Theoretical restrictions on the parameters of the indirect addilog system revisited

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Abstract

The correct parameter restrictions- less restrictive than commonly thought-of the indirect addilog system (IAS) are derived. Under correct restrictions, the IAS is superior to the linear expenditure system in computable general equilibrium models with data scarcity.

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1. Introduction

Because of data scarcity, the linear expenditure system (LES) is often adopted in computable general equilibrium (CGE) models for the description of household preferences. The disadvantages of LES are that Engel curves are linear (preferences are quasi homothetic, see Deaton and Muellbauer, 1980, section 5.4), that only inelastic demand (absolute value of the own price elasticity smaller than one) is permitted, that all commodities are gross complements (cross price elasticities positive), and that inferior commodities (expenditure elasticities smaller than zero) are ruled out (see, Chung, 1994, chapter 2). A competing model for the description of consumer preferences is the indirect addilog system (IAS). It has been proved by Somermeyer and Langhout (1972) that IAS exhibits Engel curves that are non-linear, that elastic demand is permitted, that inferior goods are allowed for, and that cross price elasticities can be negative, permitting commodities to be gross substitutes as well.

De Boer and Missaglia (2005) prove that IAS needs exactly the same outside information (a social accounting matrix, all income elasticities and the Frisch parameter) as the LES in order to assign a numerical value to its parameters (i.e., the models have the same data requirement for calibration) and, in view of the richer description of consumer behavior, they propose to use IAS rather than LES in CGE modeling.

Houthakker (1960) has shown that IAS can be derived from an indirect utility function. An important question is under which parameter restrictions the indirect utility function satisfies the theoretical properties given in Varian (1992, p. 102): homogeneity of degree zero in prices and total expenditure, nonincreasing in prices and nondecreasing in total expenditure, quasi-convexity in prices, and continuity in all prices and total expenditure.

In the literature, there is no consensus, as rightly pointed out by Murty (1982), which is caused by the fact that there are (at least) two alternative ways of defining the indirect utility function. Murty gives, without proof, the correct parameter restrictions. We did not find the proof in the literature, either, so that the goal of this paper is to supply the full proof.

The rest of the paper is organized as follows: in section 2, we review the parameter restrictions presented in literature, section 3 is devoted to the proof that the parameter restrictions by Murty are the correct ones, whereas section 4 gives some concluding remarks.

2. Parameter restrictions in literature

Let Y_i denote the quantity that a consumer demands from commodity i (=1,…, N) and P_i the corresponding price. We assume that a consumer maximizes his utility subject to his budget constraint:

$$C = \sum_{i=1}^{N} P_i Y_i \qquad (1)$$

where C denotes expenditure. Let the vector of prices be denoted by $P' = (P_1 ... P_N)$.

Houthakker (1960), who does not give general parameter restrictions, specifies the indirect utility function as:

$$V^*(P,C) = \sum_{i=1}^{N} \alpha_i^* \left(C / P_i\right)^{\beta_i} \quad (2)$$

Hanoch (1975) derives (2) as special case from a more general function, the so-called specialized CDE model, and imposes as restrictions: $\alpha_i^* > 0$ (page 411, his B_i which is our α_i^*) and $\beta_i > 0$ (page 416, his b_i which is equivalent to $-\beta_i$ in our notation). These parameter restrictions are also mentioned, a.o., in Deaton and Muellbauer (1980, p. 84) and in Chung (1991, p.42).

Akin and Stewart (1979) give as parameter restrictions: $\alpha_i^* > 0$; $\beta_i > 0$ or: $\alpha_i^* < 0$; $\beta_i < 0$; no more than one β_i having a value less than -1.

On the other hand, Leser (1941), Somermeyer and Wit (1956), Somermeyer and Langhout (1972), Somermeyer (1974) and Gamaletsos (1974) start directly from the demand equations and impose as restriction on β_i that it should be larger than -1, allowing for negative values as well.

Murty (1982) correctly points out that the confusion about the signs of the parameters β_i has arisen due to the two ways of specifying the indirect utility function. He proposes the reparametrization $\alpha_i = \alpha_i^* \beta_i$ so that his specification reads:

$$V(P,C) = \sum_{i=1}^{N} \frac{\alpha_i (C / P_i)^{\beta_i}}{\beta_i} \quad (3)$$

He states, without proof, that the parameter restrictions are:

$\alpha_i \geq 0$ and $\beta_i \geq -1$ for all i, the equality holding for at most N-1 items for α_i and for at most one commodity for β_i.

In the next section, we slightly generalize Murty's specification by subtracting the constant $\sum_{i=1}^{N} \alpha_i / \beta_i \, (= \sum_{i=1}^{N} \alpha_i^*)$ to yield:

$$V(P,C) = \sum_{i=1}^{N} \alpha_i \frac{(C / P_i)^{\beta_i} - 1}{\beta_i} \quad (4)$$

The reason is that in the Box-Cox form (4) the special case that $\beta_i = 0$ is defined to be equal to $\ln(C / P_i)$, whereas it is not defined in the formulation (3). Because utility is ordinal, (3) and (4) represent the same preference ordering.

We note that if $\alpha_i = 0$, commodity i does not appear in the indirect utility function (4) so that, for at least one item, it should hold true that $\alpha_i \neq 0$ (or, equivalently, that the equality sign holds for at most N-1 items for α_i). In the section 3, we only consider items for which $\alpha_i \neq 0$ and give the proof that Murty's restrictions are the correct ones.

3. Proof

An indirect utility function is well behaved (see Varian, 1992, p. 102) if it is:

(i) homogeneous of degree zero in prices (P_i) and total expenditure (C)

(ii) nonincreasing in prices (P_i), and nondecreasing in total expenditure (C)

(iii) quasi-convex in prices (P_i), but since we want to obtain a *unique* solution to the problem of maximization utility under the budget constraint, we impose that the indirect utility function should be *strictly* quasi-convex.

(iv) continuous in all prices $P_i > 0$ and in $C > 0$

It is straightforward to verify from (4) that (i) and (iv) are met.

To verify (ii), we take the derivative of the indirect utility function (4) with respect to C:

$$\frac{\partial V(P,C)}{\partial C} = \sum_i \alpha_i P_i^{-\beta_i} C^{\beta_i - 1} \tag{5}$$

which is a positive function of C for all $P_i > 0$, if:

$$\alpha_i > 0 \tag{6}$$

(In case $\alpha_i = 0$, commodity i is not in the consumption bundle, as noted in the previous section).

We derive from (4) that under (6):

$$\frac{\partial V(P,C)}{\partial P_i} = -\alpha_i P_i^{-\beta_i - 1} C^{\beta_i} < 0 \tag{7}$$

i.e. the indirect utility function is decreasing in prices.

Consequently, under parameter restrictions (6), (ii) is met.

Finally, we turn to (iii). We have to prove that the indirect utility function is strictly quasi-convex in prices if, besides (6), $\beta_i \geq -1$ for all i, the equality holding for at most for one commodity.

To alleviate the notation, we put $C = 1$, so that the indirect utility function reads:

$$V(P) = \sum_{i=1}^{n} \alpha_i \frac{P_i^{-\beta_i} - 1}{\beta_i} \tag{8}$$

We will give the proof in three steps:

(i) if $\beta_i > -1$ for all i, then V (P) is strictly quasi-convex;

(ii) if one $\beta_i = -1$ and $\beta_j > -1$ for all $j \neq i$, then V (P) is strictly quasi-convex;

(iii) if two (or more) $\beta_i = -1$, then V (p) is not strictly quasi-convex.

Ad (i) From (7) we derive:

$$\frac{\partial^2 V(P)}{\partial P_i \partial P_j} = (\beta_i + 1)\alpha_i P_i^{-\beta_i - 2} \quad \text{for } i = j$$

$$= 0 \quad \text{for } i \neq j$$

Consequently, if $\beta_i > -1$ for all i, V (P) is strictly convex, hence strictly quasi-convex.

Ad (ii) Without loss of generality, we consider the case that:

$\beta_1 = -1$ and $\beta_i > -1 \;\; i = 2,...,N$

We partition the price vector as:

$P' = [P_1 \; P'_{-1}]$ where $P'_{-1} = (P_2 ... P_N)$

Then, we can rewrite the indirect utility function (8) as:

$$V(P) = -\alpha_1(P_1 - 1) + \sum_{i=2}^{N} \alpha_i \frac{P_i^{-\beta_i} - 1}{\beta_i} = -\alpha_1 P_1 + \varphi(P_{-1}) \qquad (9)$$

Since $\beta_i > -1$, $i = 2,...,N$, the function $\varphi(P_{-1})$ is strictly convex (see (i)).
Let two price vectors P^1 and P^2 ($P^1 \neq P^2$) be given and let λ be a number between 0 and 1, and substitute the convex combination $\lambda P_{-1}^1 + (1-\lambda)P_{-1}^2$ into (9):

$$V[\lambda P^1 + (1-\lambda)P^2] = -\alpha_1[\lambda P_1^1 + (1-\lambda)P_1^2] + \varphi[\lambda P_{-1}^1 + (1-\lambda)P_{-1}^2] \qquad (10)$$

Since $P^1 \neq P^2$, we have to consider two cases:

(iia) $P_{-1}^1 \neq P_{-1}^2$

(iib) $P_{-1}^1 = P_{-1}^2$, but $P_1^1 \neq P_1^2$, without loss of generality: $P_1^1 > P_1^2$

Ad (iia) Because of the strict convexity of $\varphi(P_{-1})$ we have:

$$\varphi[\lambda P_{-1}^1 + (1-\lambda)P_{-1}^2] < \lambda\varphi(P_{-1}^1) + (1-\lambda)\varphi(P_{-1}^2)$$

Then it follows from (10), using (9), that:

$$V[\lambda P^1 + (1-\lambda)P^2] < \lambda[-\alpha_1 P_1^1 + \phi(P_{-1}^1)] + (1-\lambda)[-\alpha_1 P_1^2 + \phi(P_{-1}^2)]$$
$$= \lambda V(P^1) + (1-\lambda)V(P^2) \leq \max[V(P^1), V(P^2)]$$

Consequently, V (P) is strictly quasi-convex.

Ad (iib) Because $P_{-1}^1 = P_{-1}^2$, we have:

$$\lambda\varphi(P_{-1}^1) + (1-\lambda)\varphi(P_{-1}^2) = \varphi(P_{-1}^2) \tag{11}$$

and because $\alpha_1 > 0$ and $P_1^1 > P_1^2$, we have:

$$-\lambda\alpha_1 P_1^1 - (1-\lambda)\alpha_1 P_1^2 < -\alpha_1 P_1^2 \tag{12}$$

Substitution of (11) and (12) into (10) yields:

$$V[\lambda P^1 + (1-\lambda)P^2] < -\alpha_1 P_1^2 + \varphi(P_{-1}^2) = V(P^2) = \max[V(P^1), V(P^2)]$$

by virtue of $P_1^1 > P_1^2$ (and $P_{-1}^1 = P_{-1}^2$).

Consequently, V (P) is strictly quasi-convex.

(iii) Suppose, without loss of generality that $\beta_1 = \beta_2 = -1$.

Let P^1 and P^2 be two price vectors which only differ in the first two elements and which yield the same value of indirect utility. Then any convex combination of these two vectors yields that value as well. Consequently, V (P) is not strictly quasi-convex.
This completes the proof that Murty's (1982) parameter restrictions:

$\alpha_i \geq 0$ and $\beta_i \geq -1$ for all i, the equality holding for at most N-1 items for α_i and for at most one commodity for β_i

are the correct ones.

It follows from this proof that Leser (1941), Somermeyer and Wit (1956), Somermeyer and Langhout (1972), Somermeyer (1974) and Gamaletsos (1974) only deal with strict convexity, rather than with strict quasi-convexity.

4. Concluding remarks

(i) The demand equations are obtained from (3) or (4) using Roy's identity:

$$Y_i = -\frac{\partial V(P,C)/\partial P_i}{\partial V(P,C)/\partial C} = \frac{\alpha_i (C/P_i)^{\beta_i+1}}{\sum_{j=1}^{N} \alpha_j (C/P_j)^{\beta_j}} \tag{13}$$

It follows from (13) that the parameters α_i are not identified: multiplication with the same scalar leads to the same demand equations. That is the reason why we impose the identifying restriction:

$$\sum_{i=1}^{N} \alpha_i = 1 \tag{14}$$

(ii) Application of Roy's identity to the Houthakker version of the indirect utility, (2), leads to the same functional form as (13), with α_i replaced by $\alpha_i^* \beta_i$. As correctly pointed out by Murty (1982), estimation of the parameters α_i and β_i from (13) implies estimation of α_i^* and β_i (and vice versa).

(iii) From (13) we derive own price and cross price elasticities:

$$E(Y_i, P_i) = -\beta_i (1 - e_i) - 1 \tag{15}$$

$$E(Y_i, P_j) = \beta_j e_j \tag{16}$$

where $e_i = P_i Y_i / C$ denotes the budget share of commodity i.

It follows from (15) and (16) that the parameter restrictions $\beta_i > 0$, given by Hanoch (1975), Deaton and Muellbauer (1980) and Chung (1991), imply that:

$E(Y_i, P_i) < -1$ and $E(Y_i, P_j) > 0$

Consequently, these restrictions rule out the existence of inelastic demand ($-1 < E(Y_i, P_i) < 0$) and of gross complements ($E(Y_i, P_j) < 0$), which - in the context of CGE modeling - is empirically inadequate.

It follows from (15) and (16) that Murty's restrictions ($\beta_i \geq -1$, with the equality sign for at most one item) do allow for inelastic demand, and for commodities to be gross complements as well.

Consequently, the indirect addilog model is more flexible than is commonly suggested in the literature.

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