Symmetry groups for beta-lattices

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Abstract

We present a construction of symmetry plane-groups for quasiperiodic point-sets named beta-lattices. The framework is issued from beta-integers counting systems. The latter are determined by Pisot-Vijayaraghavan (PV) algebraic integers $\beta > 1$. Beta-lattices are vector superpositions of beta-integers. The sets of beta-integers can be equipped with abelian group structures and internal multiplicative laws. When $\beta = (1+\sqrt{5})/2$, $1+\sqrt{2}$ and $2+\sqrt{3}$, we show that these arithmetic and algebraic structures lead to freely generated symmetry plane-groups for beta-lattices. These plane-groups are based on repetitions of discrete "adapted rotations and translations". Hence beta-lattices, endowed with these adapted rotations and translations, can be viewed like lattices. The quasiperiodic function $\rho_S(n)$, defined on the set of beta-integers as counting the number of small tiles between the origin and the n^{th} beta-integer, plays a central part in these new group structures. In particular, this function behaves asymptotically like a linear function. As an interesting consequence, beta-lattices and their symmetries behave asymptotically like lattices and lattice symmetries.

Key words: Beta-lattices, Pisot numbers, quasicrystals, tilings, plane groups
PACS: 52C22, 52C23

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1 Introduction

Underlying the notion of a *tiling* there is the notion of a *point-set*. In this paper we assume point-sets to be *Delaunay sets* [16,17]. There exist infinitely many possibilities to build a tiling from a Delaunay set, and conversely, there are infinitely many ways to build a Delaunay set from its associated tiling. A possible method is to consider the set of vertices of a tiling as an associated Delaunay set [12], which is the correspondence we will assume in the following. We will indifferently mention a tiling or its associated Delaunay set, displaying or not the edges in the figures.
In general, there does not exist a symmetry group for a tiling nor for its associated Delaunay set, except for periodic tilings and *lattices*. Historically, the latter merge from Crystallography, and are associated with crystals. Note that in 1991, after the discovery of modulated phases and of quasicrystals, Crystallography have been divided in two categories: periodic Crystallography, and aperiodic Crystallography [10]. Let us sketch the general algebraic frame of periodic Crystallography.

Definition 1 *A crystallographic group in $\mathbb{R}^d$, or a space-group in $\mathbb{R}^d$, is a discrete group of isometries whose maximal translation subgroup is of rank d, hence isomorphic to $\mathbb{Z}^d$.*

Definition 2 *A periodic crystal is the orbit under the action of a crystallographic group of a finite number of points of $\mathbb{R}^d$.*

We can illustrate these definitions with the square lattice $\Lambda = \mathbb{Z}+\mathbb{Z}e^{i\frac{\pi}{2}}$, which is a classical lattice case. This set presents a 4-fold rotational symmetry. The symmetry space-group G associated with Λ is the semi-direct product of the translation-group of Λ by its rotation-group

$$G = \Lambda \rtimes \{1, -1, e^{i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}\}\,,$$

its internal law being

$$(\lambda, R)(\lambda', R') = (\lambda + R\lambda', RR')\,,$$

with $\lambda, \lambda' \in \Lambda$ and $R, R' \in \{1, -1, e^{i\frac{\pi}{2}}, e^{-i\frac{\pi}{2}}\}$.
In the context of the 18th problem of Hilbert, Bieberbach has shown that the number of isomorphism classes (equivalently of conjugation classes) of crystallographic groups is finite for all d [25]. Therefore the number of crystallographic groups leaving invariant a fixed crystal of $\mathbb{R}^d$ is finite.

For quasicrystals, as a consequence of aperiodicity, we do not have such a convenient algebraic structure of symmetry space-groups, as in the periodic case. For quasicrystals determined by some quadratic Pisot-Vijayaraghavan (PV) units, generically denoted by $\beta > 1$, we can introduce an underlying structure, the so-called *beta-lattice* [1]. Experimentally observed quasicrystals are related to well known PV numbers [11], namely for $\beta = \tau = \frac{1+\sqrt{5}}{2}$, $\beta = \delta = 1 + \sqrt{2}$, and $\beta = \theta = 2 + \sqrt{3}$.

Beta-lattices are based on beta-integers. The set of beta-integers, denoted by $\mathbb{Z}_\beta$, is a self-similar Meyer set, with self-similarity factor β. Recall that a Meyer set is a Delaunay set $\Lambda \in \mathbb{R}^d$ if

$\Lambda - \Lambda \subset \Lambda + F$, where F is a finite set. We generically define a beta-lattice $\Gamma = \Gamma(\beta) \in \mathbb{R}^d$ by

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i \,,$$

with $(\mathbf{e}_i)$ a base of $\mathbb{R}^d$. Therefore, Γ is a self-similar Meyer set with self-similarity factor β. With this respect, beta-lattices are eligible frames in which one could think of the properties of quasiperiodic point-sets and tilings, thus generalizing the notion of lattice in periodic cases.

The aim of the present work is to extend the algebraic frame of periodic crystals to beta-lattices: we construct a space-group matching Definition 1 such that the beta-lattice is the orbit under the action of this space-group of a finite set of points of $\mathbb{R}^2$, as in Definition 2. In other words, we show that a beta-lattice is at least a "crystal" for a "space-group" that we determine explicitly.

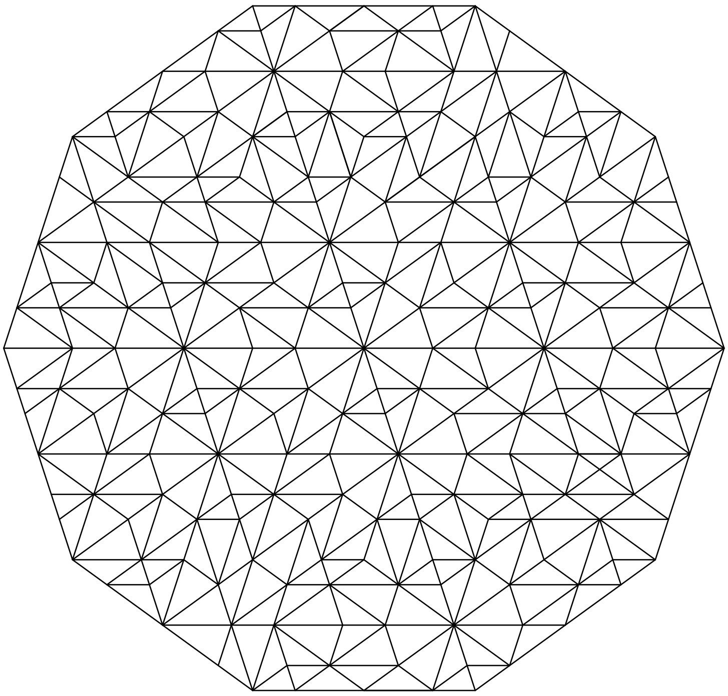

Fig. 1. A tiling of the τ-lattice $\Gamma_1(\tau)$.

We consider the cases where β is one of the "quasicrystallographic" number mentioned above. Moreover we restrict ourselves to the case $d = 2$. Therefore we rather talk of "plane-groups". We proceed by, first recalling the internal additive and multiplicative laws on the set of beta-integers $\mathbb{Z}_\beta \subset \mathbb{R}$, which "almost" endow this set with a structure of ordered ring (order induced by that of $\mathbb{R}$) [6], then by establishing a set of algebraic operations, acting on the given beta-lattice by leaving it invariant. We report on the algebraic constructions of such extended plane-groups, leaving aside the delicate questions of compatible metrics and of the number (finite or infinite) of possible "space-groups" leaving invariant a given beta-lattice. However we show that the internal transformations defined on beta-lattices are *compatible* with Euclidian transformations. Compatibility property is given by the following definition.

Definition 3 *Let $\top$ be an internal law defined on $\mathbb{R}^d$, and let $\Lambda \subset \mathbb{R}^d$ be a Delaunay set. We say that an internal operation $\widehat{\top}$ defined on Λ is* $\top$-compatible *with the operation $\top$ if for all λ, $\lambda' \in \Lambda$, $\lambda \top \lambda' \in \Lambda$ implies $\lambda \top \lambda' = \lambda \widehat{\top} \lambda'$.*

The article is organized as follows. In Section 2 we recall some definitions on Delaunay sets, Meyer sets, and on cyclotomic PV numbers. In Section 3 we recall results on the arithmetics and the internal laws on $\mathbb{Z}_\beta$. Most of this material can be found in [6], and is essential for the understanding of the present article. In Section 4 we give the definition of beta-lattices in the plane, together with their rotational and translational properties. A general form for beta-lattices is $\Gamma_1(\beta) = \mathbb{Z}_\beta + \mathbb{Z}_\beta e^{i\frac{2\pi}{N}}$ for β a cyclotomic PV unit of symmetry N. Figure 1 is a possible tiling of such a beta-lattice with $\beta = \tau$, the golden mean, namely a τ-lattice. Section 5 is the central part of the article, with its main result: the construction of the plane-groups associated with the beta-lattices. We use the internal additive and multiplicative laws on $\mathbb{Z}_\beta$ to define a symmetry point-group for $\Gamma_1(\beta)$ in Theorem 1, and the free symmetry plane-group of $\Gamma_1(\beta)$ in Theorem 2. Then we illustrate the action of the symmetry plane-group of $\Gamma_1(\tau)$ on the tiles of a τ-lattice. Section 6 is dedicated to the asymptotic properties of beta-lattices. The striking feature which is shown there is that asymptotically the set of beta-integers behaves like a ring, but with a contraction factor. We touch here the fundamental question whether a beta-lattice can be considered as a module over an ordered ring. If it were the case, the present construction would enter into the realm of the Artin-Schreier theory (Lam [13] chapter 6). Eventually, we make explicit the rotation actions for the quasicrystallographic numbers τ, δ and θ in the Appendix.

2 Preliminaries

2.1 Delaunay sets and Meyer sets

Delaunay sets were introduced as a mathematical idealization of a solid-state structure, see [12]. A set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly discrete* if there exists $r > 0$ such that $||x - y|| \geq r$, for all $x, y \in \Lambda$. We can equivalently say that every closed ball of radius r contains at most a point of Λ. A set Λ is said to be *relatively dense* if there exists $R > 0$ such that for all $y \in \mathbb{R}^d$, there exists $x \in \Lambda$ such that $||x - y|| < R$. We can equivalently say that every open ball of radius R contains at least a point of Λ. If both conditions are satisfied, Λ is said to be a *Delaunay set*. The possible range of ratios R/r is studied in [21] as a function of d. The action of the group of rigid motions (or Euclidean displacements) of $\mathbb{R}^d$ on the set of uniformly discrete sets and Delaunay sets can be found in [22].
The first models of quasicrystal were introduced by Meyer [16–18], and they are now known as *Meyer sets*. A set $\Lambda \subset \mathbb{R}^d$ is said to be a *Meyer set* if it is a Delaunay set and if there exists a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F.$$

This is equivalent to $\Lambda - \Lambda$ being a Delaunay set. A review on Meyer sets can be found in [19,20].

2.2 Crystals and Bravais lattices

Bravais lattices are used as mathematical models for crystals. A Bravais lattice is an infinite discrete point-set such that the neighborhoods of a point are the same whichever point of the set is considered. Geometrically, a Bravais lattice is characterized by all Euclidean transformations (translations and possibly rotations) that transform the lattice into itself. The condition $2\cos\frac{2\pi}{N} \in \mathbb{Z}$ characterizes Bravais lattices which are left invariant under rotation of $2\pi/N$, N-fold Bravais lattices, in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). Let us put $\zeta = e^{\frac{2\pi i}{N}}$, $\zeta^N = 1$. If we consider the $\mathbb{Z}$-module in the plane:

$$\mathbb{Z}[\zeta] \;=\; \mathbb{Z} + \mathbb{Z}\zeta + \mathbb{Z}\zeta^2 + \cdots + \mathbb{Z}\zeta^{N-1} = \mathbb{Z}[2\cos\frac{2\pi}{N}] + \mathbb{Z}[2\cos\frac{2\pi}{N}]\zeta,$$

we get the cyclotomic ring of order N. This N-fold structure is generically dense in $\mathbb{C}$, except precisely for the crystallographic cases. We indeed check that $\mathbb{Z}[\zeta] = \mathbb{Z}$ for $N = 1$ or 2, $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}i$ for $N = 4$ (square lattice), and $\mathbb{Z}[\zeta] = \mathbb{Z} + \mathbb{Z}e^{i\frac{\pi}{3}}$ for the triangular and hexagonal cases $N = 3$ and $N = 6$. Note that a Bravais lattice is a Meyer set such that $F = \{0\}$.

2.3 Non-crystallographic cases

If N is not crystallographic, $2\cos\frac{2\pi}{N}$ is an algebraic integer of degree $m = \varphi(N)/2 \leq \lfloor (N-1)/2 \rfloor$ where φ is the Euler function and $\lfloor y \rfloor$ denotes the integer part of a real number y. We shall now recall some definitions on numbers.

A *Pisot-Vijayaraghavan number*, or PV number in short, is an algebraic integer $\beta > 1$ such that all its Galois conjugates (*i.e.* other roots of the involved algebraic equation) have their moduli strictly smaller than 1. A *cyclotomic PV number* with symmetry of order N is a PV number β such that

$$\mathbb{Z}[2\cos\frac{2\pi}{N}] = \mathbb{Z}[\beta]. \tag{1}$$

Then $\mathbb{Z}[\zeta] = \mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$, with $\zeta = e^{i\frac{2\pi}{N}}$, is a ring invariant under rotation of order N (see [1]). This ring is the natural framework for two-dimensional structures having β as scaling factor, and $2\pi/N$ rotational symmetry. In this paper we will focus on quadratic PV units. They are of two kinds. The first kind is such that β is solution of

$$X^2 = aX + 1 \quad a \geq 1\,,$$

and its conjugate is $\beta' = -1/\beta$. The second kind is such that β is solution of

$$X^2 = aX - 1 \quad a \geq 3\,,$$

and its conjugate is $\beta' = 1/\beta$. Let us give some examples of those numbers, together with their respective Galois conjugates, related to non-crystallographic cyclotomic structures in the plane, and minimal polynomials, the following notations being used throughout the article:

$$N=5 \quad \beta=\tau=\frac{1+\sqrt{5}}{2}=1+2\cos\frac{2\pi}{5} \quad \tau'=-\frac{1}{\tau}=1-\tau\,, \quad X^2-X-1 \quad \text{(pentagonal case)},$$
$$N=10 \quad \beta=\tau=\frac{1+\sqrt{5}}{2}=2\cos\frac{2\pi}{10} \quad \tau'=-\frac{1}{\tau}=1-\tau\,, \quad X^2-X-1 \quad \text{(decagonal case)},$$
$$N=8 \quad \beta=\delta=1+\sqrt{2}=1+2\cos\frac{2\pi}{8} \quad \delta'=-\frac{1}{\delta}=2-\delta\,, \quad X^2-2X-1 \quad \text{(octogonal case)},$$
$$N=12 \quad \beta=\theta=2+\sqrt{3}=2+2\cos\frac{2\pi}{12} \quad \theta'=\frac{1}{\theta}=4-\theta\,, \quad X^2-4X+1 \quad \text{(dodecagonal case)}.$$

Note that in the case $N=7$, we have $\beta=1+2\cos\frac{2\pi}{7}$ which is solution of the cubic equation $X^3-2X^2-X+1=0$. At this point, we should be aware that finding a PV number such that the cyclotomic condition (1) is fulfilled for $N\geq 16$ is an open problem!

3 Additive and multiplicative properties of beta-integers

3.1 Beta-expansions

When a number $\beta>1$ appears as a kind of fundamental invariant in a given structure, it is tempting to introduce into the procedure of understanding the latter a *counting system* based precisely on this β. Let us explain here what we mean by counting system.
Among all *beta-representations* of a real number $x\geq 0$, *i.e.* infinite sequences $(x_i)_{i\leq k}$, such that $x=\sum_{i\leq k}x_i\beta^i$ for a certain integer k, there exists a particular one, called the *beta-expansion*, which is obtained through the "greedy algorithm" (see [24] and [23]). Recall that $\lfloor y\rfloor$ is the integer part of the real number y, and denote by $\{y\}$ the fractional part of y. There exists $k\in\mathbb{Z}$ such that $\beta^k\leq x<\beta^{k+1}$. Let $x_k=\lfloor x/\beta^k\rfloor$ and $r_k=\{x/\beta^k\}$. For $i<k$, put $x_i=\lfloor\beta r_{i+1}\rfloor$, and $r_i=\{\beta r_{i+1}\}$. Then we get the expansion $x=x_k\beta^k+x_{k-1}\beta^{k-1}+\cdots$. If $x<1$ then $k<0$, and we put $x_0=x_{-1}=\cdots=x_{k+1}=0$. The beta-expansion of x is denoted by

$$\langle x\rangle_\beta = x_kx_{k-1}\cdots x_1x_0\cdot x_{-1}x_{-2}\cdots$$

The digits x_i obtained by this algorithm are integers from the set $A=\{0,\ldots,\lceil\beta\rceil-1\}$, called the *canonical alphabet*, where $\lceil\beta\rceil$ denotes the smallest integer larger than β. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted. For instance, if $\beta=\tau\approx 1\cdot 618\cdots$, then $x_i\in\{0,1\}$. The τ-expansion of, say, $4=\tau^2+1+1/\tau^2$ is $\langle 4\rangle_\tau=101\cdot 01$. There is a representation which plays an important role in the theory. The *beta-expansion* of 1, denoted by $d_\beta(1)$, is computed by the following process [24]. Let the *beta-transformation* be defined on $[0,1]$ by $T_\beta(x)=\beta x \bmod 1$. Then $d_\beta(1)=(t_i)_{i\geq 1}$, where $t_i=\lfloor\beta T_\beta^{i-1}(1)\rfloor$. Bertrand has proved that if β is a PV number, then $d_\beta(1)$ is eventually periodic [2]. For instance, $d_\tau(1)=11$, $d_\delta(1)=21$, and $d_\theta(1)=322\cdots=3(2)^\omega$, where $(\cdot)^\omega$ means that the digit between parenthesis is repeated an infinite number of times. A number β such that $d_\beta(1)$ is eventually periodic is traditionally called a *beta-number*. Since these numbers were introduced by Parry [23], we propose to call them *Parry numbers*. When $d_\beta(1)$ is finite, β is said to be a *simple* Parry number.

3.2 The set of beta-integers

We now come to the notion of *beta-integer*. The set of beta-integers is the set of real numbers whose beta-expansions are polynomial,

$$\begin{aligned}\mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+)\end{aligned}$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative beta-integers. The set $\mathbb{Z}_\beta$ is self-similar and symmetrical with respect to the origin

$$\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta, \quad \mathbb{Z}_\beta = -\mathbb{Z}_\beta.$$

It has been shown in [3] that if β is a PV number then $\mathbb{Z}_\beta$ is a Meyer set. This means that there exists a finite set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. This beta-dependent set F has to be characterized in order to see to what extent beta-integers differ from ordinary integers with respect to additive and multiplicative structures. This problem is solved in [3,4,6] for all quadratic PV units and for a few higher-degree cases (see also [27]). We now restrict the presentation to quadratic PV units. There are two cases to consider.

Case 1. β is solution of $X^2 = aX + 1$, $a \geq 1$. The Galois conjugate is $\beta' = -1/\beta$. The canonical alphabet is equal to $A = \{0, \ldots, a\}$, the beta-expansion of 1 is finite, equal to $d_\beta(1) = a1$, and every positive number of $\mathbb{Z}[\beta]$ has a finite beta-expansion [7]. Denote $\mathbb{A} = \{L, S\}$. Define the substitution σ_β by

$$\sigma_\beta : \begin{cases} L \mapsto L^a S \\ S \mapsto L. \end{cases}$$

The fixed point of the substitution, denoted by $\sigma_\beta^\infty(L)$, is associated with a tiling of the positive real line, made with the two tiles L and S, where the lengths of the tiles are $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - a = 1/\beta$, see [26,5]. The nodes of this tiling are the positive beta-integers.

Case 2. β is solution of $X^2 = aX - 1$, $a \geq 3$. The Galois conjugate is $\beta' = 1/\beta$. The canonical alphabet is equal to $A = \{0, \ldots, a-1\}$, the beta-expansion of 1 is eventually periodic, equal to $d_\beta(1) = (a-1)(a-2)^\omega$, and every positive number of $\mathbb{Z}[\beta]$ has an eventually periodic beta-expansion, which is finite for numbers from $\mathbb{N}[\beta]$, [7]. The substitution σ_β is defined on $\mathbb{A} = \{L, S\}$ by

$$\sigma_\beta : \begin{cases} L \mapsto L^{a-1} S \\ S \mapsto L^{a-2} S. \end{cases}$$

As in Case 1, the fixed point of the substitution is denoted by $\sigma_\beta^\infty(L)$, and is associated with a tiling of the positive real line, made with the two tiles L and S. The lengths of the tiles are $\ell(L) = 1$, $\ell(S) = T_\beta(1) = \beta - (a-1) = 1 - 1/\beta$ [26,5]. The nodes of this tiling are the positive beta-integers.

In both cases we shall denote by $|\sigma_\beta^q(L)|$ the number of letters in the word generated by $\sigma_\beta^q(L)$, and by $|\sigma_\beta^q(L)|_L$, respectively $|\sigma_\beta^q(L)|_S$, the number of letters L, respectively S, in the later word.

3.3 Beta-integers arithmetics

Since $\mathbb{Z}_\beta$ is a Meyer set symmetrical with respect to the origin, we have $\mathbb{Z}_\beta - \mathbb{Z}_\beta = \mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$. Hence the set $\mathbb{Z}_\beta$ can be qualified as "quasi-additive". It can also be qualified as "quasi-multiplicative". Accordingly, addition and multiplication of beta-integers are characterized below.

- In Case 1 we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm(1 - \frac{1}{\beta})\} \subset \mathbb{Z}_\beta/\beta^2, \tag{2}$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}, \dots, \pm\frac{a}{\beta}\} \subset \mathbb{Z}_\beta/\beta^2. \tag{3}$$

  For instance, for $\beta = \tau$, $1 + 1 = 2 = \tau + (1 - \frac{1}{\tau})$, and $(\tau^2+1)(\tau^2+1) = \tau^5 + \tau^2 - \frac{1}{\tau}$.

- In Case 2 we have

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\} = \widetilde{\mathbb{Z}}_\beta, \tag{4}$$

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \mathbb{Z}_\beta^+/\beta,$$

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}, \dots, \pm\frac{a-1}{\beta}\} \subset \mathbb{Z}_\beta/\beta. \tag{5}$$

  For instance, for $\beta = \theta$, $2 + 2 = \theta + \frac{1}{\theta} = 2 \times 2$.

The set $\widetilde{\mathbb{Z}}_\beta$, introduced in Equation (4), is called the set of decorated beta-integers. This set plays an important role in the theory of algebraic model sets, see for instance [6], and is to be mentioned in the two-dimensional case (Figure 4).

3.4 Beta-integers as an additive group

Let b_m and b_n be the m^{th} and n^{th} beta-integers.

Definition 4 *We define the* beta-addition *as the internal additive law on the set of beta-integers, as*

$$b_m \oplus b_n = b_{m+n} \,.$$

The beta-substraction is defined by

$$b_m \ominus b_n = b_{m-n} = b_m \oplus (-b_n) \,.$$

The set of beta-integers endowed with the beta-addition has an abelian group structure [4,6]. Actually, we can endow any countable strictly increasing sequence $\mathcal{S} = (s_n)_{n\in\mathbb{Z}}$ of real numbers, $s_0 = 0$, with such an internal additive law by simple isomorphic transport of the additive group structure of the integers, the additive law of $\mathcal{S}$ being defined by

$$s_m \oplus s_n \stackrel{def}{=} s_{m+n}.$$

Recall that the internal additive law $\oplus$ defined on $\mathcal{S}$, is said to be *compatible* with addition of real numbers if for all $(m,n) \in \mathbb{Z}^2$, $s_m + s_n \in \mathcal{S}$ implies $s_m + s_n = s_m \oplus s_n$, and obviously, for an arbitrary sequence $\mathcal{S}$, the law $\oplus$ *is not* compatible with the addition of real numbers. Yet this property holds true for $\mathbb{Z}_\beta$!

Lemma 1 *Beta-addition is compatible with addition if β is a quadratic PV unit.*

Proof. It has been proven in [4] and [6] that beta-addition has the following *minimal distortion property* with respect to addition: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β a quadratic PV unit,

$$b_m + b_n - (b_m \oplus b_n) \in \begin{cases} \{0, \pm(1-\frac{1}{\beta})\} & \text{in Case 1,} \\ \{0, \pm 1/\beta\} & \text{in Case 2.} \end{cases} \tag{6}$$

Put $b_m + b_n = b_q$. Then $b_q - (b_m \oplus b_n)$ verifies Equation (6), which implies $b_q - (b_m \oplus b_n) = 0$, since the distances between two consecutive beta-integers are $\ell(L) = 1$ or $\ell(S) = 1/\beta$ in Case 1 and $\ell(S) = 1 - 1/\beta$ in Case 2. Since $b_m \oplus b_n = b_{m+n}$, we have $q = m + n$. ∎

For instance, if $\beta = \tau$, then $1 \oplus 1 = \tau$ and $2 - \tau = 1 - 1/\tau$, and if $\beta = \theta$, then $2 \oplus 2 = \theta$ and $4 - \theta = 1/\theta$.

3.5 Internal multiplicative law for beta-integers

We could attempt to play the same game with multiplication by defining

$$b_m \text{``}\times\text{''} b_n \stackrel{def}{=} b_{mn},$$

for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$. However, we reject this definition of an internal multiplicative law since *it is not* consistent with multiplication in $\mathbb{R}$. For instance, for $\beta = \tau$, $b_2 \times b_2 = \tau \times \tau = \tau^2 = b_3 \neq b_4$.

Definition 5 *We define the* quasi-multiplication *as the internal multiplicative law on the set of beta-integers, as*

$$b_m \otimes b_n = \begin{cases} b_{(mn - a\rho_S(m)\rho_S(n))} & \textit{in Case 1,} \\ b_{(mn - \rho_S(m)\rho_S(n))} & \textit{in Case 2,} \end{cases} \tag{7}$$

where, for $n \geq 0$, $\rho_S(n)$ denotes the number of tiles S between $b_0 = 0$ and b_n [6]. For instance, for τ, $\rho_S(5) = 2$ while for θ, $\rho_S(5) = 1$. Geometrically, the n^{th} beta-integer is the right vertex of the n^{th} tile of the tiling associated with $\mathbb{Z}_\beta$, which can be expressed by $b_n = n + (-1 + l(S))\rho_s(S)$ and from which we derive the following

$$\begin{aligned} \rho_S(n) &= \frac{1}{1 - 1/\beta}(n - b_n)\,, && \text{Case } 1\,, \\ \rho_S(n) &= \beta(n - b_n)\,, && \text{Case } 2\,. \end{aligned}$$

For $n < 0$, $\rho_S(n) = -\rho_S(-n)$.

Lemma 2 *Quasi-multiplication is compatible with multiplication of real numbers if β is a quadratic PV unit.*

Proof. Quasi-multiplication has minimal distortion property with respect to multiplication [4,6]: for all $(b_m, b_n) \in \mathbb{Z}_\beta^2$ with β quadratic PV unit,

$$b_m b_n - (b_m \otimes b_n) \in \begin{cases} \{(0, \pm 1, \ldots, \pm a)(1 - \frac{1}{\beta})\} & \text{Case 1,} \\ \{(0, 1, \ldots, a-1)\frac{\text{sgn}(b_m b_n)}{\beta}\} & \text{Case 2.} \end{cases} \tag{8}$$

Put $b_m b_n = b_q$. Then $b_q - (b_m \otimes b_n)$ verifies Equation (8), which implies $b_q - (b_m \otimes b_n) = 0$, since the distances between two consecutive beta-integers are $\ell(L) = 1$ or $\ell(S) = 1/\beta$ in Case 1 and $\ell(S) = 1 - 1/\beta$ in Case 2. Since $b_m \otimes b_n = b_{mn - a\rho_S(m)\rho_S(n)}$ in Case 1 and $b_m \otimes b_n = b_{mn - \rho_S(m)\rho_S(n)}$ in Case 2, we have $q = mn - a\rho_S(m)\rho_S(n)$ in Case 1, and $q = mn - \rho_S(m)\rho_S(n)$ in Case 2. ∎

An interesting outcome of this multiplicative structure is the following explicit result concerning self-similarity properties of the set of beta-integers.

Let $U = (u_q)_{q \in \mathbb{N}}$ be the linear recurrent sequence of integers associated with β. In Case 1, the u_q are defined by $u_{q+2} = au_{q+1} + u_q$ with $u_0 = 1$, $u_1 = a + 1$. In Case 2, the u_q are defined by $u_{q+2} = au_{q+1} - u_n$ with $u_0 = 1$, $u_1 = a$. The recurrence is possibly extended to negative indices.

Proposition 1 *Let β be a quadratic PV unit, and $\mathbb{Z}_\beta$ the corresponding set of beta-integers. Then for $q \in \mathbb{N}$ and $b_n \in \mathbb{Z}_\beta$ we have the self-similarity formulas:*

$$\begin{aligned} \beta^q\, b_n = b_{u_q} b_n = b_{u_q} \otimes b_n &= b_{u_q\, n - a\rho_S(u_q)\rho_S(n)} = b_{u_q\, n - (u_q - u_{q-1})\,\rho_S(n)} && (\textit{in Case 1}), \\ \beta^q\, b_n = b_{u_q} b_n = b_{u_q} \otimes b_n &= b_{u_q\, n - \rho_S(u_q)\rho_S(n)} = b_{u_q\, n - (u_{q-1})\,\rho_S(n)} && (\textit{in Case 2}). \end{aligned}$$

The proof is a direct consequence of the definition of the quasi-multiplication and of the following lemma giving some of the properties of the counting function ρ_S.

Lemma 3 *The values assumed by the counting function $\rho_S(n)$ when $n = u_q \in U$ are*

$$\rho_S(u_q) = \frac{u_q - u_{q-1}}{a}, \qquad (\textit{in Case 1}),$$
$$\rho_S(u_q) = u_{q-1}, \qquad (\textit{in Case 2}).$$

Proof. Case 1. Let $w_q = \rho_S(u_q)$. By construction, $u_q = |\sigma_\beta^q(L)|$ and $w_q = |\sigma_\beta^q(L)|_S$. Therefore the sequence (w_q) satisfies the same linear recurrence as (u_q), that is, $w_q = aw_{q-1}+w_{q-2}$, with $w_0 = 0$, $w_1 = 1$. Thus $w_2 = aw_1 + w_0 = (u_2 - u_1)/a = a$ and $w_3 = aw_2 + w_1 = (u_3 - u_2)/a = a^2 + 1$. The recurrence is proved through $w_{q+1} = aw_q+w_{q-1} = a(u_q-u_{q-1})/a+(u_{q-1}-u_{q-2})/a = (u_{q+1}-u_q)/a$. **Case 2**. Let $w_q = \rho_S(u_q)$. We have $w_q = aw_{q-1} - w_{q-2}$, with $w_0 = 0$ and $w_1 = 1$. Then $w_2 = aw_1 - w_0 = u_1 = a$ and $w_3 = aw_2 - w_1 = u_2 = a^2 - 1$. The recurrence is proved through $w_{q+1} = aw_q - w_{q-1} = au_{q-1} - u_{q-2} = u_q$. ■

It should be noticed that quasi-multiplication does not define a group for not being associative and is not distributive with respect to beta-addition. So it seems hopeless to obtain a ring structure, like we have with integers, with such an internal multiplicative law. Note that beta-addition and quasi-multiplication are related to some operations in numeration systems studied in [9,14,15]. Nevertheless, the set of beta-integers recovers a ring structure asymptotically, see Section 6 for details.

4 Beta-lattices in the plane

4.1 General considerations

We have seen that the condition $2\cos(2\pi/N) \in \mathbb{Z}$, *i.e.* $N = 1, 2, 3, 4$ and 6, characterizes N-fold Bravais lattices in $\mathbb{R}^2$ (and in $\mathbb{R}^3$). We would like to generalize this notion when N is quasicrystallographic *i.e.* $N = 5, 10, 8$ and 12, respectively associated with one of the cyclotomic Pisot units $\tau = 2\cos(2\pi/10)$, $\delta = 1 + 2\cos(2\pi/8)$ and $\theta = 2 + 2\cos(2\pi/12)$. As a consequence of the results presented above, if $(\mathbf{e}_i)$ is a base of $\mathbb{R}^d$

$$\Gamma = \sum_{i=1}^{d} \mathbb{Z}_\beta \mathbf{e}_i$$

is a Meyer set and a lattice for the law $\oplus$. Moreover $\mathbb{Z}_\beta \otimes \Gamma \subset \Gamma$. We shall adopt the generic name of *beta-lattice* for such a Γ. Examples of beta-lattices in the plane are point-sets of the form

$$\Gamma_q(\beta) = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q\,,$$

with $\zeta = e^{i\frac{2\pi}{N}}$, for $1 \leq q \leq N-1$. Note that the latter are not rotationally invariant. Examples of rotationally invariant point-sets based on beta-integers are

$$\Lambda_q \stackrel{def}{=} \bigcup_{j=0}^{N-1} \Gamma_q \zeta^j\,, \quad 1 \leq q \leq N-1\,,$$

and

$$\mathbb{Z}_\beta[\zeta] \stackrel{def}{=} \sum_{j=0}^{N-1} \mathbb{Z}_\beta \zeta^j.$$

Note that the sets Λ_q and $\mathbb{Z}_\beta[\zeta]$ are Meyer sets.

Let us now focus on the simplest case, namely $N = 5$ or 10. It is more convenient to introduce the root of unity $\zeta = e^{\frac{i\pi}{5}}$, since $\tau = 2\cos\pi/5 = \zeta + \zeta^c$, where ζ^c is the complex conjugate of ζ. We obtain the set

$$\mathbb{Z}_\tau[\zeta] \;\equiv\; \mathbb{Z}_\tau + \mathbb{Z}_\tau\zeta + \mathbb{Z}_\tau\zeta^2 + \mathbb{Z}_\tau\zeta^3 + \mathbb{Z}_\tau\zeta^4.$$

Consider now the following τ-lattices in the plane,

$$\Gamma_q = \mathbb{Z}_\tau + \mathbb{Z}_\tau\zeta^q, \quad q = 1,\ 2,\ 3, \text{or } 4,$$

The following inclusions were proven in [3]

$$\Gamma_q \subset \mathbb{Z}_\tau[\zeta] \subset \frac{\Gamma_q}{\tau^4}.$$

It has been shown that a large class of aperiodic sets can be embedded in beta-lattices such as $\Gamma_q(\beta)$ (see [3]).

On Figures 2, 3 and 4, we displayed the τ-lattice $\Gamma_1(\tau)$, the δ-lattice $\Gamma_1(\delta)$ and the decorated θ-lattice $\Gamma_1(\theta)$, respectively, both as point-sets, and as tilings.

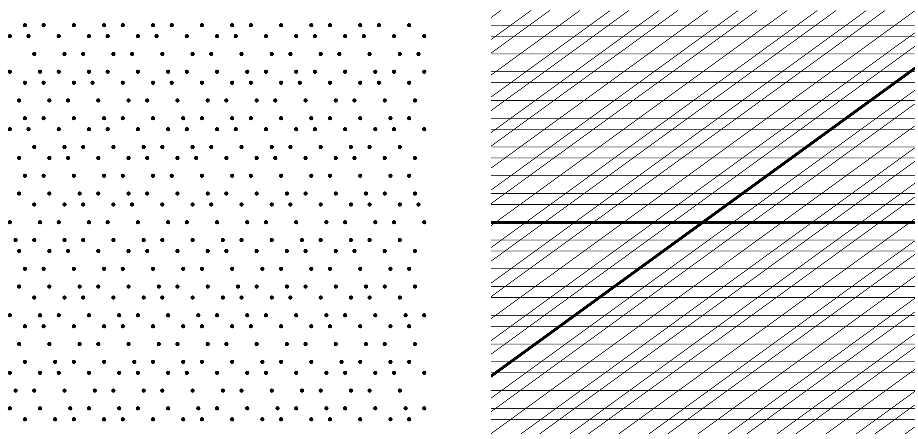

Fig. 2. The τ-lattice $\Gamma_1(\tau)$ with points (left), and its trivial tiling made by joining points along the horizontal axis, and along the direction defined by ζ.

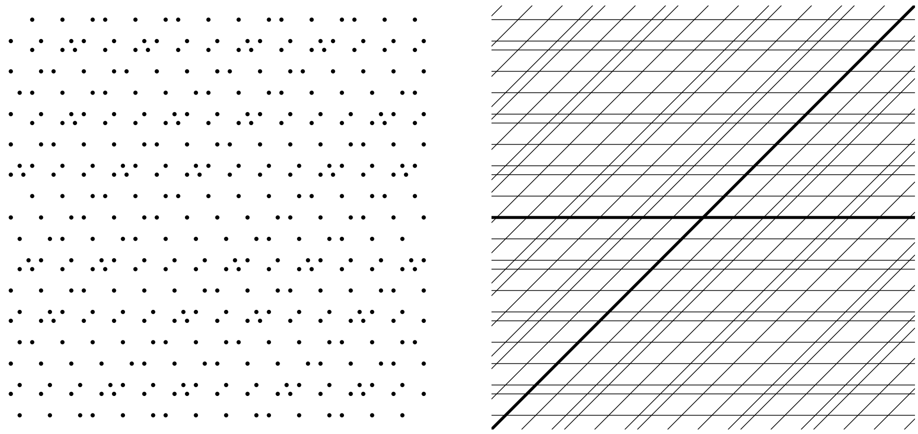

Fig. 3. The δ-lattice $\Gamma_1(\delta)$ with points (left), and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

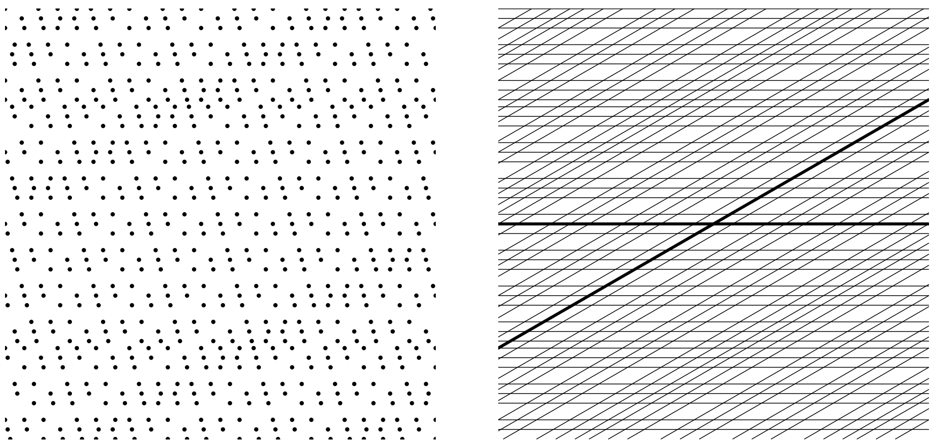

Fig. 4. The decorated θ-lattice $\widetilde{\Gamma}_1(\theta)$ with points (left), and its trivial tiling obtained by joining points along the horizontal axis, and along the direction defined by ζ.

4.2 Rotational properties of the beta-lattices $\Gamma_1(\beta)$

Although beta-lattices are not rotationally invariant, we can nevertheless study the action of rotations on them. In this section, and throughout the rest of the article, we focus on $\Gamma_1(\beta)$. Note that for $\beta = \tau$ and δ, any beta-lattice $\Gamma_q(\beta)$ is a subset of the properly scaled beta-lattice $\Gamma_1(\beta)$. Therefore, the rotational properties of $\Gamma_q(\beta)$ can always be reexpressed in terms of the rotational properties of $\Gamma_1(\beta)$. Note that since θ is a quadratic PV unit of the second kind, the game is slightly different, since the θ-lattices $\Gamma_q(\theta)$ are not subsets of the properly scaled $\Gamma_1(\theta)$, for $q \neq 1$, but of its decorated version $\widetilde{\Gamma}_1(\theta)$.

We introduce the algebraic integer associated with ζ, $\chi = \zeta + \overline{\zeta} = 2\cos(2\pi/N)$, which entails $\zeta^2 = -1 + \chi\zeta$, and

$$\zeta^q = \eta_q + \nu_q \zeta\,, \quad q \in \{0, 1, \ldots, N-1\}\,. \tag{9}$$

A rotation by $q2\pi/N$ on an arbitrary element $b_m + b_n\zeta$ of $\Gamma_1(\beta)$ gives

$$\zeta^q(b_m + b_n\zeta) = (\eta_q b_m - \nu_q b_n) + (\nu_q b_m + (\eta_q + \nu_q\chi)b_n)\zeta\,, \tag{10}$$

which is not an element of $\Gamma_1(\beta)$ in general, but belongs to a deflated version of $\Gamma_1(\beta)$ by a certain factor. If we consider the values of the pairs (η_q, ν_q) and of $\eta_q + \nu_q\chi$, when β assumes the specific values τ and δ, we can determine this deflation factor. When $\beta = \theta$, $\zeta^q(b_m + b_n\zeta)$ is included in the twice decorated θ-lattice $\tilde{\tilde{\Gamma}}_1(\theta)$, as will be shown explicitly.

- When $\beta = \tau$, the results are given for $\zeta = e^{i\frac{2\pi}{10}}$, $\chi = \tau$.

$$\begin{array}{rcccccc} q & = & 0 & 1 & 2 & 3 & 4 \\ (\eta_q, \nu_q) & = & (1,0) & (0,1) & (-1,\tau) & (-\tau,\tau) & (-\tau,1) \\ \eta_q + \nu_q\chi & = & 1 & \tau & \tau & 1 & 0\,, \end{array}$$

together with $(\eta_{q+5}, \nu_{q+5}) = (-\eta_q, -\nu_q)$. Hence

$$\begin{aligned} \zeta^q\Gamma_1(\tau) &\subset \Gamma_1(\tau) + \Big(\{0, \pm(1-\frac{1}{\tau})\} + \{0, \pm(1-\frac{1}{\tau})\}\zeta\Big) \\ &\subset \frac{\Gamma_1(\tau)}{\tau^2}\,. \end{aligned}$$

Note that since $\chi = \tau$, $\Gamma_1(\tau)$ is endowed with specific properties which are not encountered in other cases, namely when $\beta = \delta$, and $\beta = \theta$.

Lemma 4 *For $\zeta = e^{i\frac{\pi}{5}}$, all elements of the cyclic group $\{\zeta^q, q \in \{0, 1, 2, \ldots, 9\}\}$ are elements of the τ-lattice $\Gamma_1(\tau)$.*

Proof. The demonstration is trivial from the values assumed by η_q and ν_q in the case of τ,

$$\zeta^q = \eta_q + \nu_q\zeta, \text{ with } \eta_q, \nu_q \in \{0, \pm 1, \pm\tau\}.$$

■

Also note that from the self-similarity property of $\mathbb{Z}_\tau$ we have $\eta_q b_n \in \mathbb{Z}_\tau$, $\nu_q b_n \in \mathbb{Z}_\tau$ and $(\eta_q + \tau\nu_q)b_n \in \mathbb{Z}_\tau$, for all q and n.

- When $\beta = \delta$, $\zeta = e^{i\frac{2\pi}{8}}$ and $\chi = \delta - 1$.

$$\begin{array}{rccccc} q & = & 0 & 1 & 2 & 3 \\ (\eta_q, \nu_q) & = & (1,0) & (0,1) & (-1,\delta-1) & (-\delta+1,1) \\ \eta_q + \nu_q\chi & = & 1 & \delta-1 & 1 & 0\,, \end{array}$$

together with $(\eta_{q+4}, \nu_{q+4}) = (-\eta_q, -\nu_q)$. Hence

$$\begin{aligned}\zeta^q \Gamma_1(\delta) &\subset \Gamma_1(\delta) + \Big(\{0, \pm(1-\frac{1}{\delta}), \pm 2(1-\frac{1}{\delta})\} + \{0, \pm(1-\frac{1}{\delta}), \pm 2(1-\frac{1}{\delta})\}\zeta\Big) \\ &\subset \frac{\Gamma_1(\delta)}{\delta^3} \,.\end{aligned}$$

Note that $\delta - 1 = \sqrt{2}$ is not a δ-integer. Its δ-expansion is $\langle \delta - 1 \rangle_\delta = 1 \cdot 1$. It turns out that only ζ, ζ^5, ζ^4 and 1 are in $\Gamma_1(\delta)$.

- When $\beta = \theta$, $\zeta = e^{i\frac{2\pi}{12}}$ and $\chi = \theta - 2$.

$$\begin{array}{rcllllll} q & = & 0 & 1 & 2 & 3 & 4 & 5 \\ (\eta_q, \nu_q) & = & (1,0) & (0,1) & (-1, \theta-2) & (-\theta+2, 2) & (-2, \theta-2) & (-\theta+2, 1) \\ \eta_q + \nu_q \chi & = & 1 & \theta - 2 & 2 & \theta - 2 & 1 & 0 \,, \end{array}$$

together with $(\eta_{q+6}, \nu_{q+6}) = (-\eta_q, -\nu_q)$. Note that $\theta - 2 = \sqrt{3}$ is not a θ-integer. Moreover, the θ-expansion of $\theta - 2$ is infinite: $\langle \theta - 2 \rangle_\theta = 1 \cdot (2)^\omega$. Then, only ζ, ζ^6, ζ^7 and 1 are in $\Gamma_1(\theta)$. Let us introduce the decorated θ-lattice $\widetilde{\Gamma}_1(\theta)$, as we have done in the one-dimensional case (Equation (4)),

$$\Gamma_1(\theta) \subset \widetilde{\Gamma}_1(\theta) = \widetilde{\mathbb{Z}}_\theta + \widetilde{\mathbb{Z}}_\theta \zeta \,.$$

Since $\theta - 2 = 2 - 1/\theta$, then all ζ^q are in $\widetilde{\Gamma}_1(\theta)$, and

$$\begin{aligned}\zeta^q \Gamma_1(\theta) &\subset \Gamma_1(\theta) + \Big(\{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\} + \{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\}\zeta\Big) \\ &\subset \widetilde{\widetilde{\Gamma}}_1(\theta) \equiv \widetilde{\widetilde{\mathbb{Z}}}_\theta + \widetilde{\widetilde{\mathbb{Z}}}_\theta \zeta \,,\end{aligned} \tag{11}$$

where $\widetilde{\widetilde{\mathbb{Z}}}_\theta = \mathbb{Z}_\theta + \{0, \pm 1/\theta, \pm 2/\theta\}$.

4.3 Translational properties

They are deduced from Equations (2) and (4). In Case 1,

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \Gamma_q(\beta)/\beta^2 \,,$$

and in Case 2,

$$\Gamma_q(\beta) + \Gamma_q(\beta) \subset \widetilde{\Gamma}_q(\beta) \,.$$

5 A plane-group for beta-lattices

Since beta-lattices of the type $\Gamma_q(\beta)$ are not rotationally and translationally invariant, we shall enforce invariance by changing the usual additive and multiplicative laws by the beta-addition and the quasi-multiplication.

5.1 A point-group for beta-lattices in the plane

Explicit calculations of internal rotation actions on $\Gamma_1(\beta)$, referred to as beta-rotations, are given in the Appendix. Note that since the quasi-multiplication is not distributive with respect to beta-addition, we find several candidates for internal rotational operators on $\Gamma_1(\beta)$. The choice for the beta-rotations presented in the following proposition is driven by consistency property. Other internal rotational operator *are not* consistent with Euclidian rotations!

We formally imitate the expressions of successive rotations given by Equation (10), by replacing in the equations, $+$ and $-$ by $\oplus$ and $\ominus$, and $\times$ by $\otimes$, when necessary. Proposition 2 below defines the beta-rotations on $\Gamma_1(\beta)$.

Proposition 2 • *When $\beta = \tau$, with the notations of (9), the following 10 operators r_q, $q = 0, 1, \ldots, 9$, leave $\Gamma_1(\tau)$ invariant:*

$$r_q \odot (b_m + b_n\zeta) = \eta_q b_m \ominus \nu_q b_n + (\nu_q b_m \oplus (\eta_q + \tau\nu_q) b_n)\zeta.$$

• *When $\beta = \delta$, the following operators leave $\Gamma_1(\delta)$ invariant:*

$$\begin{aligned}
r_1 \odot (b_m + b_n\zeta) &= -b_n + (b_m \oplus \delta b_n \ominus b_n)\zeta &&= -b_n + b_{m+2n-2\rho_S(n)}\zeta \\
r_2 \odot (b_m + b_n\zeta) &= -(b_m \oplus \delta b_n \ominus b_n) + (\delta b_m \ominus b_m \oplus b_n)\zeta &&= -b_{m+2n-2\rho_S(n)} + b_{2m+n-2\rho_S(m)}\zeta \\
r_3 \odot (b_m + b_n\zeta) &= -(\delta b_m \ominus b_m \oplus b_n) + b_m\zeta &&= -b_{2m+n-2\rho_S(m)} + b_m\zeta\,.
\end{aligned}$$

• *When $\beta = \theta$, the following operators leave $\Gamma_1(\theta)$ invariant:*

$$\begin{aligned}
r_1 \odot (b_m + b_n\zeta) &= -b_n + (b_m \oplus \theta b_n \ominus 2b_n)\zeta &&= -b_n + b_{m+2n-\rho_S(n)}\zeta \\
r_2 \odot (b_m + b_n\zeta) &= -(b_m \oplus \theta b_n \ominus 2b_n) + (\theta b_m \ominus 2b_m \oplus 2b_n)\zeta &&= -b_{m+2n-\rho_S(n)} + b_{2m+2n-\rho_S(m)}\zeta \\
r_3 \odot (b_m + b_n\zeta) &= -(\theta b_m \ominus 2b_m \oplus 2b_n) + (2b_m \oplus \theta b_n \ominus 2b_n)\zeta &&= -b_{2m+2n-\rho_S(m)} + b_{2m+2n-\rho_S(n)}\zeta \\
r_4 \odot (b_m + b_n\zeta) &= -(2b_m \oplus \theta b_n \ominus 2b_n) + (\theta b_m \ominus 2b_m \oplus b_n)\zeta &&= -b_{2m+2n-\rho_S(m)} + b_{2m+n-\rho_S(n)}\zeta \\
r_5 \odot (b_m + b_n\zeta) &= -(\theta b_m \ominus 2b_m \oplus b_n) + b_m\zeta &&= -b_{n+2m-\rho_S(m)} + b_m\zeta\,.
\end{aligned}$$

For $\beta = \tau$, δ or θ, let the composition rule of these operators on $\Gamma_1(\beta)$ be defined by

$$(rr') \odot z = r \odot (r' \odot z) \,,$$

and denote by Id the identity and by ι the space inversion

$$\iota \odot z = -z \,.$$

Then, the composition rule $(r, r') \to rr'$ is associative and the following identities hold: $r_0 = Id$ and $r_{q+N/2} = \iota r_q = r_q \iota$ for $q = 0, 1, \ldots, \frac{N}{2} - 1$, where N is the symmetry order of β.

Lemma 5 *Beta-rotations defined in Proposition 2 on $\Gamma_1(\beta)$ are compatible with rotations when β assumes one of the specified values τ, δ and θ.*

Proof. We deduce from Equation (6) and Equation (8) that beta-rotations have minimal distortion property with respect to rotation: let $z_{m,n} = b_m + b_n\zeta \in \Gamma_1(\beta)$, then

- $\beta = \tau$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm(1 - \frac{1}{\tau})\} + \{0, \pm(1 - \frac{1}{\tau})\}\zeta$,
- $\beta = \delta$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm(1 - \frac{1}{\delta}), \pm 2(1 - \frac{1}{\delta})\} + \{0, \pm(1 - \frac{1}{\delta}), \pm 2(1 - \frac{1}{\delta})\}\zeta$,
- $\beta = \theta$, $\zeta^q z_{m,n} - r_q \odot z_{m,n} \in \{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\} + \{0, \pm\frac{1}{\theta}, \pm\frac{2}{\theta}\}\zeta$.

Proposition 2 shows that beta-rotations can be decomposed in terms of beta-additions and quasi-multiplications. Compatibility of beta-rotation with euclidian rotation is thus a consequence of $+$-compatibility of beta-addition and $\times$-compatibility of quasi-multiplication. ■

Computing of composition of any two of such beta-rotations r_q yields the following important result.

Proposition 3 *For $\beta = \tau$, δ and θ and for $N = 10$, 8 and 12 respectively, let $\Re_N = \Re_N(\beta)$ denote the semi-group freely generated by all r_q, $q \in \{0, 1, \ldots, N-1\}$. Among all beta-rotations, only $r_0, r_1, r_{N/2-1}, r_{N/2+1}, r_{N-1}, \iota$ have their inverse in $\Re_N$.*

Proof. The following identities are straightforwardly checked

$$\begin{aligned} r_1 r_{N/2-1} &= r_{N/2-1} r_1 = r_{N/2+1} r_{N-1} = r_{N-1} r_{N/2-1} = \iota, \\ r_1 r_{N-1} &= r_{N-1} r_1 = r_{N/2-1} r_{N/2+1} = r_{N/2+1} r_{N/2-1} = r_0 \,. \end{aligned}$$

A case study of all possible combinations of r_q shows that no other such operators are invertible. ■

An immediate consequence is the existence of a symmetry group for $\Gamma_1(\beta)$, *i.e.* a group of planar transformations leaving $\Gamma_1(\beta)$ invariant.

Theorem 1 *For $\beta = \tau$, δ and θ, the group $\mathcal{R}_N = \mathcal{R}_N(\beta)$, freely generated by the four element set $\{r_0, \iota, r_1, r_{N/2-1}\}$, is a symmetry group for the beta-lattice $\Gamma_1(\beta)$. It is called the* symmetry point-group *of $\Gamma_1(\beta)$.*

Proof. An easy computation shows that the elements of $\mathcal{R}_N$ are invertible. Associativity of the law of internal composition of elements of $\mathcal{R}_N$ is a consequence of Proposition 2. ■

5.2 A plane-group for beta-lattices $\Gamma_1(\beta)$

We now introduce into the present formalism the beta-translations acting on $\Gamma_1(\beta)$.

Proposition 4 *Let $z_0 = b_{m_0} + b_{n_0}\zeta$ be an element of the beta-lattice $\Gamma_1(\beta)$. There corresponds to it the internal action $t_{z_0} : \Gamma_1(\beta) \mapsto \Gamma_1(\beta)$*

$$t_{z_0}(z) = z \oplus z_0 \stackrel{def}{=} b_m \oplus b_{m_0} + (b_n \oplus b_{n_0})\zeta.$$

The set of beta-translations forms an abelian group isomorphic to the beta-lattice $\Gamma_1(\beta)$ considered itself as a group for the law $\oplus$. For this reason it will be also denoted by $\Gamma_1(\beta)$.

Proof. The beta-translation is a simple two-dimensional generalization of the one-dimensional beta-addition. ■

As a direct generalization of one-dimensional beta-addition, it is obvious that beta-translation has minimal distortion property with respect to translation, and is compatible with it. Using Proposition 4, we come to the main result of this article.

Theorem 2 *For $\beta = \tau$, δ and θ, and for $N = 10$, 8 and 12 respectively, the group $\mathcal{S}_N = \mathcal{S}_N(\beta)$ freely generated by the five-element set $\{r_0, \iota, r_1, r_{\frac{N}{2}-1}, t_1\}$ is a symmetry group for the beta-lattice $\Gamma_1(\beta)$. This group is the semi-direct product of $\Gamma_1(\beta)$ and $\mathcal{R}_N$*

$$\mathcal{S}_n = \Gamma_1(\beta) \rtimes \mathcal{R}_N$$

with the composition rule

$$(b, R)(b', R') = (b \oplus R \odot b', RR').$$

In the present context, $\mathcal{S}_N$ is called the symmetry plane-group *of $\Gamma_1(\beta)$.*

The action of an element of $\mathcal{S}_N$ on $\Gamma_1(\beta)$ is thus defined as

$$(b, R) \centerdot z = b \oplus R \odot z = t_b(R \odot z) \in \Gamma_1(\beta).$$

Proof. An easy computation shows that the elements of $\mathcal{S}_N$ are invertible. Associativity of the law of internal composition of elements of $\mathcal{R}_N$ is a consequence of Proposition 2 and of Theorem 1. ■

5.3 Tile transformations using internal operations on $\Gamma_1(\tau)$

We would like to illustrate the action of $\mathcal{S}_n$ on $\Gamma_1(\beta)$, in the case of τ, by showing how a tile of $\Gamma_1(\tau)$ is transformed under the action of an element of $\mathcal{S}_{10}$.

Let $z = b_m + b_n\zeta \in \Gamma_1(\tau)$. An *elementary quadrilateral tile on* z is the following

$$\mathbf{T}(z) = \{z, z \oplus 1, z \oplus \zeta, z \oplus (1+\zeta)\}\,.$$

From the definition of $\Gamma_1(\tau)$, we trivially see that their exist four kinds of elementary tiles, which we shall denote by LL, LS, SL and SS, as a reference to the length of their edges (see Figure 5).

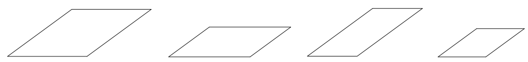

Fig. 5. Elementary quadrilateral tiles for the τ-lattice $\Gamma_1(\tau)$. From left to right: LL, LS, SL, SS. See also Figure 2.

In case of a translation operation by z_0, t_{z_0}, the elementary quadrilateral tile $\mathbf{T}(z)$ is transformed into another elementary quadrilateral tile, whether of the same kind or of another kind, according to

$$t_{z_0}(\mathbf{T}(z)) = \mathbf{T}(z \oplus z_0) = z_0 \oplus \{z, z \oplus 1, z \oplus \zeta, z \oplus (1+\zeta)\}\,.$$

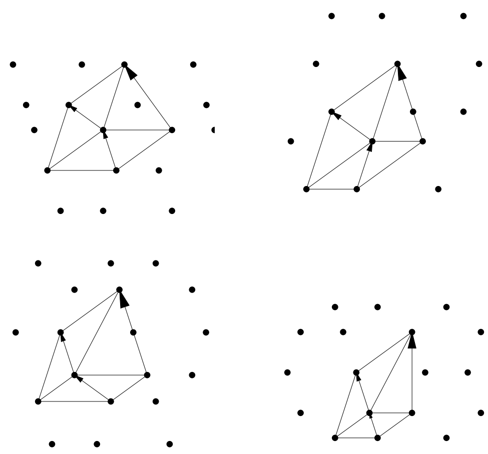

Fig. 6. Rotation operator r_1 applied to elementary tiles of the τ-lattice $\Gamma_1(\tau)$, $\mathbf{T}(0)$, $\mathbf{T}(1)$ (up), $\mathbf{T}(\zeta)$ and $\mathbf{T}(1+\zeta)$ (down). Note how the tiles are deformed, by this operation, in order for the vertices to remain in $\Gamma_1(\tau)$. The arrows indicate the vertices of the new tile in which are mapped the vertices of the original tile.

Another interesting transformation arises when one applies the rotation operator r_1 on $\mathbf{T}(z)$ and around one of the vertex of $\mathbf{T}(z)$. For instance, the rotation around z is given by

$$t_z(r_1 \odot t_{-z}(\mathbf{T}(z))) = \{z, z \oplus \zeta, z \oplus (-1+\tau\zeta), z \oplus (-1+\tau^2\zeta)\} .$$

Examples of such rotation operations are displayed on Figure 6. This operation not only rotates, but distorts the tiles, in general. Therefore, the beta-rotated tile is not elementary anymore.

6 Asymptotic properties

An interesting feature of beta-lattices is that they behave like lattices asymptotically.

Lemma 6 *The asymptotic behavior of the counting function ρ_S is given by*

$$\rho_S(n) \underset{|n|\to\infty}{\approx} \left(1 - \frac{1}{\beta}\right)\frac{n}{a}, \qquad (\textit{Case 1}),$$

$$\rho_S(n) \underset{|n|\to\infty}{\approx} \frac{n}{\beta}, \qquad (\textit{Case 2}).$$

Proof. Case 1. The proof is based on the development of integers in the linear system $U = (u_q)_{q\in\mathbb{N}}$. We have $n = \sum_{i=0}^{k} u_i d_i$. Then $\rho_S(n) = \sum_{i=0}^{k} \rho_S(u_i) d_i = \sum_{i=0}^{k} \frac{u_i}{a}(1 - u_{i-1}/u_i) d_i$. When $n \to \infty$ we know that $u_{i-1}/u_i \to 1/\beta$ and $\rho_S(n) \approx \frac{1}{a}(1-1/\beta)\sum_{i=0}^{k} u_i d_i \approx \frac{n}{a}(1-1/\beta)$, as $n \to \infty$.
Case 2. As in the first case, the proof is based on the development of integers in (u_q): $n = \sum_{i=0}^{k} u_i d_i$, $\rho_S(n) = \sum_{i=0}^{k} \rho_S(u_i) d_i = \sum_{i=0}^{k} u_{i-1} d_i = \sum_{i=0}^{k} \frac{u_i}{a}(1 + u_{i-2}/u_i) d_i$. When $n \to \infty$ we know that $u_{i-2}/u_i = \frac{u_{i-2}}{u_{i-1}}\frac{u_{i-1}}{u_i} = 1/\beta^2 = \frac{a}{\beta} - 1$. therefore $\rho_S(n) \approx \frac{1}{\beta}\sum_{i=0}^{k} u_i d_i = \frac{n}{\beta}$, as $n \to \infty$. ■

Lemma 6 tells us what is the asymptotic behavior of beta-integers for large n, and of the multiplication $\otimes$ for large m and n. From Equation (7) and Lemma 6 is deduced the following result.

Proposition 5 *Let β be a quadratic PV unit number. Then the following asymptotic behaviour of beta-integers holds true*

$$b_n \underset{|n|\to\infty}{\approx} \gamma n ,$$

$$b_m \otimes b_n \underset{|m|,|n|\to\infty}{\approx} \gamma^2 mn .$$

where

$$\gamma = \begin{cases} 1 - \frac{1}{a}\left(1 - \frac{1}{\beta}\right)^2 = \frac{(a+2)\beta - a^2 - a - 2}{a} & (\textit{Case 1}), \\ 1 - \frac{1}{\beta^2} = a(\beta - a) + 2 & (\textit{Case 2}). \end{cases}$$

Proof. Case 1. Any beta-integer b_n can be written $b_n = n - \rho_S(n)(1 - 1/\beta)$. When n becomes large, we can replace $\rho_S(n)$ by its asymptotic value. We then have $b_n \approx n(1 - \frac{1}{a}(1 - 1/\beta)^2) = \gamma n$.

Case 2. In the same fashion, we have $b_n = n - \rho_S(n)1/\beta$, and by replacing $\rho_S(n)$ by its asymptotic value for large n we obtain $b_n = n(1 - 1/\beta^2) = \gamma n$.
The second part of the proposition is a direct consequence of the first part. ■

We then *almost* recover the definition of multiplication we were thinking about at the beginning of Section 3.5, left alone that in both cases we have a contraction of the resulting index by a factor $\gamma < 1$. We should notice that the multiplication $\otimes$ is *asymptotically* associative and distributive with respect to the addition $\oplus$. In this sense we can say that $\mathbb{Z}_\beta$ is asymptotically a ring

$$\begin{aligned} b_m \otimes (b_n \oplus b_p) - b_m \otimes b_n \oplus b_m \otimes b_p &\underset{|m|,|n|,|p| \to \infty}{\approx} 0\,, \\ b_m \otimes (b_n \otimes b_p) - (b_m \otimes b_n) \otimes b_p &\underset{|m|,|n|,|p| \to \infty}{\approx} 0\,. \end{aligned}$$

Note that m, n and p must be such that $m+n$ and $m+p$ are large numbers, otherwise the above equations are not true.

Consequently we compute the asymptotic behavior of rotational internal laws of beta-lattices, as defined in Section 5.1 in the studied cases.

- When $\beta = \tau$, we have for invertible operators

$$\begin{aligned} r_1 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-n + (m + \tau n)\zeta)\,, \\ r_4 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-\tau m - n - m\zeta)\,. \end{aligned}$$

- When $\beta = \delta$, we have for invertible operators

$$\begin{aligned} r_1 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-n + (m + (\delta - 1)n)\zeta)\,, \\ r_3 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-(\delta - 1)m - n + m\zeta)\,. \end{aligned}$$

- When $\beta = \theta$, we have for invertible operators

$$\begin{aligned} r_1 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-n + (m + (\theta - 2)n)\zeta) \\ r_5 \odot (b_m + b_n\zeta) &\underset{|m|,|n| \to \infty}{\approx} \gamma(-(\theta - 2)m - n + m\zeta)\,. \end{aligned}$$

At this point one should be aware that these asymptotic beta-rotations are equivalent to rotations for large $|m|$ and $|n|$, and an easy computation shows, $z_{m,n} \in \Gamma_1(\beta)$

$$\zeta z_{m,n} - r_1 \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0\,,$$
$$\zeta^{N/2-1} z_{m,n} - r_{N/2-1} \odot z_{m,n} \underset{|m|,|n|\to\infty}{\approx} 0\,,$$

with $N = 10, 8$ and 12.

7 Conclusion

The main result of this article is the construction of a symmetry plane-group for beta-lattices for three quadratic PV units. Though preliminary, this study shows the richness of the beta-lattices as far as all the operations of the plane-group can be made arithmetically explicit. Many questions seem to be open, such as the number of possible plane-groups leaving a beta-lattice invariant. Another important issue is to determine whether there is or not a metric left invariant under the action of such groups. It has been shown that a large class of point sets, such as model sets, can be embedded in beta-lattices [8]. A question related to distortion of distances is the action of beta-rotations and beta-translations over a point set embedded in a beta-lattice and over the tiling associated to this point set. The point group $\mathcal{R}_N(\beta)$ also deserves to be carefully studied. The link between beta-lattices and the class of finitely generated modules over ordered rings would deserves to be handled nicely in the framework of the Artin-Schreier theory. The case of PV of higher degree remains open. The present contribution shows the potentiality offered by the class of beta-lattices to provide structure models of more general quasiperiodic crystals, and possibly to predict new crystals.

Appendix: Explicit internal rotations actions on beta-lattices

In this section we make the beta-rotation explicit for the quasicrystallographic numbers τ, δ, and θ, and for all the corresponding q, the remaining beta-rotation being deduced from them by combining with space inversion. We give the resulting integer indexes in terms of m, n, and the counting function ρ_S as all involved relations have been introduced in Equation (6) and Equation (7).

Case of the τ-lattice $\Gamma_1(\tau)$

$$\begin{aligned}
r_1 \odot (b_m + b_n\zeta) &= b_{-n} + b_{m+2n-\rho_S(n)}\zeta \\
r_2 \odot (b_m + b_n\zeta) &= b_{-m-2n+\rho_S(n)} + b_{2(m+n)-\rho_S(m)-\rho_S(n)}\zeta \\
r_3 \odot (b_m + b_n\zeta) &= b_{-2(m+n)+\rho_S(m)+\rho_S(n)} + b_{2m+n-\rho_S(m)}\zeta \\
r_4 \odot (b_m + b_n\zeta) &= b_{-2m-n+\rho_S(m)} + b_m\zeta
\end{aligned}$$

Case of the δ-lattice $\Gamma_1(\delta)$

For the δ-rotations we would like to play the same game of formal imitation of Equation (10) as in the case of τ. The case of δ however is slightly more complicated since $\eta_q b_n$ and $\nu_q b_n$ are not in $\mathbb{Z}_\delta$. When we compute the rotation of an arbitrary element of $\Gamma_1(\delta)$, we need to determine the value of $(\delta-1)b_n$, which is of course not a δ-integer in the general case. Recall that $\otimes$ is not distributive with respect to $\oplus$. Therefore, we have to replace $(\delta-1)b_n$ either by $(\delta\ominus 1)\otimes b_n = b_{2n}$ or by $\delta b_n \ominus b_n = b_{2n-\rho_S(n)}$ (recall that from self similarity of δ-integers we have $\delta \otimes b_n = \delta b_n$). We then have to make a choice about which operation to choose to build the point-group of $\Gamma_1(\delta)$. We chose to replace $(\delta-1)b_n$ by $\delta b_n \ominus b_n$ in Section 5, since this case satisfies the consistency property. Other operations may be interesting. For example, the other internal rotation laws do not satisfy the consistency property and do not have the same asymptotic behavior.

$$r_1 \odot (b_m + b_n\zeta) = \begin{cases} -b_n + b_{m+2n-\rho_S(n)}\zeta \\ -b_n + b_{m+2n}\zeta \end{cases}$$

$$r_2 \odot (b_m + b_n\zeta) = \begin{cases} -b_{m+2n-\rho_S(n)} + b_{2m+n-\rho_S(m)}\zeta \\ -b_{m+2n} + b_{2m+n-\rho_S(m)}\zeta \\ -b_{m+2n-\rho_S(n)} + b_{2m+n}\zeta \\ -b_{m+2n} + b_{2m+n}\zeta \end{cases}$$

$$r_3 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+n-\rho_S(m)} + b_m\zeta \\ -b_{2m+n} + b_m\zeta \end{cases}$$

Case of the θ-lattice $\Gamma_1(\theta)$

As in the case of the δ-lattice, we have to decide which operation to use to build the point-group of $\Gamma_1(\theta)$ because of the factor $(\theta-2)b_n$, introduced in the computation of rotations of $\Gamma_1(\theta)$. Once again, we have replaced $(\theta-2)b_n$ by $\theta b_n \ominus 2 \otimes b_n = b_{2n-\rho_S(n)}$ in Section 5.2. We give now all possibilities.

$$r_1 \odot (b_m + b_n\zeta) = \begin{cases} -b_n + b_{m+2n-\rho_S(n)}\zeta \\ -b_n + b_{m+2n}\zeta \end{cases}$$

$$r_2 \odot (b_m + b_n\zeta) = \begin{cases} -b_{m+2n-\rho_S(n)} + b_{2m+2n-\rho_S(m)}\zeta \\ -b_{m+2n} + b_{2m+2n-\rho_S(m)}\zeta \\ -b_{m+2n-\rho_S(n)} + b_{2m+2n}\zeta \\ -b_{m+2n} + b_{2m+2n}\zeta \end{cases}$$

$$r_3 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+2n-\rho_S(m)} + b_{2m+2n-\rho_S(n)}\zeta \\ -b_{2m+n} + b_{2m+2n-\rho_S(n)}\zeta \\ -b_{2m+2n-\rho_S(m)} + b_{2m+2n}\zeta \\ -b_{2m+2n} + b_{2m+2n}\zeta \end{cases}$$

$$r_4 \odot (b_m + b_n\zeta) = \begin{cases} b_{2m+2n-\rho_S(n)} + b_{n+2m-\rho_S(m)}\zeta \\ b_{2m+2n} + b_{n+2m-\rho_S(m)}\zeta \\ b_{2m+2n-\rho_S(n)} + b_{n+2m}\zeta \\ b_{2m+2n} + b_{n+2m}\zeta \end{cases}$$

$$r_5 \odot (b_m + b_n\zeta) = \begin{cases} -b_{2m+n-\rho_S(m)} + b_m\zeta \\ -b_{2m+n} + b_m\zeta \end{cases}$$

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