EXTREMAL BEHAVIOUR OF SOLUTIONS TO A STOCHASTIC DIFFERENCE EQUATION WITH APPLICATIONS TO ARCH PROCESSES

Laurens de HAAN*

Econometric Institute, Erasmus Universiteit Rotterdam, Postbus 1738, 3000 DR Rotterdam, The Netherlands

Sidney I. RESNICK*†

Department of Operations Research, Upson Hall, Cornell University, Ithaca, NY 14853, USA

Holger ROOTZÉN*‡

Department of Mathematical Statistics, University of Lund, S-22100 Lund, Sweden

Casper G. de VRIES

Department of Economics, Erasmus Universiteit Rotterdam, Postbus 1738, 3000 DR Rotterdam, The Netherlands

We consider limit distributions of extremes of a process $\{Y_n\}$ satisfying the stochastic difference equation

$$Y_n = A_n Y_{n-1} + B_n, \quad n \geq 1, \ Y_0 \geq 0,$$

where $\{A_n, B_n\}$ are i.i.d. $\mathbb{R}_+^2$-valued random pairs. A special case of interest is when $\{Y_n\}$ is derived from a first order ARCH process. Parameters of the limit law are exhibited; some are hard to calculate explicitly but easy to simulate.

AMS 1980 Subject Classifications: Primary 60F05; Secondary 60G10, 60G17.

extreme values * ARCH process * stochastic difference equation with random coefficients

1. Introduction

Consider a process $\{Y_n, n \geq 1\}$ which satisfies the stochastic difference equation

$$Y_n = A_n Y_{n-1} + B_n, \quad n \geq 1, \ Y_0 \geq 0, \tag{1.1}$$

where $\{(A_n, B_n), n \geq 1\}$ are i.i.d. $\mathbb{R}_+^2$-valued random pairs (cf. Vervaat, 1979). We study the extremal behaviour of $\{Y_n\}$ under rather mild assumptions. Our interest

in this problem was stimulated by the desire to understand extremal characteristics of the autoregressive conditional heteroscedastic (ARCH) processes introduced by Engle (1982). The first order ARCH process is defined by

$$\xi_n = X_n(\beta + \lambda\xi_{n-1}^2)^{1/2}, \quad n \geq 1, \tag{1.2}$$

where $\{X_n\}$ are i.i.d. $N(0, 1)$ random variables, $\beta > 0, 0 < \lambda < 1$. Thus $\{\xi_n^2\}$ satisfies (1.1) with $A_n = \lambda X_n^2$, $B_n = \beta X_n^2$. (Higher order ARCH processes, considered for example by Engle, 1982, and Milhøj, 1985, or Bollerslev, 1986, would satisfy the higher order version of (1.1).)

ARCH processes were introduced in econometric modelling because the usual linear time series models, with constant conditional variance and Gaussian tails, are inadequate for many types of financial data. Typically such data exhibit clusters of high and low volatility. The ARCH processes have been successfully applied in modelling exchange rate yields (Domowitz and Hakkio, 1985, and Hsieh, 1988), and stock returns (Engle et al., 1987, and Bollerslev, 1987). Bollerslev and Engle (1986) provide a good review; Bollerslev (1986) and Weiss (1986) give some extensions.

Extreme behaviour is of obvious interest in economics. For example, extreme yields may characterize the occurrence of bankruptcy (McCulloch, 1981) or foreign exchange rate realignments (Flood and Garber, 1984). Because of the importance of extremes, it is natural to inquire into the statistical properties of extremes of the ARCH process (and more generally of solutions of (1.1)). In particular we want to find limiting distributions and to resolve whether or not there is clustering associated with such extremes. These issues are taken up in Section 2. Section 3 is concerned with the numerical computation of some constants appearing in the limiting distributions.

It is rather striking that the building blocks of the ARCH process in (1.2) are normal variates but yet ξ_n has Pareto-like tails. The reason for this is the following result of Kesten (1973) given as (iv) of the next theorem which collects some needed information as given in Vervaat (1979).

Theorem 1.1. *Suppose* (1.1) *holds and that there is a* $\kappa > 0$ *with*

$$EA_1^\kappa = 1, \qquad EA_1^\kappa \log^+ A_1 < \infty, \qquad 0 < EB_1^\kappa < \infty, \tag{1.3}$$

that $B_1/(1 - A_1)$ *is nondegenerate and that the conditional distribution of* $\log A_1$ *given* $A_1 \neq 0$ *is nonlattice.*

(i) *The equation*

$$Y_\infty \stackrel{d}{=} A_1 Y_\infty + B_1,$$

Y_∞ *and* (A_1, B_1) *independent, has a solution unique in distribution, given by*

$$Y_\infty \stackrel{d}{=} \sum_{j=1}^\infty B_j \prod_{i=1}^{j-1} A_i.$$

(ii) *If in* (1.1) *we take* $Y_0 \stackrel{d}{=} Y_\infty$, *then the process* $\{Y_n\}$ *is stationary.*

(iii) *No matter how the process* $\{Y_n\}$ *is initialized*

$$Y_n \xrightarrow{d} Y_\infty,$$

where "$\xrightarrow{d}$" denotes convergence in distribution.

(iv) *There exists a constant* $c > 0$ *such that as* $t \to \infty$,

$$P(Y_\infty > t) \sim ct^{-\kappa}. \quad \square \tag{1.4}$$

Remark. Unfortunately it is difficult to get hold of the constant c in (1.4) explicitly, but as we shall argue later, this is of small practical consequence.

In the case of the ARCH process $\{\xi_n^2\}$, where $A_1 = \lambda X_1^2$, $B_1 = \beta X_1^2$, the conditions (1.3) are readily seen to hold and κ is the solution of

$$E(\lambda X_1^2)^\kappa = 1, \tag{1.5}$$

where $X_1 \sim N(0, 1)$. For a specific value of λ $(0 < \lambda < 1)$, the value κ is readily found by solving for κ in the equation

$$\Gamma(\kappa + \tfrac{1}{2}) = \pi^{1/2}(2\lambda)^{-\kappa}. \tag{1.6}$$

For instance, when $\lambda = \tfrac{1}{2}$, the nonzero root of (1.6) is approximately 2.365.

The occurrence of the Pareto type tail in (1.4) is more understandable after an outline of Kesten's (1973) argument: By iterating (1.1) we find for $n \geq 1$,

$$Y_n = \sum_{k=1}^n \left(\prod_{j=k+1}^n A_j \right) B_k + \left(\prod_{j=1}^n A_j \right) Y_0$$

(where $\prod_{j=n+1}^n A_j = 1$). Assumptions (1.3) imply $E \log A_1 < 0$. Kesten shows $\sum_{k=0}^\infty (\prod_{j=1}^k A_j) B_{k+1}$ has a tail comparable to $\bigvee_{k=0}^\infty (\prod_{j=1}^k A_j) B_{k+1}$, and the tail of this variable is determined by $\bigvee_{k=0}^\infty \sum_{j=1}^k \log A_j$. Now results for the distribution of the maximum of a random walk with negative drift by means of defective renewal theory (cf. Feller, 1971, Section XII.5, Example c) give (1.4). The occurrence of the Pareto type tail in (1.4) has been further studied in a recent preprint by Goldie (1988).

2. Extremal behaviour

Assume the conditions (1.3) of Theorem 1.1 hold. We show below that $M_n = \bigvee_{i=1}^n Y_i$ has a type II extreme value limit law. This is the same type of limit as would occur if the Y's were i.i.d. with marginal distribution satisfying (1.4). However, the norming constants are different in the present dependent case. We will express this by means of the extremal index θ of the Y-process. Loosely speaking, large values of the Y-process have a tendency to come in small clusters, which makes M_n have the same limit distribution as the maximum of $n\theta$, rather than of n, i.i.d. variables with the same marginal distribution (cf. Leadbetter et al., 1983, Section 3.7, and Rootzén, 1988). To describe the clustering of extremes in more detail, we also show that the

time-normalized point process N_n of exceedances of a suitably chosen high level u_n, defined by

$$N_n(A) = \#\{k/n \in A: X_k > u_n\}$$

converges to a compound Poisson process N. Specifically, in the limiting compound Poisson process events occur as an (ordinary) Poisson process with intensity $\eta = c\theta x^{-\kappa}$ and the multiplicities of the events are independent and with compounding probabilities $\{\pi_k\}$ given in the theorem below, π_k being the probability that an event has multiplicity k. Further, convergence in distribution of point processes (denoted $\xrightarrow{d}$) is as defined e.g. in the appendix of Leadbetter et al. (1983).

Without loss of generality, we suppose throughout this section that $Y_0 \overset{d}{=} Y_\infty$ so that $\{Y_n, n \geq 1\}$ is stationary. If $Y_n(Y_0)$ is the solution of (1.1) initialized by some Y_0 other than Y_∞, we have as in Vervaat (1979) that

$$Y_n(Y_0) - Y_n(Y_\infty) = \left(\prod_{j=1}^{n} A_j\right)(Y_0 - Y_\infty). \tag{2.1}$$

Since $\prod_{j=1}^{n} A_j \to 0$ a.s., we have for any $a_n \to 0$ that

$$a_n\left(\bigvee_{j=1}^{n} Y_j(Y_0)\right) = a_n \bigvee_{j=1}^{n} \left\{Y_j(Y_\infty) + \left(\prod_{i=1}^{j} A_i\right)(Y_0 - Y_\infty)\right\}$$

$$\leq a_n \bigvee_{j=1}^{n} Y_j(Y_\infty) + a_n \bigvee_{j=1}^{n} \left(\prod_{i=1}^{j} A_i\right)(Y_0 - Y_\infty)$$

$$= a_n \bigvee_{j=1}^{n} Y_j(Y_\infty) + o(1)$$

with a similar inequality in the reverse direction, showing that $a_n \bigvee_{j=1}^{n} Y_j(Y_0)$ and $a_n \bigvee_{j=1}^{n} Y_j(Y_\infty)$ have identical limit laws if one of them has a limit law. The same comment applies to the point process convergence.

Theorem 2.1. *If* (1.3) *holds, then* $\{Y_n\}$ *has an extremal index* θ *given by*

$$\theta = \int_1^\infty P\left(\bigvee_{j=1}^{\infty} \prod_{i=1}^{j} A_i \leq y^{-1}\right)\kappa y^{-\kappa-1}\,dy \tag{2.2}$$

and with $a_n = n^{-1/\kappa}$ *we have for* $x > 0$,

$$\lim_{n \to \infty} P(a_n M_n \leq x) = \exp\{-c\theta x^{-\kappa}\}. \tag{2.3}$$

Further, let N_n *be the time-normalized point process of exceedances of the level* $u_n = x/a_n = xn^{1/\kappa}, x > 0$, *as defined above, and let* N *be a compound Poisson process with intensity* $c\theta x^{-\kappa}$ *and compounding probabilities* $\pi_k = (\theta_k - \theta_{k+1})/\theta, k = 1, 2, \ldots,$ *for*

$$\theta_k = \int_1^\infty P\left[\#\left\{j \geq 1: \prod_{i=1}^{j} A_i > y^{-1}\right\} = k-1\right]\kappa y^{-\kappa-1}\,dy$$

(in particular $\theta_1 = \theta$*). Then* $N_n \xrightarrow{d} N$ *as* $n \to \infty$.

Remarks. (1) Hence the distribution function of $M_n/(nc\theta)^{1/\kappa}$ is approximately $\exp\{-1/x^\kappa\}$. However, the value of c is at present only known for integer values of κ and for general κ only bounds are available, see Goldie (1988). The main result of the theorem is therefore the existence and representation of the extremal index θ and of the limits. Since we know θ and the shape parameter κ of the limiting extreme value distribution, all that remains is estimating the unknown scale parameter. This can be done by adapting known results for i.i.d. variables.

(2) An alternative expression for θ_k can be obtained as follows: Let $\{S_j = \sum_{i=1}^j \log A_i, j \geq 1\}$ be the random walk with negative drift. Then

$$\theta_k = \int_1^\infty P\left[\sum_{j=1}^\infty 1_{\{S_j > -\log y\}} = k - 1 \right] \kappa y^{-\kappa-1} \, dy.$$

Let $\{T_j, j \geq 1\}$ be the sequence $\{S_j\}$ written in decreasing order with repetitions allowed if there are ties, set $T_0 = \infty$ and let T_1 be the largest of $\{S_j, j \geq 1\}$, T_2 be the second largest of $\{S_j, j \geq 1\}$, and so on. Then

$$\sum_{j=1}^\infty 1_{\{S_j > -\log y\}} = k - 1$$

if and only if $T_{k-1} > -\log y$ and $T_k \leq -\log y$, so that

$$\theta_k = \int_1^\infty (P(T_{k-1} > -\log y) - P(T_k > -\log y))\kappa y^{-\kappa-1} \, dy$$

$$= \int_1^\infty (P(e^{\kappa T_{k-1}} > y^{-\kappa}) - P(e^{\kappa T_k} > y^{-\kappa}))\kappa y^{-\kappa-1} \, dy$$

$$= \int_0^1 (P(e^{\kappa T_{k-1}} > u) - P(e^{\kappa T_k} > u)) \, du$$

$$= E(e^{\kappa \min(T_{k-1}, 0)} - e^{\kappa \min(T_k, 0)}).$$

Proof of Theorem 2.1. The proof is an application of Theorem 4.1 of Rootzén (1988). For the first part, concerning the extremal index θ, we are required to show that $D(u_n)$ holds for $u_n = x/a_n = xn^{1/\kappa}$, $x > 0$, (cf. Leadbetter et al., 1983, p. 53, and Rootzén, 1988) and that

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} |P(M_{[n\varepsilon]} \leq a_n^{-1} \mid Y_0 > a_n^{-1}) - \theta| = 0. \tag{2.4}$$

The mixing condition $D(u_n)$ is obtained similarly as Lemma 3.1 of Rootzén (1986), so we only briefly indicate the changes needed. For $\nu > 0$, let

$$Y_t'' = Y_t - A_t \cdots A_{t-n\nu+1} Y_{t-n\nu},$$

so that Y_s and Y_t'' are independent for $t - s \geq n\nu$, and write $Y_t' = Y_t$, $a_n = n^{-1/\kappa}$, $b_n = 0$, and $G(x) = \exp\{-cx^{-\kappa}\}$. Inserting this in the proof of the cited lemma, with X's replaced by Y's, it is readily seen that $D(u_n)$ holds provided

$$\limsup_{n \to \infty} nP(|Y_0 - Y_0''| > \varepsilon n^{1/\kappa}) = 0, \tag{2.5}$$

for any ν, $\varepsilon > 0$. However, for $\delta > 0$, the quantity in (2.5) can be bounded by

$$\limsup_{n \to \infty} nP(Y_0 \cdot A_1 \cdots A_{n\nu} > \varepsilon n^{1/\kappa})$$

$$\leqslant \limsup_{n \to \infty} nP(Y_0 > \delta^{-1} \varepsilon n^{1/\kappa}) + \limsup_{n \to \infty} nP(A_1 \cdots A_{n\nu} > \delta)$$

$$\leqslant c(\delta/\varepsilon)^\kappa + \limsup_{n \to \infty} n(EA_0^t)^n / \delta^t,$$

for any $t \geqslant 0$, by (1.4) and Markov's inequality. Now, (1.3) can be seen to imply (cf. Section 3 below) that there is a $t > 0$ with $EA_0^t < 1$, so that the last expression equals zero for this choice of t, and since $\delta > 0$ is arbitrary, this implies (2.5), and hence that $D(u_n)$ is satisfied.

Now we concentrate on verifying (2.4). We need an auxiliary process

$$Y_n^\# = \prod_{j=1}^n A_j Y_0, \qquad Y_0^\# = Y_0,$$

which thus satisfies

$$Y_n^\# = A_n Y_{n-1}^\#, \quad n \geqslant 1, \tag{2.6}$$

and hence

$$\Delta_n := Y_n - Y_n^\#, \qquad \Delta_0 = 0 \tag{2.7}$$

satisfies

$$\Delta_n = A_n \Delta_{n-1} + B_n, \quad n \geqslant 1, \qquad \Delta_0 = 0,$$

i.e. $\{\Delta_n\}$ satisfies (1.1) with a different initial condition. Set $M_n = \bigvee_{j=1}^n Y_j$, $M_n^\# = \bigvee_{j=1}^n Y_j^\#$. Since the A_n and B_n are nonnegative, we have that

$$P(M_{[n\varepsilon]} > a_n^{-1} \mid Y_0 > a_n^{-1})$$

$$\geqslant P(M_{[n\varepsilon]}^\# > a_n^{-1} \mid Y_0 > a_n^{-1})$$

$$= \int_1^\infty P\left(a_n \bigvee_{j=1}^{[n\varepsilon]} \prod_{i=1}^j A_i Y_0 > 1 \mid a_n Y_0 > y\right) \frac{P(a_n Y_0 \in dy)}{P(a_n Y_0 > 1)}$$

and since $P(Y_0 > a_n^{-1} y)/P(Y_0 > a_n^{-1}) \to y^{-\kappa}$ uniformly for $y \geqslant 1$ we find

$$\liminf_{n \to \infty} P(M_{[n\varepsilon]} > a_n^{-1} \mid Y_0 > a_n^{-1})$$

$$\geqslant \int_1^\infty P\left(\bigvee_{j=1}^\infty \prod_{i=1}^j A_i > y^{-1}\right) \kappa y^{-\kappa-1} \, dy = \theta. \tag{2.8}$$

For an inequality in the reverse direction, write $Y_j = Y_j^\# + \Delta_j$ so that

$$P(M_{[n\varepsilon]} > a_n^{-1} \mid Y_0 > a_n^{-1}) = P\left(\bigvee_{j=1}^{[n\varepsilon]} (Y_j^\# + \Delta_j) > a_n^{-1} \mid Y_0 > a_n^{-1}\right)$$

$$\leqslant P\left(M_{[n\varepsilon]}^\# + \bigvee_{j=1}^{[n\varepsilon]} \Delta_j > a_n^{-1} \mid Y_0 > a_n^{-1}\right)$$

and for any $\delta > 0$ we find the above bounded by

$$P(M_{[n\varepsilon]}^{\#} > a_n^{-1}(1-\delta) \mid Y_0 > a_n^{-1}) + P\left(\bigvee_{j=1}^{[n\varepsilon]} \Delta_j > \delta a_n^{-1} \mid Y_0 > a_n^{-1}\right)$$

$$= (A) + (B),$$

say. Now

$$(B) \leq \sum_{j=1}^{[n\varepsilon]} P(\Delta_j > \delta a_n^{-1}).$$

Examining (2.1) we realize that the solutions of (1.1) are monotone with respect to the initial value. Since $\Delta_0 = 0 \leq Y_0 = Y_\infty$, we find $\Delta_j \leq Y_j(Y_\infty) \stackrel{d}{=} Y_\infty$ and thus

$$(B) \leq \sum_{j=1}^{[n\varepsilon]} P(Y_\infty > \delta a_n^{-1}) = [n\varepsilon] P(Y_\infty > \delta a_n^{-1}) \to \varepsilon c \delta^{-\kappa}.$$

For (A) we find upon examining the logic which led to (2.8) that as $n \to \infty$,

$$(A) \to \int_1^\infty P\left(\bigvee_{j=1}^\infty \left(\prod_{i=1}^j A_i\right) > y^{-1}(1-\delta)\right) \kappa y^{-\kappa-1} \, dy$$

and thus

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} P(M_{[n\varepsilon]} > a_n^{-1} \mid Y_0 > a_n^{-1})$$

$$\leq \int_1^\infty P\left(\bigvee_{j=1}^\infty \left(\prod_{i=1}^j A_i\right) > y^{-1}(1-\delta)\right) \kappa y^{-\kappa-1} \, dy$$

$$= (1-\delta)^{-\kappa} \int_{(1-\delta)^{-1}}^\infty P\left(\bigvee_{j=1}^\infty \left(\prod_{i=1}^j A_i\right) > z^{-1}\right) \kappa z^{-\kappa-1} \, dz$$

$$\to \int_1^\infty P\left(\bigvee_{j=1}^\infty \left(\prod_{i=1}^j A_i\right) > z^{-1}\right) \kappa z^{-\kappa-1} \, dz \tag{2.9}$$

as $\delta \to 0$. We now get (2.4) by combining (2.8) and (2.9).

The second part of the theorem is obtained similarly, with only straightforward changes of the arguments, now using (ii) of Theorem 4.1 of Rootzén (1988) instead of (i). $\square$

3. Computing the extremal index

The extremal index θ given by

$$1 - \theta = \int_1^\infty P\left(\bigvee_{j=1}^\infty \left(\prod_{i=1}^j A_i\right) > y^{-1}\right) \kappa y^{-\kappa-1} \, dy$$

will in general be difficult to compute analytically in closed form. However, it is easy to simulate this quantity. Let

$$S_j = \sum_{i=1}^{j} \log A_i$$

and suppose E_κ is a random variable with exponential density and parameter κ, which is independent of $\{S_j\}$. Then we may write

$$1 - \theta = E 1_{\{\bigvee_{j=1}^{\infty} S_j > -E_\kappa\}}, \tag{3.1}$$

which suggests running replications of the random walk and counting the number of replications where the random walk exceeds the level $-E_\kappa$. A mechanism is needed to determine how many steps the random walk is allowed to run on each replication, so consider the following: For any $t > 0$,

$$\begin{aligned}
P(S_j > -E_\kappa) &= P(\exp\{t(S_j + E_\kappa)\} > 1) \\
&\leq E(\exp\{t(S_j + E_\kappa)\}) \\
&= E(\exp\{tS_1\})^j E(\exp\{tE_\kappa\}) \\
&\leq \varphi^j(t) \kappa/(\kappa - t), \quad 0 \leq t < \kappa,
\end{aligned} \tag{3.2}$$

where $\varphi(t) = E \exp\{tS_1\} = EA_1^t$. A convenient choice of t is the value t_0 which minimizes $\varphi(t)$. Since $\varphi'(0) = E \log A_1 < 0$, this exists in $(0, \kappa)$ and can be found by solving $\varphi'(t) = 0$. Using the value of t_0 and summing (3.2) yields

$$P\left(\bigvee_{j=m+1}^{\infty} S_j > -E_\kappa \right) \leq \left[\frac{\varphi^{m+1}(t_0)}{1 - \varphi(t_0)} \right] \kappa/(\kappa - t_0) =: b(t_0). \tag{3.3}$$

For $m > 0$ and large, set

$$1 - \theta_\# = E 1_{\{\bigvee_{j=1}^{m} S_j > -E_\kappa\}}$$

and for N a large number of replications, we set

$$1 - \hat{\theta}_\# = N^{-1} \sum_{i=1}^{N} 1_{\{\bigvee_{j=1}^{m} S_j^{(i)} > -E_\kappa^{(i)}\}}, \tag{3.4}$$

where the superscript i refers to the replication number. We know

$$\frac{(1 - \hat{\theta}_\#) - (1 - \theta_\#)}{\{\theta_\#(1 - \theta_\#)/N\}^{1/2}}$$

is approximately $N(0, 1)$ so an approximate $100(1 - \gamma)\%$ confidence interval for $\theta_\#$ is $\hat{\theta}_\# \pm z_{\gamma/2}(4N)^{-1/2}$. Furthermore

$$\begin{aligned}
0 &\leq (1 - \theta) - (1 - \theta_\#) \\
&= P\left(\bigvee_{j=1}^{\infty} S_j > -E_\kappa \right) - P\left(\bigvee_{j=1}^{m} S_j > -E_\kappa \right) \leq P\left(\bigvee_{j=m+1}^{\infty} S_j > -E_\kappa \right),
\end{aligned}$$

which from (3.3) is bounded by the geometric bound $b(t_0)$ and so the approximate $100(1-\gamma)\%$ confidence interval for θ is

$$(\hat{\theta}_{\#} - z_{\gamma/2}(4N)^{-1/2} - b(t_0), \hat{\theta}_{\#} + z_{\gamma/2}(4N)^{-1/2}).$$

Since

$$\theta_k = P\left(\sum_{j=1}^{\infty} 1_{\{S_j > -E_\kappa\}} = k - 1\right)$$

we can estimate θ_k in a similar way from the same simulations by counting the number of replications where the number of exceedances of $-E_\kappa$ is $k-1$. Now the obvious estimator of θ_k is

$$\hat{\theta}_{\#k} = N^{-1} \sum_{i=1}^{N} 1_{\{\sum_{j=1}^{m} 1_{\{S_j > -E_\kappa\}} = k - 1\}}, \tag{3.5}$$

so that a $100(1-\gamma)\%$ confidence interval for

$$\theta_{\#k} = P\left(\sum_{j=1}^{m} 1_{\{S_j > -E_\kappa\}} = k - 1\right)$$

is $\hat{\theta}_{\#k} \pm z_{\gamma/2}(4N)^{-1/2}$. As before,

$$0 < \theta_k - \theta_{\#k}$$

$$= P\left(\sum_{j=1}^{\infty} 1_{\{S_j > -E_\kappa\}} = k - 1\right) - P\left(\sum_{j=1}^{m} 1_{\{S_j > -E_\kappa\}} = k - 1\right)$$

$$\leqslant P\left(\sum_{j=m+1}^{\infty} 1_{\{S_j > -E_\kappa\}} > 0\right) \leqslant \sum_{j=m+1}^{\infty} P(S_j + E_\kappa > 0),$$

and as before this has the bound $b(t_0)$. Hence the approximate $100(1-\gamma)\%$ confidence interval for θ_k is

$$(\hat{\theta}_{\#k} - z_{\gamma/2}(4N)^{-1/2}, \hat{\theta}_{\#k} + z_{\gamma/2}(4N)^{-1/2} + b(t_0)).$$

Finally we return to the ARCH process (1.2). Clearly $\{\xi_n^2, n \geq 1\}$ satisfies (1.1) and the conditions (1.3). Hence the extremal index and compounding probabilities for ξ_n^2 are given by Theorem 2.1 and can be computed from (3.4) and (3.5). Since an exceedance of u^2 by ξ_t^2 is the same as an exceedance of u by $|\xi_n|$, the process $\{|\xi_n|\}$ has the same extremal index and compounding probabilities. Table 3.1 gives values of the extremal index θ and the compounding probabilities $\pi_k = (\theta_k - \theta_{k+1})/\theta$ for these processes, based on the described simulation. The length m of the random walk and the number N of replications are $m = 1000$, $N = 1000$. Different rows of the table are based on separate simulations. The values obtained for $\varphi(t_0)$ in the table clearly render $b(t_0)$ negligible for the given value of m.

Table 3.1
Extremal index $\hat{\theta}$ and compounding probabilities $\hat{\pi}_k$ for the absolute value $|\xi_k|$ of the ARCH process, based on $N = 1000$ simulations of length $m = 1000$

λ	κ	$\hat{\theta}$	$\hat{\pi}_1$	$\hat{\pi}_2$	$\hat{\pi}_3$	$\hat{\pi}_4$	$\hat{\pi}_5$	t_0	$\varphi(t_0)$
0.1	13.24	0.997	0.997	0.003	0.000	0.000	0.000	4.99	0.009
0.3	4.180	0.887	0.892	0.094	0.011	0.003	0.003	1.64	0.261
0.5	2.365	0.727	0.751	0.168	0.055	0.014	0.008	0.962	0.500
0.7	1.586	0.579	0.623	0.208	0.081	0.046	0.022	0.663	0.655
0.9	1.152	0.460	0.527	0.209	0.119	0.046	0.042	0.494	0.757
0.95	1.072	0.438	0.524	0.184	0.118	0.063	0.037	0.462	0.777
0.99	1.014	0.422	0.51	0.20	0.11	0.06	0.04	0.439	0.791

The table describes the occurrence of large, positive *or* negative values of the ARCH process. However, the behaviour of large (positive) values, i.e. the extremal index and compounding probabilities for the ARCH process $\{\xi_n\}$ itself, can also be deduced from the same simulations. Clearly $\{\xi_n\} \stackrel{d}{=} \{C_n\sqrt{\xi_n^2}\}$, where the $\{C_n\}$ are i.i.d., independent of $\{\xi_n\}$ and $P(C_1 = 1) = P(C_1 = -1) = \frac{1}{2}$. Hence the point process of exceedances by ξ_n is obtained from the corresponding process for $|\xi_n|$ by independent thinning, and this easily gives the extremal index and compound Poisson limit of the ξ-process itself. For $x > 0$, let $u_n = xn^{1/(2\kappa)}$. Then

$$P(\xi_1 > u_n) = \tfrac{1}{2}P(\xi_1^2 > u_n^2) \sim \tfrac{1}{2}c(u_n^2)^{-\kappa} = \tfrac{1}{2}cx^{-2\kappa}n^{-1},$$

where c and κ are the constants appearing in (1.4) for the ξ_n^2-process. Hence the probability that the maximum of n independent variables with this distribution is less than u_n is given by

$$P(\xi_1 \leq u_n)^n \to \exp\{-\tfrac{1}{2}cx^{-2\kappa}\} \quad \text{as } n \to \infty. \tag{3.6}$$

Next, let N_n be the time-normalized point process of exceedances of u_n^2 by $\{\xi_t^2\}$, and let $1 \leq \tau_1 < \tau_2 < \cdots$, be the times of occurrence of these exceedances. Then

$$P(\max\{\xi_1, \ldots, \xi_n\} \leq u_n)$$

$$= \sum_{k=0}^{\infty} P(N_n((0, 1]) = k, C_{\tau_1} = \cdots = C_{\tau_k} = -1)$$

$$= \sum_{k=0}^{\infty} P(N_n((0, 1]) = k)2^{-k}$$

$$\to \sum_{k=0}^{\infty} P(N((0, 1]) = k)2^{-k} \quad \text{as } n \to \infty, \tag{3.7}$$

where N is the limiting compound Poisson process for $\{\xi_t^2\}$ given in Theorem 2.1. Let N' be the Poisson process with intensity $\eta = c\theta x^{-2\kappa}$ which governs the occurrence of points in N, let $\{\pi_k\}$ be the compounding probabilities and introduce their probability generating function $\Pi(u) = \sum_{k=1}^{\infty} \pi_k u^k$. Further, let $\{\pi^{*l}(j)\}_{j=l}^{\infty}$ be the

l-fold convolution of $\{\pi_k\}$, i.e. $\pi^{*l}(j)$ is the probability that the sum of l independent variables with point probabilities π_k assumes the value j. It then follows that

$$\sum_{k=0}^{\infty} P(N((0,1]) = k)2^{-k}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} P(N'((0,1]) = l)P(N((0,1]) = k \mid N'((0,1]) = l)2^{-k}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\eta^l}{l!} e^{-\eta} \pi^{*l}(k)2^{-k}$$

$$= \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \pi^{*l}(k)2^{-k} \frac{\eta^l}{l!} e^{-\eta}$$

$$= \sum_{l=0}^{\infty} \Pi(\tfrac{1}{2})^l \frac{\eta^l}{l!} e^{-\eta}$$

$$= \exp\{-\eta(1 - \Pi(\tfrac{1}{2}))\}.$$

Inserting $\eta = c\theta x^{-2\kappa}$ it follows that

$$P(\max\{\xi_1, \ldots, \xi_n\} \le u_n) \to \exp\{-c\theta x^{-2\kappa}(1 - \Pi(\tfrac{1}{2}))\}$$

and comparing with (3.6) it is seen that the extremal index θ', say, for the ARCH process $\{\xi_n\}$ itself is

$$\theta' = 2\theta(1 - \Pi(\tfrac{1}{2})), \tag{3.8}$$

where θ is the extremal index for $\{\xi_n^2\}$. Since $\Pi(\tfrac{1}{2}) < \tfrac{1}{2}$ we have $\theta < \theta' < 1$.

It is now readily seen that also the compounding probabilities π_k' for the ARCH process can be obtained from the π_k's for $\{\xi_t^2\}$ as

$$\pi_k' = (1 - \Pi(\tfrac{1}{2}))^{-1} \sum_{l=k}^{\infty} \pi_l \binom{l}{k} 2^{-l}. \tag{3.9}$$

Table 3.2 contains the extremal index and compounding probabilities for the ARCH process, computed from the simulations in Table 3.1 by means of (3.8), (3.9) (in this we of course have used π_k's also for larger values of k than those listed in Table 3.1).

Table 3.2
Extremal index $\hat{\theta}'$ and compounding probabilities $\hat{\pi}_k'$ for the ARCH process, computed from the simulations in Table 3.1

λ	κ	$\hat{\theta}'$	$\hat{\pi}_1'$	$\hat{\pi}_2'$	$\hat{\pi}_3'$	$\hat{\pi}_4'$	$\hat{\pi}_5'$
0.1	13.24	0.999	0.998	0.002	0.000	0.000	0.000
0.3	4.180	0.939	0.941	0.054	0.004	0.001	0.000
0.5	2.365	0.835	0.844	0.124	0.025	0.006	0.001
0.7	1.586	0.721	0.742	0.176	0.054	0.018	0.007
0.9	1.152	0.612	0.651	0.203	0.079	0.034	0.016
0.95	1.072	0.589	0.631	0.203	0.088	0.040	0.019
0.99	1.014	0.571	0.621	0.202	0.088	0.042	0.021

Acknowledgement

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