Canonical substitutions tilings of Ammann-Beenker type

E.O. Harriss and J.S.W. Lamb

Department of Mathematics, Imperial College London, London SW7 2AZ, UK

Abstract

We present a large class of parallelogram tilings that have two important features in common with the Ammann-Beenker tiling: they can be constructed by canonical projection from $\mathbb{Z}^4 \subset \mathbb{R}^4$ to a plane, and they admit substitution rules.

Key words: Quasicrystals, substitution rules, tilings
MCS(2000) 52C23

1 Introduction

In 1981 De Bruijn [14] discovered that the beautiful aperiodic substitution tilings of Penrose [16] can be constructed from the projection of a slice of a five dimensional lattice. This led to the development of the canonical projection and more general projection methods, which have been studied in great detail, for example see [5]. The projection setting can be used to study a wide variety of substitution tilings [6–8,31,39].

The method of De Bruijn was used by Beenker [11] and Socolar [41] to construct aperiodic tilings with eight- and twelve-fold rotational symmetry, which later turned out to have been discovered independently by Ammann. Like the Penrose tilings, these tilings can be obtained by canonical projection, but also admit a substitution rule. This leads us to pose the following general question:

[Q] *What substitution rules do canonical projection tilings admit?*

Interestingly, despite the elementary nature of the question, and the amount of research the Penrose, Ammann and Socolar examples have generated, this particular question has received relatively little attention.

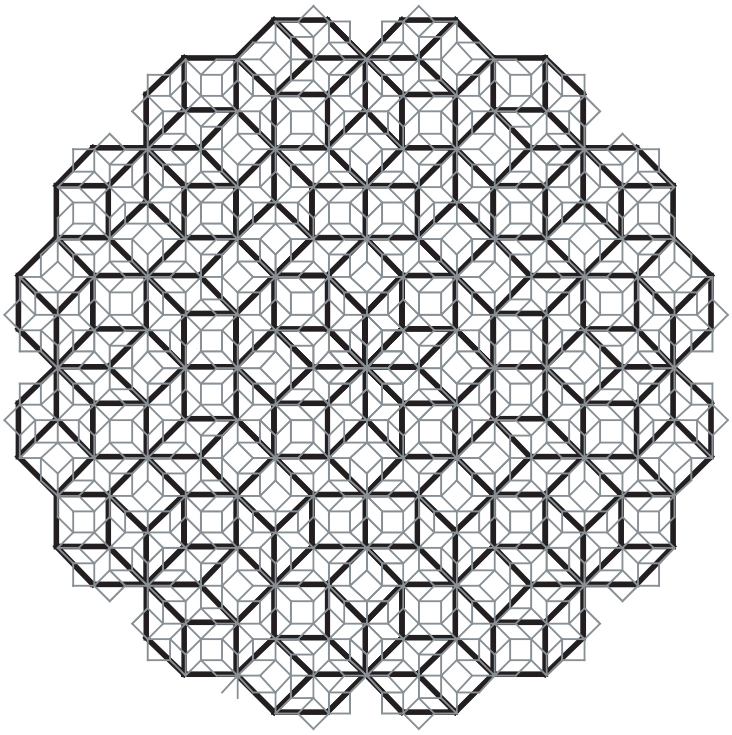


Fig. 1. Section of an Ammann-Beenker tiling (grey tiles) and a locally isomorphic expansion tiling (black lines). The edges and vertices of this picture are obtained by canonical projection. The window for the tiling with grey tiles is the projection of the hypercube with it center at the origin onto the window space. The window for the expansion tiling with black lines is the subwindow at the center of this window. This is shown in Figure 3. The central vertex of this figure is the origin.

A partial answer to [Q] was formulated in the setting of 2 to 1 canonical projection tilings by Lamb [28], combining results of [43] and [40], detailing all 2-letter substitutions. We will present a general answer to [Q] in the 2 to 1 setting in a forthcoming publication [20].

The current paper contains the first step of a program to answer [Q] and thus classify *canonical substitution tilings* and the *canonical substitution rules* that they admit. Here we present a class of planar canonical substitution tilings, where the canonical projection is from $\mathbb{Z}^4 \subset \mathbb{R}^4$ to a plane. In fact, the class of canonical projection tilings we present is precisely the set of all 4 to 2 canonical projection tilings that admit substitution rules, but a proof is beyond the scope of this paper. We address this issue elsewhere [19].

Our main result is Theorem 1.12. Before stating and discussing this theorem in Section 1.4, we first describe our setting in Sections 1.1, 1.2 and 1.3 .

The rest of the paper is organised as follows. In Section 2 we illustrate our method of constructing substitution rules for canonical projection tilings with the Ammann-Beenker tilings. Our treatment provides an alternative proof to that of Beenker. Thereafter, in Section 3 we prove our main result: Theorem 1.12. This constructs general 4 to 2 canonical projection tilings, including the Ammann-Beenker tiling. Finally, to illustrate our general result, in Section 4 we present some new examples of canonical substitution tilings.

1.1 Parallelogram tilings

We consider planar parallelogram tilings. Although the idea may be intuitively clear, we provide some definitions following the settings of [17] and [36] (cf. also [18]). After that we provide the more important details of our notions of substitution tilings and canonical projection tilings.

As usual, we let the sum $P+Q$ of two subsets $P, Q \subset \mathbb{R}^n$, denote the set of points

$$P+Q=\{p+q|p\in P, q\in Q\}\subset \mathbb{R}^n. \qquad (1)$$

We also let $\mathbb{E}^2$ denote $\mathbb{R}^2$ with the Euclidean metric.

We consider tiles that are parallelograms: a *parallelogram* is the set of points lying within the polygonal region with vertices $0, a, a+b, b$, where a and b are vectors in $\mathbb{E}^2$. For such a parallelogram S, we define its vertices to be $vert(S)=\{0,a,b,a+b\}$. As we are interested in tilings that are constructed from a finite set of tiles, we introduce the notion of the set of proto-tiles for a tiling.

Definition 1.1 (Proto-tiles) *Let P be a finite set of parallelograms which are pairwise non-equivalent up to translation. For each $S\in P$ let L_S be a finite set of labels for the parallelogram S. Then, the collection*

$$\tau=\{(S,l)|S\in P, l\in L_S\} \qquad (2)$$

is a set of proto-tiles. *The pairs (S,l) provide the* labelling *of the set of proto-tiles.*

For any proto-tile $T=(S,l)\in\tau$, we denote its support $supp(T)=supp(S)$, its label $label(T)=(S,l)$ and its vertices $vert(T)=vert(S)$.

With tiles from a given set of proto-tiles, one may (try to) construct a tiling of the plane. A patch is a part of such a tiling.

Definition 1.2 (Tilings and patches) *Given a set of proto-tiles τ. A* tile*, R, is the translation of a proto-tile $T=(S,l)\in\tau$. That is. $R=T+r$, where $supp(R)=supp(S)+r$, $label(R)=(S,l)$ and $vert(R)=vert(S)+r$.*

A tiling *is a set of tiles $\mathcal{T}$ that do not intersect except on their edges and that cover the plane:*

(1) For distinct $R_1, R_2 \in \mathcal{T}$, interior$(R_1) \cap$ interior$(R_2) = \emptyset$.
(2) $\bigcup_{R\in\mathcal{T}} supp(R) = \mathbb{E}^2$.

The vertices of the tiling are denoted $vert(\mathcal{T}) = \bigcup_{R\in\mathcal{T}} vert(R)$.

Let Z be a bounded subset of $\mathbb{E}^2$. Then the patch *of the tiling $\mathcal{T}$ covering Z is*

$$P_Z^{\mathcal{T}} = \{R \in \mathcal{T} | Z \cap R \neq \emptyset\} \tag{3}$$

Note that we consider here only tilings with tiles that are translations of a finite set of proto-tiles. We thus exclude certain examples, such as the pinwheel tiling [38], that have an infinite set of translation classes of tiles, but only a finite set of proto-tiles if one considers an equivalence relation that not only considers translations but also rotations.

1.2 Substitution tilings

We now define what we mean by a substitution rule. A substitution rule consists of two steps. First, we stretch a tiling by some uniform constant factor and then replace each inflated tile by a patch of tiles from the set of proto-tiles.

Definition 1.3 (Substitution rule) *Consider a set of proto-tiles τ, and a real number $\lambda > 1$. A* substitution rule *σ associates a tile (translation of a proto-tile from τ) to a patch of tiles. We first consider the action of σ on a single proto-tile (S, l). The substitution rule consists of two parts. First, we uniformly inflate the tile by the factor λ. Then we replace the inflated tile by a patch of tiles from the original (non-inflated) set of proto-tiles. The substitution rule naturally extends to all tiles $R = (S + r, l)$ as $\sigma(R) = \sigma((S, l)) + r$. Analogously, the action of σ can be defined for patches of tiles, by first inflating the patch and then replacing each tile from the patch according to the substitution rule for single tiles.*

We call σ a substitution rule *if for all proto-tiles $T \in \tau$ and all $n \in \mathbb{N}$ the following conditions hold:*

(1) The substitution patch $\sigma^n(T)$ covers the inflated tile: $supp(\lambda^n T) \subset supp(\sigma^n(T))$.
(2) The substitution patch $\sigma^n(T)$ is minimal: for all tiles $R \in \sigma^n(T)$ we have $supp(R) \cap supp(\lambda^n T) \neq \emptyset$.
(3) The substitution patches are consistent. Consider a tile R in $\sigma^n(T)$. This tile R is contained in $\sigma(R')$ for every tile R' of $\lambda\sigma^{n-1}(T)$ that intersects R.

Note that in our definition of a substitution rule, we explicitly allow (in point 3.) for the patches replacing tiles to overlap, as long as individual tiles in these patches match (either have no intersection of their interior or overlap entirely). For some examples, see Figure 2.

A substitution rule is called *primitive* if for all $T \in \tau$ there is an $n \in \mathbb{N}$ such that $\sigma^n(T)$ contains instances of translations of every proto-tile in τ.

Before we define what we mean by a substitution tiling, we first recall the concept of a local isomorphism class of a tiling, which contains all tilings that are locally indistinguishable from a given one.

Definition 1.4 (Local isomorphism class) *Two tilings $\mathcal{T}$ and $\mathcal{T}'$ are* locally isomorphic *if for any patch P_Z of $\mathcal{T}$ there exists a vector t such that $P_Z + t$ is in $\mathcal{T}'$ and vice versa. The* local isomorphism class *of a tiling $\mathcal{T}$ is the set of all tilings that are locally isomorphic to $\mathcal{T}$.*

A substitution tiling is a tiling whose patches are defined by iteration of a substitution rule.

Definition 1.5 (Substitution tiling) *A tiling $\mathcal{T}$, with proto-set τ is a substitution tiling with substitution rule σ if for every patch $P_Z^{\mathcal{T}}$, there are $T \in \tau$, $n \in \mathbb{N}$ and $t \in \mathbb{E}^2$ such that:*

$$P_Z^{\mathcal{T}} \subset \sigma^n(T) + t \tag{4}$$

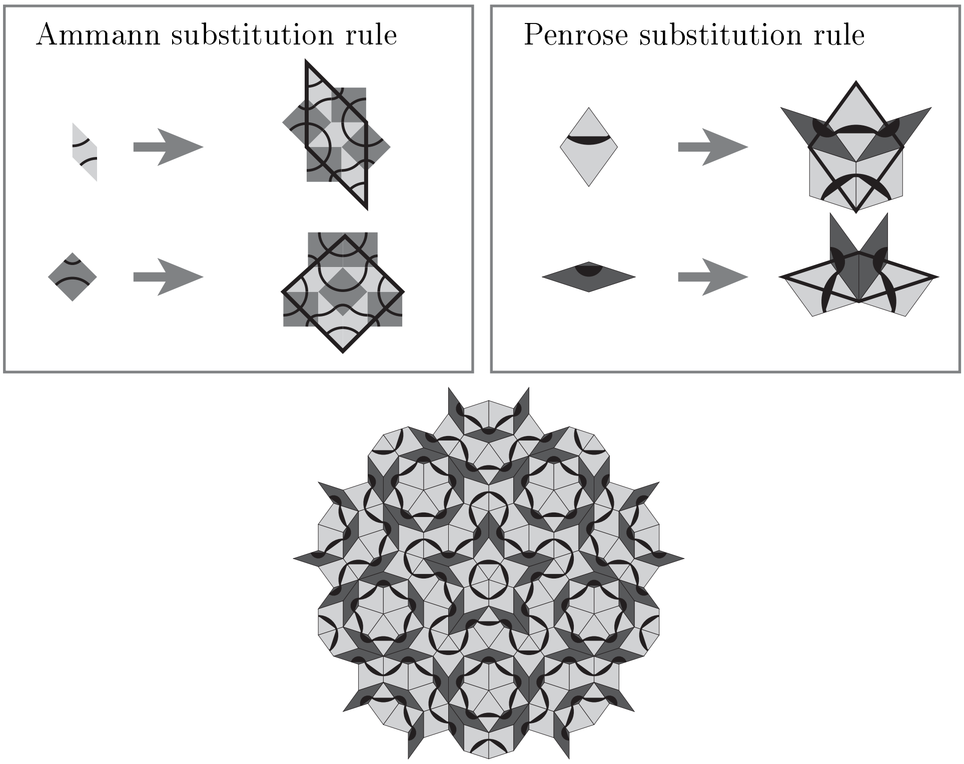
If $vert(\lambda\mathcal{T}) \subset vert(\sigma\mathcal{T})$ then the tiling is called *vertex hierarchic*. If σ is primitive then (4) holds for all T, and the set of substitution tilings for σ forms a local isomorphism class.

Two of the best known substitution tilings are those by Penrose and Ammann-Beenker.

Example 1.6 (Ammann-Beenker and Penrose substitution rules) *The Ammann-Beenker and Penrose substitution rules, shown in Figure 2, are associated to two of the simplest and most beautiful non-trivial tilings on the plane. Figure 1 shows a patch of the tiling associated to the Ammann substitution rule.*

Finally, in this section, we discuss the concept of a vertex hierarchic expansion tiling. As will be shown in this paper, the existence of expansion tilings is necessary for the existence of substitution rules.

Definition 1.7 (Vertex hierarchic expansion tiling) *Let $\mathcal{T}$ be a tiling with labelled set of proto-tiles τ. Let $\overline{\mathcal{T}}$ be a tiling with set of proto-tiles $\overline{\tau} = \{\lambda T | T \in \tau\}$.*



Patch of the Penrose tiling

Fig. 2. The substitution rules which generate the Ammann-Beenker and Penrose tilings. The curved lines break the symmetry of the tiles. The other proto-tiles will be rotations of the ones shown. In the Penrose substitution each tile has 5 rotated versions, each in two orientations as given by the pattern, so there are 20 proto-tiles. In the Ammann-Beenker substitution the rhomb has 4 rotated versions and the square 8 so there are 12 proto-tiles.

(1) $\overline{\mathcal{T}}$ *is an* expansion tiling *of* $\mathcal{T}$ *if* $\lambda^{-1}\overline{\mathcal{T}}$ *is locally isomorphic to* $\mathcal{T}$*.*
(2) $\overline{\mathcal{T}}$ *is a* vertex hierarchic *expansion tiling, if moreover*

$$vert(\overline{\mathcal{T}}) \subset vert(\mathcal{T}) \tag{5}$$

In this paper, whenever we refer to expansion tilings, we in fact refer to vertex hierarchic expansion tilings .

1.3 Quadratic expansion matrices and associated canonical projection tilings

We now introduce the concept of canonical projection tilings. To be more precise, we consider a specific construction of such tilings, using a certain type of 4×4 integer matrices.

Definition 1.8 (4×4 **Quadratic expansion matrix)** *A* 4×4 quadratic expansion matrix *is a unimodular,* 4×4 *integer matrix which has two planar eigenspaces* V *and* W*, with* $Mv = \lambda v$ *for all* $v \in V$ *and* $Mw = \pm\lambda^{-1}w$ *for all* $w \in W$*, with* $\lambda \in \mathbb{R}, \lambda > 1$.

The eigenspace V *is called the* expanding*, or* tiling *eigenspace, the eigenspace* W *the* contracting*, or* window *eigenspace.*

We now define our version of the canonical projection method, in relation to quadratic expansion matrices. We first use it to define a point set. We then add an additional condition to the matrix requiring that the point set can be considered as the set off vertices of a parallelogram tiling.

Definition 1.9 (Canonical projection point set) *Consider* $\mathbb{R}^4$ *containing the lattice* $\mathcal{L} = \mathbb{Z}^4$ *and a* 4×4 *quadratic expansion matrix* M*. Let* a_i *be the standard generators of* $\mathcal{L}$*. Let* V *and* W *denote the expanding and contracting two-dimensional eigenspaces of* M*.*

We equip $\mathbb{R}^4$ *with the inner product:*

$$< a, b >= a_V \cdot b_V + a_W \cdot b_W \tag{6}$$

where $a = a_V + a_W, b = b_V + b_W$*,* $a_V, b_V \in V, a_W, b_W \in W$ *and* $\cdot$ *denotes the standard (Euclidean) inner product on* V *and* W*.*

Let Π_A *denote orthogonal projection to* $A \subset \mathbb{R}^4$*, with respect to the inner product* $< \cdot, \cdot >$*.*

Let $\Omega_0 = \{(a, b, c, d)|a, b, c, d \in [0, 1]\}$ *denote a unit hypercube in* $\mathcal{L}$*, and* $\Omega_t = \Omega_0 + t$ *be its translation by a vector* $t \in \mathbb{R}^4$ *with vertices* $vert(\Omega_t) = vert(\Omega_0) + t$*.*

We now consider the cylinder $V + \Omega_t$*, and the intersection of this cylinder with the lattice* $\mathcal{L}$*. The* canonical projection point set *is defined as the projection of this set to* V*:*

$$\Pi_V(\mathcal{L} \cap (V + \Omega_t)) \tag{7}$$

We refer to the set $\Pi_W(\Omega_t)$ *as the window for the canonical projection point set.*

A canonical projection point set is called singular if $\Pi_W(t) \in \Pi_W(\mathcal{L})$*, and non-singular otherwise.*

The intersection of the lattice $\mathcal{L}$ and the cylinder $V + \Omega_t$ in the above definition, forms a set of vertices of a *discrete plane*, or *staircase*.

Definition 1.10 (Canonical projection staircase) *Let* $line(a, b)$ *denote the line segment joining a point* a *to a point* b*, and* $facet(r, c, d) = line(r, r + c) + line(r, r + d)$ *be the parallelogram plane segment with vertices at* $r, r + c, r +$

$d, r+c+d$. *Then, the* canonical projection staircase *is the set of facets in the cylinder:*

$$\mathcal{S}_t = \{facet(r,p,q) \mid r,p,q \in (\mathcal{L} \cap (V + \Omega_t)),\ |r-q| = |r-p| = 1,\ p \neq q\} \quad (8)$$

We note that the notion of staircase is not restricted to that given in this definition. In fact, we use other staircases later in this paper.

The aim is to obtain a tiling from projecting $\mathcal{S}_t$ to V, with the facets of $\mathcal{S}_t$ projecting to tiles of this tiling.

Definition 1.11 (Tiling property) *A* 4×4 *quadratic expansion matrix* M *has the* tiling property, *if the projection to* V *of the canonical projection staircases associated to* M *are all tilings, whose tiles are the projections of the facets constituting the staircase.*

We note that in order to check if a matrix has the tiling property, it is sufficient to check the projection of only finite sets of two-dimensional facets of the hypercube meeting at one point of the $\mathbb{Z}^4$ lattice.

The specific class of canonical projection tilings we deal with in this paper are *canonical projection tilings associated to* 4×4 *quadratic expansion matrices with the tiling property* which are the projections to V of the canonical projection staircases associated to such matrices.

1.4 Statement and discussion of the main result

The main result of this paper is summarized in the following theorem.

Theorem 1.12 *The canonical projection tiling of a* 4×4 *quadratic expansion matrix* M *with the tiling property is a substitution tiling.*

In fact, we establish the existence of a countable infinity of different vertex hierarchic substitution rules for each such canonical projection tiling, see Theorem 3.15.

We refer to the tilings described in Theorem 1.12 as canonical substitution tilings. Our result is constructive in the sense that we provide an algorithm that constructs substitution rules for any such canonical projection tiling.

In the literature substitution rules are frequently used to define a tiling or a set of tilings; in fact it has been shown that many substitution tilings can be realised as projection tilings by lifting the tiling to a higher dimensional space and then constructing a suitable window, see for instance [31,39,6,8,7]. However, in the theoretical construction of the window it has proven difficult to control its properties. For example, it is not easy to determine from the substitution rule whether the associated window will be connected, or whether the window is in fact the projection of a hyper-cube, such as in the canonical projection method. In this paper we approach the study of the connection between the substitution and projection properties from the opposite direction: starting with canonical projection tilings (and thus imposing a window structure) and then establishing the existence of substitution rules.

In related work, Pleasants [35] has studied projection point sets, establishing, under certain conditions, the mutual local derivability (MLD) property. If the canonical projection point sets is interpreted as the set of vertices of a canonical projection tiling, then MLD implies the existence of a substitution rule as shown in [9]. The substitution rules we consider in this paper all have the MLD property. However, it is not true that all possible substitution rules admit the MLD property [19]. In contrast to Pleasant's work, we obtain an algorithm to explicitly construct a countable infinity of non-equivalent substitution rules, rather than establishing the existence of (at least) one substitution rule.

Another property of interest is that of matching rules. These are local decorations on the tiles that enforce aperiodicity of the tiling. It is known that the canonical projection tilings with quadratic slope admit matching rules [29]. In this paper we begin to answer a question of Goodman-Strauss in [17], which asked whether these were also the canonical projection tilings admitting a substitution rule. The canonical substitution tilings in this paper all have quadratic slope. In fact, our class of tilings consists of all 4 to 2 canonical projection tilings that admit a substitution rule. As there exist planes with quadratic slope that are not fixed by a quadratic expansion matrix with the tiling property, the set of canonical projection tilings with matching rules is therefore strictly larger than the set of canonical projection tilings with substitution rules.

In this paper we develop a method for constructing substitution rules, that uses a set of parallel lines on the plane. These lines appear superficially similar to lines Ammann used to construct the Ammann tiling. Such lines are nowadays refered to as Ammann lines, and have been used to help understand individual examples in [41] and [18, Chapter 10]. They are considered in greater generality in [32]. The lines have primarily been used to find tilings with a high degree of symmetry, for example tilings with eight-, ten- and twelve-fold rotational symmetries have been constructed in [23,22,21]. More general examples have been considered in [10, and references therein].

It is important to stress that the lines we use are different to the Ammann lines. The lines that we use are such that intersections of four of them them correspond to vertices of our tiling. The Ammann lines create the edges of a planar tiling that is dual to the Ammann tiling (with each of the tiles generated by the lines corresponding to a tile of the Ammann tiling and vice-versa).

Some examples from the class of tilings we describe in this paper have been discussed before in the literature. Apart from the Ammann-Beenker tiling, we mention in particular the work of Katz *et al.* [25] and Baake *et al.* [3,4,2].

While studying matching rules, Katz [25] considered the set of tilings with the edge, but not vertex, matching rules of the Ammann tiling. Some of the non-periodic tilings that he considered in fact turn out to be topologically equivalent to canonical projection tilings for 4×4 quadratic expansion matrices with the tiling property. For more details see section 4.

In [3,4,2], Baake *et al.* consider the effects of Schur rotations on the planes V and W, in order to try to explain the formation of quasicrystals. These tilings are the projection from $\mathbb{Z}^4$ onto the plane V. The planes V and W are rotations of the spaces for the Ammann tiling. A similar set of canonical substitution tilings are also considered in [42].

Our insistence on the quadratic nature of the expansion matrices is motivated by the well established fact that canonical substitution rules require quadratic scaling, see for example [35,15]. As a byproduct of quadratic scaling, our work on canonical projection tilings is firmly rooted on the theory of canonical projection tilings from $\mathbb{Z}^2 \subset \mathbb{R}^2$ to a line. The canonical projection tilings in this setting are the geometric representation of Sturmian words that are studied in combinatorics [30,37] and their 2-letter substitution properties are by now well understood. Substitution properties of other 2 to 1 interval projection tilings can be found in [34] and [20].

It is worth noting that the matrices used to contruct canonical substitution tilings in this paper, induce hyperbolic automorphisms on the lattice $\mathbb{Z}^4$ in $\mathbb{R}^4$. As this hyperbolic structure arises in combination the Euclidean nature of V, this observation might be explored further in the context of multi-metrical crystallography, in the spirit of Janner [24]. In particular, it is precisely the hyperbolic structure that gives rise to the existence of substitution rules.

As mentioned above the results of this paper are the first step in a larger program. Two important examples that are beyond the scope of the present paper are the Penrose tiling, which can be obtained by canonical projection from $\mathbb{Z}^5 \subset \mathbb{R}^5$ to a plane, and the icosahedral tiling in three dimensions [27,26] which can be obtained by canonical projection from $\mathbb{Z}^6 \subset \mathbb{R}^6$ to a three-dimensional subspace. We aim to consider these and other examples in the future as we work towards answering [Q].

As an extension of our result, one may consider the substitution properties of canonical projection tilings associated to matrices without the tiling property. In this case, one can derive a (in general non-parallelogram) tiling from the canonical projection point set and we expect that substitution rules can be constructed for these tilings. Alternatively, one may project the canonical projection staircase in such a way to V that one obtains a parallelogram tiling. In that case, one may find substitution rules with a nonuniform (tile-dependent) inflation factor.

Finally, we expect that the geometric techniques used in this paper can be adapted to study more general settings for substitution, as discussed for example in [1,12].

2 Example: The Ammann-Beenker tiling

In this section we illustrate our method for obtaining substitution rules for canonical projection tilings in the case of the Ammann-Beenker tiling. In the process we give a new proof, alternative to that given by Beenker [11], of the fact that the Ammann-Beenker tiling is a canonical projection tiling.

We consider the 4×4 quadratic expansion matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \tag{9}$$

and construct a substitution rule for the associated canonical projection tiling. It is readily verified that M has determinant 1 and doubly degenerate eigenvalues $\lambda = 1 + \sqrt{2}$ and $-\lambda^{-1} = \sqrt{2} - 1$. Let V and W be the corresponding expanding and contracting eigenspaces.

We now consider the set $\{vert(\mathcal{T}_t) \mid t \in \mathbb{R}^4\}$ of canonical projection point sets associated to this matrix M. Recall that the window used for canonical projection is $\Pi_W(\Omega_t)$ where Ω_t is a translation by t of the 4-dimensional unit hypercube Ω_0. A patch of such a point set is the set of vertices of the tiling with grey tiles in Figure 1.

From this figure it is clear that the matrix has the tiling property and thus fits the context of Theorem 1.12. We consider the canonical projection tilings $\mathcal{T}_t$ associated to M.

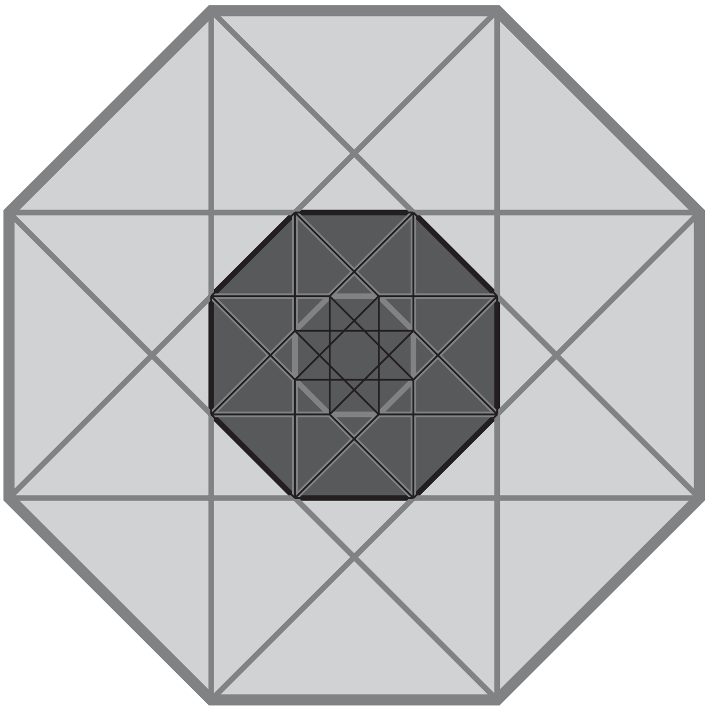


Fig. 3. The window for the Ammann-Beenker tiling and the subwindow corresponding to the Ammann-Beenker substitution. The projection of the hypercube Ω_t to W, which gives the window, is shown in light grey. The dark grey lines are the projection of the edges of the hypercube. The projection of $M\Omega_0 + t - a_2 + a_3$, which gives the subwindow, is shown in dark grey. The black lines are the projection of the edges of $M\Omega_0 + t - a_2 + a_3$.

We now detail the construction of a substitution rule for this example in several steps. Later, in Section 3, we show that this construction can be generalised to give the result of Theorem 1.12.

2.1 Finding expansion tilings

The first step in our search for substitution rules concerns the finding of a suitable vertex hierarchic expansion tiling. We consider the hypercube Ω_t and its image under a transformation by M and a subsequent translation

$$\overline{\Omega}_{t,s} = M\Omega_0 + t + s$$

with $s = a_3 - a_2$. The associated windows for objects are $\Pi_W(\Omega_t)$ and the subwindow are shown in Figure 3. As M fixes the lattice $\mathbb{Z}^2$, the projection tiling associated to the subwindow is a uniform inflation of the tiling with window $\Omega_0+M^{-1}(t+s)$. We denote this tiling $\overline{\mathcal{T}}_{t,s}$. It turns out (see Proposition 3.1) that this tiling is in fact also an expansion tiling of $\mathcal{T}_t$: the tiling $\overline{\mathcal{T}}_{t,s}$ lies

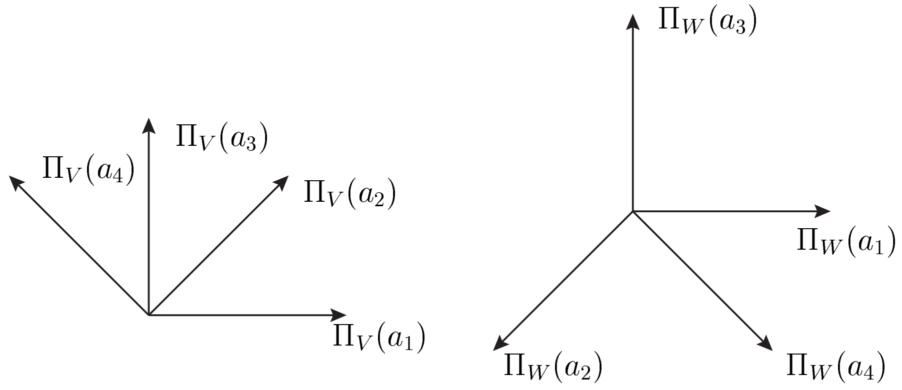


Fig. 4. Projections of the lattice generators onto V and W.

on top of $\mathcal{T}_t$ such that $vert(\overline{\mathcal{T}}_{t,s}) \subset \mathcal{T}_t$. In Figure 1, the expansion tiling is depicted with black edges.

2.2 The plane A_1 and its geometry

The generators $\{a_1, ..., a_4\}$ of $\mathcal{L}$ project to V and W as shown in Figure 4. One observes that a_1 projects parallel to the projection of $a_2 - a_4$ in both V and W. We now define the plane A_1 orthogonal to $\Pi_V(a_1)$ and $\Pi_W(a_1)$. Note that this is precisely the plane $[\Pi_V(a_3), \Pi_W(a_3)]_{\mathbb{R}}$. Let Π_{A_1} denote orthogonal projection to A_1 with respect to the inner product $< \cdot, \cdot >$, which in this case is actually identical to the standard (Euclidean) inner product on $\mathbb{R}^4$. An illustration of this projection is given in Figure 5.

We now observe the following:

Claim 2.1

(1) $\Pi_{A_1}(V)$ *is a line* $V_1 = [\Pi_V(a_3)]_{\mathbb{R}}$.
(2) $\Pi_{A_1}(W)$ *is a line* $W_1 = [\Pi_W(a_3)]_{\mathbb{R}}$.
(3) $\Pi_{A_1}(\mathcal{L})$ *is a lattice* $\mathcal{L}_1$ *of rank* 2 *(*$\mathcal{L}_1 \simeq \mathbb{Z}^2$*).*
(4) $M|_{\mathcal{L}_1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

PROOF. The first two points hold as $\Pi_V(a_1)$ and $\Pi_W(a_1)$ are orthogonal to A_1. For the third, consider $a_2 - a_4$. This is orthogonal to A_1 as $\Pi_V(a_2 - a_4)$ and $\Pi_W(a_2 - a_4)$ are parallel to $\Pi_V(a_1)$ and $\Pi_W(a_1)$. Thus there are two integer

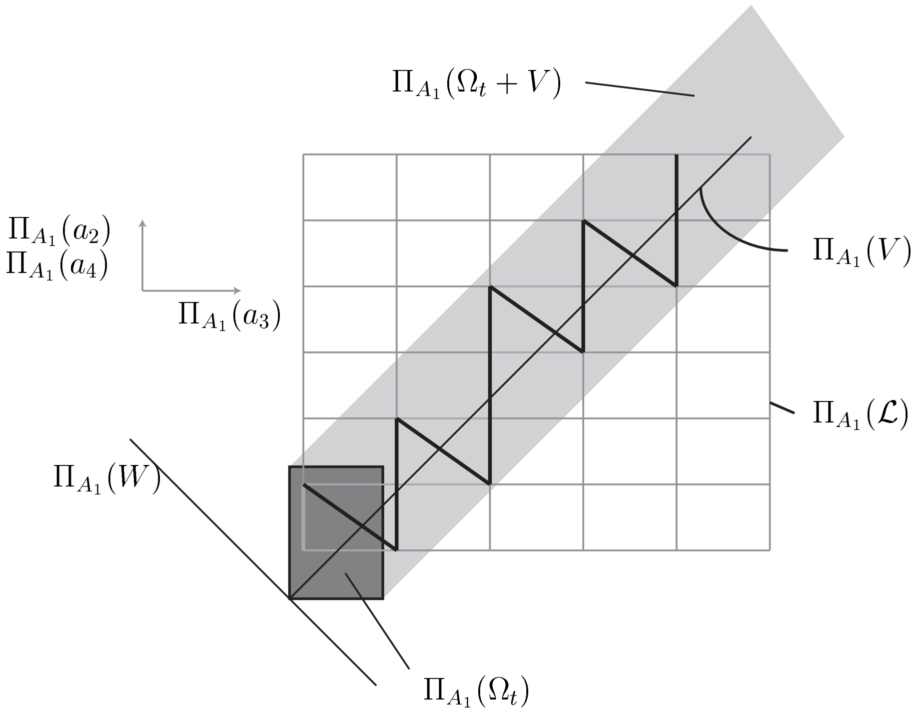


Fig. 5. The plane A_1 for the Ammann-Beenker substitution. The projection $\Pi_{A_1}(\Omega_t + V)$ is colored light grey. The projected lattice points in $\mathcal{L}_1$ are linked by a staircase S_t^1, which is indicated in black.

relations on the $\mathbb{Z}$-module $\Pi_{A_1}(\mathcal{L})$ and the projection is therefore a lattice of rank 2. For the final claim, note that M induces an automorphism of the lattice $\mathcal{L}$, and thus also induces an automorphism of the projected lattice $\mathcal{L}_1$. By considering M and noting that $\Pi_V(a_1) = 0$ and $\Pi_W(a_1) = 0$ one easily verifies the form precise of $M|_{\mathcal{L}_1}$. □

The vertices of the canonical projection staircase project to A_1 are the set of points $S_t^1 = \mathcal{L}_1 \cap (V_1 + \Pi_{A_i}(\Omega_t))$. These points can be joined up in a staircase with two steps; a vertical step, which we shall call p, and a diagonal step d, as shown in Figure 5. Note that this staircase is not the projection of the canonical projection staircase. We aim to find a substitution rule on the projection to V_1 of the staircase with vertices at S_t^1.

Consider the expansion tiling. The vertices of this tiling project to A_1 as the set of points $\mathcal{L}_1 \cap (V_1 + \Pi_{A_1}(M\Omega_t - a_2 + a_3))$. The projection of Ω_t and $M\Omega_t - a_2 + a_3$, along with the regions that they generate when added to V_1, are shown in Figure 6. This is also a two-step staircase, the steps of which are the images of the steps d and p under the action induced by $M|_{\mathcal{L}_1}$ on A_1.

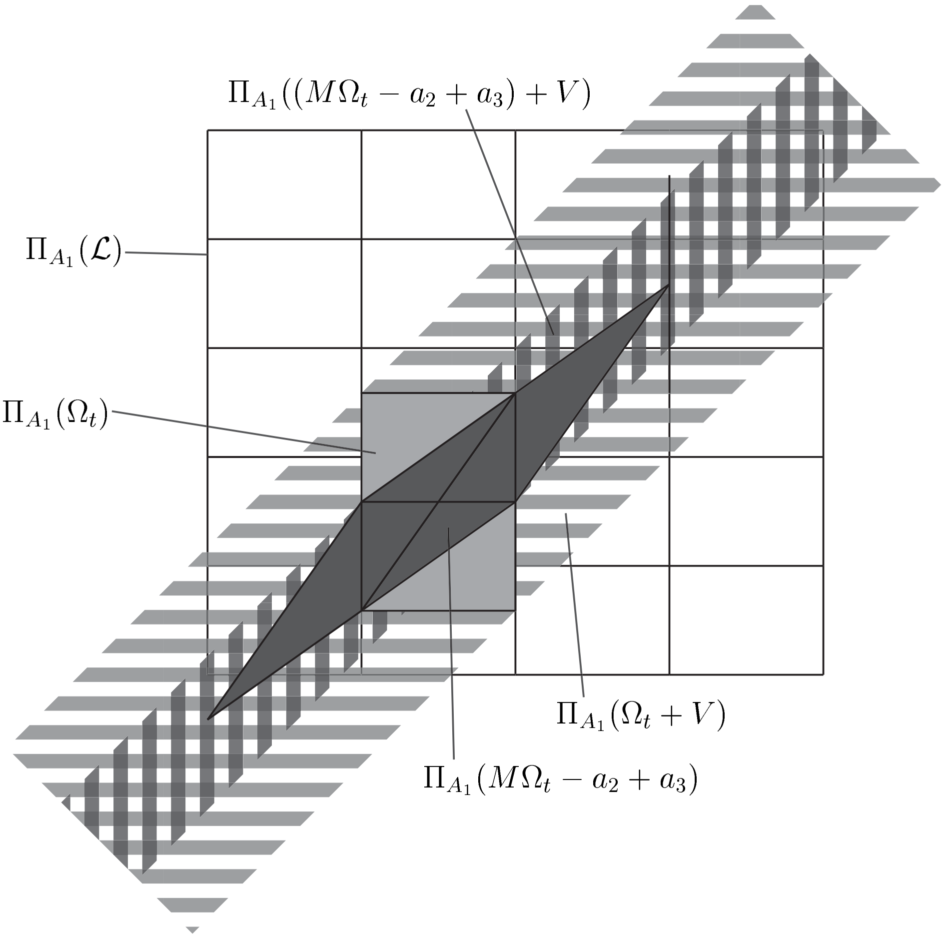


Fig. 6. The projection of the hypercube Ω_0 and $M\Omega_t - a_2 + a_3$ onto the A_1 plane of the Ammann-Beenker tiling, and the associated cylinders

.

The image, $\overline{d}$, of d is a now a vertical step, the image, $\overline{p}$, of p is the transverse diagonal. We shall denote the vertices of this staircase $\overline{S}^1_{t,s}$. The two staircases are shown in Figure 7.

2.3 The substitution rule on A_1

To find the substitution rule, we consider the window $\Pi_{W_1}(\Omega_t)$. As the projections of the lattice $\mathcal{L}_1$ to the window space W_1 is totally irrational, every point in S^1_t projects to a unique point in the window. We aim to partition the window into intervals, so that vertices of the staircase projecting to the same interval are similar (in some sense). In particular, we would like to use such intervals of the window to label corresponding vertices and steps of the staircase (and hence taking projection into account, tiles). In this context we

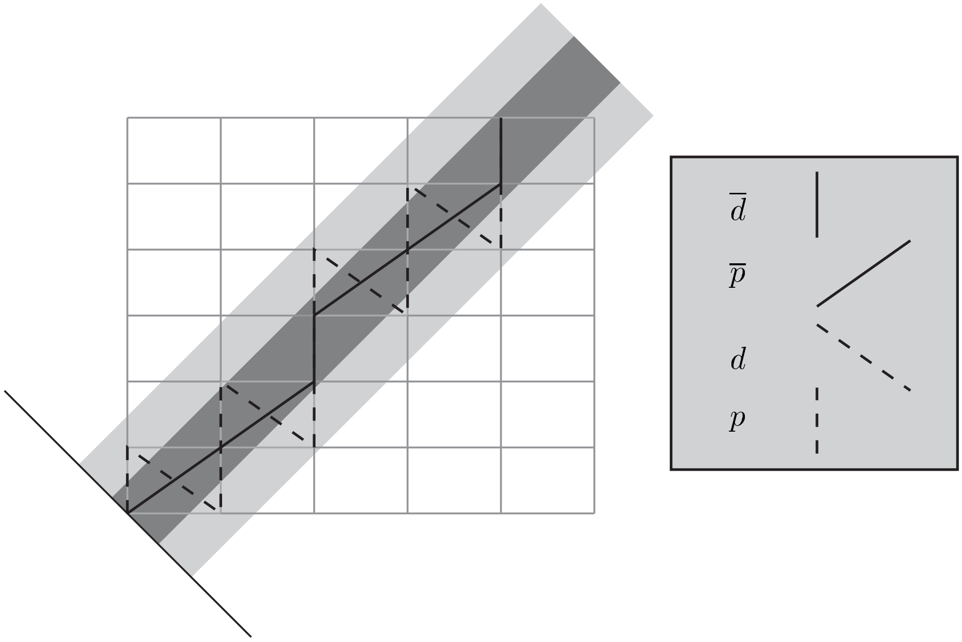


Fig. 7. The staircase, the expansion staircase and their steps for the projection of the Ammann-Beenker staircase to A_1. The steps p and d make up the staircase for the original tiling and the steps $\overline{p} = M|_{\mathcal{L}_1} p$ and $\overline{d} = M|_{\mathcal{L}_1} d$ make up the staircase for the expansion tiling.

.

will refer to such a window interval partition as labelling intervals.

To divide up the window into labelling intervals, we consider the lattice $\mathcal{M} = \mathcal{L}_1 + \Pi_{A_1}(t)$. The set $\Pi_{A_1}(\Omega_t)$ is singular with respect to this lattice. As a result, the boundary points of the window coincide with the projections of lattice points in $\mathcal{M}$. We choose a rectangle $R \subset V_1 + \Pi_{A_1}(\Omega_t)$, take the intersection with $\mathcal{M}$ and project to W_1. This induces a partition of $\Pi_{W_1}(\Omega_t)$ into intervals. Our choice of the rectangle is depicted in Figure 8(a), as well as the induced partition on $\Pi_{W_1}(\Omega_t)$.

We now construct an alternative partition of the window into two intervals by grouping together the lowest three intervals in the previously defined partition (see fig 8(a)). The second partition has the following property: let $l \in S_t^1$, $\Pi_{W_1}(l)$ lies in the top interval if and only if there is a line segment d to the right side of l in the staircase with vertices S_t^1. Analogously, projection to the lower interval indicates that a line segment p lies to the right side of l in the staircase with vertices S_t^1.

Now observe (Figure 8(b)) that our first partition restricts nicely to a partition of $\Pi_{W_1}(\overline{\Omega}_{t,s})$, into two intervals. This partition has a property similar to the second partition of $\Pi_{W_1}(\Omega_t)$: let $l \in \overline{S}_{t,s}^1$, $\Pi_{W_1}(l)$ lies in the lower interval if

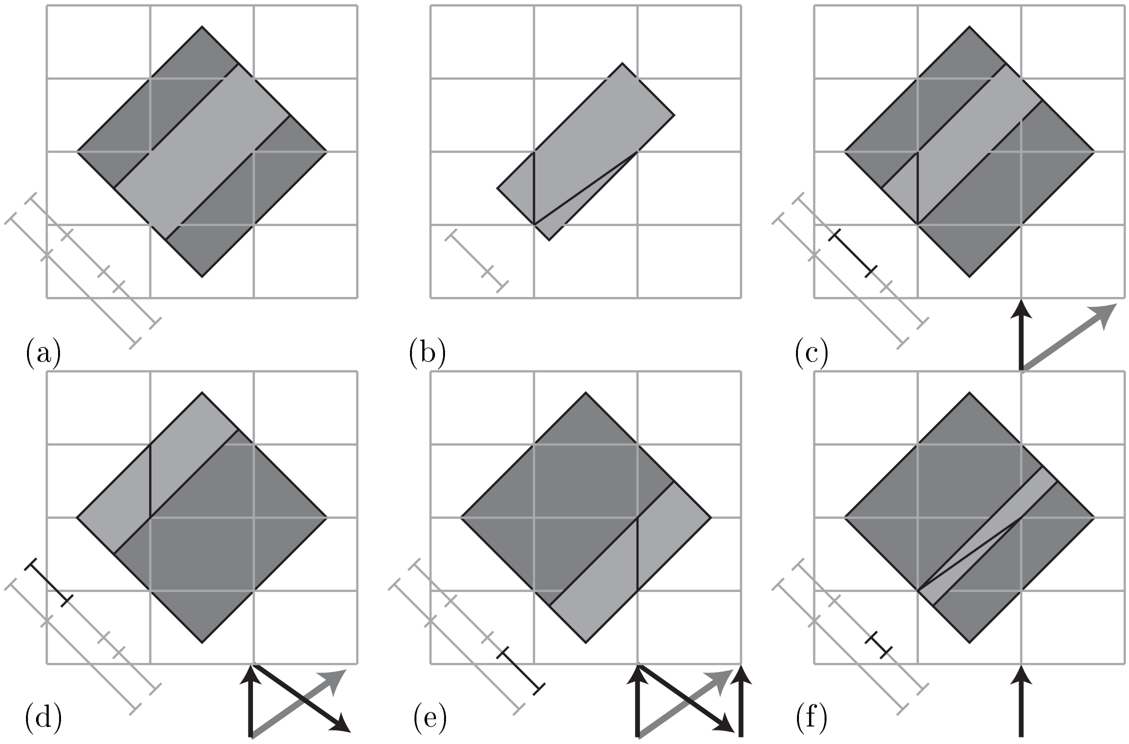


Fig. 8. This figure demonstrates the substitution rule on the plane A_1 for the Ammann tiling. A section of the cylinder for the original and expansion tilings are shown in (a). The edges of this region are parallel to V and W. At the bottom left of (a) we consider the window in W. The lower left partition shows the intervals corresponding to the two tiles. The higher partition shows the intervals for the tiles of the expansion tiles in the full window. The intervals for the expansion tiles are shown on their own in (b). Parts (c) to (d) show the construction of the replacement rule for the tile $\overline{p}$. Note that at each step the corresponding particular region of the window (shown in black in the window division) lies within precisely one of the two tile intervals. In the bottom right of each figure is the step $\overline{p}$ in grey, and the steps of the original tiling in black. In (f) we show that $\overline{d}$ is replaced by p.

and only if there is a line segment $\overline{d}$ to the right side of l in the staircase with vertices $\overline{S}^1_{t,s}$. Analogously, projection to the top interval indicates a line segment $\overline{p}$ lies to the right side of l in the staircase with vertices $\overline{S}^1_{t,s}$.

Our partitions of the window have the important property that for $l \in S^1_t$, $\Pi_{W_1}(l)$ always lies inside an interval and never on the edge of an interval. This is due to the fact that Ω_t is non-singular with respect to $\mathcal{L}$, whereas $\Pi_{A_1}(\Omega_t)$ is singular with respect to $\mathcal{M}$.

We can now construct the substitution rule. We start by considering a step of type $\overline{p}$ in the expansion staircase with vertices $\overline{S}^1_{t,s}$. Such a step is shown on the bottom left of Figure 8(c). The vertex at the left of this step is the point

$l \in \mathcal{L}_1$. We shall now show that l can be joined to $l + \overline{p}$ with a unique segment of the staircase S_1^t. In order to see this, we recall that such $\Pi_{W_1}(l)$ will lie in the region shown in Figure 8(c) if and only if l is the left-most point of a $\overline{p}$ step. From the second partition of $\Pi_{W_1}(\Omega_t)$ we now see that the staircase with vertices S_1^t has a p step to the right of l.

We can, therefore, consider $l + p$. As $\Pi_{W_1}(l + p)$ always lies in the top interval (Figure 8(d)) of the second partition we conclude that in the staircase with vertices S_1^t, there is a step d to the right of $l + p$.

Finally, as $l + p + d$ always lies in the lowest interval of the second partition, this point is followed by a p step (Figure 8(e)). As the staircase segment pdp connects l to $\overline{p}$ substitutes to pdp. By a similar argument we can obtain that $\overline{d}$ substitutes to p (Figure 8(f)).

We have therefore obtained a substitution rule that takes the points $\overline{S}^1_{t,s}$ to the points S_1^t. On the intervals between the points we have the substitution rule $(\overline{p}, \overline{d}) \to (pdp, p)$.

2.4 The relationship between the A_i planes and the tiling

In this section we show how the plane A_1 defined above, and similar planes for the other lattice directions, can be put together to provide information about the full tiling.

We have the substitution rule $(p, d) \to (pdp, p)$ rule acting on intervals of the line V_1. These intervals can be extended to strips of V, with edges parallel to $\Pi_V(a_1)$. Every point in the tiling projects to a point on A_1, so every point lies on one of these lines.

In analogy to the definition of A_1, we define the planes A_i as the planes orthogonal to $\Pi_V(a_i)$ and $\Pi_W(a_i)$. As the window and the subwindow we are using are symmetric under a rotation of order 8 the same substitution rule may be used for the projections to planes A_2, A_3 and A_4. We thus find substitution rules in these four directions. The labellings of these substitution rules can be extended to strips of V in the manner analogous to that used for a_1.

We call the lines corresponding to the tiling $\mathcal{T}_t$ thin lines. The call the subset of these lines, which concern the expansion tiling $\overline{\mathcal{T}}_{t,s}$, thick lines. Note that every thick line lies on top of a thin line as the substitution rules found above are vertex hierarchic.

Importantly, every vertex of $\mathcal{T}_t$ corresponds exactly to the intersection point of four thin lines. This is proven in Lemma 3.14.

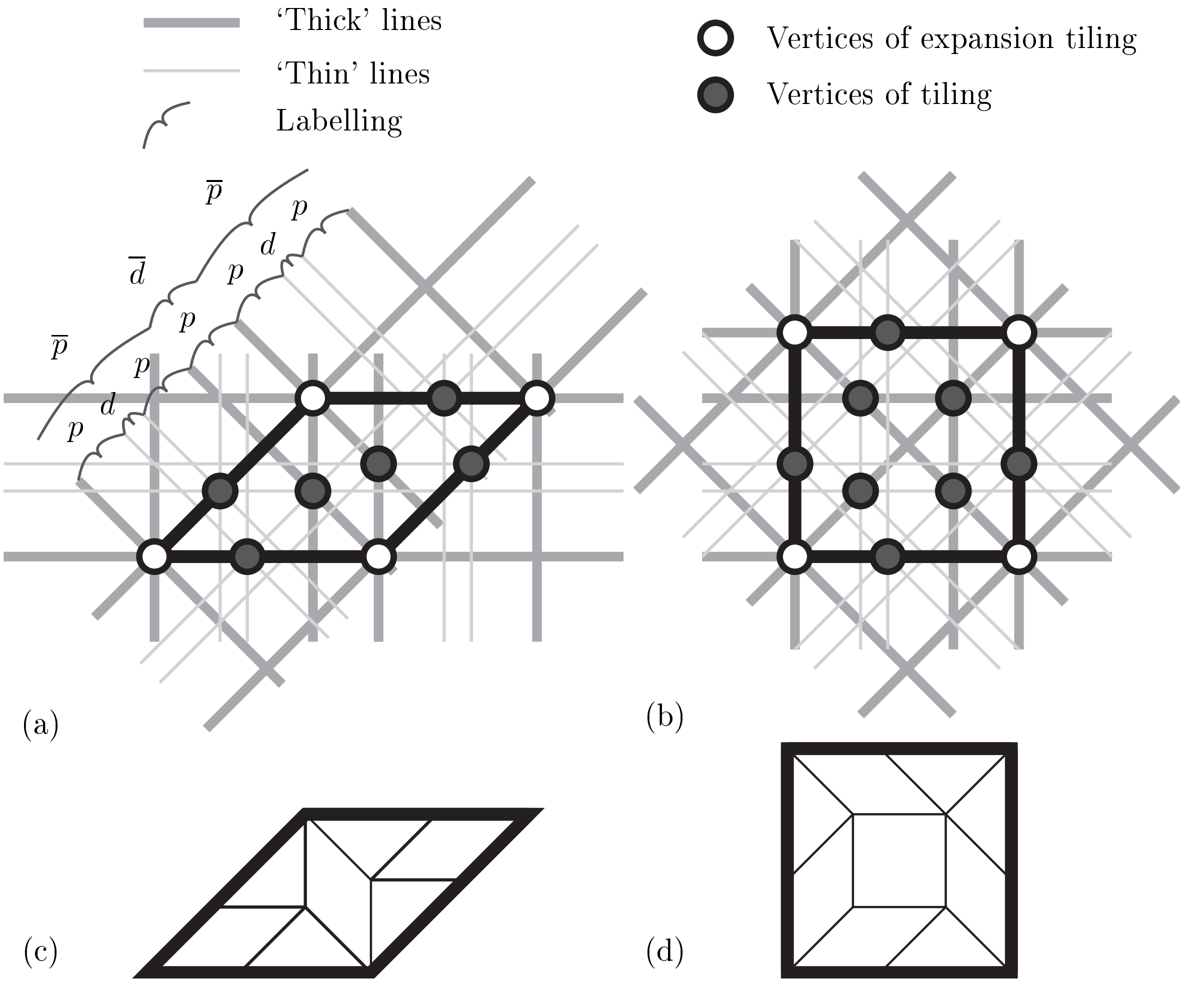


Fig. 9. The substitution rule for tiles in the Ammann-Beenker tiling. The thick grey lines show the a_i strips before substitution, the thin grey lines show the strips after substitution by the rules derived on the A_i plane. The points of the tiling after substitution are the points where grey lines from all four directions intersect. These are joined by edges of the tiling, shown in black.

2.5 Finding the substitution rule

We can now construct the actual substitution rule for the tiling. The idea is to consider the strips, and the thick lines between them lying over an expansion tile. The substitution rule is applied to those strips, producing the set of thin lines. This information can then be used to derive the section of the tiling lying over that expansion tile.

Consider a rhomb tile in the expansion tiling $\overline{\mathcal{T}}_{t,s}$. Due to the symmetry we need only to consider one tile. Let us consider the tile with edges parallel to $\Pi_V(a_1)$ and $\Pi_V(a_2)$. In the directions of these vectors the tile lies over a $\overline{p}$

strip, in the other two directions it lies over the strips $\overline{pdp}$. This determines the position of thick lines as shown in Figure 9(a). The situation for the other three rhombs is the same apart from a rotation. The application the substitution rule on the strips yields the thin lines as shown in the figure. In the figure, the vertices of the expansion tiling and the vertices of the tiling are now found as the intersection points of respectively four thick lines or four thin lines. The vertices can then be connected by projections of the lattice generators to yield the tiles, as shown in Figure 9(c).

Now consider a square with edges parallel to $\Pi_V(a_1)$ and $\Pi_V(a_3)$ in the expansion tiling. In the directions $\Pi_V(a_2)$ and $\Pi_V(a_4)$, this is covered by the strips $\overline{pp}$. In the other two directions things are not defined by the vertices. There are two options for the strips, $\overline{pd}$ and $\overline{dp}$ for each of the two directions. Inspection of Figure 1 shows that all four options can occur. Each of the two squares, therefore has four possible orientations. One of these orientations is shown in Figure 9(b), the other seven possibilities are rotations of this tile. The figure shows the thick lines, together with the thin lines generated by the substitution rule. Once again the vertices are determined by the points where four thin lines cross. The crossings of four thick lines and four thin lines are shown. Figure 9(d) shows how the crossings of four thin lines are linked together.

This construction produces the vertices inside the expansion tiles. To finish we wish to find all the tiles that intersect the expansion tile. Observe that the pattern of thin lines over the edge of the expansion tiles in Figure 9(a) and (b) is only compatible with one orientation of the square tile. Alternatively, one can consider the line pattern just outside the exapnsion tiles. Observe that, using the coding of the strips as introduced before, the first and last symbol of the images of the substitution on $\overline{p}$ and $\overline{d}$, must be equal to p. Every thick line, therefore lies between two strips of type p in the original tiling. This adds extra thin lines outside the expansion tiles. These lines intersect with lines from the other directions at the position of the missing vertices for the square. We have thus retrieved the Ammann substitution rule, shown in Figure 2.

3 Proof of Theorem 1.12

In this section we prove our main theorem. The proof is essentially algorithmic, constructing substitution rules from a suitable matrix. Our start point is a matrix M and an associated canonical projection tiling $\mathcal{T}_t$.

In subsection 3.1 we find a set of expansion tilings for $\mathcal{T}_t$. For a tiling $\mathcal{T}_t$, these are tilings with window $M\Omega_0 + s + t$ such that $\Pi_W(M\Omega_0 + s + t) \subset \Pi_W(\Omega_t)$. We require in addition that $s + t \notin \Pi_W(\mathcal{L})$, as we require that the two tilings be locally isomorphic up to expansion.

Then in subsection 3.2 we consider the A_i planes. It turns out that we can define substitution rules on the lines V_i which are the projections of V to these planes.

In subsection 3.3 we show how the tiling on V can be reconstructed from the tilings on the lines V_i.

Finally in subsection 3.4 we show how the substitution rules on the lines V_i can be combined to find the substitution rules on the plane V.

3.1 Vertex Hierarchic Expansion tilings

In this section we describe a set of vertex hierarchic expansion tilings for $\mathcal{T}_t$.

Proposition 3.1 *Let $\mathcal{T}_t$, be a canonical projection tiling for a 4×4 quadratic expansion matrix M with non-singular window $\Pi_W(\Omega_t)$. Any tiling with non-singular window $\Pi_W(M\Omega_0 + s + t) \subset \Pi_W(\Omega_t)$ is a vertex hierarchic expansion tiling for $\mathcal{T}_t$.*

In order to prove Proposition 3.1, we first formulate some Lemmas. The first Lemma asserts that application of the matrix to $\mathbb{R}^4$ expands the tiling and contracts the window.

Lemma 3.2 *The set of points $\Pi_V(\mathcal{L} \cap (M\Omega_t) + V)$ is the set of vertices of the tiling $\lambda\mathcal{T}_t$.*

PROOF. Consider a point $r \in vert(\mathcal{S}_t)$ so that $\Pi_V(r) \in vert(\mathcal{T}_t)$ and $\Pi_W(r) \in \Pi_W(\Omega_t)$. The matrix M acts as an expansion by λ on V so that $\Pi_V(Mr) = \lambda\Pi_V(r)$, and as a contraction by $\pm\lambda^{-1}$ on W so that $\Pi_W(Mr) = \pm\lambda^{-1}\Pi_W(r)$. Therefore $\Pi_V(Mr) \in vert(\lambda\mathcal{T}_t)$ and $\Pi_W(Mr) \in \Pi_W(M\Omega_t)$. As M is an automorphism of the lattice, the image under M contains all lattice points that project to $\Pi_W(M\Omega_t) + V$, concluding the proof. □

Lemma 3.3 *The tilings $\mathcal{T}_t$ and $\mathcal{T}_r$ are locally isomorphic for non-singular t and r.*

PROOF. Consider some vertex $v \in vert(\mathcal{T}_t)$, where $v = \Pi_V(p)$ for $p \in \mathcal{L}$, and some ball $B_r(v) \subset V$ about v. The set of points $vert(\mathcal{T}_t) \cap B_r(v)$ are the projection of some set of points in the lattice $L \subset \mathcal{L}$, say. We therefore have some set of points in the lattice which project to the vertices of some region of the tiling. These points project to the set $\Pi_W(L)$ in $\Pi_W(\Omega_t)$. This set $\Pi_W(L)$ is a finite subset of $\Pi_W(\Omega_t)$. Recall that we are assuming the non-singular case so we can assume that $\Pi_W(\Omega_t)$ is an open set. We can, therefore, place open balls of radius s about each of the points.

Every point in $\mathcal{L}$ which projects to the open ball about $\Pi_W(p)$, $B_s(\Pi_W(p))$ has the same local region up to a radius of r. This is because the same lattice generator steps will project to the balls about the other points in $\Pi_W(L)$. In other words the points in the ball $B_s(\Pi_W(p))$ correspond to the points at the center of a translation of $\mathcal{T}_t \cap B_r(v)$. Now consider $B_s(\Pi_W(p)) + r - t \subset \Omega_r$. As it is an open ball and the lattice projects densely to W, there will be the projection of lattice points in this ball. Each of these will have the same local region as part of $\mathcal{T}_t$.

Thus any region of $\mathcal{T}_t$ will occur in $\mathcal{T}_r$. □

PROOF. [Proof of Proposition 3.1] Consider the projection tiling with a window obtained by projecting the transformed hypercube $M\Omega_0 + s + t$ to W. By Lemma 3.2, this is an expansion of the tiling $\Omega_0 + M^{-1}(t+s)$. The window for this tiling is the projection to W of a translation of Ω_t. By assumption, $\Omega_0 + M^{-1}(t + s)$ is non-singular. Hence, by Lemma 3.3, the tiling is locally isomorphic to $\mathcal{T}_t$. If $M\Omega_0 + s + t \subset \Omega_t$, using that fact that M fixes the lattice, we find that the vertices of the tiling with transformed hypercube $M\Omega_0 + s + t$ is a subset of the vertices of $\mathcal{T}_t$. □

For a substitution rule we need to have a unique relationship between a tiling and its predecessor. We fix a relative location of the subwindow with respect to the original window. For the window $\Pi_W(\Omega_t)$ let the subwindow be $\Pi_W(M\Omega_0 + s + t)$, with s chosen so that the entire subwindow fits within the window. Let $\overline{\mathcal{T}}_{t,s}$ denote the corresponding expansion tiling with vertices $\Pi_V(\mathcal{L} \cap (M\Omega_0 + t + s) + V)$ and subwindow hypercube $\overline{\Omega}_{t,s} = M\Omega_0 + t + s$.

We aim to show that if $\Pi_W(s) \in \Pi_W(\mathcal{L})$ show that there is a substitution rule relating the expansion to the original tiling. Note that if $\Pi_W(s) \in \Pi_W(\mathcal{L})$ we may choose without loss of generality that $s \in \mathcal{L}$.

3.2 The planes A_i and their geometry

In this section we define the A_i planes. These planes are positioned in such a way that the 4 to 2 canonical projection structure projects to the structure of a 2 to 1 projection tiling. This latter situation is well studied and relatively easy to work with. In this section we generalise the results of Sections 2.2 and

2.3. In the generalisation of Section 2.2 we need to take care of that fact that in contrast to the Ammann example, where the projected staircase has two types of steps, in the more general setting the staircase may have three types of steps. This is the contents of Lemma 3.9. From there we then continue to generalise the result of Section 2.3, demonstrating that all tilings oobtained from such staircases admit substitution rules.

Definition 3.4 *For each lattice direction a_i, let A_i be the plane orthogonal to $[\Pi_V(a_i), \Pi_W(a_i)]_{\mathbb{R}}$. Let Π_{A_i} be the orthogonal projection to A_i with respect to the inner product $< \cdot, \cdot >$.*

We first show that these planes have some useful properties.

Lemma 3.5 *The projection to A_i of the 2 to 4 projection structure contains the following:*

(1) The tiling space: $\Pi_{A_i}(V)$ is a line V_i.
(2) The window space: $\Pi_{A_i}(W)$ is a line W_i.
(3) $\Pi_{A_i}(\mathcal{L})$ is a lattice $\mathcal{L}_i$ of rank 2.
(4) The matrix M fixes A_i setwise. We denote $M_i = M|_{A_i}$. The action of M_i on $\mathcal{L}_i$ induces an action as that of a unimodular integer matrix, with leading eigenvalue $\lambda > 1$.
(5) The hypercube: $\Pi_{A_i}(\Omega_t) = \Omega_t^i$.
(6) The subwindow translation: $\Pi_{A_i}(s) = s_i \in \mathcal{L}_i$.
(7) The transformed hypercube: From definitions $\Pi_{A_i}(\overline{\Omega}_{0,s}) = \Pi_{A_i}(M\Omega_0 + s) = M_i\Omega_0^i + s_i$.
(8) The vertices of the tiling's staircase: $(V_i + \Omega_t^i) \cap \mathcal{L} = S_t^i$.
(9) The vertices of the expansion tiling's staircase: $(V_i + \overline{\Omega}_{t,s}^i) \cap \mathcal{L} = \overline{S}_{t,s}^i$.

PROOF. For points 1 and 2, note that $\Pi_{A_i}(\Pi_V(a_i)) = 0$ and $\Pi_{A_i}(\Pi_W(a_i)) = 0$, so the projection is at most one-dimensional. The projection cannot be a point as neither V nor W is orthogonal to A_i.

For point 3, recall that the matrix acts on the planes V and W by uniform expansion and contraction. Thus $M\Pi_V(a_i)$ is parallel to $\Pi_V(a_i)$ for $i \in \{1, \ldots, 4\}$. Now we note that the orthogonal projection with respect to the inner product $< \cdot, \cdot >$ commutes with M. Hence, $\Pi_V(Ma_i)$ is parallel to $\Pi_V(a_i)$. By an analogous argument, $\Pi_W(Ma_i)$ is parallel to $\Pi_W(a_i)$ and consequently $Ma_i \in [\Pi_V(a_i), \Pi_W(a_i)]_{\mathbb{R}}$. There are thus two integer relations in the $\mathbb{Z}$-module $\Pi_{A_i}(\mathcal{L})$, which is therefore a lattice of rank two. As $\mathbb{Z}$-module is relatively dense in A_i, the rank cannot be less than two.

For point 4, consider the eigenvalues of M, which are λ and $\pm\lambda^{-1}$. In addition, λ is a quadratic algebraic integer as M is an integer matrix. The matrix M acts a multiplication on V and W. On $\mathcal{L}$, M acts as an automorphism.

For point 4, consider the matrix M. The action of M on A_i is linear as $MA_i = A_i$, because M acts as multiplication on V and W. M therefore acts as a matrix M_i on A_i. As M acts as an automorphism on $\mathcal{L}$, the matrix M_i is an unimodular integer matrix. The eigen-values of M_i will be λ and $\pm\lambda^{-1}$, the eigenvalues of M. □

Having observed the 2 to 1 projection structure on the A_i planes, we now address the question of the existence of substitution rules on the A_i planes. Our result that follows is related to the results of [28] and [34]. Our approach is close to that of [34]. However, in contrast to the treatment in [**?**], we approach the problem from a geometric rather than a number theoretic viewpoint. In addition, we construct the substitution rules for a large class of sub-windows, whereas Masakova *et al.* focus on the existence of only one particular substitution rule. Our geometric technique is related to that of [31].

The remainder of this section is devoted to the proof of the following proposition:

Proposition 3.6 *The interval projection tiling with vertices* $\Pi_{V_i}((V+\Omega_t)\cap\mathcal{L})$ *admits a countable set of substitution rules. For each* $s \in \mathcal{L}$*, so that* $\Pi_{W_i}(\Omega_t) \supset \Pi_{W_i}(\overline{\Omega}_{t,s})$*, there is a substitution rule so that the associated expansion predecessor is the projection tiling whose vertices are* $\Pi_{V_i}((V + \overline{\Omega}_{t,s}) \cap \mathcal{L})$*.*

This proposition provides a countable infinity of substitution rules for the interval projection tiling. These substitution rules are inequivalent in the sense that they have different predecessors and consequently different substitution rules. It is interesting to note, however, that all the predecessors considered here are equivalent up to translation.

In order to prepare the proof of this proposition, we first discuss some intermediate results.

We choose a point $z \in \mathcal{L}_i$ such that $0 \in interior(\Pi_{V_i}(\Omega^i_z))$. It is readily verified that one can always find such a point. In line with the above discussion, we consider a transformation of this hypercube in the following manner: $M_i\Omega^i_z + (s_i + z - M_i z)$. To simplify notation, we define s_i by $s'_i = s_i + z - M_i z$. Note that s'_i is a lattice vector.

Lemma 3.7 *There exists an* $n \in \mathbb{N}$ *such that*

$$[\Pi_{V_i}(M_i(\Omega^i_z + s'_i)) \cup \Pi_{V_i}(-M_i^{-1}s'_i)] \subset \Pi_{V_i}(M_i^n\Omega^i_z). \tag{10}$$

PROOF. The linear subspace V is the expanding eigenspace of M. In addition, we have $0 \in interior(\Pi_{V_i}(\Omega^i_z))$. Thus $lim_{n\to\infty}\Pi_{V_i}(M_i^n\Omega^i_z) = V$. As the set of points $\Pi_{V_i}(M_i(\Omega_z^i + s_i')) \cup \Pi_{V_i}(-M_i^{-1}s_i')$ is bounded, it will therefore be a subset of $\Pi_{V_i}(M_i^n \Omega_z^i)$ for some $n \in \mathbb{N}$. □

Lemma 3.8 *Let M be a unimodular, integer 2×2 matrix with quadratic real eigenvalue $\lambda > 1$. Then, either $\lambda = (1+\sqrt{5})/2$, or $\lambda > 2$.*

PROOF. Let M have determinant $D = \pm 1$ and trace T, we have:

$$\lambda = \frac{T + \sqrt{T^2 - 4D}}{2}$$

Consider $1 < \lambda < 2$, this implies:

$$1 < \frac{T + \sqrt{T^2 - 4D}}{2} < 2$$

so $2 - T < \sqrt{T^2 - 4D} < 4 - T$. As λ is real, $\sqrt{T^2 - 4D} > 0$. Thus $T^2 - 4D < 16 - 8T + T^2$. As $D \leq 1$, $1 < 4 - 2T$, we therefore conclude that $T \leq 1$. As $T \leq 1$, $2 - T > 0$, thus $4 - 4T + T^2 < T^2 - 4D$. This gives $T > 0$, so we have $T = 1$, as T is an integer. If $T = 1$, $\lambda = (1+\sqrt{5})/2$, as if $D = 1$, λ is not real. □

We now show that the set of points S_t^i can be considered as the vertices of a staircase, with at most three types of steps. This result is well known, see for example [33]. We include a proof for completeness.

Lemma 3.9 *Let $P \subset W_i$ be an interval such that $|P| = |\Pi_{W_i}(z)|$ for some $z \in \mathcal{L}_i$ Then the set of points $(P+V) \cap \mathcal{L}_i$ are the vertices of a staircase S with three types of steps. These steps can be taken to be a, b and $a+b$. Furthermore, P can be divided into three subintervals, whose lengths are $|\Pi_{W_i}(z)|$ for some $z \in \mathcal{L}_i$.*

Consider a vertex point $v \in S$. Then the step in the staircase from that point in the positive direction can be deduced from considering to which of the three subintervals $\Pi_{W_i}(v)$ lies.

PROOF. Consider the leftmost point e_1 of the interval P. Consider a translation f of the $\mathcal{L}_i$ such that $e_1 \in \mathcal{L}_i + f$. Since $|P| = \Pi_{W_i}(z)$ for some $z \in \mathcal{L}_i$, there is a point $e_2 \in \mathcal{L}_i + f$ such that $\Pi_{W_i}(e_2)$ is the other end point of P. Now consider the set of points $(P + V_i) \cap (\mathcal{L}_i + f) \setminus \{e_2\}$, and its projection $\Pi_{V_i}((P + V_i) \cap ((\mathcal{L}_i + f) \setminus \{e_2\})$. Since P is bounded, the projection will be a discrete set of point in V_i.

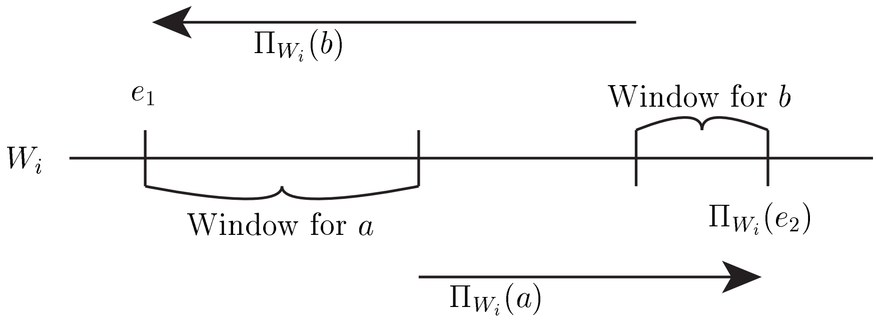


Fig. 10. The division of the window for a staircase with three steps. The arrows show the projection of steps of type a and b to the window. The central window is the window for $a + b$.

Consider now the two points in $\mathcal{L}_i + f$ that project to either side of $\Pi_{V_i}(e_1)$ in V_i. Let us denote these points $e_1 + a$ and $e_1 - b$. The vectors a and b therefore are among the steps of the staircase. Indeed, the lattice is periodic and e_1 is in $\mathcal{L}_i + f$. If there were a closer point in the projection to V_i then there would be a translation of that point closer to e_1.

The windows for the steps a and b in P are points $p \in P$ such that $p + \Pi_{W_i}(a) \in P$ or $\Pi_{W_i}(b) + p \in P$, respectively. These windows are precisely the intervals $[e_1, \Pi_{W_i}(e_2 - a)]$ for a and $[\Pi_{W_i}(e_1 - b), \Pi_{W_i}(e_2)]$ for b. Hence, if $\Pi_{W_i}(e_2 - a) = \Pi_{W_i}(e_1 - b)$, then for every vertex in the staircase the step in the positive direction is either a or b, and we have a staircase with two types of steps.

If this is not the case, the situation is as in Figure 10. For any vertex q of the staircase whose projection lies in the central window interval $\Pi_{W_i}(q) \in [\Pi_{W_i}(e_2 - a), \Pi_{W_i}(e_1 - b)]$, consider the point $q + a + b$. The projection of $q + a + b$ lies in the interval $[\Pi_{W_i}(e_2 + b), \Pi_{W_i}(e_1 + a)]$, which by the definition of a and b lies within P. We now claim that $q + a + b$ is the nearest point in the positive direction for the point q the central window interval. If there are vertices of the staircase whose projection to V lies between $\Pi_{V_i}(q)$ and $\Pi_{V_i}(q + a + b)$, then there must be a piece of the staircase connecting q and $q + a + b$ consisting of at least two steps. Suppose there are two steps, then by the earlier argument one of them must project to a line segemnt of V_i of length greater than or equal to $\Pi_{V_i}(a)$ and one to a line segment of length greater than or equal to $\Pi_{V_i}(b)$. Hence, the two steps must be equal to a and b, which we ruled out before by the definition of q. If there are more steps, it is immediately clear that at least one of them would need to project to a line segement of length smaller than $\Pi_{V_i}(a)$ or $\Pi_{V_i}(b)$, while at the same time it should be longer.

The boundary points for the intervals of the window are $e_1, \Pi_{W_i}(e_2 - a), \Pi_{W_i}(e_1 - b)$ and $\Pi_{W_i}(e_2)$. These are all projections of points in $\mathcal{L}_i + f$, thus the lengths of the three intervals will be in $\Pi_{W_i}(\mathcal{L}_i)$. □

We now further develop our geometric approach towards finding substitution rules.

The projection $\Pi_{A_i}(\Omega_t)$ is a convex polygon whose vertices are separated by elements of $\mathcal{L}_i$. Thus the edges of the window in the projection to W_i are separated by the projection of an element of this lattice. The length of the window in the projection to W_i will therefore be $|\Pi_{W_i}(z)|$ for some $z \in \mathcal{L}_i$. By Lemma 3.9, the set of points S_t^i can be considered as the vertices of a staircase with three steps. We introduce a labelling for these steps, and show that two steps with the same label in the expansion staircase will be covered in the projection to V_i by the same patch of labelled steps in the original staircase. This is the essential ingredient implying the existence of a substitution rule. In section 2.3, when discussing the Ammann-Beenker tiling, it was sufficient to simply consider the labelling given by the type of each step. In general, however, the labelling can be more complicated.

We now discuss the construction of the labelling intervals. As in the Ammann example, we divide up the window into intervals. The boundary points of these intervals, which will be used as labelling intervals, are defined in the same manner as in the Ammann example and in the proof of Lemma 3.9: the boundaries for the intervals are the projections of a finite set of points in a translated copy of the lattice. Considering the tiling $\mathcal{T}_t$, we define the lattice

$$\mathcal{M}_i = \mathcal{L}_i + \Pi_{A_i}(t) + z \tag{11}$$

where z is the vector mentioned above Lemma 3.7. Ω_t^i, the projection of the hypercube Ω_t, is singular with respect to $\mathcal{M}_i$, and as $\mathcal{M}_i$ depends on t, is in the same position with respect to Ω_t^i for all t. Let $\tilde{M}_i$ be the conjugation of M_i by a translation, so that $\tilde{M}_i$ fixes the point $z + t \in \mathcal{M}_i$, and consequently fixes the lattice $\mathcal{M}_i$ setwise:

$$\tilde{M}_i \mathcal{M}_i = \mathcal{M}_i. \tag{12}$$

We now construct a set of points which, when projected to W_i, define the labelling intervals in the window. Let R^n be the parallelogram:

$$R^n = \Pi_{V_i}(\tilde{M}_i^n \Omega_t^i) + \Pi_{W_i}(\Omega_t^i) \tag{13}$$

Note that $R^{n+1} \supset R^n$.

By Lemma 3.7 there exists an $N \in \mathbb{N}$ such that $((\tilde{M}_i \Omega_t^i + s_i') \cup (-\tilde{M}_i^{-1} s_i')) \subset R^N$. Thus both Ω_t^i and $\tilde{M}_i \Omega_t^i + s_i'$ lie in R^N.

Let the lattice point in $\mathcal{M}_i$ that projects to the rightmost point of $(\Pi_{V_i}(\tilde{M}_i^n \Omega_t^i))$ be denoted $max(R^n, V_i)$. Consider $\Pi_{W_i}((\mathcal{M}_i \cap R^{N+1}) \backslash max(R^{N+1}, V_i))$. This is a set of points in the window $\Pi_{W_i}(\Omega_t^i)$. We now use these points as the boundary points for the labelling intervals in W_i for the vertices of the staircase S_t^i. Let

C_i be the set of subintervals of the window in W_i that provides the labelling B_i.

It is now useful to consider this labelling in another way. The points of the lattice $\mathcal{M}_i$ lying in the region $R^{N+1} \cap \mathcal{M}_i$ can be joined by line segments so that each labelling interval is the projection of a line segment. By Lemma 3.9, this construction prodces a staircase with at most three types of steps. It is important to note that this is a very different staircase than the one discussed before: the staircase progresses in the W_i direction and not in the V_i direction, like before. Also, the staircase we consider here has only a finite number of steps. In Figure 11 there is an illustration of the situation with the set of points in $R^{N+1} \cap \mathcal{M}_i$ linked up into such a staircase. The partition induced on the window on W_i is also indicated.

In the definition of the points that define the intervals, we have used the parallelogram R^{N+1}. Later in Claim 3.11, it turns out that in one special case we actually need to consider R^{N+2} instead of R^{N+1}. We continue the general discussion with R^{N+1}, and point out later when the modification to R^{N+2} needs to be made.

As the tiling $\mathcal{T}_t$ is non-singular every vertex point of S_t^i projects to the interior of a labelling interval on W_i. Thus every vertex point in S_t^i has a unique B_i label. As $\tilde{M}_i \Omega_t^i + s_i' \subset R^N$, the B_i-labelling of a point identifies whether it lies in the subwindow $\Pi_{W_i}(\tilde{M}_i \Omega_t^i + s_i')$. We can, therefore, consider a labelling of the points in the expanded staircase $\overline{S}_{t,s}^i$, given by the intervals of C_i in the subwindow. We call this labelling L_i. Note that this labelling is induced by the projection to W_i of the points of the lattice lying in the region:

$$\Pi_{V_i}(\tilde{M}_i^{N+1} \Omega_t^i) + \Pi_{W_i}(\tilde{M}_i \Omega_t^i + s_i') \tag{14}$$

To show that there is a substitution rule on the set of tiles labelled by L_i we now state two claims. These claims concern tha fact that staircase steps at vertex points in the staircases S_t^i and $\overline{S}_{t,s}^i$ are determined by their B_i-labelling.

Claim 3.10 *The step in the positive direction from a vertex point v in S_t^i or $\overline{S}_{t,s}^i$ is completely determined by the B_i-label of $\Pi_{W_i}(v)$.*

PROOF. By Lemma 3.9 the window can be divided into subintervals which determine the step from a vertex point in the positive direction.

The points of $\mathcal{M}_i$ that project to the boundaries of these intervals are one staircase step from the points of $\mathcal{M}_i$ that project to the boundaries of the window on W_i. In the projection to $\mathcal{M}_i \cap (V_i + \Omega_t^i)$ to V_i, these points are precisely

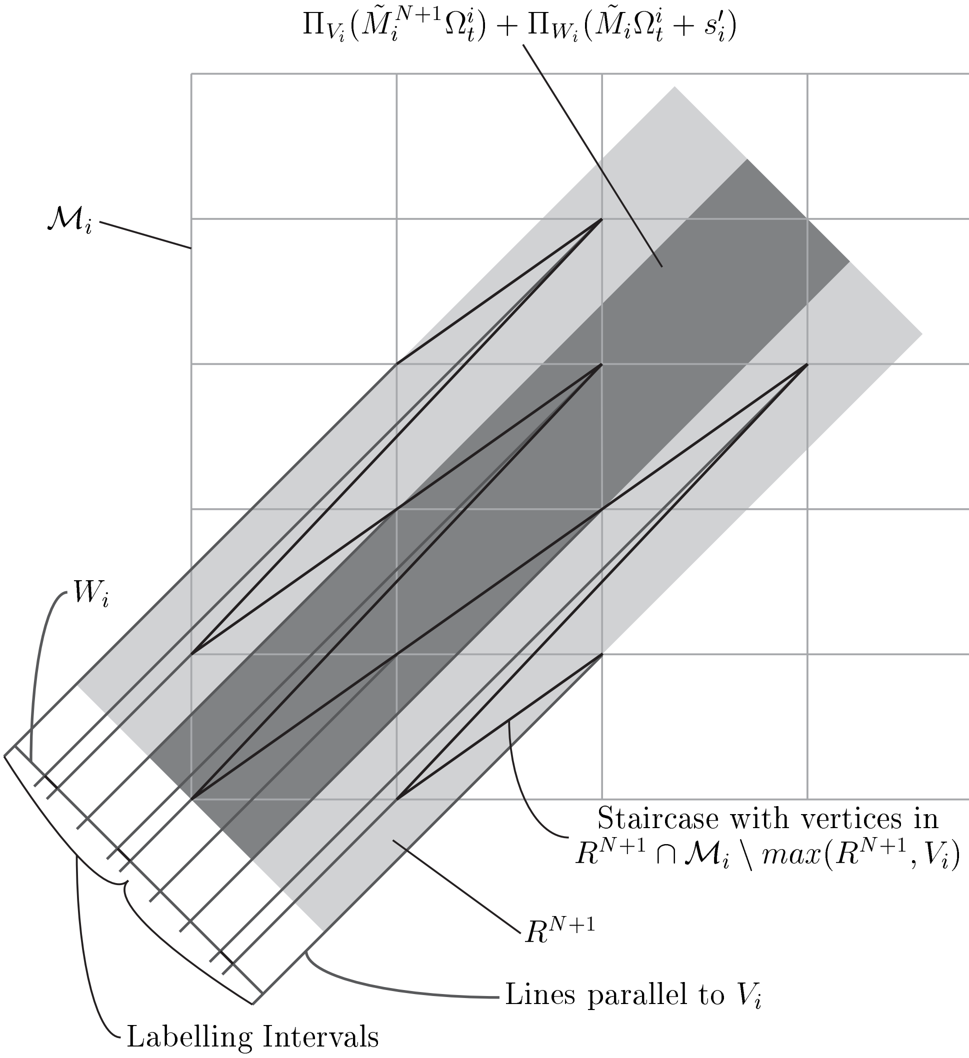


Fig. 11. The region R^{N+1} of the lattice $\mathcal{M}_i$, showing the staircase linking the points in the direction of W_i. The partition induced on the window, which can be used to label the vertices of a non-singular staircase in the V_i direction, with lattice $\mathcal{L}_i$, is also shown.

neighbours. Note that points in $\mathcal{M}_i \cap R^{N+1}$ corresponding to the extremal points in V_i projection are different from the lattice points corresponding to the extremal points in the W_i projection.

We thus assign B_i labels to vertices of the staircase S_t^i, according to which of the three (or two) W_i-window intervals they project to, that determine the staircase step from a vertice in the positive direction.

The arguments for the expansion staircase are analogous. □

We now aim to apply the labelling L_i, defined on $\overline{S}^i_{t,s}$, to the staircase S^i_t. Recall that the labelling L_i is determined by the projection of the $\mathcal{M}_i$ lattice points in:

$$\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t) + \Pi_{W_i}(\tilde{M}_i\Omega^i_t + s'_i) \tag{15}$$

In order to produce the L_i labelling we rescale the subwindow so that it fits over the entire (original) window. The existence of a substitution rule relies on the L_i labelling intervals consisting of unions of entire B_i labelling intervals. To show this property we invoke the inverse of the map taking Ω^i_t to $\overline{\Omega}^i_{t,s}$. This is translation by $-s'_i$, followed by multiplication by the inverse matrix $\tilde{M}_i^{-1}$. This gives the set of points of $\mathcal{M}_i$ in:

$$\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t - \tilde{M}_i^{-1}s'_i) + \Pi_{W_i}(\Omega^i_t) \tag{16}$$

We now show that the obtained L_i labelling is in fact determined by the B_i labellinga.

Claim 3.11 *The L_i labelling of a vertex point in S^i_t is completely determined by its B_i labelling.*

PROOF. We need to show that the set $\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t - \tilde{M}_i^{-1}s'_i) + \Pi_{W_i}(\Omega^i_t)$ is a subset of R^{N+1}. It is sufficient to demonstrate that

$$\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t - \tilde{M}_i^{-1}s'_i) \subset \Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t). \tag{17}$$

We first assume that $\Pi_{V_i}(s_i) \neq 0$. With this assumption and the fact that

$$\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t) \subset \Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t) \tag{18}$$

it is sufficient to show that the leftmost point of $\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t - \tilde{M}_i^{-1}s'_i)$ lies within $\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t)$. Let the leftmost point of $\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t)$ be m. As 0 lies in $\Pi_{V_i}(\tilde{M}_i^N\Omega^i_t)$, the leftmost point of $\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t)$ is λm (recall that λ is the expanding eigenvalue of M on V). In addition, we have $-\tilde{M}_i^{-1}s \in R^N$ so $m < \Pi_{V_i}(-\tilde{M}_i^{-1}s'_i)$. We thus have proven our claim if $\lambda m < m + \Pi_{V_i}(-\tilde{M}_i^{-1}s'_i) < 2m$, which, recalling that $m < 0$, requires $\lambda > 2$. The latter is assured by Lemma 3.8, with the one exception of the case that $\lambda = \frac{1+\sqrt{5}}{2}$.

In the latter case, we need to repeat our argument using R^{N+2} instead of R^{N+1}, and are led to verify that

$$\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega^i_t - \tilde{M}_i^{-1}s'_i) \subset \Pi_{V_i}(\tilde{M}_i^{N+2}\Omega^i_t). \tag{19}$$

By similar considerations as set out above, this leads to the requirement that $\lambda^2 m < \lambda m + \Pi_{V_i}(-M^{-1}s)$. As $\lambda^2 = \lambda + 1$ and $\Pi_{V_i}(-M^{-1}s) > m$, this requirement holds. □

We are now ready to present the proof of Proposition 3.6. Recall that the $\mathcal{M}_i$ lattice points that define the set of intervals C_i can be joined together in a (finite) staircase in the W_i direction, as shown in Figure 11. In the following proof, we denote this staircase S_{C_i}.

PROOF. [Proof of Proposition 3.6] In the discussion below we will be considering three different staircases. The first is the finite staircase S_{C_i}, from which the labellings are induced. The other two staircases are the staircase S_t^i and the expansion staircase $\overline{S}_{t,s}^i$.

Consider the set of vertices in the expansion staircase with a certain label in L_i. These points project to a subinterval of the subwindow in W_i. This interval is the projection to W_i of a step of the staircase S_{C_i}, that we denote P. The label determines, by Claim 3.10, the step in the positive direction of the expansion staircase $S_{t,s}^i$. We aim to show that it also determines the set of steps in S_t^i that connect the same two points that are connected by the step of the expansion staircase. By Claim 3.10, the first step in S_t^i is determined. Suppose the type of the first step in the staircase S_t^i is g and the type of the first step in the expansion staircase with vertices $\overline{S}_{t,s}^i$ is $\overline{g}$. The window for the set of points after the first step in the staircase with vertices S_t^i will be $\Pi_{W_i}(P+g)$, where $P+g$ here denotes the line segment P translated by the vector g.

If $P+g$ is a step in the staircase S_{C_i} then the following step is again determined. The aim is to show that this condition is satisfied for a finite sequence of subsequent steps of the staircase S_t^i (according to the B_i label associated to the Pi_{W_i} projection of the translated copy of P), so that we start at P and end at $P+\overline{g}$. In fact, this construction defines a rule replacing all expansion tiles with label corresponding to P, with a set of tiles that corresponds to the staircase segment of S_t^i that was followed.

It is readily verified that the end points of the line segment $P+\overline{g}$ are contained in $R^{N+1}\cap\mathcal{M}_i$. Thus the end points of every intermediate step (along the staircase S_t^i) of P are also in $R^{N+1}\cap\mathcal{M}_i$. We now recall that the staircase S_{C_i} contains two or three types of steps. Let us consider first the case where P is one of the two shorter steps (called a and b in Lemma 3.9). In that case, $P+g$ is a line segment with end points in $R^{N+1}\cap\mathcal{M}_i$, which must be connected by a part of the S_{C_i} staircase. This connecting path can be either equal to a translation of P, or a translation of a number of concatenated steps of the other shortest type, since we know that the longer step type projects to a longer interval in W_i. As the second option is also impossible, because the two shorter steps lie in different directions, this step must be a translation of P.

This leaves us to discuss the case where P is a step of the longest type.

In this case consider the windows along V_i for the three steps in S_{C_i}. By Lemma 3.9 the boundary points for these windows are projectio of points in the lattice $\mathcal{M}_i$ to V_i. In fact they are points in

$$\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega_t^i) + \Pi_{W_i}(\tilde{M}_i\Omega_t^i + s_i') \subset R^{N+1}$$

. The boundary points of the one-step labelling intervals in the window for S_{C_i} for the shorter steps are precisely projections of two points whose projections to W_i are neighbours. This follows from the definitions of the shorter steps and their windows as before. In fact points lie in $\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega_t^i) + \Pi_{W_i}(\tilde{M}_i\Omega_t^i + s_i')$, as $\tilde{M}_i\Omega_t^i$ is not a line, so that its extremal points with respect to V_i are not the extremal points with respect to W_i.

We now consider the process of construction the sequence of steps in the staircase S_t^i, to get from P to $P+\overline{g}$, where $\overline{g}$ is the first step in the expansion staircase $\overline{S}_{t,s}^i$. Let P be the line segment from $p_1 \in \mathcal{M}_i$ to $p_2 \in \mathcal{M}_i$. In the expansion tiling in V_i, $\Pi_{V_i}(p_1+\overline{g})$ is the neighbour of $\Pi_{V_i}(p_1)$. The point $\Pi_{V_i}(p_1)$ is in the central subwindow for the longest type step (type "$a+b$") of the staircase S_{C_i}. As the maximum for this window is the projection of a $\mathcal{M}_i$ lattice point in $\Pi_{V_i}(\tilde{M}_i^{N+1}\Omega_t^i) + \Pi_{W_i}(\tilde{M}_i\Omega_t^i + s_i')$ then the most extreme position of this point is the leftmost point of the line segment $P+\overline{g}$. Thus every step as we go from P to $P+\overline{g}$ is of the longest type, which is precisely what we set out to show.

Note that we are not concerned whether at the end of the walk on the S_t^i staircase, $P+\overline{g}$ coincides a step of the staircase S_{C_i}. The next step in the original tiling after an expansion tile may be different for different tiles. □

3.3 The relationship between the A_i planes and the tiling

In the previous sections we have identified substitution tilings on each of the lines V_i. In this section we consider how these tilings and their substitutioon rules can be related to the full projection tiling in V.

Information about the position of the projection of a vertex of the tiling in V in one of the planes A_i is equivalent to information about the position of the vertex in the direction orthogonal to $\Pi_V(a_i)$. We thus decide to treat all points along lines parallel to $\Pi_V(a_i)$ as being equivalent with respect to A_i. The following definition provides us with a structure that contains this information.

Definition 3.12 *Define the* extension *of a set of points $P \subset V$, in the direc-*

tion $v \in V$, as a set of lines parallel to v as follows:

$$ext_V(P, v) := \{t + [v]_{\mathbb{R}} | t \in P\} \quad (20)$$

For a tiling $\mathcal{T}_t$ on V and a vector $v \in V$. Let the extension *of $\mathcal{T}_t$ by v be defined as:*

$$ext_V(\mathcal{T}_t, v) = ext_V(vert(\mathcal{T}_t), v) \quad (21)$$

We define ext_W in the analogous manner.

Now consider the extensions $ext_V(\Pi_V(\mathcal{L}), \Pi_V(a_i))$ for all i. Clearly the projection of each lattice point in $\mathcal{L}$ to V lies in the intersection of four such lines. the converse is also holds.

Lemma 3.13 *Let $v \in V$ be such that for all i, $ext(v, \Pi_V(a_i)) \in ext(\Pi_V(\mathcal{L}), \Pi_V(a_i))$. Then, $v \in \Pi_V(\mathcal{L})$.*

PROOF. As $\Pi_V(\mathcal{L})$ is relatively dense in V, there exists a j such that $V = [\Pi_V(a_1), \Pi_V(a_j)]_{\mathbb{R}}$. Now consider the set

$$D = [\Pi_V(a_1)]_{\mathbb{R}} \cap ext(\Pi_V(\mathcal{L}), \Pi_V(a_j)). \quad (22)$$

This is a $\mathbb{Z}$-module of rank 2, as there are two integrally-independent lattice points that project parallel to $\Pi_V(a_j)$, by Lemma 3.5. In addition this $\mathbb{Z}$-module contains the points $[\Pi_V(a_1)]_{\mathbb{Z}}$ as these are the projection of lattice points. We therefore have:

$$D = [\Pi_V(a_1), k]_{\mathbb{Z}} \quad (23)$$

where k is a vector in $[\Pi_V(a_1)]_{\mathbb{R}}$, that is incommensurate to $\Pi_V(a_1)$. The $\mathbb{Z}$-module D will also contain the projection of a lattice point with 0 component in the a_1 direction, again by Lemma 3.5. Some integer multiple of k would be the projection of this point, as k is the generator of D which is not in $[\Pi_V(a_i)]_{\mathbb{Z}}$. We may therefore consider k to be the projection of a point in the space generated by the lattice generators with the exception of a_i:

$$k = \Pi_V((0, k_2, k_3, k_4)^T) \quad (24)$$

Now consider the point v, that lies at the crossing of four extension lines. This implies that every point in $v + \Pi_V(\mathcal{L})$ lies at the crossing of four extension lines, as the translation of an extension line by a lattice vector gives an extension line, thus all of the four lines crossing v can be translated to any point of $v + \Pi_V(\mathcal{L})$. In particular there is a point $v' \in [\Pi_V(a_1)]_{\mathbb{R}} \cap v + \Pi_V(\mathcal{L})$. By the definition of D, this point must be in D. Thus $v' = q_1\Pi_V(a_1) + q_2 k$, for some $q_1, q_2 \in \mathbb{Z}$. We therefore have

$$v' = \Pi_V((q_1, q_2k_2, q_2k_3, q_2k_4)^T). \quad (25)$$

This means that v' is the projection of a point with integer a_1 coefficient. Every point in $v + \Pi_V(\mathcal{L})$ is therefore the projection of a point with integer a_1 component. By an analogous argument every point in $v + \Pi_V(\mathcal{L})$ is the projection of a point with all integer coefficients. Thus:

$$v + \Pi_V(\mathcal{L}) = \Pi_V(\mathcal{L}) \tag{26}$$

and so $v \in \Pi_V(\mathcal{L})$ as claimed. □

We are now able to prove the main result of this section, i.e. that the vertices of the tiling are precisely the points where four extension lines intersect.

Lemma 3.14 *For any point $v \in V$, and canonical projection tiling $\mathcal{T}_t$ $v \in vert(\mathcal{T}_t)$ if and only if $v \in ext_V(\mathcal{T}_t, \Pi_V(a_i)$ for all $i \in 1, .., 4$.*

PROOF. If four extension lines intersect, then by Lemma 3.13 the intersection point is the projection of a point in the lattice $\mathcal{L}$. It therefore corresponds to the intersection of four lines in the window space. Each of these lines passes through some points in the window, as they are in the extension of the tiling. We therefore have the intersection of four lines all intersecting the window $\Pi_W(\Omega_t)$. Furthermore there is one line parallel to every edge of this window. Thus the intersection point must lie in the window. The point is therefore in the tiling. □

3.4 Finding the substitution rule

To complete the proof of the theorem we now wish to show that the substitution rules on the lines V_i, given by Proposition 3.6, can be combined into a substitution rule on the tiling. Some of the general considerations here are not required in the Ammann example. This is because the labelling on the squares lying over the edges of the tiles in the Ammann tiling is determined by the information given within the tile. This is not true in general, however. We first give a general discussion of the ideas required, then give the proof of the theorem.

We have the set of lines on the plane V, given by the extension of the tiling $\mathcal{T}_t$ in the four directions. We also have a second set of lines given by the extensions of the expansion tiling $\overline{\mathcal{T}}_{t,s}$. We shall denote the first set the thin lines and the second set the thick lines. Note that, as the substitution is vertex hierarchic, every thick line lies on top of a thin line. The substitution rule labels the intervals between these lines. We shall call this set of labels V_i labels.

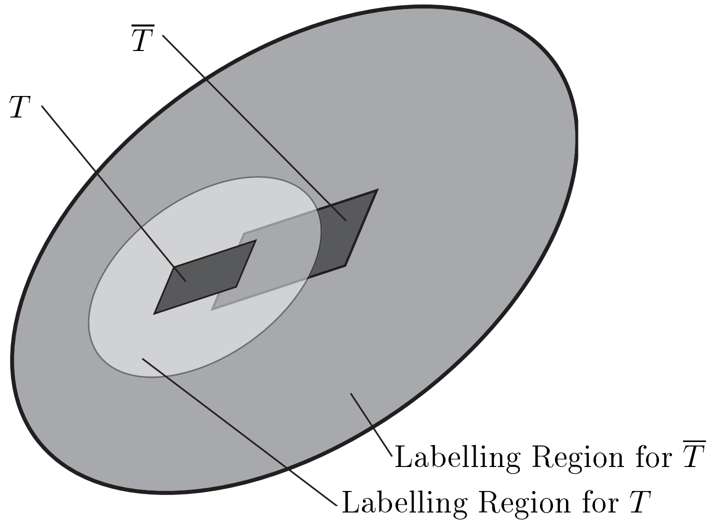


Fig. 12. A tile $\overline{T}$, in an expansion tiling is shown with the surrounding patch which is sufficient to give the labelling region. A similar tile T in the tiling $\mathcal{T}_t$ is also shown. The labelling region of $\overline{T}$ covers the labelling region for T.

We can consider the intervals in each of the four directions as words on the alphabet defined by the substitution rule. There are only finitely many possible subwords of any given length. Thus, given a point with some bounded subset of the plane around it, there will only be finitely many ways of covering the region so that the point lies at a vertex of the tiling. We now wish to construct a set of labels for the expansion tiles. This labels are given by the V_i labels of a local neighbourhood of the expansion tiling. We shall call this neighbourhood the labelling region. We define a labelling region for each tile shape.

We now consider a tile in the expansion tiling $\overline{T} \in \overline{\mathcal{T}}_{t,s}$ and the labelling region around it. We can apply the substitution rules on the intervals to this region. This gives the thin lines and the V_i-labelling of the labelling region associated with the tiling $\mathcal{T}_t$. Now, by lemma 3.14 the vertices of $\mathcal{T}_t$ that lie in the region are the points where four thin lines cross. These points can be joined by the projections of the lattice generators to V. This gives the tiles that lie within the labelling region for $\overline{T}$, as well as some of the edges of the tiles that lie over the border. To have a substitution rule we must know the labelling of every tile of $\mathcal{T}_t$ that intersects $interior(supp(\overline{T}))$. To determine the labelling of a tile in $\mathcal{T}_t$ we need to know the V_i-labels of its labelling region. Thus, if the labelling region for every tile in $\mathcal{T}_t$ that intersects $\overline{T}$, is in the labelling region of $\overline{T}$, then we have a substitution rule as the tiles can be labelled consistently. In Figure 12 we show the tile $\overline{T}$ and its labelling region. We also show a tile T which intersects $\overline{T}$. The labelling for $\overline{T}$ determines the labelling for T, as the labelling region around T is a subset of the labelling region around $\overline{T}$.

We now wish to consider how large the region around a tile needs to be. We therefore consider a tile in the expansion tiling $\overline{T} \in \overline{\mathcal{T}}_{t,s}$. For the substitution

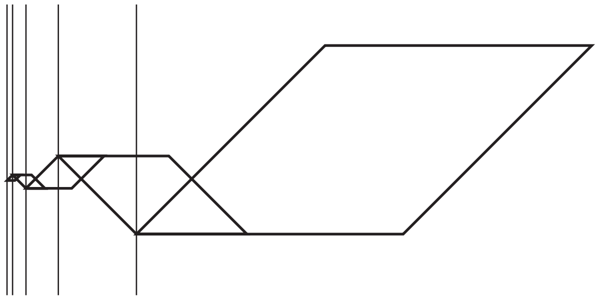


Fig. 13. A possible chain of associated tiles. The vertical lines show the increased region of the tiling that one must consider in order to label the original tile $\overline{T}$

rule, however, we need to know the labelling of every tile in $\mathcal{T}_t$ that intersects $\overline{T}$. We can therefore start by considering the region within one step of $\overline{T}$. This gives the shape of every tile that intersects $\overline{T}$. We also need to know the shapes of the tiles that these tiles substitute to, after a second application of the substitution on the intervals. We must therefore consider a chain of tiles, each intersecting the previous one. The tiles will get increasingly small as we apply the substitution rule at each stage without rescaling. A possible chain of such tiles is shown in Figure 13.

As the diameters of the tiles in the chain decrease in a geometric fashion, all tiles of the chain will lie within a bounded region. A sufficient labelling region around a tile is therefore given by expanding the tile by at most two copies of each of the lattice generators at each scale.

We can now give a formal treatment along the lines discussed above. As discussed we define the thin lines as $\bigcup_{i=1}^{4} ext(\mathcal{T}_t, \Pi_V(a_i))$ and the thick lines as the set of lines $\bigcup_{i=1}^{4} ext(\overline{\mathcal{T}}_{t,s}, \Pi_V(a_i))$. By Proposition 3.6 there is a substitution rule on the strips between the extension lines in each of the four directions. This rule induces the V_i-labelling on each set of the strips from a finite alphabet $\mathcal{A}_i$.

We obtain the following theorem, of which Theorem 1.12 is a corollary.

Theorem 3.15 *The canonical projection tiling of a* 4×4 *quadratic expansion matrix* M *with the tiling property has a primitive substitution rule associated to every* $s \in \mathcal{L}$ *such that* $\Pi_W(M\Omega_0 + s + t) \subset \Pi_W(\Omega_t)$. *As there are a countable number of distinct subwindows defined by different choices of* s, *this gives rise to a countable infinity of different substitution rules.*

PROOF. We first construct a labelling region for the tile shape. To do this we require a ball that contains two edges away from the tile at every scale. As the edges of $\overline{\mathcal{T}}_{t,s}$ all have length less than λ, and the scaling is by λ it suffices to consider the ball with radius $r = \frac{2\lambda}{1-\lambda}$. For particular examples one can find

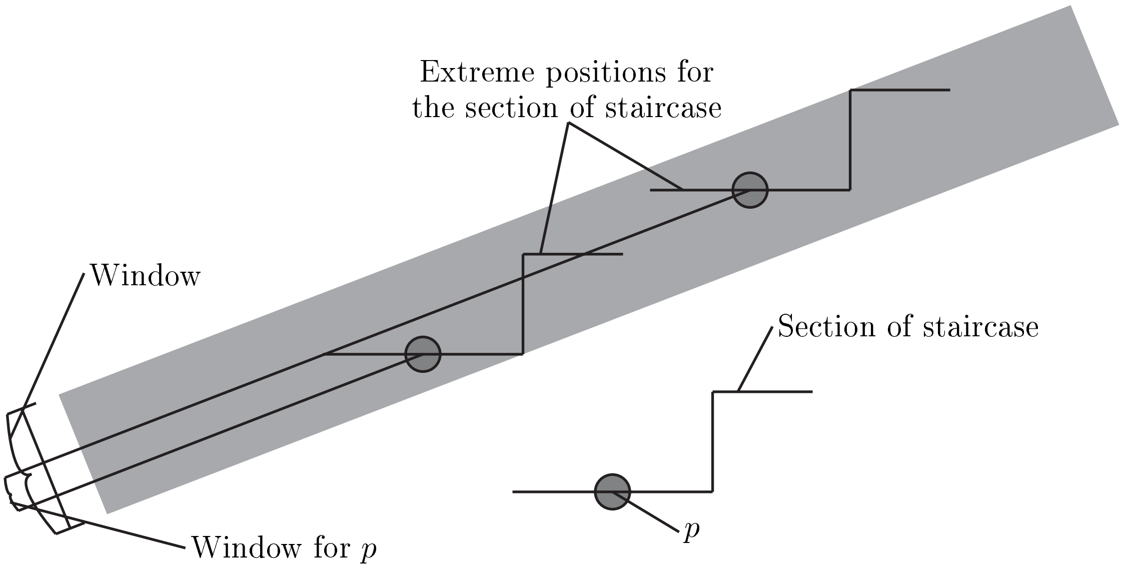


Fig. 14. The two extreme positions for a section of staircase, giving the window for a labelled point p

a more efficient labelling region.

We now digress to discuss the labelling of a closed bounded neighbourhood $U \subset V$ of a point p, using V_i-labels. We require that Π_{V_i} lies exactly on the boundary of two V_i-labelled intervals in V_i for all i. Let $I_i = \Pi_{V_i}(U)$ be the bounded interval containing $\Pi_{V_i}(p)$. We decompose $I_i = I_i^l \cup I_i^r$ the two intervals to either side of $\Pi_{V_i}(p)$ in V_i. Let w_i^l be the word representing the set of V_i-labelled intervals covering I_i^l. As such I_i^l is labelled by w_i^l. We define w_i^r analogously.

We define the label of U as the collection of words $\{w_1^l, w_1^r \ldots w_4^l, w_4^r\}$. The possible ways a neighbourhood U may be labelled is bounded in an elementary way by the slice of the intervals $\{I_1^l, I_1^r \ldots I_4^l, I_4^r\}$. In fact to each such set of interval sizes we associate a finite set of possible labels.

Hence to any bounded neighbourhood U of a point p with the required projection properties we may associate the finite set of possible labels, determined by the set of interval sizes, $\{I_1^l(U), I_1^r(U) \ldots {}_4^l(U), I_4^r(U)\}$. Importantly for our purposes the possible set of labels is independent of the choice of p.

By this discussion we find the set of possible of labels for a ball of radius r, centered at p. We now wish to refine this set of labels so that we only consider the labeled balls that occur in the tiling. We can first restrict the labels to words in the language generated by the substitution rule for each V_i. These choices, however, are not completely independent. We must therefore consider which collections of labels can occur.

To do this we consider the window spaces W_i. Each label corresponds to a section of staircase on A_i. As the window is an interval, and the projection of the point in $\mathcal{L}$ to W_i gives a dense set, this section of staircase can occur in various positions in the full staircase. The requirement for these positions is that every vertex of the section of staircase lies in the window. We may therefore find a window for the point p and a label. Figure 14 shows the window for a particular section of staircase.

We can therefore define interval windows for p in each of the four lines W_i. These correspond to strips of the full window space W. A set of labels corresponds to a section of the tiling if and only if the intersection of the corresponding four strips is not empty. Note that this intersection will always be in the window as the window for p will always be a subinterval of the window.

This set of labelled balls can be used to define a proto-set. We first note that in addition to p, the labelling gives the position of every tiling vertex in the ball, by lemma 3.14. In particular the vertices of every tile around p is defined, as the ball contains two edges away from p in each direction. We now choose a direction d on the plane that is not parallel to any of the projections of the lattice generators $\Pi_V(a_i)$. We may now associate every vertex of the tiling to the tile in the direction d. Note that this associates a vertex to every tile in a unique manner.

We have therefore identified a unique tile in each labelled ball, that is the tile associated to p. We may now apply the substitution rule to the set of strips in each of the V_i directions. We may therefore give a new set of lines covering the ball (these were described as 'thin' lines in the example). Again by lemma 3.14 this defines the vertices of the new tiling covering the ball, and thus the tiles with all four vertices in the ball are also defined. The edges in the new tiling have length less than 1, thus for every point within two of these new edges of the tiles vertices, we have defined the ball of radius $\frac{r}{\lambda}$. This is the labeling ball for the new tile. As a result any tile in the new patch of tiling that intersects the original tile can be identified as one of the proto-tiles.

We can therefore find the set of tiles that cover every proto-tile of the expansion tiling. As these tiles are all labelled, this gives a substitution rule, and the tiling is therefore a substitution tiling. Note that we have, in fact, found an infinite set of (possibly very complex) substitution rules for every such tiling. This is because we only required that the subwindow translation s lie in some infinite subset of the lattice. □

4 Some patches of other examples

In this section we show some of the canonical substitution rules tilings that can be constructed by our methods. The examples all show a patch of the tiling about the origin for the tiling with hypercube centered at the origin (which is non-singular). Each patch shows a section of the tiling (with grey lines) and a section of an expansion tiling (with black lines) the expansion tiling has transformed hypercube, and thus window centered at the origin. The tiling patches are drawn by *Mathematica*®. For some of the examples the substitution rules have been constructed. In these cases the substitution rule is also shown.

4.1 Tilings with the alternating condition

The first two examples are the tilings associated to the matrices:

$$M_1 = \begin{pmatrix} 3 & 2 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 0 & -2 & 3 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & -1 & 0 \\ 0 & -2 & 1 & 2 \\ 1 & 0 & 1 & 3 \end{pmatrix} \tag{27}$$

These tilings are topologically equivalent to tilings considered in [25], as they satisfy the alternating condition. This means that topologically equivalent tilings can be constructed using the edge matching rules for the Ammann tiling. In this setting we show that some such tilings can be associated to a substitution tiling.

The tilings are shown in Figures 15 and 16. Both these substitutions have scaling $2 + \sqrt{5}$.

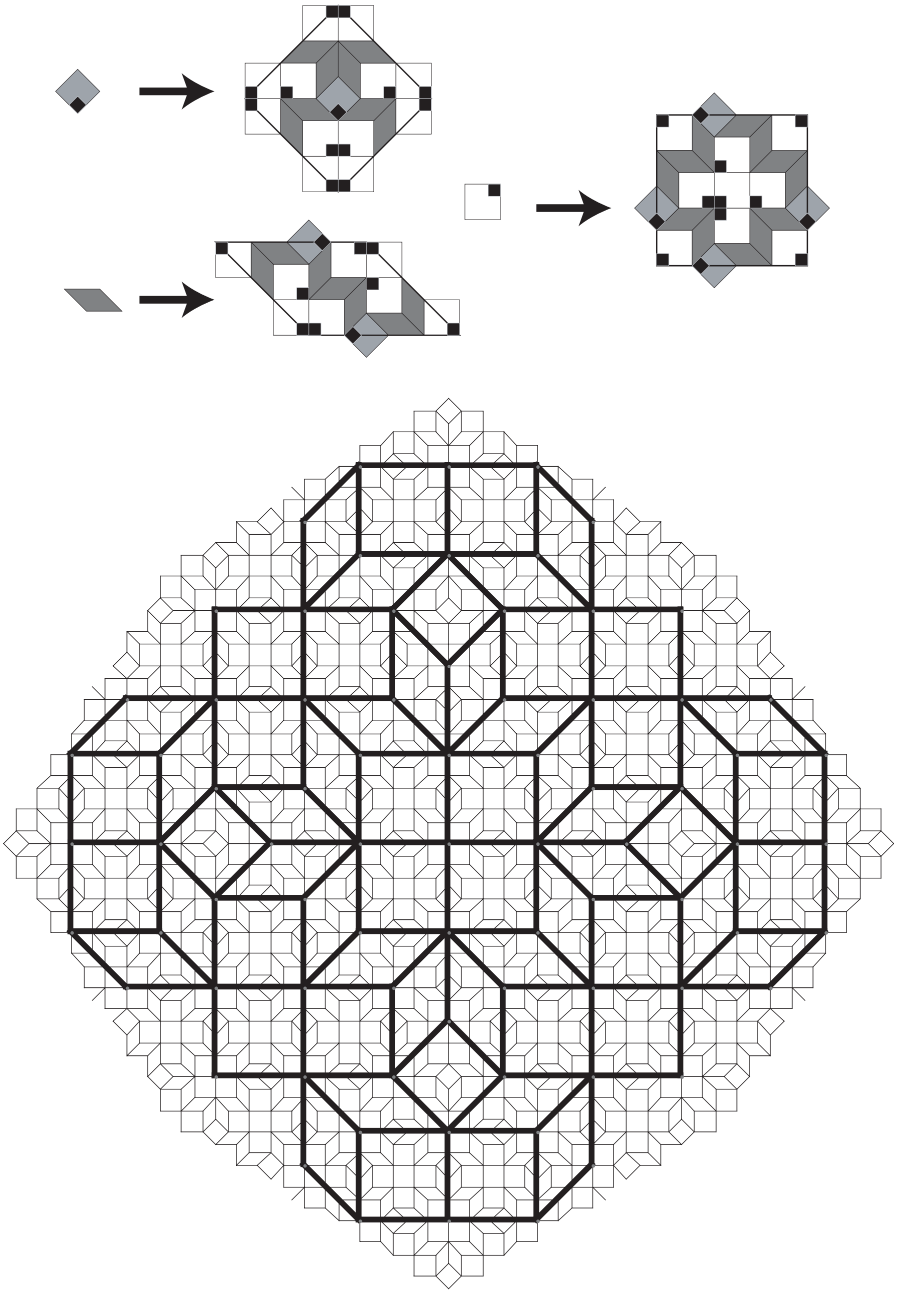


Fig. 15. Patch and substitution rules for the canonical substitution tiling with matrix M_1

4.2 Other tilings with two pairs of orthogonal edges

The second pair of examples are the tilings associated to the matrices:

$$M_3 = \begin{pmatrix} 2 & 1 & 0 & 2 \\ 1 & 2 & -2 & 0 \\ 0 & -2 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{pmatrix}, M_4 = \begin{pmatrix} 3 & 2 & 0 & 4 \\ 1 & 3 & -2 & 0 \\ 0 & -4 & 3 & 2 \\ 2 & 0 & 1 & 3 \end{pmatrix} \tag{28}$$

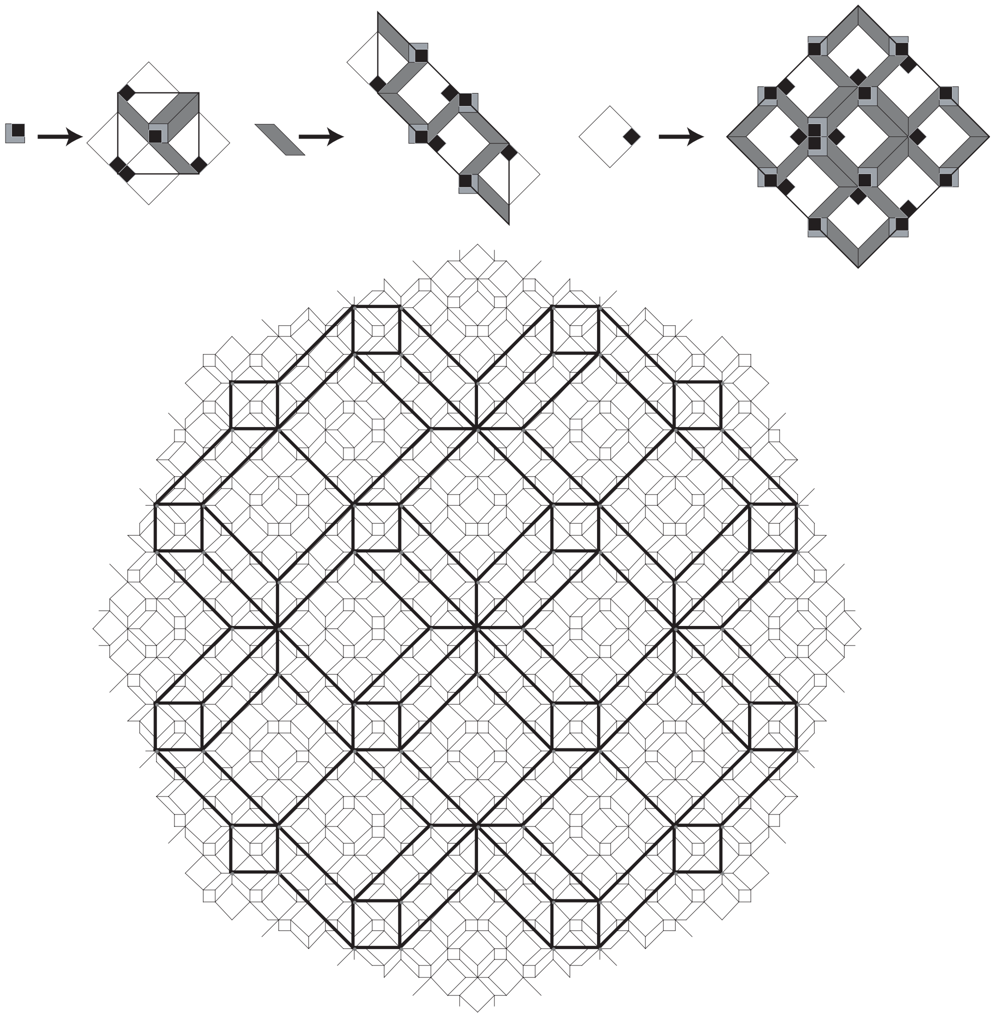


Fig. 16. Patch and substitution rules for the canonical substitution tiling with matrix M_2

Like the first two examples, the two examples in this section have two pairs of orthogonal edges. This means that they are invariant under a rotation of order 4. Unlike the Ammann tiling these tilings are not invariant under a rotation of order 8.

The tilings are shown in Figures 17 and 18. The substitution associated to M_3 has scaling $2+\sqrt{5}$, and the substitution associated to M_4 has scaling $7+\sqrt{53}$.

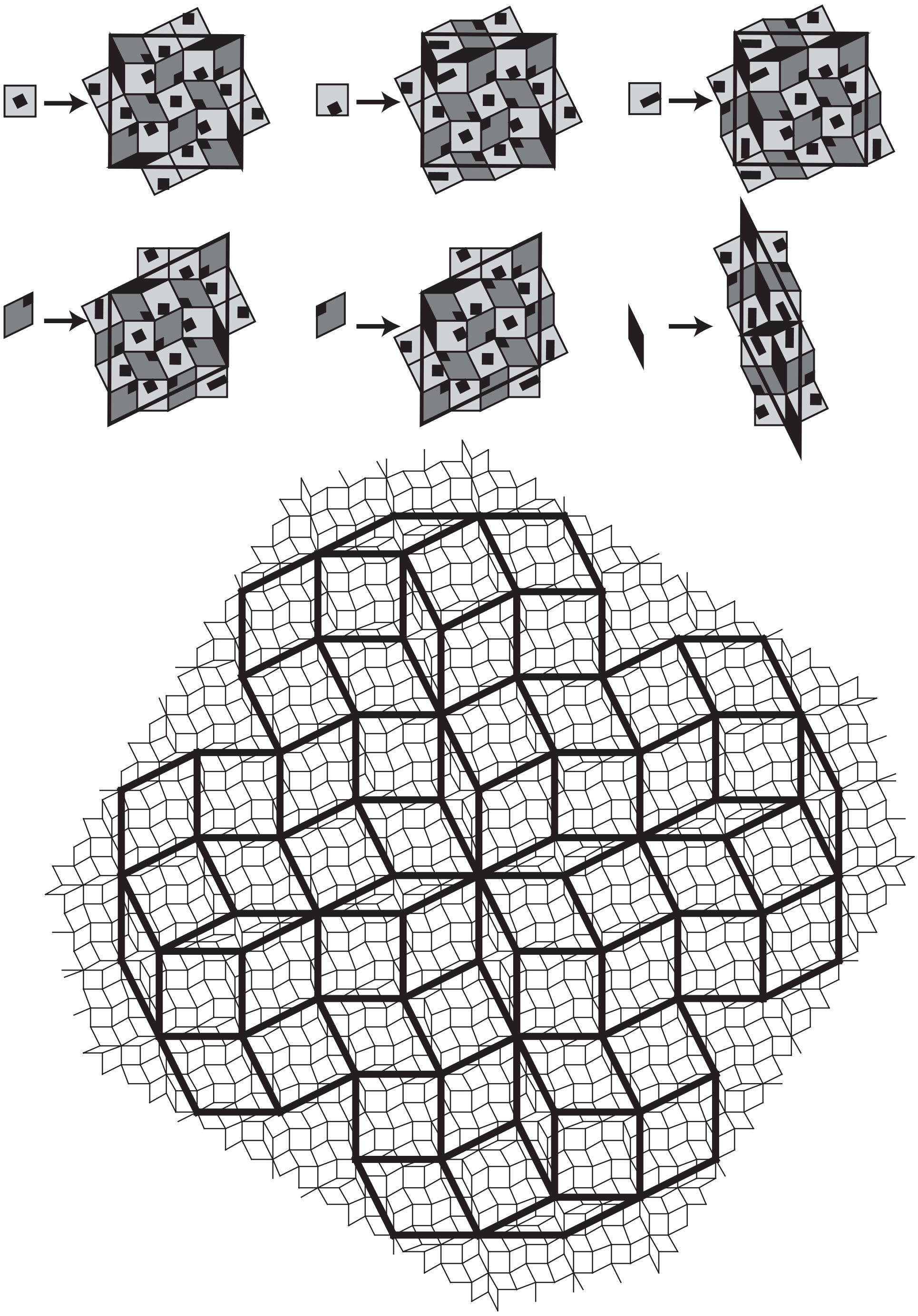


Fig. 17. Patch of canonical substitution tiling with matrix M_3

4.3 Further tilings

The final two examples are the tilings associated to the matrices:

$$M_5 = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 1 & 3 \end{pmatrix}, M_6 = \begin{pmatrix} 1 & 2 & 2 & -2 \\ 1 & 4 & 5 & -1 \\ 0 & 1 & 2 & 1 \\ -1 & 0 & 2 & 5 \end{pmatrix} \tag{29}$$

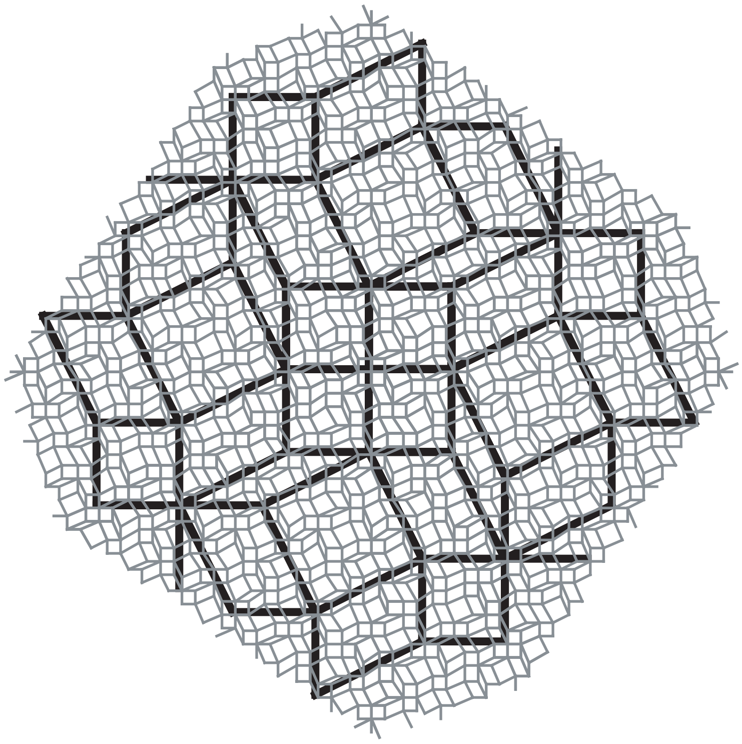


Fig. 18. Patch of canonical substitution tiling with matrix M_4

These tilings have very little symmetry other than the substitution rule. The only symmetry is a rotation of order 2. All the rhomb tilings discussed in this paper will have this symmetry, as it is a symmetry of any zonotope on the plane, the window is therefore invariant under this symmetry.

The tilings are shown in Figures 19 and 20. The substitution associated to M_5 has scaling $2+\sqrt{3}$, and the substitution associated to M_6 has scaling $3+2\sqrt{2}$.

Acknowledgements

It is a great pleasure to thank IMPA (Rio de Janeiro) for their hospitality during the visit of the authors in which this paper was written. The authors gratefully acknowledge the financial support of the UK EPSRC (through a PhD studentship (EOH) and an Advanced Research Fellowship (JSWL)) and the UK Royal Society (JSWL). We would like to take the opportunity to thank the referees, Michael Baake, and Peter Pleasants for providing useful comments and suggestions on the originally submitted version of this paper.

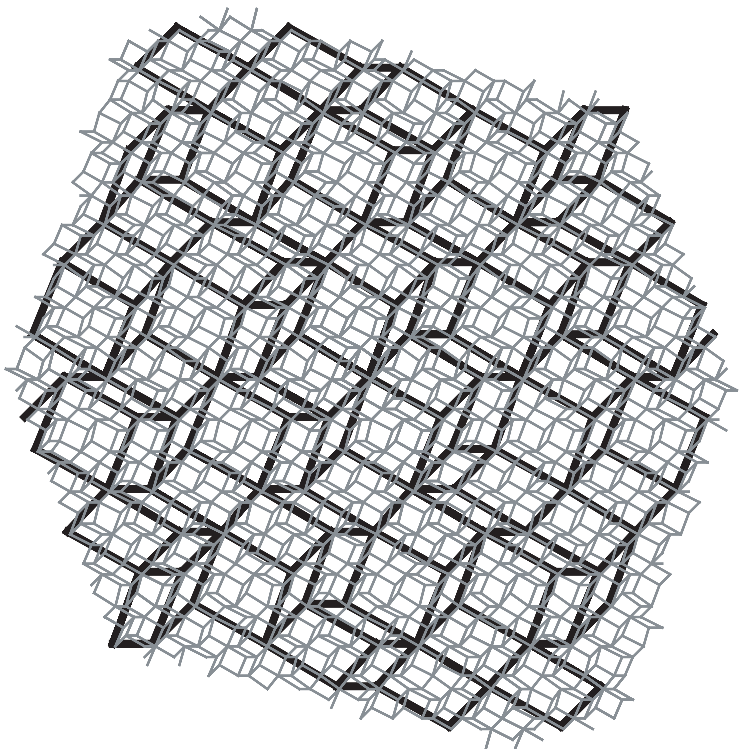


Fig. 19. Patch of canonical substitution tiling with matrix M_5

References

[1] Arnoux, P., Berthé, V., Ei, H., Ito, S., 2001. Tilings, quasicrystals, discrete planes, generalized substitutions, and multidimensional continued fractions. In: Discrete models: combinatorics, computation, and geometry (Paris, 2001). Discrete Math. Theor. Comput. Sci. Proc., AA. Maison Inform. Math. Discrèt. (MIMD), Paris, pp. 059–078 (electronic).

[2] Baake, M., Joseph, D., Kramer, P., 1991. The Schur rotation as a simple approach to the transition between quasiperiodic and periodic phases, J. Phys. A 24 (17), L961–L967.

[3] Baake, M., Joseph, D., Kramer, P., Schlottmann, M., 1990. Root lattices and quasicrystals, J. Phys. A 23 (19), L1037–L1041.

[4] Baake, M., Joseph, D., Schlottmann, M., 1991. The root lattice D_4 and planar quasilattices with octagonal and dodecagonal symmetry, Internat. J. Modern Phys. B 5 (11), 1927–1953.

[5] Baake, M., Moody, R. V. (Eds.), 2000. Directions in mathematical quasicrystals. American Mathematical Society, Providence, RI.

[6] Baake, M., Moody, R. V., 2000. Self-similar measures for quasicrystals. In: Directions in mathematical quasicrystals. Amer. Math. Soc., Providence, RI, pp. 1–42.

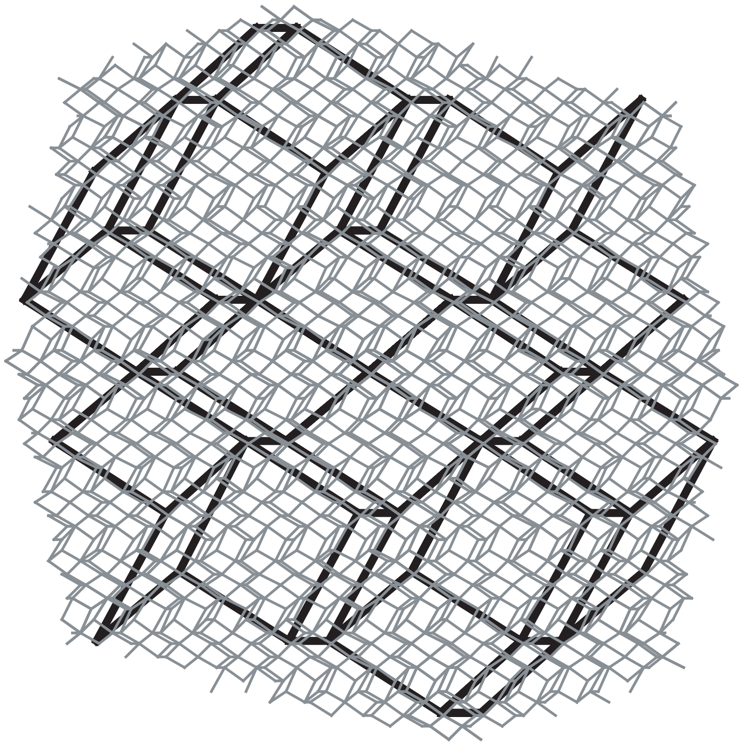


Fig. 20. Patch of canonical substitution tiling with matrix M_6

[7] Baake, M., Moody, R. V., 2001. Self-similarities and invariant densities for model sets. In: Algebraic methods in physics (Montréal, QC, 1997). CRM Ser. Math. Phys. Springer, New York, pp. 1–15.

[8] Baake, M., Moody, R. V., Schlottmann, M., 1998. Limit-(quasi)periodic point sets as quasicrystals with p-adic internal spaces. J. Phys. A 31 (27), 5755–5765.

[9] Baake, M., Schlottmann, M., Jarvis, P. D., 1991. Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability. J. Phys. A 24 (19), 4637–4654.

[10] Baranidharan, S., Gopal, E.S.R., Sasisekharan,V., 1992. Quasiperiodic tilings with fourfold symmetry, Acta Cryst. Sect. A 48 (5), 782–784.

[11] Beenker, F. P. M., 1982. Algebraic theory of non-periodic tilings of the plane by two simple building blocks: a square and a rhombus. TH-Report 82-WSK04, Eindhoven University of Technology.

[12] Berthé, V., Vuillon, L., 2000. Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences. Discrete Math. 223 (1-3), 27–53.

[13] Danzer, L., 1989. Three-dimensional analogs of the planar Penrose tilings and quasicrystals. Discrete Math. 76 (1), 1–7.

[14] de Bruijn, N. G., 1981. Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II. Nederl. Akad. Wetensch. Indag. Math. 43 (1), 39–52, 53–66.

[15] Forrest, A. H., Hunton, J. R., Kellendonk, J., 2002. Cohomology of canonical projection tilings. Comm. Math. Phys. 226 (2), 289–322.

[16] Gardner, M., 1977. Extraordinary nonperiodic tiling that enriches the theory of tiles. Sci. Am. November, 110–119.

[17] Goodman-Strauss, C., 1998. Matching rules and substitution tilings. Ann. of Math. (2) 147 (1), 181–223.

[18] Grünbaum, B., Shephard, G. C., 1987. Tilings and patterns. W. H. Freeman and Company, New York.

[19] Harriss, E. O., 2003. On canonical substitution tilings. PhD thesis, Imperial College London.

[20] Harriss, E. O., Lamb, J. S. W., 2003. Substitution rules for 2 to 1 interval projection tilings. In preparation.

[21] Haussler, D., Nissen, H.U., Lück, R., 1994. Dodecagonal tilings derived as duals from quasi-periodic ammann-grids, Phys. Status Solidi A-Appl. Res. 146, 425–435.

[22] Ingalls, R., 1992. Decagonal quasi-crystal tilings, Acta Crystallogr. Sect. A A48, 533–541.

[23] Ingalls, R., 1993. Octagonal quasi-crystal tilings, J. Non-Cryst. Solids 153, 177–180.

[24] Janner, A., 1995. Elements of a multimetrical crystallography. In: Beyond quasicrystals (Les Houches, 1994). Springer, Berlin, pp. 33–54.

[25] Katz, A., 1995. Matching rules and quasiperiodicity: the octagonal tilings. In: Beyond quasicrystals (Les Houches, 1994). Springer, Berlin, pp. 141–189.

[26] Katz, A., Duneau, M., 1986. Quasiperiodic patterns and icosahedral symmetry, J. Physique 47 (2), 181–196.

[27] Kramer, P., Neri, R., 1984. On periodic and non-periodic space fillings of $\mathbb{E}^m$ obtained by projection, Acta. Cryst. A40, 580–587.

[28] Lamb, J. S. W., 1998. On the canonical projection method for one-dimensional quasicrystals and invertible substitution rules. J. Phys. A 31 (18), L331–L336.

[29] Le, T. T. Q., 1997. Local rules for quasiperiodic tilings. In: The mathematics of long-range aperiodic order (Waterloo, ON, 1995). Vol. 489 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Kluwer Acad. Publ., Dordrecht, pp. 331–366.

[30] Lothaire, M., 2002. Algebraic combinatorics on words. Vol. 90 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, with a preface by Jean Berstel and Dominique Perrin.

[31] Luck, J. M., Godrèche, C., Janner, A., Janssen, T., 1993. The nature of the atomic surfaces of quasiperiodic self-similar structures. J. Phys. A 26 (8), 1951–1999.

[32] Lück, R., 1993. Basic ideas of Ammann bar grids, Internat. J. Modern Phys. B 7 (6-7), 1437–1453.

[33] Masáková, Z., Patera, J., Pelantová, E., 1998. Minimal distances in quasicrystals. J. Phys. A 31 (6), 1539–1552.

[34] Masáková, Z., Patera, J., Pelantová, E., 2000. Substitution rules for aperiodic sequences of the cut and project type. J. Phys. A 33 (48), 8867–8886.

[35] Pleasants, P. A. B., 2000. Designer quasicrystals: cut-and-project sets with pre-assigned properties. In: Directions in mathematical quasicrystals. Amer. Math. Soc., Providence, RI, pp. 95–141.

[36] Priebe, N., Solomyak, B., 2001. Characterization of planar pseudo-self-similar tilings. Discrete Comput. Geom. 26 (3), 289–306.

[37] Pytheas Fogg, N., 2002. Substitutions in dynamics, arithmetics and combinatorics. Vol. 1794 of Lecture notes in mathematics. Springer.

[38] Radin, C., 1994. The pinwheel tilings of the plane. Ann. of Math. (2) 139 (3), 661–702.

[39] Sano, Y., Arnoux, P., Ito, S., 2001. Higher dimensional extensions of substitutions and their dual maps. J. Anal. Math. 83, 183–206.

[40] Séébold, P., 1998. On the conjugation of standard morphisms. Theoret. Comput. Sci. 195 (1), 91–109, mathematical foundations of computer science (Cracow, 1996).

[41] Socolar, J. E. S., 1989. Simple octagonal and dodecagonal quasicrystals. Phys. Rev. B (3) 39 (15), 519–551.

[42] Soma, T., Watanabe, Y., 1992. A class of patterns generated by modification of Beenker's pattern, Acta Crystallogr. A48, 470–475.

[43] Wen, Z. X., Wen, Z. Y., 1994. Local isomorphisms of invertible substitutions. C. R. Acad. Sci. Paris Sér. I Math. 318 (4), 299–304.