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Does Risk Seeking drive Asset Prices?

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A Stochastic Dominance Analysis of Aggregate Investor Preferences

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We investigate whether risk seeking or non-concave utility functions can help to explain the cross-sectional pattern of stock returns. For this purpose, we analyze the stochastic dominance efficiency classification of the value-weighted market portfolio relative to benchmark portfolios based on market capitalization, book-to-market equity ratio and momentum. We use various existing and novel stochastic dominance criteria that account for the possibility that investors exhibit local risk seeking behavior. Our results suggest that Markowitz type utility functions, with risk aversion for losses and risk seeking for gains, can capture the cross-sectional pattern of stock returns. The low average yield on big caps, growth stocks and past losers may reflect investors' twin desire for downside protection in bear markets and upside potential in bull markets.

THE TRADITIONAL MEAN-VARIANCE CAPITAL ASSET PRICING MODEL (MV CAPM) by Sharpe (1964) and Lintner (1965) fares poorly in explaining observed cross-sectional stock returns. Specifically, market beta seems to explain only a small portion of the cross-sectional variation in average returns, while factors like market capitalization (size), book-to-market equity ratio (BE/ME) and momentum systematically appear to affect asset prices (see e.g. Fama and French, 1992, and Jagadeesh and Titman, 1993). Related to this, the value-weighted market portfolio of risky assets seems highly mean-variance inefficient, and it is possible to achieve a substantially higher mean and/or a substantially lower variance with portfolios with a higher weight of small caps, value stocks (high BE/ME stocks) and past winners.

One way to extend the MV CAPM is by changing the maintained assumptions on investor preferences. If we do not restrict the shape of the return distribution, then MV CAPM is consistent with expected utility theory only if utility takes a quadratic form. (Less restrictive assumptions are obtained if we do restrict the shape of the return distribution; see e.g. Berk (1997)). Extensions of the MV CAPM can be obtained by using alternative classes of utility. For example, Friend and Westerfield (1980) and Harvey and Siddique (2000) assume that utility can be approximated using a third-order polynomial, and Dittmar (2002) uses a fourth-order polynomial.[2] The higher-order polynomials better fit stock return data than the standard quadratic utility functions do. Still, the market portfolio remains inefficient and size, value and momentum effects remain. Two potential problems of the extended models may help to explain this result:

1. *Risk seeking.* The extended models typically maintain the assumption that investors are globally risk averse and that utility is everywhere concave (i.e. marginal utility is diminishing). However, there is evidence that decision-makers are not globally risk averse, but rather they exhibit local risk seeking behavior (i.e. the utility function has convex segments). For example, Friedman and Savage (1948) and Markowitz (1952) argue that the willingness to purchase both insurance and lottery tickets implies that marginal utility is increasing over a range (see Hartley and Farrell, 2001, for a recent discussion). Similarly, active stock traders seem to play negative-sum games and their behavior is sometimes best described as 'gambling' (see e.g. Statman, 2002). In addition, psychologists led by Kahneman and Tversky (1979) find experimental evidence for local risk seeking behavior.

[2] Similarly, Bansal and Viswanathan (1993) and Chapman (1997) use polynomial approximations in the context of Arbitrage Pricing Theory and consumption-based CAPM respectively.

2. *Specification error.* A difficulty in changing the preference assumptions is the need to give a parametric specification of the functional form of the utility function or to specify the maximum order of the approximating polynomial. Unfortunately, economic theory gives minimal guidance for functional specification, and there is a substantial risk of specification error. For example, the fourth-order polynomial used in Dittmar (2002) implies that investors care only about the first four central moments of the return distribution (mean, variance, skewness and kurtosis). This approach is problematic if investors care about the higher central moments or about lower partial moments (see e.g. Bawa and Lindenberg, 1977), which generally cannot be expressed in terms of the first four central moments. Another problem associated with low order polynomials is the difficulty to impose restrictions on the derivatives that apply globally. For example, we cannot impose non-satiation by restricting a quadratic polynomial to be monotone increasing and we cannot impose risk aversion by restricting a cubic polynomial to be globally concave (see e.g. Levy, 1969).

To circumvent these problems, we may use criteria of Stochastic Dominance (SD; see e.g. Levy, 1992, 1998). Attractively, SD criteria do not require a parameterized utility function, but rather they rely only on general preference assumptions.[3] Put differently, SD effectively considers the full house of all moments of the return distribution rather than a finite set. The SD literature involves a multitude of different criteria, associated with different sets of preference assumptions. The traditional First-order Stochastic Dominance (FSD) criterion assumes only non-satiation i.e. utility is monotone increasing. The Second-order Stochastic Dominance (SSD) criterion adds the assumption of global risk-aversion. Recently, a number of intermediate criteria have been developed based on non-concave utility functions. Most notably, Levy (1998) developed Prospect Stochastic Dominance (PSD), which assumes a S-shaped utility function that is convex for losses and concave for gains. In addition, Levy and Levy (2002) developed Markowitz Stochastic Dominance (MSD), which assumes a reverse S-shaped utility function that is concave for losses and convex for gains.

In this paper, we use various existing and novel SD criteria to analyze asset pricing. To focus on the role of preference assumptions, we largely adhere to the remaining assumptions of the MV CAPM: we use a single-period, portfolio-oriented model of a frictionless and competitive capital market with a large number of expected utility investors. Within this model, we test whether the value-weighted market portfolio is efficient relative to benchmark portfolios formed on size, BE/ME and momentum. For this purpose, we assume a simple data generating process with a serially independent and identical distribution for the excess returns. Of course, there are good reasons to doubt our maintained assumptions, and to believe that our results are affected by these assumptions in a non-trivial way. Still, we believe that our approach is useful, as we have to 'walk before we can run', and the analysis can form

[3] For the sake of analytical simplicity, we phrase in terms of expected utility theory. However, SD rules are economically meaningful also for many non-expected utility theories that account for e.g. subjective probability distortion (see e.g. Starmer, 2002). For example, it is easily verified that the MSD efficiency criterion is not affected by subjective transformations of the CDF that are increasing and concave over losses and increasing and convex over gains, and hence MSD allows for subjective overweighing of small probabilities of large gains and losses and underweighing of small and intermediate probabilities of small and intermediate gains and losses. Interestingly, empirical studies suggest that this reverse S-shape is the most common pattern of probability transformation (see e.g. Tversky and Kahneman, 1992).

the starting point for further research based on more general economic and statistical assumptions.

The remainder of this paper is structured as follows. Section I introduces a general SD efficiency criterion and it discusses the special cases used in our study. Section II discusses the issue of empirical testing. Specifically, we develop a general Linear Programming (LP) test for fitting SD efficiency criteria to empirical data and we derive the asymptotic sampling distribution of the test results. This section effectively generalizes Post's (2001) treatment of SSD efficiency towards our general SD efficiency criterion. Section III presents the empirical application. Finally, Section IV gives concluding remarks and suggestions for further research. The Appendix gives the formal proofs for our theorems.

I. STOCHASTIC DOMINANCE EFFICIENCY CRITERIA

We consider a single-period, portfolio-based model of a competitive capital market that satisfies the following assumptions:

Assumption 1 The investment universe consists of N assets, one of which is a riskless asset. Throughout the text, we index the assets by $\mathrm{I} \equiv \{i\}_{i=1}^{N}$. The excess returns $\boldsymbol{x} \in \Re^N$ are treated as random variables with a continuous joint cumulative distribution function (CDF) $G : \Re^N \rightarrow [0,1]$.[4] Investors may diversify between the assets, and we will use $\boldsymbol{l} \in \Re^N$ for a vector of portfolio weights. The portfolio possibilities are represented by the simplex $\Lambda \equiv \left\{\boldsymbol{l} \in \Re_+^N : \boldsymbol{e}^{\mathrm{T}}\boldsymbol{l} = 1\right\}$.[5]

Assumption 2 Investors select investment portfolios $\boldsymbol{l} \in \Lambda$ to maximize the expected value of a once directionally differentiable, strictly increasing utility function $u : \Re \rightarrow \Re$ that is defined over portfolio return $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{l}$.[6] We represent all admitted utility functions by $U_0 \equiv \{u : \partial u(x) \geq 1 \quad \forall x \in \Re\}$, with $\partial u(x) \equiv \lim_{l \downarrow 0} \dfrac{u(x+l) - u(x)}{l}$ for the directional derivative or 'marginal utility' at $x \in \Re$.[7]

We may characterize different SD efficiency criteria by different classes of utility functions, characterized by different sets of (conditional) linear restrictions on marginal utility. Formally, we will denote different classes of utility functions by

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[4] Throughout the text, we will use $\Re^N$ for an N-dimensional Euclidean space, and $\Re_+^N$ denotes the positive orthant. To distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars. Further, all vectors are column vectors and we use $\boldsymbol{x}^{\mathrm{T}}$ for the transpose of $\boldsymbol{x}$. Finally, $\boldsymbol{e}$ is a unity vector with dimensions conforming to the rules of matrix algebra.

[5] By using the simplex Λ, we exclude short selling. Short selling typically is difficult to implement in practice due to margin requirements and explicit or implicit restrictions on short selling for institutional investors (see e.g. Sharpe, 1991, and Wang, 1998). Still, we may generalize our analysis to include (bounded) short selling. In fact, the analysis applies for an arbitrary polytope if we replace I with the set of extreme points of the polytope.

[6] We use a directional derivative to allow for kinked utility functions, including piecewise-linear utility functions (see the proof to Theorem 1).

[7] U_0 restricts marginal utility to exceed unity. This restriction effectively standardizes the test statistic $x(t, \Psi)$ (see Section II) and forces it away from zero if the evaluated portfolio is inefficient. Still, the restriction is harmless because SD rules are invariant to strictly positive affine transformation i.e. $u \in U(\Psi) \Rightarrow bu \in U(\Psi)$ for all $b > 0$

$U(\Psi) \equiv \{u \in U_0 : \partial u(x) \geq \partial u(y) \quad \forall (x, y) \in \Psi_r \quad r = 1, \cdots, R\}$. Here, $\Psi \equiv \{\Psi_r\}_{r=1}^{R}$, with $\Psi_r \subseteq \Re^2$ for a polyhedron that represents the r-th restriction on marginal utility (special cases are given below). Using this notation, we may define the following general SD efficiency criterion:[8]

DEFINITION 1 *Portfolio* $t \in \Lambda$ *is* $U(\Psi)$*-SD efficient if and only if it is optimal relative to some utility functions* $u \in U(\Psi)$*, i.e.*

(1) $$\min_{u \in U(\Psi)} \left\{ \max_{l \in \Lambda} \left\{ \int u(\mathbf{x}^{\mathrm{T}} l)\partial G(\mathbf{x}) - \int u(\mathbf{x}^{\mathrm{T}} t)\partial G(\mathbf{x}) \right\} \right\} = 0 .$$

Assumption 3 The value-weighted market portfolio of risky assets, say $m \in \Lambda$, is $U(\Psi)$-SD efficient.

In asset pricing theories, efficiency of the market portfolio generally is not an *assumption* but rather a *prediction* that follows from underlying assumptions on the return distribution and investor preferences. For example, following Rubinstein (1974), efficiency of the market follows from assuming that the preferences of the different investors are sufficiently similar. In this case, we may use the utility function of a *representative agent* whose preferences are an aggregate of the preferences of the actual investors. In this paper, we do not take this route, because detailed distribution and preference assumptions are not consistent with the SD approach of using minimal assumptions. Rather, our analysis builds on *revealed preferences*. Specifically, our motivation for assuming market efficiency lies in the popularity of passive mutual funds and exchange traded funds that track broad value-weighted equity indexes. In other words, (some) investors reveal a preference for market indexes, and our objective is to rationalize their choice and to analyze their preferences.[9] Of course, we could directly analyze the efficiency of actual funds. Still, for the sake of data availability and comparability, we focus on the Fama and French market portfolio (see Section III), which is used in many comparable studies (e.g. Harvey and Siddique (2000) and Dittmar (2002)). Further, many actual funds, including total market index funds based on the very broad Wilshire 5000 index (e.g. the Vanguard Total Stock Market Index Fund) are likely to be very highly correlated with the Fama and French market portfolio.

Special Cases

[8] We focus on a definition in terms of utility functions, because we analyze the role of preference assumptions. SD may be defined equivalently in terms of distribution functions or their quantiles (see e.g. Levy, 1992, 1998). In fact, the LP dual of test statistic (9) formulates in terms of quantiles; associated with each primal variable β_t, $t \in \Theta$, is a dual restriction on the running mean $\sum_{s=1}^{t} \mathbf{x}_s^{\mathrm{T}} t / t$.

[9] If the investment universe includes a riskless asset, then a sufficient condition for Assumption 3 is that some investor holds a combination of the market portfolio and the riskless asset. Let $m' = mk + (1-k)d_F$ for some $k \in \langle 0,1]$, with d_F to denote a coordinate vector of zeros with a unity value for the riskless asset $F \in \mathrm{I}$. If m' is the optimal solution for $u \in U(\Psi)$, then m is optimal relative to $v(x) \equiv u(kx + (1-k)x_F) \in U(\Psi)$ and hence efficient.

In our empirical analysis we will use various different SD criteria based on different sets of preference assumptions. Table 1 summarizes the criteria, the assumptions and the associated restrictions on marginal utility, as represented by Ψ_r.

(Insert Table 1 about here)

The traditional criterion of *Second-order Stochastic Dominance* (SSD) assumes risk aversion for the entire domain of returns, or equivalently risk aversion for losses, risk aversion for gains and 'loss aversion' (marginal utility of losses exceeds marginal utility for gains):

(2) $\Psi_{SSD} \equiv \{\Psi_1, \Psi_2, \Psi_4\}$.[10]

Models of decision-making and equilibrium under uncertainty traditionally use utility functions from $U(\Psi_{SSD})$. However, as discussed in the Introduction, there is compelling evidence that many decision-makers are risk seeking over a range.

The psychological experiments by Kahneman and Tversky (1979) and Kahneman and Tversky (1992) suggest that preferences are best described by a S-shaped function that is convex for losses and concave for gains, and that is steeper for losses than for gains ('loss aversion'). In recent years, Prospect Theory has attracted much attention as a framework for understanding investor behavior and for explaining financial market anomalies (see e.g. Benartzi and Thaler (1995), Barberis *et al.* (2001), and Barberis and Huang (2001)). In this study, we consider two SD criteria that are based on Prospect Theory.[11] *Prospect Stochastic Dominance with Loss Aversion* (PSDL) assumes a S-shaped utility function with risk seeking for losses, risk aversion for gains and loss aversion:

(3) $\Psi_{PSDL} \equiv \{\Psi_1, \Psi_3, \Psi_4\}$.

If we drop loss aversion, then we obtain *Prospect Stochastic Dominance* (PSD; Levy, 1998):

[10] Gains and losses are typically measured relative to a subjective reference point. For simplicity, we set the reference point at zero. The use of excess returns implies that the reference point that distinguishes gains from losses effectively equals the riskless rate. However, a follow-up analysis demonstrates that the empirical results are not significantly affected by using a lower reference point of zero or a higher reference point the average return on the market portfolio. If the reference point is not known, it can be included as an additional model variable (at the cost of a possible loss of power).

[11] In contrast to Expected Utility Theory, Prospect Theory uses value functions with subjective decision weights that overweight or underweight the true probabilities. Levy and Wiener (1998) demonstrate that the PSD efficiency criterion is not affected by subjective transformations of the CDF that are increasing and convex over losses and increasing and concave over gains, and hence PSD allows for subjective underweighing of small probabilities of large gains and losses and overweighing of small and intermediate probabilities of small and intermediate gains and losses. However, this pattern is counterfactual (see e.g. Footnote 3) and rejection of PSD efficiency may mean rejection of S-shaped probability transformations rather than S-shaped preferences. Still, there are good reasons to expect that subjective probability distortion is less severe for investment choices than for some other choices. For example, investors can extract information about the return distribution from historical return data and from fundamental economic data. In addition, if large amounts of money are at stake, then there is a large incentive to gather and process such data so as to eliminate subjective probability distortions.

(4) $\Psi_{PSD} \equiv \{\Psi_3, \Psi_4\}$.

Recent experiments by Levy and Levy (2002) suggest that the Kahneman and Tversky experiments may be biased by the research design. Specifically, bias may originate from framing effects (unlike actual investments, the prospects involve either only positive or only negative outcomes, but no mixed outcomes) and subjective probability distortion (the prospects involve extremely small and large probabilities). After correcting for these sources of bias, Levy and Levy find evidence *against* Kahneman and Tversky type preferences; a large majority of subjects in their experiments select PSD *in*efficient prospects. In fact, the Levy and Levy results support a *reverse* S-shaped utility function with risk aversion for losses and risk seeking for gains, i.e. exactly the opposite of Kahneman and Tversky type preferences. Interestingly, Markowitz (1952) already suggested this type of utility function.[12] In this study, we consider two SD criteria that build on Markowitz type preferences. *Markowitz Stochastic Dominance with Loss Aversion* (MSDL) assumes a reverse S-shaped utility function with risk aversion for losses, risk seeking for gains and loss aversion:

(5) $\Psi_{MSDL} \equiv \{\Psi_1, \Psi_2, \Psi_5\}$.

If we drop loss aversion, then we obtain *Markowitz Stochastic Dominance* (MSD; Levy and Levy, 2002):

(6) $\Psi_{MSD} \equiv \{\Psi_2, \Psi_5\}$.

Figure 1 shows examples of non-concave utility functions that satisfy the assumptions of PSDL, PSD, MSDL and MSD.

(Insert Figure 1 about here)

II. EMPIRICAL TESTING

In practical applications, the CDF generally is not known. Rather, information typically is limited to a discrete set of time series observations, say $C \equiv (\boldsymbol{x}_1 \cdots \boldsymbol{x}_T)$ with $\boldsymbol{x}_t \equiv (x_{1t} \cdots x_{Nt})^{\mathrm{T}}$, and indexed by $\Theta \equiv \{t\}_{t=1}^{T}$.

Assumption 4 The observations are independent random draws from the CDF.[13]

[12] Markowitz (1952) actually proposed a utility function with two S-shaped segments. Such a utility function has a convex segment for 'large' losses and a convex segment for 'small and moderate' gains. Hence, MSD is consistent with Markowitz type utility only if all outcomes are 'small or moderate' gains or losses. This assumption seems reasonable for our application, because we compare well-diversified benchmark portfolios; for the full sample period (July 1963 – October 2001), the minimum monthly excess return is -34.32 percent and the maximum excess return is 38.82 percent (see Table 2A-B).

[13] There are compelling theoretical and empirical arguments in favor of a time varying return distribution. Unfortunately, the search for a satisfactory specification of the return dynamics is still far from accomplished. In fact, Ghysels (1998) finds that ill-specified conditional asset pricing models in many cases yield greater pricing errors than unconditional models. For this reason, we use an unconditional model here. Still, further research could relax the IID assumption. One possible approach

Since, by assumption, the timing of the draws is inconsequential, we are free to label the observations by their ranking with respect to the evaluated portfolio, i.e. $\boldsymbol{x}_1^{\mathrm{T}}t < \boldsymbol{x}_2^{\mathrm{T}}t < \cdots < \boldsymbol{x}_T^{\mathrm{T}}t$.[14] Using the observations, we can construct the empirical distribution function (EDF):

(7) $\quad F(\boldsymbol{x}) \equiv \mathrm{card}\{t \in \Theta : \boldsymbol{x}_t \leq \boldsymbol{x}\}/T$.[15]

By using EDF $F(\boldsymbol{x})$ in place of CDF $G(\boldsymbol{x})$, we arrive at the following empirical definition of SD efficiency:

DEFINITION 2 *Portfolio* $t \in \Lambda$ *is empirically* $U(\Psi)$*-SD efficient if and only if*:

(8)
$$\min_{u \in U(\Psi)} \left\{ \max_{l \in \Lambda} \left\{ \int u(\boldsymbol{x}^{\mathrm{T}} l) \partial F(\boldsymbol{x}) - \int u(\boldsymbol{x}^{\mathrm{T}} t) \partial F(\boldsymbol{x}) \right\} \right\} =$$
$$\min_{u \in U(\Psi)} \left\{ \max_{l \in \Lambda} \left\{ \sum_{t \in \Theta} (u(\boldsymbol{x}_t^{\mathrm{T}} l) - u(\boldsymbol{x}_t^{\mathrm{T}} t)) / T \right\} \right\} = 0 .$$

Linear Programming Formulation

We will test for efficiency by using the following test statistic:

(9)
$$x(t, \Psi) \equiv \min_{b \in \mathrm{B}(\Psi), q} \left\{ q : \sum_{t \in \Theta} b_t (\boldsymbol{x}_t^{\mathrm{T}} t - x_{it}) / T + q \geq 0 \quad \forall i \in \mathrm{I} \right\},$$

with

(10)
$$\mathrm{B}(\Psi) \equiv \left\{ b \in [1, \infty\rangle^T : b_t \geq b_s \quad \forall t, s \in \Theta : (\boldsymbol{x}_t^{\mathrm{T}} t, \boldsymbol{x}_s^{\mathrm{T}} t) \in \Psi_r \quad r = 1, \cdots, R \right\}.$$

We can derive the following result for this test statistic:

THEOREM 1 *Portfolio* $t \in \Lambda$ *is empirically* $U(\Psi)$*-SD efficient if and only if* $x(t, \Psi) = 0$

The test statistic $x(t, \Psi)$ basically checks if there exists a vector $b \equiv (b_1 \cdots b_T)^{\mathrm{T}}$ that represents the gradient vector at $\boldsymbol{X}^{\mathrm{T}} t$ for a utility function $u \in U(\Psi)$ that satisfies the first-order condition for portfolio optimality (see the proof in the Appendix).[16]

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is to use econometric time series estimation techniques to estimate a conditional CDF. Our empirical tests can then be applied to random samples from the estimated CDF rather than the EDF.

[14] Since we assume a continuous return distribution, ties do not occur. Still, the analysis can be extended in a straightforward way to cases where ties do occur e.g. due to a discrete return distribution or due to measurement problems or rounding; see Post (2001).

[15] We use $\mathrm{card}\{\cdot\}$ for the cardinality or the number of elements of a set.

[16] The theorem extends Post's (2001) Theorem 2 for SSD, and it exploits the result that the necessary and sufficient first-order optimality condition applies not only for concave utility functions, but also for more general pseudoconcave utility functions (see e.g. Mangasarian, 1969).

The test statistic involves a linear objective function and a finite set of linear constraints. Hence, the test statistic can be solved using straightforward Linear Programming (LP). The model always has a feasible solution (e.g. $\boldsymbol{ß} = \boldsymbol{e}$) and the solution is bounded from below by zero (the case of efficiency) and bounded from above by $\max_{i \in \mathrm{I}} \sum_{t \in \Theta} (\boldsymbol{x}_{it} - \boldsymbol{x}_t^{\mathrm{T}} t) / T$ (the case with $\boldsymbol{ß} = \boldsymbol{e}$). For small data sets up to hundreds of observations and/or assets, the problem can be solved with minimal computational burden, even with desktop PCs and standard solver software (like LP solvers included in spreadsheets). Still, the computational complexity, as measured by the required number of arithmetic operations, and hence the run time and memory space requirement, increases progressively with the number of variables and restrictions. Therefore, specialized hardware and solver software is recommended for large-scale problems involving thousands of observations and/or assets.

The polyhedron $\mathrm{B}(\Psi)$ involves constraints on pairs of observations $s, t \in \Theta$. This suggests that the number of constraints increases quadratically with the number of observations. However, many of the constraints are redundant, and the effective number of constraints increases linearly with the number of observations. For example, using $z \in \mathrm{I}$ for the first observation in the domain of gains, i.e. $\boldsymbol{x}_{z-1}^{\mathrm{T}} t < 0 \leq \boldsymbol{x}_z^{\mathrm{T}} t$, it is easily verified that:

(11) $\mathrm{B}(\Psi_{SSD}) = \{ b \in \Re^T : b_1 \geq \cdots \geq b_T \geq 1 \}$;

(12) $\mathrm{B}(\Psi_{PSDL}) = \{ b \in \Re^T : b_T \leq b_1 \leq \cdots \leq b_{z-1}; b_z \geq \cdots \geq b_T \geq 1 \}$;

(13) $\mathrm{B}(\Psi_{PSD}) = \{ b \in \Re^T : 1 \leq b_1 \leq \cdots \leq b_{z-1}; b_z \geq \cdots \geq b_T \geq 1 \}$;

(14) $\mathrm{B}(\Psi_{MSDL}) = \{ b \in \Re^T : b_1 \geq \cdots \geq b_{z-1} \geq b_T ; 1 \leq b_z \leq \cdots \leq b_T \}$;

(15) $\mathrm{B}(\Psi_{MSD}) = \{ b \in \Re^T : b_1 \geq \cdots \geq b_{z-1} \geq 1 \leq b_z \leq \cdots \leq b_T \}$.

Statistical Inference

We have thus far discussed SD efficiency relative to the EDF $F(\boldsymbol{x})$ rather than the CDF $G(\boldsymbol{x})$. Generally, the EDF is very sensitive to sampling variation and the test results are likely to be affected by sampling error in a non-trivial way. The applied researcher must therefore have knowledge of the sampling distribution in order to make inferences about the true efficiency classification. The remainder of this section therefore develops an asymptotic hypothesis testing procedure for the test statistic $x(t, \Psi)$.[17] The test procedure is based on two simplifications: (1) the use of a restrictive null hypothesis and (2) the use of the limiting least favorable distribution. Our null (H_0) is that the assets have an equal and finite mean, i.e. $E[\boldsymbol{x}] = m\boldsymbol{e}$, $m < \infty$,

[17] The results extend Post's (2001) Theorem 3 for SSD, and they exploits the result that $x(t, \Psi)$ is bounded from above by $w(t) \equiv \max_{i \in \mathrm{I}} \left\{ \sum_{t \in \Theta} (x_{it} - \boldsymbol{x}_t^{\mathrm{T}} t) / T \right\}$ (see the proof in the Appendix). This argument applies not only for Ψ_{SSD}, but also more generally for all Ψ.

and a covariance matrix $E[(\boldsymbol{x}-m)(\boldsymbol{x}-m)^{\mathrm{T}}]=\mathrm{W}$ with finite elements $w_{ij}<\infty$, $i,j\in\mathrm{I}$. This null is restrictive, because it gives a sufficient but not necessary condition for efficiency. In fact, under the null, all portfolios $l\in\Lambda$ are efficient, because they optimize expected utility for $u(x)=x$, i.e. for risk neutral investors. The shape of the distribution of $x(t)$ under the null generally depends on the shape of $G(x)$. Our approach will be to focus on the least favorable distribution, i.e. the distribution that maximizes the size or relative frequency of Type I error (rejecting the null when it is true). This approach stems from the desire to be protected against Type I error. For each $G(x)$, the size is always smaller than the size for the least favorable distribution. Naturally, this approach comes at the cost of a high frequency of Type II error (accepting the null when it is not true) or a low power (1- the relative frequency of Type II error).

Using $\mathbf{C}\equiv(\mathrm{I}-\boldsymbol{e}t^{\mathrm{T}})$, we can summarize the asymptotic sampling distribution of our SSD test statistic as follows: [18]

THEOREM 2 *For the asymptotic least favorable distribution, the p-value* $\Pr[x(t,\Psi)>y|H_0]$, $y\geq 0$, *equals the integral* $(1-\int_{z\leq ye} d\Phi_{\mathrm{S}}(\boldsymbol{z}))$, *with* $\Phi_{\mathrm{S}}(\boldsymbol{z})$ *for a N-dimensional multivariate normal distribution function with zero means and covariance matrix* $\mathrm{S}\equiv(\mathbf{C}\mathrm{W}\mathbf{C}^{\mathrm{T}})/T$.

The theorem shows the crucial role of the length of the time series (*T*) and the length of the cross-section (*N*); the *p*-values decrease as the time series grows, and increase as the cross section grows. We may test efficiency by comparing the *p*-value for the observed value of $x(t,\Psi)$ with a predefined level of significance; we may reject efficiency if the *p*-value is smaller than or equal to the significance level.

Computing the *p*-value requires the unknown covariance terms w_{ij}. We may estimate these parameters in a distribution-free and consistent manner using the sample equivalents:

(16) $$\hat{w}_{ij}\equiv\sum_{t\in\Theta}(x_{it}-\sum_{t\in\Theta}x_{it}/T)(x_{jt}-\sum_{t\in\Theta}x_{jt}/T)/T.$$

We stress that the asymptotic *p*-values rely on a restrictive null and on the least favorable distribution. For this reason, the *p*-values may involve low power in small samples. An alternative approach is to approximate the sampling distribution by means of bootstrapping. Bootstrapping can deal directly with the true null (SD efficiency) and with the true distribution, and hence it can yield more powerful results. Of course, this benefit has to be balanced against the additional computational burden of using computer simulations.

III. ANALYZING AGGREGATE INVESTOR PREFERENCES

We analyze investor preferences by testing if different SD criteria (characterized by different sets Ψ) can rationalize the market portfolio. For this purpose, we need a

[18] We use I for an identity matrix with dimensions conforming to the rules of matrix algebra.

proxy for the market portfolio and proxies for the individual assets in the investment universe. We will use the Fama and French market portfolio, which is the value-weighted average of all NYSE, AMEX, and NASDAQ stocks. Further, we use the one-month US Treasury bill as the riskless asset. Finally, for the individual risky assets, we use three sets of benchmark portfolios:

1. The 25 Fama and French benchmark portfolios constructed as the intersections of 5 quintile portfolios formed on size and 5 quintile portfolios formed on BE/ME. We use data on monthly returns (month-end to month-end) from July 1963 to October 2001 (460 months) obtained from the data library on the homepage of Kenneth French.[19,20] Also, we analyze the stability of the results by applying the test to 2 non-overlapping subsamples of equal length (230 months): (1) July 1963-August 1982 and (2) September 1982–October 2001.

2. A set of 30 industry momentum portfolios constructed from 30 Fama and French benchmark portfolios based on industry classification. The momentum portfolios are based on the ranking of the returns over the past 6 months; momentum portfolio no. 1 equals the Fama and French industry portfolio with the lowest past return, and portfolio no. 30 equals the industry portfolio with the highest past return.[21] The portfolios are rebalanced yearly in July.[22] As for the set of 25 size-value portfolios, we use data on monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 months), and we also consider the subsamples July 1963-August 1982 and September 1982–October 2001. The raw data on the industry portfolios are obtained from the homepage of Kenneth French.

3. The set of 27 benchmark portfolios described in Carhart *et al.* (1996) and used in Carhart (1997). The portfolios are formed using a three-way classification based on size, BE/ME and momentum. These portfolios are formed by dividing all stocks into thirds based on B/M. Each of the three first-level portfolios is then divided into three portfolios based on size. Finally, each of the nine second-level portfolios is divided into 'past losers', 'middle' and 'past winners'. We use data on monthly returns (month-end to month-end) from July 1963 to December 1994 (378 months), as well as the subsamples from July 1963 to March 1979 (189 months) and from April 1979 to December 1994 (189 months).

These three sets of benchmark portfolios provide a challenge, because many studies indicate that the cross-sectional pattern of returns across size, BE/ME and momentum portfolios cannot be explained by the traditional approach based on concave utility functions. Tables 2A-C show some descriptive statistics for the excess returns of the benchmark portfolios for the full sample.

(Insert Table 2A-C about here)

[19] The data library is found at *http://mba.tuck.dartmouth.edu/pages/faculty/ken.french*.

[20] The data set starts in 1963 because the COMPUSTAT data used to construct the benchmark portfolios are biased towards big historically successful firms for the earlier years (see Fama and French, 1992).

[21] Similar results are obtained for portfolios based on the returns over the past 3, 9 and 12 months.

[22] The literature generally analyzes momentum in firm-specific returns. However, Moskowitz and Grinblatt (1999) demonstrate that momentum also exists in industry portfolios and they suggest that industry momentum in fact accounts for much of firm-specific momentum.

To give some feeling for the data, Figures 2A-C show mean-variance diagrams including the individual benchmark portfolios (the bright dots), the market portfolio (M) and the mean-variance efficient frontier (OA).[23] Clearly, the market portfolio is highly inefficient in terms of mean-variance analysis in all cases. Roughly speaking, there exist portfolios with the same standard deviation as the market portfolio but twice the average excess return, and there exist portfolios with the same average excess return as the market portfolio but only half the standard deviation. This result violates the central prediction of the MV CAPM: mean-variance efficiency of the market portfolio.

(Insert Figure 2A-C about here)

The maintained preference assumptions of MV CAPM are a possible explanation for this violation; variance does not fully capture the risk profile of assets unless investor utility is quadratic. This provides the motivation for testing if the market portfolio is efficient in terms of SD criteria. We employ the SD efficiency criteria discussed in Section II: SSD, PSDL, PSD, MSDL and MSD. For each criterion, we compute the value of the test statistic $x(m,\Psi)$ and the associated asymptotic least favorable *p*-value. We reject efficiency if the *p*-value is smaller than or equal to the significance level of 10 percent. Tables 3A-C show the test results for the 3 sets of benchmark portfolios. The results vary across the different sets of benchmark portfolios and the different sample periods. However, the relative goodness of the different SD criteria is remarkably robust across benchmark portfolios and sample periods; PSDL and PSD perform worst and MSD performs best.

The market portfolio is significantly SSD inefficient relative to the Fama and French size-BE/ME portfolios and the Carhart size-BE/ME-momentum portfolios. Based on this finding, we reject the SSD criterion. Harvey and Siddique (2000) and Dittmar (2002) find that concave third-order and fourth-order polynomial utility functions substantially better explain the cross-sectional variation of stock returns than quadratic utility functions do. Our results suggest that *no* concave utility function can rationalize the market portfolio. Under our maintained assumptions, this implies that investors that hold the market portfolio are not globally risk averse and utility is not everywhere concave, and we have to account for (local) risk seeking behavior. This result is in line with the experimental results by Levy and Levy (2001); they find that a majority of subjects are not globally risk averse, even when controlling for effects of framing and probability distortion.

(Insert Table 3A-C about here)

Prospect Theory offers an alternative to the traditional approach based on concave utility. However, we find strong evidence against two key elements of Prospect Theory: risk seeking for losses and loss aversion. Both criteria that impose risk seeking for losses (PSD and PSDL) are rejected for all sets of benchmark portfolios. This implies that no S-shaped utility function can rationalize the market portfolio. This finding is in line with the experimental evidence by Levy and Levy (2002) that a large majority of subjects select PSD inefficient prospects. Similarly, we reject all

[23] Note that this is the frontier for the case without short selling. Again, we focus on the case where portfolio possibilities are described by all convex combinations of the individual assets (see Section I).

three criteria that assume loss aversion (SSD, PSDL and MSDL). Barberis and Huang (2001) found that a model with loss aversion couldn't reproduce the empirical cross-sectional pattern of stock returns. They used a simple utility function formed from one linear line segment for losses and one for gains, and hence investors are assumed locally risk neutral over gains and over losses. Our results suggest that their conclusion is robust for more general preference assumptions; cross-sectional stock returns are not consistent with loss aversion.

Only the MSD criterion is consistent with the data for all sets of benchmark portfolios and for all sample periods. This suggests that Markowitz type reverse S-shaped utility functions can rationalize the market portfolio, and that risk aversion over losses and risk seeking over gains helps to explain the cross-sectional pattern of stock returns. If investors are risk averse for losses and risk seeking for gains, then they will pay (ask) a premium for stocks that have low (high) downside risk in bear markets and high (low) upside potential in bull market. Hence, stocks with low (high) downside risk bear markets and high (low) upside potential provide low (high) expected returns. This explanation is consistent with the experimental evidence by Levy and Levy (2002) that a large majority of subjects select MSD efficient prospects. Our results suggest that actual stock returns are also consistent with this explanation.

For illustration, Figure 3A-C shows example utility functions that rationalize the market portfolio relative to the benchmark assets. These particular utility functions are piecewise-linear functions that are formed from the from the optimal solution $b^* \in \mathrm{B}(\Psi_{MSD})$ as

$$(17) \qquad p(x|b^*) = \begin{cases} a_1 + b_1^* x & x \le \boldsymbol{x}_1^{\mathrm{T}} t \\ a_t + b_t^* x & \boldsymbol{x}_t^{\mathrm{T}} t < x \le \boldsymbol{x}_{t+1}^{\mathrm{T}} t < 0 \\ a_{z-1} + b_{z-1}^* x & \boldsymbol{x}_{z-1}^{\mathrm{T}} t \le x < 0 \\ a_z + b_z^* x & 0 \le x < \boldsymbol{x}_z^{\mathrm{T}} t \\ a_t + b_t^* x & 0 < \boldsymbol{x}_{t-1}^{\mathrm{T}} t \le x < \boldsymbol{x}_t^{\mathrm{T}} t \\ a_T + b_T^* x & x \ge \boldsymbol{x}_T^{\mathrm{T}} t \end{cases},$$

with $a \equiv (a_1 \cdots a_T)^{\mathrm{T}}$ such that the linear line segments are connected i.e. $a_t + b_t^* x = a_{t+1} + b_{t+1}^* x$ for all $t \in \Theta \setminus T$, and $a_z = 0$. We stress that these utility functions are not unique, as we can construct alternative utility functions with the same gradient vector at $\boldsymbol{X}^{\mathrm{T}} t$. Also, these utility functions are likely to be very sensitive to estimation error and we cannot claim any confidence in their statistical estimation. Still, the utility functions do suggest that the market portfolio is optimal, and hence the benchmark assets are correctly priced, for investors that are very sensitive to large losses, say returns of less than -10 percent per month, and to large gains, say returns of more than 10 percent.

(Insert Figure 3A-C about here)

Some further results

Further support for Markowitz type preferences can be found by analyzing the relationship between average return and measures of downside risk and upside potential. Two simple measures are 'downside beta' (market beta for periods with a negative market return) and 'upside beta' (market beta for periods with a positive market return); see e.g. Bawa and Lindenberg (1977). Risk aversion for losses suggests a positive relation between expected return and downside beta, and risk seeking for gains suggests a negative relation between expected return and upside beta. For the sake of illustration, we estimated the downside and upside betas of the 27 Carhart benchmark portfolios using OLS regression for the full sample. (Very similar results are obtained for the 25 Fama and French benchmark portfolios and the 30 industry momentum portfolios). Figure 3 shows the results.

(Insert Figure 4 about here)

Obviously, beta risk generally is highly asymmetric across falling and rising markets.[24] In addition, the differences between downside and upside betas seem to support Markowitz type preferences, i.e. high downside beta portfolios generally have high average returns, while high upside betas portfolios have low average returns. For example, the 'high yield' portfolio of small value winner stocks (with an average monthly excess return of 1.341) has a downside beta of 1.409 and an upside beta of 0.883, while the 'low yield' portfolio of the big growth loser stocks (with average 0.105) has a downside beta of 0.849 and an upside beta of 1.119. We may use regression analysis to analyze if this pattern applies for the entire sample. Specifically, using OLS regression analysis, we find the following cross-sectional relation between average excess return $\hat{m}_i \equiv \sum_{t \in \Theta} x_{it} / T$ and estimated downside beta $\hat{b}_i^-$ and upside beta $\hat{b}_i^+$ (standard errors within brackets):[25]

$$(18) \qquad \underset{}{\hat{m}_i} = \underset{(0.233)}{1.633} + \underset{(0.060)}{0.856}\ \hat{b}_i^- - \underset{(0.154)}{1.938}\ \hat{b}_i^+ \qquad R^2 = 0.601 .$$

The signs of the coefficients are consistent with Markowitz type preferences: a positive coefficient for downside beta and hence risk aversion for losses, and a negative coefficient for upside beta and hence risk seeking for gains. Of course, these results may be sensitive to the parametric specification (a linear relationship between expected return and the downside and upside betas) and to the estimation method (estimation based on e.g. generalized methods of moments or maximum likelihood procedures may give different results). Also, strictly speaking, we have to reject the two-beta specification, because it does not give a perfect fit (the betas explain only 60 percent of the variation in average return) and because the intercept (the average excess return of a zero-beta portfolio) is significantly higher than zero. Still, the regression results do provide indirect support for our explanation based on risk seeking for gains.

[24] Several authors have found similar differences in upside and downside risk measures (see e.g. Ang *et al.*, 2001). These findings are sometimes interpreted as evidence for Prospect Theory. By contrast, we find evidence against two key elements of Prospect Theory -risk seeking for losses and loss aversion- and our explanation rests on risk seeking for gains.

[25] The cross-sectional regression gives a Fama and McBeth (1973) type test with betas assumed constant over time. The assumption of constant betas is needed for comparison with the SD tests that build on the assumption that the excess returns are serially IID (Assumption 4).

IV. CONCLUSIONS

We cannot reject MSD efficiency of the market portfolio relative to benchmark portfolios formed on size, BE/ME and momentum. Hence, Markowitz type reverse S-shaped utility functions may rationalize the market portfolio. By contrast, the alternative criteria of SSD, PSDL, PSD and MSDL are rejected and utility functions with global risk aversion, risk seeking over losses or loss aversion cannot rationalize the market portfolio. These results suggest that the individual choice behavior observed in the experiments by Levy and Levy (2002) can also help to explain aggregate investor behavior. If investors are risk averse for losses and risk seeking for gains, then they are willing to pay a premium for stocks that give downside protection in bear markets and upside potential in bull markets. Our results suggest that this explanation is consistent with the cross-sectional pattern of stock returns. Roughly speaking, stock returns suggests that investors are driven by the twin desires for security and potential, and that investment portfolios are designed to avoid poverty and to give a chance at riches. In this respect, our findings are consistent with the predictions of several behavioral finance models, like the behavioral portfolio theory (BPT) by Shefrin and Statman (2000). However, contrary to BPT, our results are derived within the context of expected utility theory, and we do not explicitly account for subjective probability transformation (see Footnote 3 and Footnote 11).

Of course, our results may be biased by our maintained assumptions: we used a single-period, portfolio-oriented model with a large number of expected utility maximizers and without market frictions (apart from short selling restrictions). In addition, we assumed a simple data generating process with a serially independent and identical distribution for the excess returns. There are good reasons to doubt these maintained assumptions. Further research could focus on relaxing our economic assumptions (e.g. by considering the multiperiod consumption-investment problem, imperfect competition, market imperfections like transaction costs and taxes, and non-expected utility theories with bounded rationality and/or imperfect information) and on relaxing our statistical assumptions (e.g. by using econometric time series estimation techniques to estimate a conditional CDF). Still, at the very least, our results suggest that Markowitz type utility functions are capable of capturing the cross-sectional pattern of stock returns even under very simple economic and statistical assumptions.

We hope that our results provide a stimulus for further research based on Markowitz type utility functions (and non-concave utility functions in general). Also, we hope that this study contributes to the further proliferation of the SD methodology. Since large, statistically representative samples often are available in financial economics, the 'nonparametric' SD approach is a useful complement to existing parametric approaches.

APPENDIX

Proof to Theorem 1: The problem $\max_{l\in\Lambda}\sum_{t\in\Theta} u(\boldsymbol{x}_t^{\mathrm{T}} l)/T$, $u \in U(\Psi)$, maximizes a strictly increasing and hence pseudoconcave objective over a polytope. Hence, $t \in \Lambda$ is the optimal portfolio i.e. $t \equiv \arg\max_{l\in\Lambda}\sum_{t\in\Theta} u(\boldsymbol{x}_t^{\mathrm{T}} l)/T$ if and only if all assets are enveloped by the tangent hyperplane, i.e.

(19) $\quad \sum_{t\in\Theta} \partial u(\boldsymbol{x}_t^{\mathrm{T}} t)(\boldsymbol{x}_t^{\mathrm{T}} t - x_{it})/T \geq 0 \quad \forall i \in \mathrm{I}$.

By construction, $\nabla u(\boldsymbol{X}^{\mathrm{T}} t) \equiv (\partial \boldsymbol{u}(\boldsymbol{x}_1^{\mathrm{T}} t)\cdots\partial u(\boldsymbol{x}_T^{\mathrm{T}} t))^{\mathrm{T}}$, $u \in U(\Psi)$, is a feasible solution, i.e. $\nabla u(\boldsymbol{X}^{\mathrm{T}} t) \in B(\Psi)$. Optimality condition (19) implies that this solution is associated with a solution value of zero. Hence, we find the necessary condition; t is $U(\Psi)$-SD efficient only if $x(t, \Psi) = 0$.

To establish the sufficient condition, use $b^* \equiv (b_1^* \cdots b_T^*)^{\mathrm{T}}$ for the optimal solution. It is easy to verify that we can always find some $u \in U(\Psi)$ with $\nabla u(\boldsymbol{X}^{\mathrm{T}} t) = b^*$. For example, consider the piecewise-linear function

$$p(x|b^*) = \begin{cases} a_1 + b_1^* x & x \leq \boldsymbol{x}_1^{\mathrm{T}} t \\ a_t + b_t^* x & \boldsymbol{x}_t^{\mathrm{T}} t < x \leq \boldsymbol{x}_{t+1}^{\mathrm{T}} t < 0 \\ a_{z-1} + b_{z-1}^* x & \boldsymbol{x}_{z-1}^{\mathrm{T}} t \leq x < 0 \\ a_z + b_z^* x & 0 \leq x < \boldsymbol{x}_z^{\mathrm{T}} t \\ a_t + b_t^* x & 0 < \boldsymbol{x}_{t-1}^{\mathrm{T}} t \leq x < \boldsymbol{x}_t^{\mathrm{T}} t \\ a_T + b_T^* x & x \geq \boldsymbol{x}_T^{\mathrm{T}} t \end{cases},$$

with $a \equiv (a_1 \cdots a_T)^{\mathrm{T}}$ such that the linear line segments are connected i.e. $a_t + b_t^* x = a_{t+1} + b_{t+1}^* x$ for all $t \in \Theta \setminus T$. It is easy to verify that $p(x|b^*)$ belongs to $U(\Psi_{SSD})$, $U(\Psi_{PSDL})$, $U(\Psi_{PSD})$, $U(\Psi_{MSDL})$, and $U(\Psi_{MSD})$. Applying optimality condition (19), we find that t is optimal relative to $u(x)$ i.e. $t = \arg\max_{l\in\Lambda} u(\boldsymbol{x}_t^{\mathrm{T}} l)$. Hence, portfolio $t \in \Lambda$ is $U(\Psi)$-SD efficient if $x(t, \Psi) = 0$. *Q.E.D.*

Proof of Theorem 2: Since the unity vector is a feasible solution to the primal problem, i.e. $\boldsymbol{e} \in \mathrm{B}$, we know that $x(t, \Psi) \leq w(t) \equiv \max_{i \in \mathrm{I}} \left\{ \sum_{t \in \Theta} (x_{it} - \boldsymbol{x}_t^{\mathrm{T}} t) / T \right\}$. Known results can derive the exact asymptotic sampling distribution of $w(t)$. Under the null $H_0 : E[\boldsymbol{x}] = m\boldsymbol{e}$, the $\mathbf{C}\mathbf{x}_t = ((x_{1t} - \boldsymbol{x}_t^{\mathrm{T}} t) \cdots (x_{Nt} - \boldsymbol{x}_t^{\mathrm{T}} t))^{\mathrm{T}}$, $t \in \Theta$, are serially IID random vectors with zero mean and covariance matrix $\mathbf{C}\mathrm{W}\mathbf{C}^{\mathrm{T}}$. Hence, the Lindeberg-Levy central limit theorem implies that the vector $\mathbf{C}\mathrm{C}\boldsymbol{e} / T = (\sum_{t \in \Theta} (x_{1t} - \boldsymbol{x}_t^{\mathrm{T}} t) / T \cdots \sum_{t \in \Theta} (x_{Nt} - \boldsymbol{x}_t^{\mathrm{T}} t) / T)^{\mathrm{T}}$ obeys an asymptotically joint normal distribution with zero mean and covariance matrix $\mathrm{S} \equiv (\mathbf{C}\mathrm{W}\mathbf{C}^{\mathrm{T}}) / T$. Hence, $w(t)$ asymptotically behaves as the largest order statistic of N random variables with a multivariate normal distribution, and $\Pr[w(t) > y | H_0] = 1 - \Pr[\mathbf{C}\mathrm{C}\boldsymbol{e} / T \leq y\boldsymbol{e}]$ asymptotically equals the multivariate normal integral $1 - \int_{\boldsymbol{z} \leq y\boldsymbol{e}} d\Phi_{\mathrm{S}}(\boldsymbol{z})$. Since $x(t, \Psi) \leq w(t)$, $\Pr[x(t, \Psi) > y | H_0]$ is bounded from above by $\Pr[w(t) > y | H_0]$ for all return distributions $G(x)$. Moreover, it is easily verified that there exist $G(x)$ for which $x(t, \Psi)$ approximates $w(t)$, and therefore the asymptotic distribution of $w(t)$ also represents the asymptotic least favorable distribution for $x(t, \Psi)$. *Q.E.D.*

LITERATURE

Ang, A, J. Chen and Y. Xing (2002): Downside Correlation and Expected Stock Returns, forthcoming in *Journal of Financial Economics*.

Bansal, R. and S. Viswanathan (1993): No Arbitrage and Arbitrage Pricing: A New Approach, *Journal of Finance* 48 (4), 1231-1262.

Barberis, N. and M. Huang (2001): 'Mental Accounting, Loss Aversion, and Individual Stock Returns', *Journal of Finance* 56 (4), 1247-1292.

Barberis, N. M. Huang and T. Santos (2001): 'Prospect theory and asset prices', *Quartely Journal of Economics* 116, 1-53.

Bawa, V.S. and E. B. Lindenberg (1977): Capital Market Equilibrium in a Mean-Lower Partial Moment Framework, *Journal of Financial Economics* 4: 189-200.

Benartzi, S. and R. Thaler (1995): 'Myopic loss aversion and the equity premium puzzle', *Quartely Journal of Economics* 110, 73-92.

Berk, J. (1997): Necessary Conditions for the CAPM, *Journal of Economic Theory* 73, 245-257.

Carhart, M. M. (1997), On Persistence in Mutual Fund Performance, *Journal of Finance* 52 (1), 57-82.

Carhart, M. M., R. J. Krail, R. I. Stevens, and K. E. Welch (1996), Testing the conditional CAPM, Unpublished manuscript, University of Chicago.

Chapman, D. A. (1997): Approximating the Asset Pricing Kernel, *Journal of Finance* 52 (4), 1383-1410.

Dittmar, R. (2002), Nonlinear Pricing Kernels, Kurtosis Preference, and Evidence from the Cross-Section of Equity Returns’, *Journal of Finance* 57 (1), 369-403.

Fama and McBeth (1973): Risk, Return and Equilibrium : Empirical Tests, *Journal of Political Economy* 71, 607-636.

Fama, E. and K. French (1992): The cross section of expected stock returns, *Journal of Finance* 47 (2), 427-465.

Friedman, M., and Savage, L. (1948): The utility analysis of choices involving risk, *Journal of Political Economy* 56:279-304.

Friend, I. and Westerfield, R. (1980): Co-skewness and Capital Asset Pricing, *Journal of Finance* 35, 897-913.

Ghysels, E. (1998), On Stable Factor Structures in the Pricing of Risk: Do Time-Varying Betas Help or Hurt?, *Journal of Finance* 53 (2), 549-573.

Hartley, R. and L. Farrell (2001): Can Friedman-Savage Utility Functions Explain Gambling?, forthcoming in *American Economic Review*.

Harvey, C. R. and A. Siddique (2000): Conditional Skewness in Asset Pricing Tests, *Journal of Finance* 55 (3), 1263-1295.

Jagadeesh, N., and S. Titman (1993): Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65-91.

Kahneman, D. and A. Tversky (1979): Prospect Theory: An analysis of decision under risk, *Econometrica* 47:263-291.

Levy, H. (1969), Comment: A Utility Function Depending on the First Three Moments, *Journal of Finance* 24 (4), 715-719.

Levy, H. (1992): Stochastic Dominance and Expected Utility: Survey and Analysis, *Management Science* 38(4), 555-593.

Levy, H. (1998): *Stochastic Dominance,* Kluwer Academic Publishers, Norwell, MA.

Levy, M. and H. Levy (2001), Testing for Risk Aversion: A Stochastic Dominance Approach, *Economics Letters* 71 (2), 233-240.

Levy, H. and Levy, M. (2002): Prospect Theory: Much Ado About Nothing?, forthcoming in *Management Science*.

Levy, H. and Z. Wiener (1998), Stochastic Dominance and Prospect Dominance with Subjective Weighting Functions, *Journal of Risk and Uncertainty* 16 (2), 147-163.

Lintner, J. (1965): Security prices, risk and maximal gains from diversification, *Journal of Finance* 20 (2), 587-615.

Mangasarian, O. L. (1969): *Nonlinear Programming*, McGraw-Hill, New York.

Markowitz, H. (1952): The Utility of Wealth, *Journal of Political Economy* 60, 151-158.

Moskowitz, T. J. and M. Grinblatt (1999), Do Industries Explain Momentum?, *Journal of Finance* 54 (4), 1249-1290.

Post, G. T. (2001): Empirical Tests for Stochastic Dominance Efficiency, Erasmus Research Institute of Management (ERIM) Report ERS-2001-38-F&A, forthcoming in *Journal of Finance*.

Rubinstein, M. (1974): An Aggregation Theorem for Securities Markets, *Journal of Financial Economics* 1, 225-244.

Sharpe, W. (1964): Capital asset prices: a theory of market equilibrium under conditions of risk, *Journal of Finance* 19, 425-442.

Sharpe, W. (1991): Capital asset prices with and without negative holdings, *Journal of Finance* 64, 489-509.

Shefrin, H. and M. Statman (2000): Behavioral Portfolio Theory, *Journal of Financial and Quantitative Analysis* 35 (2), 127-151.

Starmer, C. (2000): Developments in Non-expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk, *Journal of Economic Literature* 38(2), 332-82.

Statman, M. (2002), Lottery Players/Stock Traders, *Financial Analysts Journal* January/February 2002, 14-21.

Tversky, A. and D. Kahneman (1992), Advances in Prospect Theory: Cumulative Representation of Uncertainty, *Journal of Risk and Uncertainty* 5, 297-323.

Wang, Z. (1998): Efficiency loss and constraints on portfolio holdings, *Journal of Financial Economics* 48, 359-375.

Table 1: Preference Assumptions

The preference assumptions underlying the criteria of Second-order Stochastic Dominance (SSD), Prospect Stochastic Dominance with Loss Aversion (PSDL), Prospect Stochastic Dominance (PSD), Markowitz Stochastic Dominance with Loss Aversion (MSDL), and Markowitz Stochastic Dominance (MSD). Each assumption, $r = 1,\cdots,5$, is represented by a polyhedron $\Psi_r \in \Re^2$; each assumption effectively imposes the restriction $\partial u(x) \geq \partial u(y)$ for all $(x, y) \in \Psi_r$.

r	Assumption	Ψ_r	Criterion				
			SSD	PSDL	PSD	MSDL	MSD
1	Loss Aversion	$\{(x, y) \in \Re_- \times \Re_+\}$	X	X		X	
2	Risk Aversion for Losses	$\{(x, y) \in \Re_-^2 : x \leq y\}$	X			X	X
3	Risk Seeking for Losses	$\{(x, y) \in \Re_-^2 : x \geq y\}$		X	X		
4	Risk Aversion for Gains	$\{(x, y) \in \Re_+^2 : x \leq y\}$	X	X	X		
5	Risk Seeking for Gains	$\{(x, y) \in \Re_+^2 : x \geq y\}$				X	X

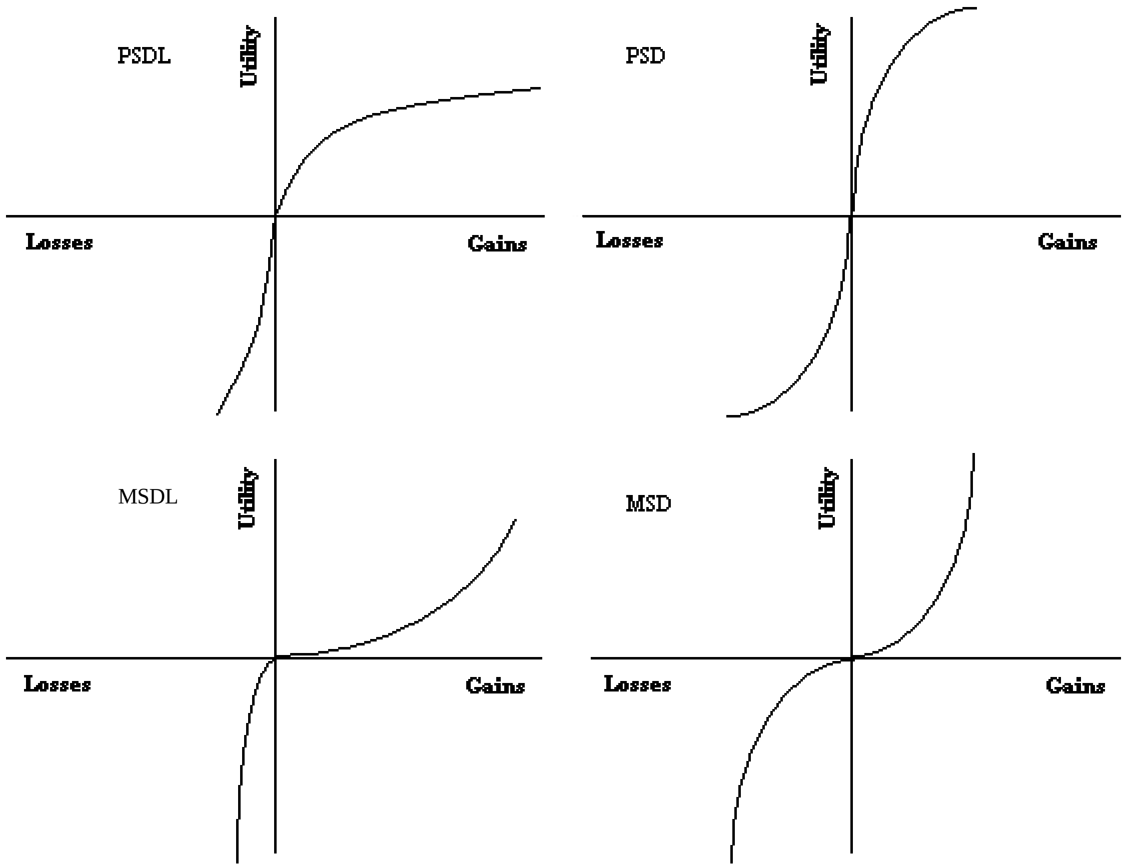

Figure 1: Example non-concave utility functions that are consistent with the assumptions of Prospect Stochastic Dominance with Loss Aversion (PSDL), Prospect Stochastic Dominance (PSD), Markowitz Stochastic Dominance with Loss Aversion (MSDL), and Markowitz Stochastic Dominance (MSD).

Table 2A: Descriptive Statistics Size-BE/ME Portfolios

Monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 months) for the value-weighted Fama and French market portfolio and 25 value-weighted benchmark portfolios based on market capitalization (size) and/or book-to-market equity ratio (BE/ME). Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. All data are obtained from the data library on the homepage of Kenneth French**.**

		Mean	St. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio		0.462	4.461	-0.498	2.176	-23.09	16.05
Benchmark Portfolios							
BE/ME	Size						
Growth	Small	0.235	8.246	0.003	2.466	-34.32	38.82
2	Small	0.733	7.064	0.037	3.338	-31.30	36.67
3	Small	0.815	6.140	-0.087	3.233	-29.17	27.86
4	Small	0.998	5.724	-0.149	3.636	-29.47	27.43
Value	Small	1.075	5.943	-0.098	4.128	-29.45	31.83
Growth	2	0.359	7.494	-0.316	1.673	-33.21	29.62
2	2	0.622	6.120	-0.472	2.778	-32.50	26.25
3	2	0.842	5.410	-0.488	3.739	-28.10	26.78
4	2	0.913	5.154	-0.385	3.979	-27.09	26.70
Value	2	0.966	5.660	-0.243	4.245	-29.83	29.43
Growth	3	0.380	6.908	-0.342	1.365	-29.96	23.39
2	3	0.665	5.511	-0.624	3.051	-29.65	23.35
3	3	0.673	4.962	-0.621	2.984	-25.56	21.14
4	3	0.818	4.710	-0.327	3.029	-23.20	22.61
Value	3	0.953	5.297	-0.423	4.384	-27.79	27.30
Growth	4	0.506	6.146	-0.182	1.683	-26.02	25.01
2	4	0.433	5.218	-0.571	3.317	-29.62	20.46
3	4	0.651	4.855	-0.423	3.348	-26.13	22.87
4	4	0.820	4.622	0.035	2.394	-18.30	24.57
Value	4	0.880	5.347	-0.198	2.697	-25.36	26.20
Growth	Big	0.442	4.841	-0.177	1.676	-22.15	21.70
2	Big	0.441	4.588	-0.334	1.897	-23.21	16.05
3	Big	0.493	4.344	-0.243	2.639	-22.47	18.22
4	Big	0.603	4.289	0.056	1.574	-15.35	18.85
Value	Big	0.583	4.645	-0.146	0.944	-19.28	15.39

Table 2B: Descriptive Statistics Industry Momentum Portfolios

Monthly excess returns (month-end to month-end) from July 1963 to October 2001 (460 months) for the value-weighted Fama and French market portfolio and 30 value-weighted benchmark portfolios formed on industry momentum. The portfolios are constructed based on the past performance ranking of 30 Fama and French industry portfolios; momentum portfolio no. 1 equals the Fama and French industry portfolio with the lowest return over the past 6 months, and portfolio no. 30 equals the industry portfolio with the highest past return. The portfolios are rebalanced yearly in July. Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. Raw data on industry portfolios are obtained from the data library on the homepage of Kenneth French**.**

	Mean	St. Dev.	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio	0.462	4.461	-0.498	2.176	-23.09	16.05
Benchmark Portfolios						
1	0.700	6.510	0.297	1.922	-26.43	29.20
2	-0.080	5.986	-0.442	2.205	-31.05	19.96
3	0.153	6.288	-0.402	3.070	-32.10	22.82
4	0.560	5.721	-0.252	2.436	-31.59	18.48
5	0.257	5.563	-0.152	1.398	-21.56	19.97
6	0.337	5.500	0.159	0.782	-19.69	19.92
7	0.218	5.558	-0.030	1.210	-21.08	21.77
8	0.259	6.276	-0.443	2.021	-28.40	25.18
9	0.333	5.828	-0.258	1.281	-21.66	19.08
10	0.408	5.925	-0.011	1.630	-22.25	25.91
11	0.623	5.486	0.021	3.714	-27.67	28.10
12	0.442	5.836	-0.404	3.033	-32.69	19.17
13	0.318	5.662	-0.239	2.394	-28.83	24.48
14	0.358	5.634	-0.098	1.616	-24.17	19.77
15	0.570	5.714	0.033	0.824	-20.90	19.56
16	0.566	5.972	0.318	1.523	-18.74	29.07
17	0.679	5.950	0.118	3.179	-28.60	31.84
18	0.775	5.776	-0.341	2.465	-31.96	22.67
19	0.169	5.523	-0.502	1.852	-27.93	16.53
20	0.475	5.819	-0.415	2.496	-29.59	23.24
21	0.850	5.576	-0.706	3.297	-32.14	17.01
22	0.809	6.460	-0.142	3.500	-33.22	30.36
23	0.491	5.676	-0.252	1.372	-26.40	18.57
24	0.485	5.586	-0.303	1.186	-25.91	17.14
25	0.657	5.529	-0.358	2.385	-28.60	21.84
26	0.680	6.077	0.096	1.901	-19.31	28.44
27	0.683	6.456	0.640	6.429	-28.70	45.92
28	0.986	6.152	-0.082	3.037	-33.02	28.19
29	1.092	7.030	-0.450	2.078	-31.76	23.56
30	0.949	7.319	0.178	2.823	-32.91	37.57

Table 2C: Descriptive Statistics Size-Value-Momentum Portfolios

Monthly excess returns from July 1963 to December 1994 (378 months) for the value weighted Fama and French market portfolio and 27 value weighted benchmark portfolios based on market capitalization (size), book-to-market equity (BE/ME) and momentum. The 27 portfolios are formed by dividing all stocks into thirds based on B/M. Each of the three first-level portfolios is then divided into three portfolios based on size. Finally, each of the nine second-level portfolios is divided into 'past losers', 'middle' and 'past winners'. Excess returns are computed from the raw return observations by subtracting the return on the one-month US Treasury bill. Data on the 27 portfolios are courtesy of Mark Carhart. The remaining data on the market portfolio and the Treasury bill. All other data are obtained from the homepage of Kenneth French.

			Mean	St. Dev	Skewness	Kurtosis	Minimum	Maximum
Market Portfolio			0.387	4.399	-0.394	2.600	-23.09	16.05
Benchmark Portfolios								
BE/ME	Size	Momentum						
Growth	Small	Loser	-0.279	6.644	0.003	2.784	-31.41	30.45
Neutral	Small	Loser	0.306	5.975	0.474	5.363	-24.81	37.51
Value	Small	Loser	0.552	6.440	0.974	7.774	-28.26	45.34
Growth	Small	Middle	0.349	6.070	-0.461	3.194	-31.33	27.54
Neutral	Small	Middle	0.631	4.957	-0.201	5.038	-27.05	27.02
Value	Small	Middle	1.062	5.583	0.199	6.219	-29.09	33.99
Growth	Small	Winner	0.933	6.755	-0.704	2.492	-33.52	19.45
Neutral	Small	Winner	1.157	6.088	-0.730	3.569	-32.60	23.17
Value	Small	Winner	1.341	6.246	-0.289	4.564	-32.00	30.20
Growth	Medium	Loser	-0.089	5.952	0.088	2.757	-26.49	27.54
Neutral	Medium	Loser	0.400	5.396	0.501	3.468	-19.08	31.20
Value	Medium	Loser	0.582	6.078	0.469	4.678	-25.79	38.07
Growth	Medium	Middle	0.143	5.341	-0.461	2.207	-26.81	18.96
Neutral	Medium	Middle	0.513	4.498	-0.344	4.720	-25.68	23.05
Value	Medium	Middle	0.909	5.303	-0.016	6.042	-28.30	31.03
Growth	Medium	Winner	0.895	6.033	-0.550	2.261	-29.90	20.73
Neutral	Medium	Winner	0.772	5.325	-0.896	3.683	-30.54	17.52
Value	Medium	Winner	1.287	5.881	-0.806	5.234	-33.86	25.76
Growth	Big	Loser	0.105	5.218	0.178	2.208	-20.40	25.40
Neutral	Big	Loser	0.438	4.801	0.477	2.287	-19.66	21.74
Value	Big	Loser	0.600	5.569	0.718	4.414	-16.99	35.34
Growth	Big	Middle	0.250	4.507	-0.161	1.885	-20.73	17.86
Neutral	Big	Middle	0.367	4.211	0.158	2.307	-15.57	21.03
Value	Big	Middle	0.589	4.752	-0.053	2.717	-23.47	20.41
Growth	Big	Winner	0.611	5.348	-0.298	2.035	-23.68	21.21
Neutral	Big	Winner	0.595	4.856	-0.347	2.565	-24.00	19.95
Value	Big	Winner	0.923	5.352	-0.254	2.846	-24.78	22.84

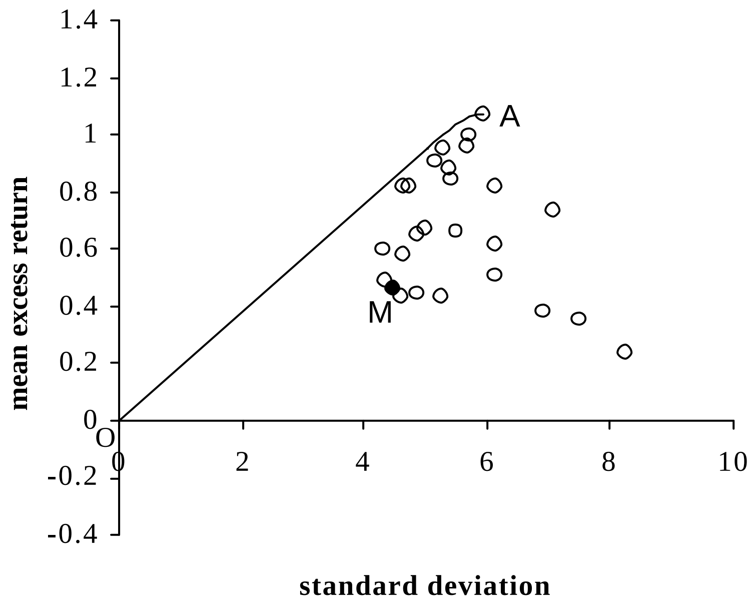

Figure 2A: Mean-variance diagram for the historical excess returns of the 25 Fama and French size-BE/ME portfolios (the bright dots) and the Fama and French market portfolio (M). OA represents the efficient frontier (with the US Treasury bill, short sales not allowed).

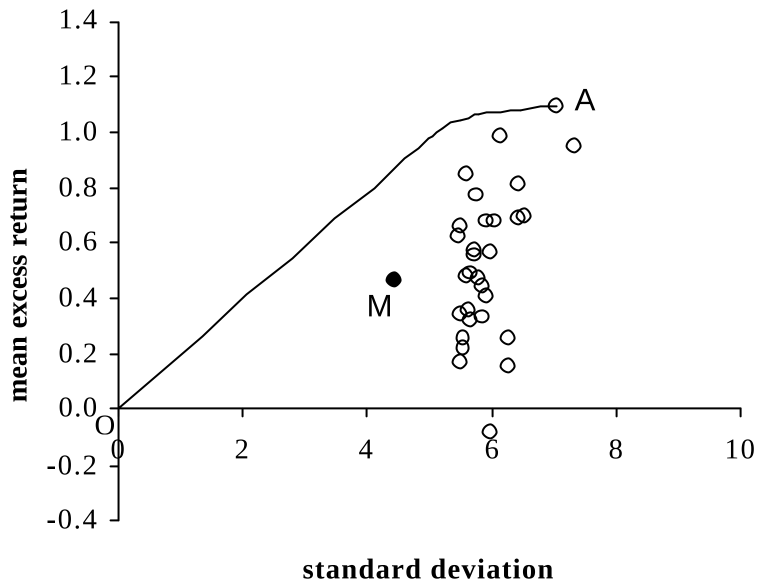

Figure 2B: Mean-variance diagram for the historical excess returns of the 30 industry momentum portfolios (the bright dots) and the Fama and French market portfolio (M). OA represents the efficient frontier (with the US Treasury bill, short sales not allowed).

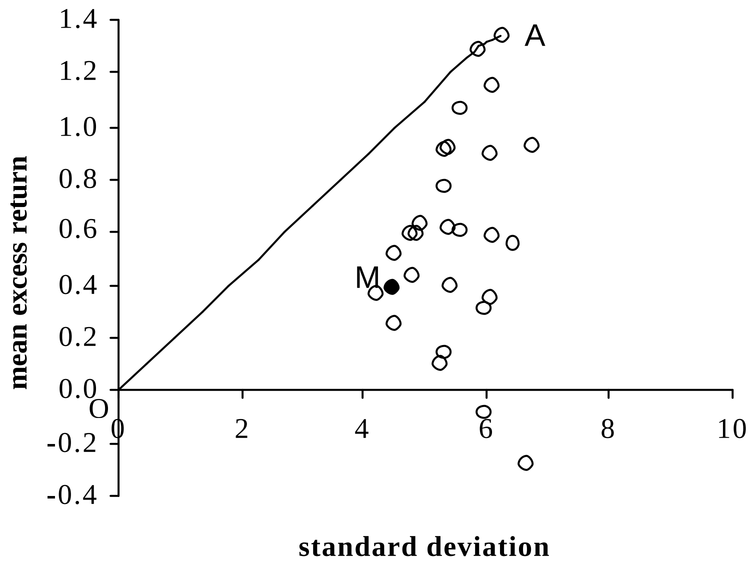

Figure 2C: Mean-variance diagram for the historical excess returns of the 27 Carhart size-BE/ME-momentum portfolios (the bright dots) and the Fama and French market portfolio (M). OA represents the efficient frontier (with the US Treasury bill, short sales not allowed).

Table 3A: Test Results Size-Value Portfolios

We test whether the Fama and French market portfolio is $U(\Psi)$-SD efficient relative to all portfolios formed from a US Treasury bill and 25 benchmark portfolios formed on market capitalization (size) and book-to-market-equity ratio (BE/ME). Each cell shows the observed value for the test statistic $x(m, \mathrm{B}(\Psi))$, as well as the asymptotic least favorable *p*-value (within brackets). We use a bold font if results are statistically significant at a level of significance of 90 percent.

	Jul 1963 - Oct 2001	Jul 1963 - Aug 1982	Sep 1982 - Oct 2001
Ψ_{SSD}	**0.434** **(0.031)**	**0.585** **(0.034)**	0.299 (0.490)
Ψ_{PSDL}	**0.613** **(0.002)**	**0.918** **(0.000)**	0.387 (0.243)
Ψ_{PSD}	**0.432** **(0.031)**	**0.905** **(0.000)**	0.356 (0.324)
Ψ_{MSDL}	**0.434** **(0.031)**	**0.585** **(0.034)**	0.299 (0.490)
Ψ_{MSD}	0.207 (0.501)	0.226 (0.748)	0.192 (0.865)

Table 3B: Test Results Industry Momentum Portfolios

We test whether the Fama and French market portfolio is $U(\Psi)$-SD efficient relative to all portfolios formed from a US Treasury bill and 30 benchmark portfolios formed on industry momentum. Each cell shows the observed value for the test statistic $x(m, \mathrm{B}(\Psi))$, as well as the asymptotic least favorable *p*-value (within brackets). We use a bold font if results are statistically significant at a level of significance of 90 percent.

	Jul 1963 - Oct 2001	Jul 1963 - Aug 1982	Sep 1982 - Oct 2001
Ψ_{SSD}	0.376 (0.375)	0.604 (0.214)	0.492 (0.479)
Ψ_{PSDL}	**0.595** **(0.026)**	**0.995** **(0.011)**	0.484 (0.508)
Ψ_{PSD}	**0.595** **(0.026)**	**0.749** **(0.057)**	0.350 (0.915)
Ψ_{MSDL}	0.376 (0.375)	0.604 (0.214)	0.492 (0.479)
Ψ_{MSD}	0.344 (0.510)	0.568 (0.295)	0.356 (0.892)

Table 3C: Test Results Size-Value-Momentum Portfolios

We test whether the Fama and French market portfolio is $U(\Psi)$-SD efficient relative to all portfolios formed from a US Treasury bill and 27 benchmark portfolios formed on size, value and momentum. Each cell shows the observed value for the test statistic $x(m, \mathrm{B}(\Psi))$, as well as the asymptotic least favorable *p*-value (within brackets). We use a bold font if results are statistically significant at a level of significance of 90 percent.

	Jul 1963 - Dec 1994	Jul 1963 - Mar 1979	Apr 1979 – Dec 1994
Ψ_{SSD}	**0.469** **(0.013)**	**0.780** **(0.004)**	0.442 (0.169)
Ψ_{PSDL}	**0.954** **(0.000)**	**1.171** **(0.000)**	**0.737** **(0.004)**
Ψ_{PSD}	**0.861** **(0.000)**	**0.930** **(0.000)**	**0.737** **(0.004)**
Ψ_{MSDL}	**0.469** **(0.013)**	**0.780** **(0.004)**	0.442 (0.169)
Ψ_{MSD}	0.291 (0.275)	0.346 (0.547)	0.384 (0.300)

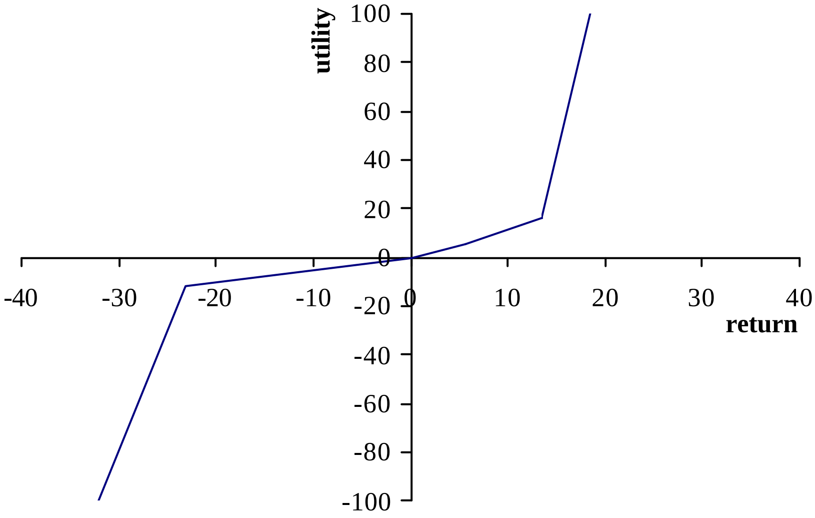

Figure 3A: Optimal piecewise-linear MSD utility function. This utility function rationalizes the Fama and French market portfolio relative to the 25 Fama and French size-BE/ME portfolios. The utility function is constructed from the optimal solution $b^* \in \mathrm{B}(\Psi_{MSD})$ to (9) as

$$p(x|b^*) = \begin{cases} a_1 + b_1^* x & x \leq \mathbf{x}_1^{\mathrm{T}} t \\ a_t + b_t^* x & \mathbf{x}_t^{\mathrm{T}} t < x \leq \mathbf{x}_{t+1}^{\mathrm{T}} t < 0 \\ a_{z-1} + b_{z-1}^* x & \mathbf{x}_{z-1}^{\mathrm{T}} t \leq x < 0 \\ a_z + b_z^* x & 0 \leq x < \mathbf{x}_z^{\mathrm{T}} t \\ a_t + b_t^* x & 0 < \mathbf{x}_{t-1}^{\mathrm{T}} t \leq x < \mathbf{x}_t^{\mathrm{T}} t \\ a_T + b_T^* x & x \geq \mathbf{x}_T^{\mathrm{T}} t \end{cases},$$

with $a \equiv (a_1 \cdots a_T)^{\mathrm{T}}$ such that the linear line segments are connected i.e. $a_t + b_t^* x = a_{t+1} + b_{t+1}^* x$ for all $t \in \Theta \setminus T$, and with $a_z = 0$.

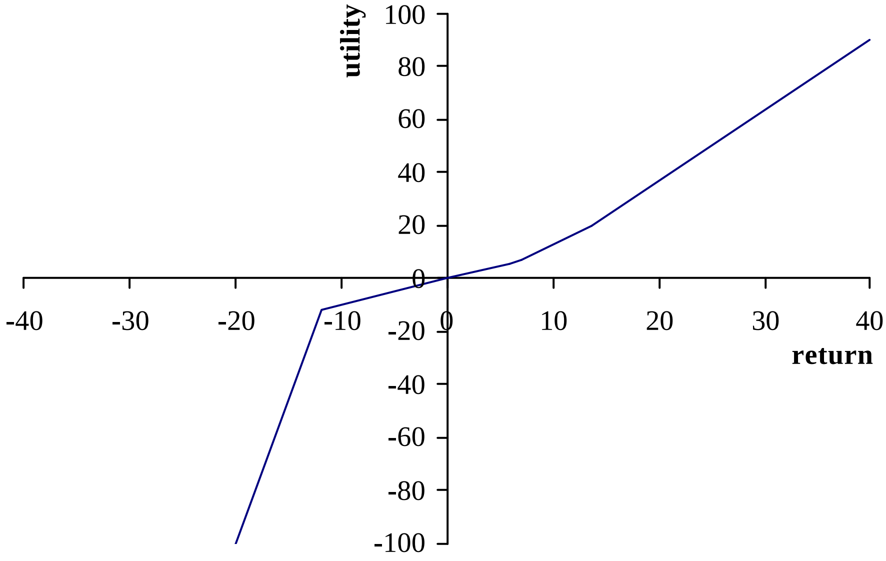

Figure 3B: Optimal piecewise-linear MSD utility function. This utility function rationalizes the Fama and French market portfolio relative to the 30 industry momentum portfolios. The utility function is constructed from the optimal solution $b^* \in \mathrm{B}(\Psi_{MSD})$ to (9) as

$$p(x|b^*) = \begin{cases} a_1 + b_1^* x & x \le \boldsymbol{x}_1^{\mathrm{T}} t \\ a_t + b_t^* x & \boldsymbol{x}_t^{\mathrm{T}} t < x \le \boldsymbol{x}_{t+1}^{\mathrm{T}} t < 0 \\ a_{z-1} + b_{z-1}^* x & \boldsymbol{x}_{z-1}^{\mathrm{T}} t \le x < 0 \\ a_z + b_z^* x & 0 \le x < \boldsymbol{x}_z^{\mathrm{T}} t \\ a_t + b_t^* x & 0 < \boldsymbol{x}_{t-1}^{\mathrm{T}} t \le x < \boldsymbol{x}_t^{\mathrm{T}} t \\ a_T + b_T^* x & x \ge \boldsymbol{x}_T^{\mathrm{T}} t \end{cases},$$

with $a \equiv (a_1 \cdots a_T)^{\mathrm{T}}$ such that the linear line segments are connected i.e. $a_t + b_t^* x = a_{t+1} + b_{t+1}^* x$ for all $t \in \Theta \setminus T$, and with $a_z = 0$.

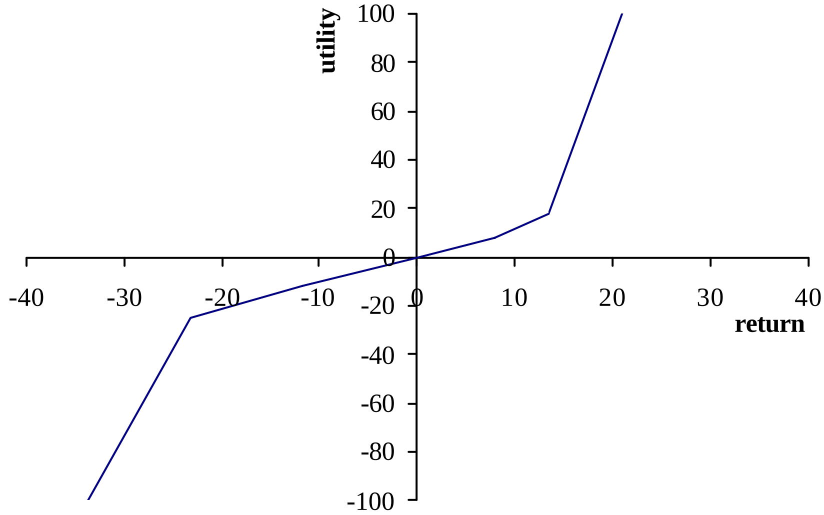

Figure 3C: Optimal piecewise-linear MSD utility function. This utility function rationalizes the Fama and French market portfolio relative to the 27 Carhart size-BE/ME-momentum portfolios. The utility function is constructed from the optimal solution $b^* \in \mathrm{B}(\Psi_{MSD})$ to (9) as

$$p(x|b^*) = \begin{cases} a_1 + b_1^* x & x \leq \mathbf{x}_1^{\mathrm{T}} t \\ a_t + b_t^* x & \mathbf{x}_t^{\mathrm{T}} t < x \leq \mathbf{x}_{t+1}^{\mathrm{T}} t < 0 \\ a_{z-1} + b_{z-1}^* x & \mathbf{x}_{z-1}^{\mathrm{T}} t \leq x < 0 \\ a_z + b_z^* x & 0 \leq x < \mathbf{x}_z^{\mathrm{T}} t \\ a_t + b_t^* x & 0 < \mathbf{x}_{t-1}^{\mathrm{T}} t \leq x < \mathbf{x}_t^{\mathrm{T}} t \\ a_T + b_T^* x & x \geq \mathbf{x}_T^{\mathrm{T}} t \end{cases},$$

with $a \equiv (a_1 \cdots a_T)^{\mathrm{T}}$ such that the linear line segments are connected i.e. $a_t + b_t^* x = a_{t+1} + b_{t+1}^* x$ for all $t \in \Theta \setminus T$, and with $a_z = 0$.

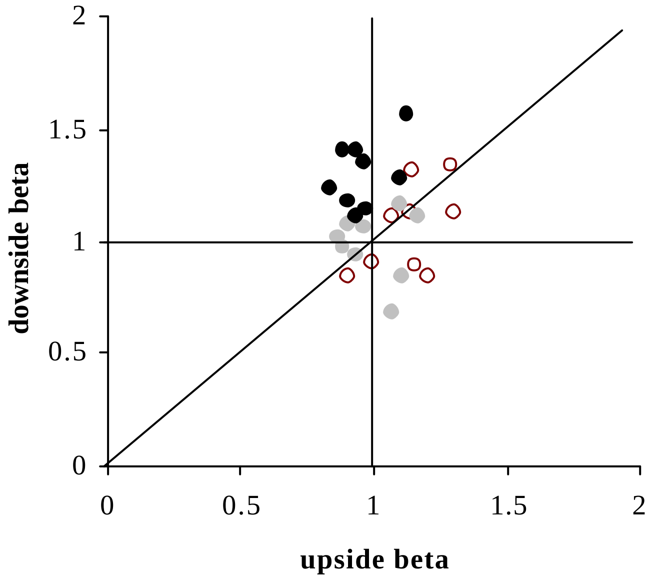

Figure 4: Upside and downside market beta's for the 27 Carhart size-BE/ME-momentum portfolios. The dark dots represent the 9 portfolios with the highest average returns, the bright dots are the 9 portfolios with the lowest average returns and the grey dots are the remaining 9 portfolios with medium average returns. Beta's are computed using OLS regression and using the full samples of monthly excess returns from July 1963 to December 1994.