pi_y = 1/\bar{y}$. It follows that production per firm, total production $m\bar{y}$, and normalised prices are determined without reference to the level of demand for the output of the industry. The output market is needed to determine price levels, however. Inverting the demand curve for industry output yields the output price $p_y = d^{-1}(m\bar{y})$, from which the level of factor prices follows. Considering (11b), marginal production costs are constant only if the industry demand curve is perfectly elastic. A perfectly elastic demand curve for output is however at odds with increasing returns in production, as it would lead to infinite expansion of the industry.

In an equilibrium with increasing returns an increase in the supply of a production factor therefore not only changes factor prices, but also induces entry of new firms. Hence, in deviation from Syrquin and Hollender (1982), marginal costs do not depend on industry output, but on the output level of the representative firm. This implies an effect of factor supply on production costs different from that implied by (5). The most obvious case is a proportional increase in all production factors, which only results in new firms entering the industry, without altering marginal costs. It follows that constancy of marginal costs is not a good assumption outside of perfect competition and should be replaced by the tangency condition (12c), which has greater generality. The next section discusses how the tangency condition fits in with a symmetric definition of the elasticity of complementarity based on the duality of cost functions and distance functions.

3.1 Quantity elasticities

To analyse the effects of factor supplies on factor prices in terms of the production function I use a distance function formulation. The distance function carries a number of advantages to production functions. It generalises naturally to multi-output production structures (Laitinen (1980)). In addition, the duality relations between the distance function and the cost function are strictly symmetric (compare Deaton (1979)).

[1] If the production structure is characterised by constant marginal cost, $\frac{\partial \ln C}{\partial \ln y} = 1/\xi$, where ξ is the returns to scale parameter, the tangency condition can determine the number of firms only if production changes affect the size of the markup. Generally, this case leads to a badly conditioned model, which has been dubbed a *fragile equilibrium* by Blanchard and Summers (1988).

Let $x \in \mathbb{R}^n, p \in \mathbb{R}^n, y \in \mathbb{R}^k$, and let T be the set of feasible input-output pairs (x,y). For $k = 1, T = \{(x,y); f(x) \geq y\}$, where f is a conventional production function, as in section 2 above. For multiple outputs, a convenient characterization of T is the (input) *distance function* g:

$$g(y,x) = \sup_{\theta}\{\theta > 0;\ (x/\theta, y) \in T\} \tag{13}$$

$g : \mathbb{R}^{k+n} \to \mathbb{R}$. Note that for $k = 1$, f and g are linked by $f(x) = y \Leftrightarrow g(y,x) = 1$ or $f(x/g(y,x)) = y$. *E.g.*, if f is homothetic, $f(x) = z(h(x))$, where h is linear homogeneous and $z' > 0$, we find $g(y,x) = h(x)z^{-1}(y)$. It is assumed that g is linear homogeneous in x, increasing and concave in x, and decreasing in y.

Associated with this distance function is a cost function $C(y,p)$, given by

$$C(y,p) = \min_{x}\left[p'x;\ g(y,x) = 1\right] \tag{14}$$

C satisfies the same homogeneity and concavity properties in p as g does in x. The distance function is also the dual of the cost function (Shephard (1953)):

$$g(y,x) = \min_{\pi}\left[\pi'x;\ C(y,\pi) = 1\right] \tag{15}$$

The dual definition of the distance function uses the normalised prices π discussed in Section 3 above. This formulation of the relation between the cost structure and the production structure is perfectly symmetric.

We now set up the equilibrium conditions of Section 3 in terms of the distance function. From (14), we obtain the first-order condition $\lambda g_x = p$, where λ is a Lagrange multiplier. Multiplying by $\overline{x}'$ and using $\overline{x}'g_x = g = 1$ gives $\lambda = p'\overline{x}$, where $\overline{x} = x/m$, the input of production factors per firm. Hence $g_x = p/p'\overline{x} = \pi$. The first-order equation for output is $M^{-1}p_y + \lambda g_y = 0$. Combined with the zero-profit condition $p_y\overline{y} - p'\overline{x} = 0$ this gives $M g_y \overline{y} = -1$. Now, because $g(y,x)$ is homogeneous of degree one in x, g_x is homogenous of degree zero in x, so $g_x(\overline{y},\overline{x}) = g_x(\overline{y},x)$. Furthermore, $g_y(\overline{y},\overline{x})$ is homogeneous of degree one in $\overline{x}$ as well, so $g_y(\overline{y},\overline{x})\overline{y} = \frac{g_y(\overline{y},\overline{x})}{g(\overline{y},\overline{x})}\overline{y} = \frac{\partial \ln g(\overline{y},x)}{\partial \ln \overline{y}}$. In terms of the distance function, the canonical form corresponding to (12a)-(12c) is therefore

$$\frac{\partial g(\overline{y},x)}{\partial x} = \pi \tag{16a}$$

$$g(\overline{y},x/m) = 1 \tag{16b}$$

$$M\frac{\partial \ln g(\overline{y},x)}{\partial \ln \overline{y}} = -1 \tag{16c}$$

The production level per firm, $\overline{y}$, is determined solely by the tangency condition (16c) and aggregate factor supply x (provided that the markup is constant). Given production per firm, the number of firms follows from the production constraint (16b). In terms of the distance function the marginal productivity conditions in (16a) therefore directly determine the normalised factor prices. Actual market prices are found from $p_y = f^{-1}(m\overline{y})$ and $p = \pi\, p'\overline{x} = \pi\, p_y\overline{y}$.

We obtain from this system:

$$\frac{\partial \pi_i}{\partial x_j} = \frac{\partial^2 g}{\partial x_i \partial x_j}$$

$$\frac{\partial \ln \pi_i}{\partial \ln x_j} = \frac{x_j}{\pi_i}\frac{g/m\, g_{ij}}{g_i g_j}\pi_i \pi_j = s_j \gamma_{ij} \tag{17}$$

were s_j is the cost share of factor j, $s_j = \frac{\pi_j(x_j/m)}{C(\bar{y},\pi)}$, and

$$\gamma_{ij} = \frac{g g_{ij}}{g_i g_j} \tag{18}$$

This result shows that it may be a good idea to define the γ_{ij} as the elasticities of complementarity between x_i and x_j. In this definition, the elasticities of complementarity are a measure of the responsiveness of the *normalised* factor prices to a change in factor supplies. As in Section 3, to deduce the effect of factor supplies on market prices, we must also take the output market into account. Market prices differ from normalised prices by a factor $p_y/\pi_y = f^{-1}(m\bar{y})\bar{y}$. That is, the market-clearing change in factor prices generally depends not just on technological constraints, but also on product market conditions. It is only in the special case of constant returns and infinitely elastic product demand that information about the technology suffices to determine the factor price response. In that case the Hicksian definition coincides with the present one.

In the general case, we obtain for the factor price change

$$\begin{aligned}\frac{\partial \ln p_i}{\partial \ln x_j} &= \frac{\partial \ln \pi_i}{\partial \ln x_j} + \frac{\partial \ln p_y}{\partial \ln y}\left(\frac{\partial \ln m}{\partial \ln x_j} + \frac{\partial \ln \bar{y}}{\partial \ln x_j}\right) + \frac{\partial \ln \bar{y}}{\partial \ln x_j} \\ &= s_j \gamma_{ij} - \frac{M-1}{M}\frac{\partial g(\bar{y},x)}{\partial x_j}\frac{x_j}{m} + \frac{1}{M}\frac{\partial \ln \bar{y}}{\partial \ln x_j} \\ &= s_j \gamma_{ij} - \frac{M-1}{M} s_j - \frac{1}{M}\,\frac{\partial^2 \ln g}{\partial \ln \bar{y}\,\partial \ln x_j} \Bigg/ \frac{\partial^2 \ln g}{(\partial \ln \bar{y})^2}\end{aligned} \tag{19}$$

(19) decomposes the effect of a change in factor supplies in three terms. The first term on the right is the elasticity of complementarity, the effect of factor supply on the normalised prices. The second term is the effect of factor supplies on factor prices through entry of new firms. The ensuing expansion of industry output causes a fall in the output price, the size of which depends on the price elasticity. The falling output price also lowers factor prices, so that the entry effect is negative. In a competitive output market $M = 1$ and the entry effect disappears. The last term on the right hand side of (19) is the firm size effect. The size of this effect depends on the curvature of the log of the distance function at the tangency point specified by (16c).

The sign of the firm size effect is theoretically ambiguous. To see this, consider a distance function of the form

$$\ln g(y,x) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i - \frac{1}{\xi}\ln\left(y + y_0\right) + \sum_{i=1}^{n} \beta_i \ln x_i \ln y \tag{20}$$

where $\sum_{i=1}^{n} \alpha_i = 1$, $\sum_{i=1}^{n} \beta_i = 0$. This production structure contains both decreasing marginal costs and fixed costs. It satisfies the curvature conditions for $(\beta_1, \ldots, \beta_n)' = \mathbf{0}$. Since $\frac{\partial^2 \ln g}{\partial \ln y \partial \ln x_j} = \beta_j$, we can choose a pair of β_i's that differ slightly from zero without violating the curvature conditions, to obtain any desired sign of the firm size effect of a factor change.

It is useful to compare (19) with (5). With constant returns and perfect competition $\gamma_{ij} = a_{ij}$. In (5), $\xi > 1$ results in a quantity elasticity *larger* than a_{ij}, because the requirement of constant marginal costs demands a larger price response, given that the increase in production lowers marginal costs as a result of increasing returns. (19) on the other hand yields a quantity elasticity *smaller* than γ_{ij}, provided that output per firm $\bar{y}$ is constant, because the increase in factor supply induces the entry of new firms, and a downward movement along the industry demand curve.

4 The relation between complements and substitutes

The relation between the elasticities of substitution and complementarity, as defined in this paper, can be derived in a simple way by noting that

$$\begin{aligned}\bar{x} &= C_\pi(\bar{y},\pi) = C_\pi(\bar{y}, g_x(\bar{y},\bar{x}))\\ \pi &= g_x(\bar{y},\bar{x}) = g_x(\bar{y}, C_\pi(y,\pi))\end{aligned}$$

Differentiating with respect to $\bar{x}$, respectively π and using the chain rule gives

$$\mathbf{I} = C_{\pi\pi} g_{xx} \tag{21a}$$

$$\mathbf{I} = g_{xx} C_{\pi\pi} \tag{21b}$$

Hence, the matrices $C_{\pi\pi}$ and g_{xx} are Moore-Penrose inverses. We can link these inverses to the bordered Hessians used by Sato and Koizumi (1973) and Syrquin and Hollender (1982) by noting that $g_x = \pi$ is an eigenvector of $C_{\pi\pi}$ for the eigenvalue zero, and $C_\pi = \bar{x}$ is an eigenvector of g_{xx} for the eigenvalue zero. Also, $C_{\pi'} g_x = \bar{x}' g_x = g(\bar{y},\bar{x}) = 1$. Hence

$$\begin{pmatrix} C_{\pi\pi} & C_\pi \\ C_{\pi'} & 0 \end{pmatrix} \begin{pmatrix} g_{xx} & g_x \\ g_{x'} & 0 \end{pmatrix} = \mathbf{I} \tag{22}$$

Furthermore, $C_{\pi'}\pi = C(\bar{y},\pi) = 1 \neq 0$ and analogously $g_{x'} x \neq 0$. Hence both bordered Hessians have full rank. The correspondence can also be formulated in terms of the elasticities of substitution and complementarity, by transforming (22) as

$$\begin{pmatrix} C_{\pi_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & C_{\pi_n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} C_{\pi_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & C_{\pi_n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} g_{x_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & g_{x_n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{\Gamma} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} g_{x_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & g_{x_n} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \mathbf{I} \tag{23}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}, \quad \mathbf{\Gamma} = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{pmatrix}$$

Note that $C_{\pi_i} g_{x_i} = s_i$, the cost share of factor i in production. Premultiplying the left-hand side of (23) with $\mathrm{diag}(g_{x_1},\ldots,g_{x_n},1)$, and postmultiplying it with $\mathrm{diag}(g_{x_1},\ldots,g_{x_n},1)^{-1}$, it follows that

$$\begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & s_n & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{\Sigma} & \iota \\ \iota' & 0 \end{pmatrix} \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & s_n & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma} & \iota \\ \iota' & 0 \end{pmatrix}^{-1} \tag{24}$$

Comparing the relation between σ_{ij} and a_{ij} in (10) in Section 2 to the relation between σ_{ij} and γ_{ij} in (24), we see that now the relation between the elasticities of complementarity and substitution is completely symmetric, both in terms of the definitions and in terms of the relation that exists between the two concepts. Furthermore, the scale elasticity no longer affects the relationship between these elasticities. An additional advantage of the proposed formulation is that it applies equally to multi-output production structures.

5 Examples

Example 1 A CES distance function

$$g(y,x) = F(x_1,x_2)\left(y+y_0\right)^{-1/\xi} \tag{25}$$
$$F(x_1,x_2) = \left[\theta_1 x_1^{1-1/\sigma} + \theta_2 x_2^{1-1/\sigma}\right]^{1/(1-1/\sigma)}$$

The elasticity of complementarity according to definition (18) is simply $\gamma_{12} = 1/\sigma$. In industry equilibrium, the optimal firm size is $\bar{y}/\left(\bar{y}+y_0\right) = \xi/M$. According to (19), the factor price response to a change in factor supply is therefore

$$\frac{\partial \ln p_i}{\partial \ln x_j} = \left(\gamma_{12} - \frac{M-1}{M}\right) s_j \tag{26}$$

Applying Sato and Koizumi's definition (5) gives $a_{ij} = \frac{1}{\xi}\frac{\bar{y}}{\bar{y}+y_0}\left(\xi - 1 + \frac{1}{\sigma}\right)$. In industry equilibrium the tangency condition requires that $\frac{\bar{y}}{\bar{y}+y_0} = 1/M$. Applying the Sato-Koizumi definition to this case therefore yields the following predicted effect of a change in factor supply:

$$\frac{\partial \ln p_i}{\partial \ln x_j} = a_{ij} s_j = \frac{1}{M}\left(\xi - 1 + \frac{1}{\sigma}\right) s_j$$

This result is not generally consistent with the theoretically correct effect given in (26). E.g., the formula suggests that factor prices do not respond to supply conditions for $M \to \infty$. The problem arises from the incorrect assumption that the output level of individual firms changes in response to the supply change. This indicates that the Sato-Koizumi definition is not useful outside of competitive equilibrium.

Example 2 A general translog distance function

$$\ln g(y,x) = \alpha_0 + \sum_{i=1}^{m} \alpha_i \ln x_i + \tfrac{1}{2}\sum_{i=1}^{m} \gamma_{ij} \ln x_i \ln x_j + \sum_{i=1}^{m} \beta_i \ln x_i \ln y + \gamma_1 \ln y + \tfrac{1}{2}\gamma_2 \ln^2 y \tag{27}$$

The tangency condition yields the following equilibrium size for the firm

$$\sum_{i=1}^{m} \beta_i \ln \bar{x}_i + \gamma_1 + \gamma_2 \ln \bar{y} = -1/M \tag{28}$$

This results in the following decomposition of the effect of factor supplies on factor prices

$$\frac{\partial \ln p_i}{\partial \ln x_j} = \underbrace{s_j \gamma_{ij}}_{\text{complementarity effect}} - \underbrace{\frac{M-1}{M} s_j}_{\text{industry size effect}} - \underbrace{\frac{1}{M}\beta_j/\gamma_2}_{\text{firm size effect}}$$

We see that for a translog distance functions the effect of factor supplies on factor prices can be decomposed in a proper complementarity effect, a negative industry size effect that depends on the elasticity of output demand, and a firm size effect that depends on the economies of scale. The Hicks-Sato-Koizumi definition (6) on the other hand combines all these effects into a single elasticity.

6 Conclusion

This paper shows that the conventional definition of the elasticity of complementarity is not suited to deal with technologies that are characterized by increasing returns to scale. With increasing returns, output markets cannot be perfectly competitive. The conventional definition assumes that marginal costs are constant in response to a change in factor supplies. This assumption is only appropriate in case of perfectly competition. With imperfectly competitive output markets the assumption should be replaced with the tangency condition. This leads to a slightly different definition, that may conveniently be cast in terms of a distance function instead of a production function. The definition proposed in this paper coincides with the Hicksian measure in case the production function displays constant returns. It is better suited for cases where returns to scale are not constant, as it disentangles entry and exit effects and substitution effects of factor supplies. In addition the new definition maintains strict symmetry in relation to the elasticity of substitution and is fully applicable to a multi-output setting.

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