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Order Statistics and the Linear Assignment Problem

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Abstract — Zusammenfassung

Order Statistics and the Linear Assignment Problem. Under mild conditions on the distribution function F, we analyze the asymptotic behavior in expectation of the smallest order statistic, both for the case that F is defined on $(-\infty, +\infty)$ and for the case that F is defined on $(0, \infty)$. These results yield asymptotic estimates of the expected optimal value of the linear assignment problem under the assumption that the cost coefficients are independent random variables with distribution function F.

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Ordnungsstatistiken und das lineare Zuordnungsproblem. Wir analysieren das asymptotische Verhalten des Erwartungswerts der kleinsten Ordnungsstatistik unter schwachen Voraussetzungen über die Verteilungsfunktion F. Dabei wird unterschieden, ob F auf ganz $(-\infty, +\infty)$ oder nur auf $(0, \infty)$ definiert ist. Die Ergebnisse liefern asymptotische Abschätzungen für den Erwartungswert des optimalen Wertes beim linearen Zuordnungsproblem, wobei angenommen wird, daß die Kostenkoeffizienten unabhängige Zufallsvariable mit Verteilungsfunktion F sind.

1. Introduction

Given an $n \times n$ matrix (a_{ij}), the *linear assignment problem* (LAP), is to find a permutation $\varphi \in S_n$ that minimizes $\sum_{i=1}^{n} a_{i\varphi(i)}$. This classical problem, which has many applications, can be solved efficiently by a variety of algorithms (see, e.g. (Lawler 1976)). It can be conveniently viewed as the problem of finding a *minimum weight perfect matching* in a *complete bipartite graph.* Here we shall be concerned with a *probabilistic analysis* of the *value* Z of the LAP, under the assumption that the coefficients a_{ij} are *independent, identically distributed* (i.i.d.) random variables with distribution function F. We shall be particularly interested in the asymptotic behavior of

$$E\underline{Z} = E \min_{\varphi \in S_n} \sum_{i=1}^{n} \underline{a}_{i\varphi(i)}. \tag{1}$$

Previous analyses of this nature have focused on several special choices for F. In the case that $\underline{a}_{ij}$ is *uniformly distributed* on $(0, 1)$, $E\underline{Z} = 0(1)$; the initial upper bound of 3 on the constant (Walkup 1979) was recently improved to 2 (Karp 1984). In the case that $-\underline{a}_{ij}$ is *exponentially distributed,* $E(-\underline{Z}) = 0\ (n \log n)$ (Loulou 1983).

We shall generalize the above results by showing that, under mild conditions on F, $E\underset{\sim}{Z}$ is asymptotic to $nF^{-1}(1/n)$. The interpretation of this result is that the asymptotic behavior of $E\underset{\sim}{Z}/n$ is determined by that of the *smallest order statistic*. In Section 2, we establish lower and upper bounds on the expected value of this statistic, that may be of interest on their own. In Section 3, we apply the technique developed in (Walkup 1979) to these bounds to arrive at the desired result. As we shall see, the condition on F under which the result is valid, is in a sense both a necessary and a sufficient one.

2. Order Statistics

Suppose that $\underset{\sim}{X}_i$ $(i=1, \ldots, n)$ is a sequence of i.i.d. random variables with distribution function F. It is well known that $\underset{\sim}{X}_i \overset{d}{=} F^{-1}(\underset{\sim}{U}_i)$, where the $\underset{\sim}{U}_i$ are independent and uniformly distributed on $(0, 1)$, and where

$$F^{-1}(y)=\inf\{v \mid F(v) \geq y\}.$$

The *smallest order statistic* (i.e., the minimum) of random variables $\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_n$ will be denoted by $\underset{\sim}{Y}_{1:n}$.

We first consider the case that

$$\lim_{n \to \infty} F^{-1}(1/n)=-\infty \tag{2}$$

under the additional assumption that

$$\int_{-\infty}^{+\infty} |x| \, F(dx)<\infty. \tag{3}$$

We start by deriving an upper bound on $E\underset{\sim}{X}_{1:n}$.

Lemma 1: (F *defined on* $(-\infty, +\infty)$).

$$E\underset{\sim}{X}_{1:n} \leq F^{-1}\left(\frac{1}{n}\right)\left(1-\left(1-\frac{1}{n}\right)^n\right)+n(1-F(0))^{n-1} \int_0^{\infty} x \, F(dx). \tag{4}$$

Proof: We observe that

$$\begin{aligned} E\underset{\sim}{X}_{1:n} &= E \min\{F^{-1}(\underset{\sim}{U}_1), \ldots, F^{-1}(\underset{\sim}{U}_n)\} \\ &= EF^{-1}(\underset{\sim}{U}_{1:n}). \end{aligned} \tag{5}$$

Let $\underset{\sim}{V}_i=\max\{\underset{\sim}{U}_i, 1/n\}$ $(i=1, \ldots, n)$. Clearly, $EF^{-1}(\underset{\sim}{U}_{1:n}) \leq EF^{-1}(\underset{\sim}{V}_{1:n})$.

Hence,

$$\begin{aligned} & EF^{-1}(\underset{\sim}{U}_{1:n}) \leq \\ & F^{-1}(1/n) \Pr\{\underset{\sim}{V}_{1:n}=1/n\}+E\left(F^{-1}(\underset{\sim}{V}_{1:n}) \cdot I_{V_{1:n}>1/n}\right)= \\ & F^{-1}(1/n)(1-\Pr\{\underset{\sim}{U}_{1:n} \geq 1/n\})+n \int_{1/n}^{1} F^{-1}(x)(1-x)^{n-1} \, dx. \end{aligned} \tag{6}$$

Now (2) and (3) imply that the latter term is bounded by

$$\begin{aligned} & n \int_{F(0)}^{1} F^{-1}(x)\,(1-x)^{n-1}\,dx \leq \\ & n\big(1-F(0)\big)^{n-1} \int_{F(0)}^{1} F^{-1}(x)\,dx = \\ & n\big(1-F(0)\big)^{n-1} \int_{0}^{\infty} x\,F(dx). \end{aligned} \tag{7}$$

Together, (6) and (7) imply (4). □

Since $1-F(0)<1$, we obtain as an immediate consequence that

$$\liminf_{n\to\infty} \frac{E\underline{X}_{1:n}}{F^{-1}(1/n)} \geq 1-\frac{1}{e}. \tag{8}$$

To derive a lower bound on $E\underline{X}_{1:n}$ of the same form (and thus an upper bound on $E\underline{X}_{1:n}/F^{-1}(1/n)$), an assumption is needed on the rate of decrease of F when $x\to-\infty$). We shall assume that F is a function of *positive decrease at* $-\infty$, i.e., that

$$\liminf_{x\to\infty} \frac{F(-x)}{F(-ax)} > 1 \tag{9}$$

for some $a>1$. It can be shown (De Haan and Resnick 1981) that this condition implies that

$$\alpha(F) = \lim_{a\to\infty} \frac{\ln\big(\liminf_{x\to\infty} F(-x)/F(-ax)\big)}{\ln a} \tag{10}$$

exists and is positive. The condition is satisfied, for instance, when $F(x)$ decreases polynomially $\big(0<\alpha(F)<\infty\big)$ or exponentially $\big(\alpha(F)=\infty\big)$ fast when $x\to-\infty$. Condition (9) implies and is equivalent with (De Haan and Resnick 1981)

$$\limsup_{y\to\infty} \frac{F^{-1}(1/ay)}{F^{-1}(1/y)} < \infty \tag{11}$$

with $a>1$. Again,

$$\lim_{a\to\infty} \frac{\ln\big(\limsup_{y\to\infty} F^{-1}(1/ay)/F^{-1}(1/y)\big)}{\ln a} \tag{12}$$

can be shown to exist and to be equal to $\beta(F)=1/\alpha(F)$.

Theorem 1: $\big($*F defined on* $(-\infty, +\infty)\big)$.

$$\limsup_{n\to\infty} \frac{E\underline{X}_{1:n}}{F^{-1}(1/n)} < \infty \tag{13}$$

if and only if F is a function of positive decrease at $-\infty$ *with* $\alpha(F)>1$.

Proof: We note that

$$EF^{-1}(\underline{U}_{1:n})=n\int_{0}^{F(0)} F^{-1}(x)(1-x)^{n-1}\,dx+n\int_{F(0)}^{1} F^{-1}(x)(1-x)^{n-1}\,dx. \tag{14}$$

The latter term is bounded by

$$n(1-F(0))^{n-1}\int_{0}^{\infty} x\,F(dx) \tag{15}$$

and hence

$$\lim_{n\to\infty}\frac{n\int_{F(0)}^{1} F^{-1}(x)(1-x)^{n-1}\,dx}{F^{-1}(1/n)}=0. \tag{16}$$

If $nF(0)>1$, the former term is bounded from below by

$$\begin{aligned}&\frac{n}{1-F(0)}\int_{0}^{F(0)} F^{-1}(x)(1-x)^{n}\,dx\geq\\&\frac{n}{1-F(0)}\int_{0}^{F(0)} F^{-1}(x)\exp(-nx)\,dx=\\&\frac{1}{1-F(0)}\int_{0}^{1} F^{-1}(x/n)\exp(-x)\,dx+\\&\frac{1}{1-F(0)}\int_{1}^{nF(0)} F^{-1}(x/n)\exp(-x)\,dx.\end{aligned} \tag{17}$$

The monotonicity of F^{-1} implies that, for large n, the latter term is at least as large as

$$\frac{F^{-1}(1/n)}{1-F(0)}\int_{1}^{\infty}\exp(-x)\,dx. \tag{18}$$

Also, (11), $\alpha(F)>1$ and (Frenk 1983, Theorem 1.1.7) imply that there exist constants $B>0$ and $\beta\in(0,1)$ such that for sufficiently large n and $x\in(0,1)$

$$0<\frac{F^{-1}(x/n)}{F^{-1}(1/n)}\leq Bx^{-\beta} \tag{19}$$

(cf. (12)), so that, for sufficiently large n,

$$\frac{\int_{0}^{1} F^{-1}(x/n)\exp(-x)\,dx}{F^{-1}(1/n)}\leq B\int_{0}^{1} x^{-\beta}\exp(-x)\,dx<\infty. \tag{20}$$

Together, (20) and (18) imply (13).

Now, suppose that (13) is satisfied, i.e., that

$$\limsup_{n\to\infty}\frac{n\int_{0}^{F(0)} F^{-1}(x)(1-x)^{n-1}\,dx}{F^{-1}(1/n)}<\infty. \tag{21}$$

If $a<nF(0)$, then

$$nF^{-1}(a/n)\int_0^{a/n}(1-x)^{n-1}\,dx\geq n\int_0^{F(0)}F^{-1}(x)(1-x)^{n-1}\,dx \tag{22}$$

and hence

$$\limsup_{n\to\infty}\frac{F^{-1}(a/n)}{F^{-1}(1/n)}=0\left(\frac{1}{1-\exp(-a)}\right). \tag{23}$$

Hence (cf. (11)) F is of positive decrease with $\alpha(F)\geq 1$, and all that has to be shown is that $\alpha(F)\neq 1$. Thus, it is sufficient to show that $\alpha(F)=1$ implies that

$$\limsup_{n\to\infty}\frac{\int_0^1 F^{-1}(x/n)\,dx}{F^{-1}(1/n)}=\frac{\int_1^\infty F^{-1}(1/xn)\,x^{-2}\,dx}{F^{-1}(1/n)}=\infty. \tag{24}$$

In (De Haan and Resnick 1981) it is shown that there exists a sequence n_k and a function $\varphi(z)\geq z\,(z\geq 1)$ such that

$$\lim_{k\to\infty}\frac{F^{-1}(1/xn_k)}{F^{-1}(1/n_k)}=\varphi(x)\geq x \tag{25}$$

for almost every $x\geq 1$, i.e., except in the (countably many) points x where φ is discontinuous. But this implies the existence of a sequence x_m, with $x_m\in(2m,2m+1)$, such that for all N

$$\limsup_{k\to\infty}\frac{\int_1^\infty F^{-1}(1/xn_k)x^{-2}\,dx}{F^{-1}(1/n_k)}\geq\sum_{m=1}^{N}\varphi(x_m)\left(\frac{1}{x_m}-\frac{1}{x_{m+1}}\right)\geq\sum_{m=1}^{N}\left(1-\frac{2m+1}{2m+2}\right) \tag{26}$$

which goes to $+\infty$ when $N\to\infty$. □

Lemma 1 and Theorem 1 imply that, under conditions (2) and (3), the following statements are equivalent:

(i) F is a function of positive decrease at $-\infty$ with $\alpha(F)>1$;

(ii) $1-e^{-1}\leq\liminf_{n\to\infty}\dfrac{E\underline{X}_{1:n}}{F^{-1}(1/n)}\leq\limsup_{n\to\infty}\dfrac{E\underline{X}_{1:n}}{F^{-1}(1/n)}<\infty$.

Now let us deal with the (much simpler) case that

$$\lim_{n\to\infty}F^{-1}\left(\frac{1}{n}\right)=0. \tag{27}$$

No additional assumption such as (3) is needed.

Lemma 2: (F *defined on* $(0,\infty)$).

$$E\underline{X}_{1:n}\geq F^{-1}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^n. \tag{28}$$

Proof: Define

$$\underline{W}_i = \begin{cases} 1/n & \text{if } \underline{U}_i > 1/n \\ 0 & \text{if } \underline{U}_i \leq 1/n. \end{cases} \tag{29}$$

Then

$$E\underline{X}_{1:n} = EF^{-1}(\underline{U}_{1:n}) \geq EF^{-1}(\underline{W}_{1:n}) = F^{-1}\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^n. \tag{30}$$

□

Again, let us assume that F^{-1} satisfies (11), or that, equivalently,

$$\liminf_{x \downarrow 0} \frac{F(x)}{F(ax)} > 1 \tag{31}$$

for some $a < 1$. Thus, F being defined on $(0, \infty)$, the function is assumed to be of positive decrease at 0.

Theorem 2: $\big(F$ *defined on* $(0, \infty)\big)$.

$$\limsup_{n \to \infty} \frac{E\underline{X}_{1:n}}{F^{-1}(1/n)} < \infty \tag{32}$$

if and only if F *is a function of positive decrease at* 0.

Proof:

$$\begin{aligned} E\underline{X}_{1:n} &= n \int_0^1 F^{-1}(x)(1-x)^{n-1}\, dx \leq \\ &n \int_0^1 F^{-1}(x) \exp(-nx)\, dx = \\ &\int_0^n F^{-1}(x/n) \exp(-x)\, dx. \end{aligned} \tag{33}$$

As before, we split the integral in two parts, corresponding to $x \in (0, 1)$ and $x \in (1, n)$ respectively. The first part is bounded by

$$F^{-1}(1/n) \int_0^1 \exp(-x)\, dx. \tag{34}$$

As in the proof of Theorem 1, we can bound

$$\frac{\int_1^n F^{-1}(x/n) \exp(-x)\, dx}{F^{-1}(1/n)} \tag{35}$$

by invoking (12). This yields the proof of (32).

Conversely, (32) implies that, since for $0 < a < 1$

$$F^{-1}(a/n) \int_{a/n}^1 (1-x)^{n-1}\, dx \leq \int_0^1 F^{-1}(x)(1-x)^{n-1}\, dx, \tag{36}$$

we may conclude that

$$\limsup_{n\to\infty} \frac{F^{-1}(a/n) \int_{a/n}^{1} (1-x)^{n-1}\,dx}{F^{-1}(1/n)} < \infty \tag{37}$$

which leads directly to (11). □

Hence, in the case that (27) holds, we have the following two equivalent conditions:

(i) F is a function of positive decrease at 0;

(ii) $\frac{1}{e} \le \liminf_{n\to\infty} \frac{E\underline{X}_{1:n}}{F^{-1}(1/n)} \le \limsup_{n\to\infty} \frac{E\underline{X}_{1:n}}{F^{-1}(1/n)} < \infty$.

We note that no condition on $\alpha(F)$ occurs in (i). We also note that the case that F is defined on (c, ∞) for any finite c can easily be reduced to the above one.

3. The Linear Assignment Problem

Our analysis of the linear assignment problem is based on a technique developed in (Walkup 1981). Very roughly speaking, this approach can be summarized as follows: if in a complete, randomly weighted bipartite graph all edges but a few of the smaller weighted ones at each node are removed, then the resulting graph will still contain a perfect matching with high probability. In that way we derive a probabilistic upper bound on the value Z of the LAP.

More precisely, assume that the LAP coefficients $\underline{a}_{ij}$ $(i,j=1,\ldots,n)$ are i.i.d. random variables with distribution function F. It is possible to construct two sequences $\underline{b}_{ij}$ and $\underline{c}_{ij}$ of i.i.d. random variables such that

$$\underline{a}_{ij} \underline{\underline{d}} \min\{\underline{b}_{ij}, \underline{c}_{ij}\}. \tag{38}$$

Indeed, since we desire that

$$\Pr\{\underline{a}_{ij} \ge x\} = \Pr\{\min\{\underline{b}_{ij}, \underline{c}_{ij}\} \ge x\} = \Pr\{\underline{b}_{ij} \ge x\}\Pr\{\underline{c}_{ij} \ge x\},$$

the common distribution function $\bar{F}$ of $\underline{b}_{ij}$ and $\underline{c}_{ij}$ will have to satisfy

$$1 - F(x) = (1 - \bar{F}(x))^2 \tag{39}$$

so that

$$\bar{F}^{-1}(x) = F^{-1}(1 - (1-x)^2). \tag{40}$$

For future reference, we again observe that $\underline{b}_{ij} \underline{\underline{d}} \bar{F}^{-1}(\underline{V}_{ij})$ and $\underline{c}_{ij} \underline{\underline{d}} \bar{F}^{-1}(\underline{W}_{ij})$, where $\underline{V}_{ij}$ and $\underline{W}_{ij}$ are i.i.d. and uniformly distributed on $(0, 1)$. If we fix any pair of indices (i,j), then the order statistics of $\underline{V}_{ij}$ $(j=1,\ldots,n)$ are independent of and distributed as the order statistics of $\underline{W}_{ij}$ $(i=1,\ldots,n)$; we shall denote these order statistics by $\underline{V}_{1:n} \le \underline{V}_{2:n} \le \ldots \le \underline{V}_{n:n}$ and $\underline{W}_{1:n} \le \underline{W}_{2:n} \le \ldots \le \underline{W}_{n:n}$ respectively.

Now, let $\underline{G}_n$ be the complete directed bipartite graph on $S=\{s_1,\ldots,s_n\}$ and $T=\{t_1,\ldots,t_n\}$ with weight $\underline{b}_{ij}$ on arc (s_i, t_j) and $\underline{c}_{ij}$ on arc (t_j, s_i). For any realization $b_{ij}(\omega), c_{ij}(\omega)$, we construct $G_n(d,\omega)$ by removing arc (s_i, t_j) unless $b_{ij}(\omega)$ is one of the d smallest weights at s_i and by removing arc (t_j, s_i) unless $c_{ij}(\omega)$ is one of the d smallest weights at t_j. Let us define $P(n,d)$ to be the probability that $\underset{\sim}{G}_n(d)$ contains a (perfect) matching. A counting argument can now be used to prove (Walkup 1981) that

$$1-P(n,2)\leq\frac{1}{5n} \tag{41}$$

$$1-P(n,d)\leq\frac{1}{122}\left(\frac{d}{n}\right)^{(d+1)(d-2)} \quad (d\geq 3). \tag{42}$$

We use these estimates to prove two theorems about the asymptotic value of $E\underset{\sim}{Z}$. Again, we first deal with the case that

$$\lim_{n\to\infty} F^{-1}(1/n)=-\infty \tag{43}$$

under the additional assumption that

$$\int_{-\infty}^{+\infty} |x|\,F(dx)<\infty. \tag{44}$$

Theorem 3: $(F$ *defined on* $(-\infty,+\infty))$.

If F is a function of positive decrease at $-\infty$ *with* $\alpha(F)>1$, *then*

$$\left(1-\frac{3}{2e^{1/2}}\right)^2\leq\liminf_{n\to\infty}\frac{E\underset{\sim}{Z}}{nF^{-1}(1/n)}\leq\limsup_{n\to\infty}\frac{E\underset{\sim}{Z}}{nF^{-1}(1/n)}<\infty. \tag{45}$$

Proof: Since

$$E\underset{\sim}{Z}\geq nE\underset{\sim}{a}_{1:n} \tag{46}$$

the upper bound in (45) is an immediate consequence of Theorem 1.

For the lower bound we apply (41) and (42) as follows.

Obviously,

$$\begin{aligned}E\underset{\sim}{Z}=&P(n,2)\,E\big(\underset{\sim}{Z}\mid \underset{\sim}{G}_n(2)\text{ contains a matching}\big)\\&+(1-P(n,2))\,E\big(\underset{\sim}{Z}\mid \underset{\sim}{G}_n(2)\text{ does not contain a matching}\big).\end{aligned} \tag{47}$$

The second conditional expectation is bounded trivially by $nE\underset{\sim}{a}_{n:n}=0(n^2)$ (cf. (44)). The first conditional expectation is bounded by

$$nE\bar{F}^{-1}(\max\{\underset{\sim}{V}_{2:n},\underset{\sim}{W}_{2:n}\}). \tag{48}$$

Hence it suffices to prove that

$$\liminf_{n\to\infty}\frac{E\bar{F}^{-1}(\max\{\underset{\sim}{V}_{2:n},\underset{\sim}{W}_{2:n}\})}{F^{-1}(1/n)}\geq\left(1-\frac{3}{2e^{1/2}}\right)^2. \tag{49}$$

To this end, define $x_n=1-(1-1/n)^{1/2}$ and note from (40) that $\bar{F}^{-1}(x_n)=F^{-1}(1/n)$ so that

$$\begin{aligned}&E\bar{F}^{-1}(\max\{\underset{\sim}{V}_{2:n},\underset{\sim}{W}_{2:n}\})\leq\\&F^{-1}(1/n)\Pr\{\underset{\sim}{V}_{2:n}\leq x_n,\underset{\sim}{W}_{2:n}\leq x_n\}+\\&E\big(\bar{F}^{-1}(\max\{\underset{\sim}{V}_{2:n},\underset{\sim}{W}_{2:n}\})\,I_{\max\{\underset{\sim}{V}_{2:n},\underset{\sim}{W}_{2:n}\}\geq x_n}\big).\end{aligned} \tag{50}$$

To bound the first term, note that

$$\begin{aligned}
&\Pr\{\underline{V}_{2:n}\le x_n,\ \underline{W}_{2:n}\le x_n\}=\\
&(\Pr\{\underline{V}_{2:n}\le x_n\})^2=\\
&\left(\sum_{k=2}^{n}\binom{n}{k}x_n^k(1-x_n)^{n-k}\right)^2=\\
&\left(1-(1-x_n)^n-n x_n(1-x_n)^{n-1}\right)^2
\end{aligned}\tag{51}$$

which tends to $(1-3/(2e^{1/2}))^2$ as $n\to\infty$.

The second term in (50) is equal to

$$\begin{aligned}
&\int_{x_n}^{1}\bar{F}^{-1}(x)\,d(\Pr\{\underline{V}_{2:n}\le x\}^2)=\\
&2n(n-1)\int_{x_n}^{1}\bar{F}^{-1}(x)\Pr\{\underline{V}_{2:n}\le x\}\,x(1-x)^{n-2}\,dx.
\end{aligned}\tag{52}$$

After a transformation $x=1-(1-y)^{1/2}$ (cf. (40)), we find that (52) for large n is bounded by

$$\begin{aligned}
&n(n-1)\int_{F(0)}^{1}F^{-1}(y)\left(1-(1-y)^{1/2}\right)(1-y)^{(n-3)/2}\,dy\le\\
&n(n-1)\left(1-F(0)\right)^{(n-3)/2}\int_{F(0)}^{1}F^{-1}(y)\,dy,
\end{aligned}\tag{53}$$

thus completing the proof of (49). □

Again, the case that

$$\lim_{n\to\infty}F^{-1}(1/n)=0\tag{54}$$

is much simpler to analyze.

Theorem 4: (F *defined on* $(0,\infty)$).

If F is a function of positive decrease at 0, *then*

$$0<\liminf_{n\to\infty}\frac{E\underline{Z}}{nF^{-1}(1/n)}\le\limsup_{n\to\infty}\frac{E\underline{Z}}{nF^{-1}(1/n)}<\infty.\tag{55}$$

Proof: We have, for all $d\ge 3$, that

$$\begin{aligned}
E\underline{Z}\le&\left(1-P(n,d)\right)E(\underline{Z}\mid\underline{G}_n(d)\text{ does not contain a matching})+\\
&+P(n,d)\,E(\underline{Z}\mid\underline{G}_n(d)\text{ does contain a matching})\\
\le&\,0(d^{d^2-d-2}n^{-d^2+d+4})+nE\bar{F}^{-1}(\max\{\underline{V}_{d:n},\underline{W}_{d:n}\}).
\end{aligned}\tag{56}$$

As in (19), we use constants $B,\beta>0$ to bound $n^{-d^2+d+4}/nF^{-1}(1/n)$ by $Bn^{-d^2+d+3+\beta}$, and choose $\bar{d}$ such that $-\bar{d}^2+\bar{d}+3+\beta<0$. For this value $\bar{d}$, we bound $E\bar{F}^{-1}(\max\{\underline{V}_{d:n},\underline{W}_{d:n}\})$ as before by

$$\bar{d}\binom{n}{\bar{d}}\int_0^1\bar{F}^{-1}(x)\Pr\{\underline{V}_{\bar{d}:n}\le x\}\,(1-x)^{n-\bar{d}}x^{\bar{d}-1}\,dx.$$

These two bounding arguments yield that $\limsup_{n\to\infty} E\underline{Z}/nF^{-1}(1/n) < \infty$. The lower bound on $\liminf_{n\to\infty} E\underline{Z}/nF^{-1}(1/n)$ follows from (46). □

The conditions of positive decrease on F turned out to be necessary as well as sufficient to describe the asymptotic behavior of the smallest order statistic (Theorems 1 and 2) that play an important role in the above theorems. It can easily be seen that this condition is necessary and sufficient in Theorem 4 as well, and one suspects that the same holds for Theorem 3.

Theorems 3 and 4 capture the behavior of the expected LAP value for a wide range of distributions. To derive almost sure convergence results under the same mild conditions of F, the results from (Walkup 1981) would have to be strengthened further. For special cases such as the uniform distribution, however, almost sure results can indeed be derived quite easily (see (Van Houweninge 1984)).

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