USC-97/006
NSF-ITP-97-055
hep-th/9705237

Investigating the BPS Spectrum of Non-Critical E_n Strings

J.A. Minahan, D. Nemeschansky

Physics Department, U.S.C.
University Park, Los Angeles, CA 90089

and

N.P. Warner

Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106-4030 *

We use the effective action of the E_n non-critical strings to study its BPS spectrum for $0 \leq n \leq 8$. We show how to introduce mass parameters, or Wilson lines, into the effective action, and then perform the appropriate asymptotic expansions that yield the BPS spectrum. The result is the E_n character expansion of the spectrum, and is equivalent to performing the mirror map on a Calabi-Yau with up to nine Kähler moduli. This enables a much more detailed examination of the E_n structure of the theory, and provides extensive checks on the effective action description of the non-critical string. We extract some universal (E_n independent) information concerning the degeneracies of BPS excitations.

USC-97/006 NSF-ITP-97-055
May, 1997

* On leave from Physics Department, U.S.C., University Park, Los Angeles, CA 90089

1. Introduction

The E_n non-critical strings are ubiquitous in the formulation of non-perturbative string theory, and understanding these highly non-trivial fixed points is becoming of increasing importance. Even though there are many ways of characterizing such strings [1,2,3], there is, as yet, no explicit action or intrinsic formulation. Descriptions of such non-critical strings are either based upon classical solutions of low energy effective actions, or involve interpolating between branes, or wrapping branes around vanishing cycles. We will consider the string in its incarnation in the type IIA theory (or M-theory) compactified to four (or five) dimensions on an elliptically fibered Calabi-Yau manifold that is also a $K3$ fibration. This corresponds to the (six-dimensional) non-critical string compactified to four (five) dimensions on a torus (circle). The non-critical string emerges, *i.e.* has excitations of low mass, when a 4-cycle in the Calabi-Yau manifold becomes extremely small [2,4]. The magnetic non-critical string is obtained by wrapping a five-brane around the collapsing 4-cycle, while the dual electric states of the non-critical string are obtained by wrapping the membrane around 2-cycles within the 4-cycle. If the collapsing 4-cycle is a B_n del Pezzo surface then the string is endowed with an E_n symmetry: the del Pezzo surface has $(n+1)$ 2-cycles, n of which are acted upon naturally by the Weyl group of E_n. The remaining 2-cycle is the canonical divisor of the del Pezzo, whose Kähler modulus, k_D, determines the scale of the del Pezzo, and hence determines the string tension. Associated with the non-critical string is the anti-self-dual tensor multiplet, which in four (five) dimensions gives rise to a vector multiplet. The vanishing canonical 2-cycle in the del Pezzo provides the harmonic 2-form need to make the $U(1)$ gauge field strength.

This compactification of the string has two natural phases that are separated by a flop transition, and are referred to as phases I and II in [5]. They are defined as follows: In addition to the Kähler modulus k_D, there is the Kähler modulus, k_E, of the elliptic fiber of the Calabi-Yau manifold. At $k_D = 0$ only a 2-cycle collapses. In order to collapse the 4-cycle one has to pass to $k_D < 0$ and go to the point where $k_D + k_E = 0$ [4]. Phase I is the region with $k_D \geq 0$, and this phase connects directly to the weakly coupled heterotic theory. Phase II is the region for which $-k_E \leq k_D \leq 0$. Common to both phases is the strong coupling singularity where $k_D = 0$, which corresponds to an $SU(2)$ gauge theory. In phase I, one can view the low mass sector as this gauge theory coupled to $N_f = 8$ hypermultiplets associated with the other 2-cycles of the del Pezzo surface. There is a Coulomb modulus and an effective coupling, and the latter is independent of the former.

From the point of view of the five dimensional $SU(2)$ gauge theory, phase II is the regime where one of the hypermultiplet masses is taken to be large compared to the expectation value of the scalar in the vector multiplet. For most of this paper, we will assume that this hypermultiplet mass is infinite, leaving an effective $SU(2)$ gauge theory with $N_f < 8$ fundamental matter. The various E_n theories then correspond to $N_f = n-1$. There is an extra mass parameter that appears because of the existence of a soliton with $SU(2)$ gauge charges for the five dimensional theory and this mass parameter combines with the bare mass parameters for the fundamental matter to fill out the full E_n global symmetry.

For the Calabi-Yau compactification of the IIA theory, the five dimensional gauge theories are compactified to four dimensions on a circle of radius R_5. Since these are $N = 2$ theories with a one-dimensional coulomb branch, there should exist Seiberg-Witten elliptic curves whose modulus is the effective coupling for the theory. In this paper we will construct the explicit curves for all values of $N_f < 8$. These curves have an R_5 dependence which generalizes the results in [6]. Varying R_5 from small radius to large radius interpolates between the four dimensional superconformal theories in [6] and the five dimensional $SU(2)$ gauge theories discussed in [7].

In the underlying string theory, phase II is rather more exotic than phase I, and was the focus of [5]. In particular, it is the phase in which one can directly access the perturbative non-critical string. One can view k_D and k_E as different combinations of the tension of the string and the compactification radii. In particular, if a membrane wraps the elliptic fiber with degree d_E, and the canonical divisor with degree d_D, then these integers represent the winding number and momentum, respectively, of the compactification of the non-critical string on a circle. The remaining Kähler moduli of the del Pezzo 2-cycles can then be interpreted as Wilson line parameters on the circle of compactification. For $d_E > 0$ one finds that the only BPS membranes have $d_E \geq d_D$. The states with $d_E = d_D = d$ are those that become massless when the 4-cycle collapses when $k_D + k_E = 0$. This is an infinite tower of states, indexed by d, and they should represent a fundamental "electric" representation of the non-critical string. It is this infinite tower that we will study extensively here.

To study these very stringy BPS states, one can use the mirror map as in [4], and obtain the degeneracies of the states. So far only the lowest level of the excitations ($d_E = 1$) has been adequately understood [4]. The problem is that it is very hard to extract detailed information about the E_n structure from a mere count of the number of BPS states at a given level. One of the purposes of this paper is to describe and utilize a technique that provides much more precise and detailed information about the E_n structure of the spectrum. We will also extract some apparently universal data about the degeneracies.

One could, in priniciple accomplish this by introducing the Kähler moduli corresponding to the Wilson lines, and then performing the mirror map. However, we will describe a much simpler approach that involves passing first to a form of consistent truncation of the type II compactification down to the essential sector of the non-critical string [5].

The basic idea of [5] was to isolate the non-critical string as a closed monodromy subsector of the type II compactification. That is, one takes the view that one can consistently truncate a theory to a subsector if one isolates a set of BPS states and moduli so that monodromies over that moduli space have a closed representation on the selected BPS states. In [5] it was proposed that one could exactly capture and model the non-critical E_n strings by using some very particular non-compact Calabi-Yau manifolds to "compactify" the IIB theory. The approach is directly parallel to the manner in which IIB compactification on ALE fibrations over $\mathbb{P}^1$ captures the exact quantum effective action of $N = 2$ supersymmetric gauge theories (completely decoupled from the rest of the original type II string theory) [8]. The claim is that by using the proper non-compact manifolds, one can study *in isolation* the non-critical strings decoupled from the "superfluous" excitations of the original and larger string theory in which the non-critical string appeared. Such a IIB compactification gives rise to an "effective action" for the non-critical string. The natural expectation for such an effective action is that it will describe the Coulomb branch of the gauge theory and indeed, such field theory actions were constructed in [6,9,7]. However, getting the effective action via a IIB compactification leads to a much deeper stringy insight as in [8]: The BPS states appear as 3-branes wrapping 3-cycles, but one can "see" the string by first wrapping the 3-branes over 2-cycles. The effective action is constructed from the period integrals of the holomorphic $(3,0)$-form on 3-cycles. For the non-compact Calabi-Yau manifolds of [5], these integrals can be reduced to integrals of a meromorphic $(1,0)$-form, or generalized Seiberg-Witten differential, on a torus. This torus can then be thought of as a compactifying space of the string, and the Seiberg-Witten differential represents a local string tension. Apart from satisfying some of the basic properties of their their gauge theory counterparts, this formulation of the quantum effective actions of the non-critical strings also has some fundamentally new features: For example, the differential (local string tension) vanishes indentically at one value of the modulus (the tensionless string point) and the asymptotic expansion of these actions yields a generating function for the counting of BPS states.

It is important to remember that this formulation of the non-critical string is not strictly derived from other formulations: it was proposed for rather general reasons, and this proposal has been checked against the results from the mirror map. One of the purposes of this paper is to perform extensive further verification of this proposal by including Wilson lines for the compactified non-critical string. In the effective action these parameters become masses for the hypermultiplets for the $SU(2)$ gauge theory. We are thus able use the techniques introduced in [10] and developed in [6]. We find that these masses enter the counting of BPS states in precisely the same manner as multiple Kähler moduli appear when one uses mirror symmetry to count rational curves. Thus at one level we have found an extremely efficient method of computing the mirror map (with up to nine parameters) by working on a torus. This not only enables the counting of curves in terms of E_n characters, but also enables us to study the flow of the effective action as one moves from E_n to E_{n-1}. The fact that the asymptotic forms of effective actions flows in exactly the proper manner provides strong confirmation of our results and of the proposal of [5].

In section 2 we briefly review the results of [5], focussing on how the effective action of the non-critical string is computed and reduced to period integrals on a torus. We also generalize this effective action to include two or three mass parameters. In section 3 we compute the instanton expansion from the effective action, and show how this expansion is refined into characters of E_n. In section 4 we focus on the E_8 theory, and reverse the previous philosophy by using the E_8 structure of the instanton expansion to determine the exact form of the torus for the complete set of eight mass parameters. In section 5 we make a much more extensive study of the characters that appear in the instanton expansion, determining degeneracies of E_n Weyl orbits up to curves of degree 5 for $n = 8$, degree 6 for $n = 6, 7$ and degree 10 for $n = 5$. We also show how one can flow from the E_8 theory down to any E_n, and that the instanton expansion behaves appropriately under this flow. We then extract some universal (E_n independent) information about degeneracies from this data. In section 6 we show how our methods should generalize to yield an effective action that includes another modulus, and this should enable the computation of the full set of excitations of the non-critical string given in [4]. Section 7 contains some brief concluding remarks. There are two appendices: The first contains details of how the tori and Seiberg-Witten differentials for the massive theories were computed. The second contains the explicit formulae for the tori for E_n, $0 < n \leq 8$.

2. The Effective action of the Non-Critical String

2.1. The massless theory

To compute the effective action of the non-critical, E_n string it was argued in [5] that one should compute the classical pre-potential of a IIB compactification on particular non-compact Calabi-Yau 3-folds that depends upon the choice of E_n. Here we will restrict our attention to E_8, but the calculation for other E_n proceeds similarly.

The appropriate 3-fold is given by the following polynomial in weighted projective space:

$$w^2 \ = \ z_1^3 \ + \ z_2^6 \ + \ z_3^6 \ - \ \frac{1}{z_4^6} \ - \ \psi \, w z_1 z_2 z_3 z_4 \ . \tag{2.1}$$

As was described in [5], the Seiberg-Witten differential for the underlying $SU(2)$ gauge theory can then be computed by (partially) integrating the holomorphic 3-form, $\Omega_{(3)}$, of this surface over suitably chosen 3-cycles. One can then express the result as

$$\begin{aligned} \lambda_{\rm SW} \ &= \ \left(\int^{\psi} \frac{d\zeta}{\sqrt{1 + x^3 + \frac{1}{4}\zeta^2 x^2}} \right) dx \\ &= \ \frac{1}{2} \ \log \ \left[\frac{\sqrt{1 + x^3 + \frac{1}{4}\psi^2 x^2} \ + \ \frac{1}{2}\psi x}{\sqrt{1 + x^3 + \frac{1}{4}\psi^2 x^2} \ - \ \frac{1}{2}\psi x} \right] \ \frac{dx}{x} \ . \end{aligned} \tag{2.2}$$

One should interpret $\lambda_{\rm SW}$ as a differential on the curve $y^2 = 1 + x^3 + \frac{1}{4}\psi^2 x^2$.

One can easily make a direct connection between this and the approach of [6], which is based on the more standard description of the E_8 torus. Make the change of variables:

$$x \ \to \ 2^8 \ \psi^{-10} \ x \ , \quad y \ \to \ 2^{12} \ \psi^{-15} \ y \ , \quad u \ \equiv \ -\tfrac{1}{32} \ \psi^6 \ . \tag{2.3}$$

One then obtains the curve

$$y^2 \ = \ x^3 \ - \ 2 \ u^5 \ + \ u^2 x^2 \ , \tag{2.4}$$

while the differential that is being integrated in (2.2) becomes

$$\Omega_{(2)} \ = \ \frac{dx \ du}{\sqrt{x^3 - 2u^5 \ + \ u^2 x^2}} \ . \tag{2.5}$$

This is very close to the starting point of [6] where the Seiberg-Witten differential is constructed by writing such a 2-form as the exterior derivative of a 1-form. Here we see that the corresponding 2-form naturally appears in the partial integration of the Calabi-Yau holomorphic 3-form. There are however some fundamental differences: the curve (2.4) contains an extra term compared to that of [6]. Upon shifting $x \to x - \frac{1}{3}u^2$ one obtains a u^6 term, that is more characteristic of the elliptic singularity, rather than of the E_8 singularity. This, combined with the fixed normalization of $\Omega_{(2)}$, also leads to a Seiberg-Witten differential that has irremovable logarithmic branch cuts (2.2) [5].

The Calabi-Yau manifold and the torus described above only depend upon one complex modulus. As was described in [5], this modulus corresponds to the Kähler modulus $t_S = i(k_D + k_E)$ of the IIA compactification. The simplest closed sub-monodromy problem is the truncation to the study of this single modulus in phase II. It is also precisely this modulus that one needs to characterize and count the fundamental massless tower of electric states with $d_E = d_D = d$. We will discuss in section 6 how to restore the second modulus to the foregoing model.

This brings us to an important technical point: We have specialized to a one parameter closed sub-monodromy problem based upon a single complex structure modulus, ψ. As was described in [5], such a single parameter truncation only exists in phase II. On the other hand we are ultimately going to look at the large complex structure limit and this corresponds taking the string tension to infinity. If one is in phase II and one takes this limit, then one necessarily crosses the boundary into phase I, *unless* one is at a point in moduli space where this boundary has been shifted infinitely far away. This means that the foregoing Calabi-Yau manifold and torus must describe a rather singular limit of the IIA compactification: one in which k_E has been shifted off to infinity. In terms of the toroidal compactification of the non-critical string, the ratio R_5/R_6 has been taken to infinity in such a manner that $\phi R_5 R_6$ remains finite, where ϕ is the non-critical string tension [1]. One can view this limit as degenerating the six-dimensional theory to five dimensions, and then compactifying the theory to four dimensions on a circle of radius R_5. Thus there is only one scale left in the theory, namely $1/R_5$.

It is important to highlight the unusual but crucial form of (2.2). As was emphasized in [5], for $\psi = 0$, the differential $\lambda_{\rm SW}$ vanishes identically over the entire Riemann surface. This is essential since the BPS states become massless when the 4-cycle collapses. Moreover, if one were to obtain the Seiberg-Witten differential by integrating the holomorphic differentials, then apart from normalization issues, the boundary condition that $\lambda_{\rm SW}$ must vanish at $\psi = 0$ provides a constant of integration that is crucial to the proper instanton

[1] In terms of the torus of [9], we have specialized to the point in moduli space with $\sigma = i\infty$.

expansion at $\psi = \infty$. The unusual feature in (2.2) is the presence of the logarithm, and the logarithmic branch cuts, which imply that it is multi-valued on the torus. Given the geodesic description of BPS states [8] the multi-valuedness, at first sight, seems extremely surprising. However, one should recall the rather singular limit that one has implicitly taken, and use the fact that the only mass scale in the problem is $1/R_5$. This scale must multiply (2.2). The multi-valuedness of the logarithm then implies that on the N^{th} sheet the differential, λ_{SW}, has a simple pole of residue $2\pi i N/R_5$. Following the rules of [10] this means that there must be an infinite tower of hypermultiplet states of masses $2\pi i N/R_5$. These are simply the Kaluza-Klein modes of the string on the compactified R_5. If the six-dimensional theory were compactified on a non-degenerate torus then the Seiberg-Witten differential must involve the inverse of a doubly periodic function, that is an inverse elliptic function whose τ-parameter is that of the torus upon which the compactification is made (see section 6).

The logarithmic branch cuts also play an important role in the Seiberg-Witten differential for the model with Wilson lines: The differential must have residues that are linear in the masses, or Wilson line parameters, m_i, while the parametrization of the relevant algebraic surface must respect the periodicity of the Wilson line parameter space, *i.e.* $m_i \to m_i + 2\pi$. This means that the coefficient functions in the Seiberg-Witten differential must involve inverse trigonometric functions (or inverse elliptic functions for the toroidal compactification). This is exactly what one finds in (2.2).

2.2. Incorporating masses: first iteration

Following [10,6] one builds the model with non-zero masses by making deformations of (2.4), and seeking the lines in the surface defined by (y, x, u). The Seiberg-Witten differential is determined by finding λ_{SW} such that $\Omega_{(2)} = d\lambda_{\text{SW}}$ on this surface with the lines excised. There is still some ambiguity in this process, but this is resolved by requiring that λ_{SW} has the proper (Weyl invariant) residues.

If one introduces p mass parameters, the E_8 symmetry is broken to $SO(16-2p)$, and this means that the discriminant of the curve must behave as $\sim u^{10-p}$ as $u \to 0$. For three or fewer masses, the general form of the curve is not very complicated. Consider the limit $R_5 = 0$. The curves in this limit were constructed in [6] and will henceforth be referred to as the *polynomial* curves because of their polynomial dependence on the masses. Correspondingly, we will refer to the curves that we are about to construct as the *trigonometric* curves. One can make an Ansatz that all coefficients of u and x that are absent in the polynomial case remain so for the trigonometric curves, with the exception of the coefficient of the u^2x^2 term, which is set to 1. The non-zero terms need to be modified, but this is done such that a Seiberg-Witten differential can be constructed whose residues are linear in the mass parameters m_i. Details of these constructions for up to two non-zero masses, along with the corresponding Seiberg-Witten differential are given in appendix A.

The two mass curve has a particularly simple form and is given by

$$y^2 = x^3 + u^2x^2 - 2u(u^2 + \sin^2(m_+)x)(u^2 + \sin^2(m_-)x), \tag{2.6}$$

where $m_\pm = (m_1 \pm m_2)/2$. This can be compared with the E_8 polynomial curve

$$y^2 = x^3 - 2u(u^2 + m_+^2 x)(u^2 + m_-^2 x), \tag{2.7}$$

For three non-zero masses we find that the curve still has the simple form

$$y^2 \;=\; x^3 \;+\; u^2x^2 \;-\; u\,\Big(\,2u^4 \;+\; T_2\,u^2\,x \;+\; 2\widetilde{T}_4\,x^2\Big) \;-\; T_6\,u^4\;, \tag{2.8}$$

where

$$\begin{aligned}
&T_2 \;\equiv\; \sum_{i=1}^{4} \sin^2(p_i)\;, \qquad \widetilde{T}_4 \;\equiv\; \prod_{i=1}^{4} \sin(p_i)\;, \qquad T_6 \;\equiv\; \prod_{i=1}^{3} \sin^2(m_i)\;;\\
&p_1 \;\equiv\; \tfrac{1}{2}(m_1 - m_2 - m_3)\;, \qquad p_2 \;\equiv\; \tfrac{1}{2}(-m_1 + m_2 - m_3)\;,\\
&p_3 \;\equiv\; \tfrac{1}{2}(-m_1 - m_2 + m_3)\;, \qquad p_4 \;\equiv\; \tfrac{1}{2}(m_1 + m_2 + m_3)\;.
\end{aligned} \tag{2.9}$$

The Seiberg-Witten differential for the two-mass curve can be found in Appendix A. Its form is rather complicated, and indeed we will not need it directly – we will only need the asymptotic form of (2.2).

3. The instanton expansion

One of the interesting things about the effective action defined in [5] is its behaviour at large u. Following [10] one defines

$$\begin{aligned}
\phi(u) \;&=\; \int_{\gamma_a} \lambda_{\rm SW} \;=\; \int \Big(\int_{\gamma_a} \omega\Big)\, du \;+\; \delta\\
\phi_D(u) \;&=\; \int_{\gamma_b} \lambda_{\rm SW} \;=\; \int \Big(\int_{\gamma_b} \omega\Big)\, du \;+\; \delta_D\;,
\end{aligned} \tag{3.1}$$

where $\omega = dx/y$ is the holomorphic differential on the torus (2.4), and δ, δ_D are integration constants. The constants of integration are crucial to the asymptotic expansion at infinity, and can be determined by a careful asymptotic expansion of the period integrals of $(2.2)^2$, or equivalently by analytically continuing and imposing the requirement that ϕ and ϕ_D vanish at $\psi = 0$.

In [5] it was shown that the Yukawa coupling, $C_{\phi\phi\phi} = \partial^3_\phi \mathcal{F}$, was exactly the one obtained in [4]:

$$-1 \ + \ 252 \ \frac{1^3 q^1}{1-q^1} \ - \ 9252 \ \frac{2^3 q^2}{1-q^2} \ + \ 848628 \ \frac{3^3 q^3}{1-q^3} \ - \ 114265008 \ \frac{4^3 q^4}{1-q^4} \ + \ \dots \ . \quad (3.2)$$

The corresponding pre-potential, $\mathcal{F}$, was also shown to be:

$$\mathcal{F} \ = = \frac{1}{6}\phi^3 \ + \ \frac{1}{4}\phi^2 \ - \ \frac{5}{12}\phi \ + \ \frac{1}{4\pi^2} \sum_{k=1}^{\infty} n^r_k \, Li_3(e^{2\pi i k\,\phi}) \ , \quad (3.3)$$

where the instanton coefficients $n^r_k = \{252, -9252, ..\}$. The fact that the torus (2.4) and the differential (2.2) replicate the counting of BPS states of the non-critical string provided confirmation that the foregoing does indeed provide a model of the non-critical string.

We now describe in a little more detail how to compute the instanton expansion from the torus, but this time we include the two or three non-zero mass parameters by using the torus (2.6) or (2.8). The first step is to recast the torus in canonical form:

$$\begin{aligned} y^2 \ &= 4x^3 \ - \ g_2(\sigma)\, x \ - \ g_3(\sigma) \ , \\ g_2(\sigma) \ &= 60 \ \omega_2^{-4} \ G_4(\sigma) \ , \quad g_3(\sigma) \ = \ 140 \ \omega_2^{-6} \ G_6(\sigma) \ , \\ G_{2k}(\sigma) \ &\equiv \ \frac{2(2\pi i)^{2k}}{(2k-1)!} \ \Big[\sigma_{2k-1}(n) \ q^n \ \Big] \ , \end{aligned} \quad (3.4)$$

where G_{2k} are the canonically normalized Eisenstein functions, ω_2 is one of the torus periods and $q = e^{2\pi i\sigma}$. The other torus period is thus $\omega_1 = \sigma\omega_2$. This gives one expressions for g_2 and g_3 in terms of u and the m_i, and substituting these into:

$$\begin{aligned} j(\sigma) \ &= \ \frac{1728 \ g_2^3}{g_2^3 - 27 g_3^2} \\ &= \ \frac{1}{q \ \prod_{n=1}^{\infty} \ (1-q^n)^{24}} \ \Big[\ 1 \ + \ 240 \ \sum_{n=1}^{\infty} \ \sigma_{2k-1}(n) \ q^n \ \Big]^3 \ , \end{aligned} \quad (3.5)$$

[2] The key observation to getting the asymptotic expansion correct is that the log branch cuts must run through the square-root branch cuts thereby connecting the log branch points on different y-sheets. The curve γ_a must not cross the log cut. All this is required to have the proper $\psi \to 0$ limit, and in the $\psi \to \infty$ limit it gives ϕ a $\log(\psi)$ divergence.

yields a relation between σ, u and the m_i. One can then expand this in a series for large u, or $\sigma \to i\infty$, and invert it to get an expansion for σ in terms of u and the m_i. Using this in G_4 in (3.4) yields an expansion for ω_2, and hence ω_1 in terms of u and the m_i. To get $\phi(u)$ of (3.1) one integrates ω_2 with respect to ψ, or u, (using the constant of integration of [5]). Inverting this one can finally determine σ, and hence $C_{\phi\phi\phi}$ as a function of ϕ and m_i.

The result is a series like (3.2) but with the integer coefficients replaced by polynomials in $\sin(m_i)$ or $\sin(p_i)$. These polynomials are then easily recognized in terms of characters of E_8, or more precisely of characters of the $SO(2k)$ subgroup of E_8 defined by the non-zero m_i, $i = 1, \ldots, k$. This information is more than adequate to reconstruct the complete E_8 characters of the first few terms of the expansion – indeed it is a highly overdetermined system providing quite a number of consistency checks.

The first few terms become

$$\begin{aligned} -1 \,+\, 1^3 \, &\big[12\ \chi_{0,1}(q;m_i) + \chi_{2,240}(q;m_i)\ \big] \\ -\ 2^3 \, &\big[\ 132\ \chi_{0,1}(q^2;m_i) + 20\ \chi_{2,240}(q^2;m_i) + 2\ \chi_{4,2160}(q^2;m_i)\ \big] \\ +\ 3^3 \, &\big[\ 4068\ \chi_{0,1}(q^3;m_i) + 927\ \chi_{2,240}(q^3;m_i) + 180\ \chi_{4,2160}(q^3;m_i) + \\ &\qquad 27\ \chi_{6,6720}(q^3;m_i) + 3\ \chi_{8,17280}(q^3;m_i)\ \big]\ +\ \ldots\ , \end{aligned} \tag{3.6}$$

where

$$\chi_{p,k}(q;m_i)\ \equiv\ \sum_{\vec{v}\in\mathcal{O}_{p,k}}\ \frac{q\ e^{2i\vec{v}\cdot\vec{m}}}{1\ -\ q\ e^{2i\vec{v}\cdot\vec{m}}}\ . \tag{3.7}$$

In this expression, $\vec{v}$ is summed over the set $\mathcal{O}_{p,k}$, consisting of vectors with length-squared p, that lie in a single Weyl orbit of order k on the root lattice of E_8.

One of the interesting features of (3.6) is the form of the terms that subtract of the multiply wrapped rational curves of lower degree. In (3.2) this subtraction was performed by the denominators $(1-q^n)$, whereas in (3.6) these denominators have been replaced by $(1-q^n e^{2i\vec{v}\cdot\vec{m}})$ for a particular $\vec{v}$ in a Weyl orbit. Expanding this denominator leads to Weyl orbit characters that are evaluated at ℓm_i, where ℓ is the multiplicity of the wrapping of the fundamental rational curve. Thus the character parameter properly reflects the multiple wrapping. Indeed, the form of (3.7) is precisely the proper form for effective potential obtained from a mirror map in which σ and the m_i are Kähler moduli.

Thus far we have only needed the curve with three non-zero Wilson lines. To get the complete curve we now reverse the foregoing procedure and determine the curve by requiring that the higher terms in the instanton expansion have the proper E_8 structure

4. Deriving the curves from the instanton expansion

In the last section we saw that, in principle, the Seiberg-Witten curves can be derived by looking for the holomorphic lines. In practice, this can be carried out for a small number of masses, but becomes exceedingly difficult beyond three masses.

We could in principle also derive the curves by solving the linear equations in [9] and then taking the appropriate limit to reduce everything to the five-dimensional theory. Unfortunately, this too cannot be easily carried out.

We propose another way to compute the curves, which takes advantage of the instanton expansion. This turns out to be an efficient method. We find that the curves are much simpler than one would expect for the full six-dimensional theory, and in fact are not much more complicated than the curves found in [6] for the polynomial mass cases.

Our strategy is to compute the instanton expansion for a curve with unknown coefficients, assuming that the curve has the correct polynomial limit. We then assume that the instanton expansion will lead to an expansion in characters for the appropriate E_n group. As we saw in the previous section, the character expansion is consistent up to two or three masses. It will turn out that we will only need to impose some rather simple constraints arising from general E_n character requirement.

We will do the calculation for the E_n theories with $n \leq 8$. One could derive each curve individually, or one could start with the E_8 curve and reduce to the lower cases by taking the masses to a certain limit. Doing the calculation both ways provides some useful consistency checks.

To compute the E_8 curve, there is a useful trick that we can employ. Once a curve has been obtained, it is a straightforward procedure to derive its instanton expansion. For the lower E_n, we can compute the instanton expansion directly from its curve, or we can derive it from the E_8 instanton expansion [3]. Since the lower E_n instanton expansions can be computed directly from the E_8 expansion, all coefficients in front of characters also appear in the E_8 expansion (although the reverse is not true). This means that some (although not all) of the E_8 coefficients can be determined from the lower E_n. This is useful since the E_8 instanton computation is much more intensive than the lower expansions.

We will also find that there is a duality in the character expansions, which is somehow related to the T-duality of the original 2-torus. This will be described further in section 5.

[3] The E_6 del Pezzo is of particular interest since it is equivalent to the space of cubics in $\mathbb{CP}_3$, so the instanton expansion is giving us information about the holomorphic curves on this surface.

4.1. The curve for E_8

The E_8 polynomial curve was derived in [6] and is given in Appendix B. The coefficients of the curve are written in terms of $SO(16)$ invariants T_n, where

$$T_{2n} = \sum_{i_1<i_2..i_n}^{8} m_{i_1}^2 .. m_{i_n}^2$$

and the m_i are some bare mass parameters. As we saw for the curve with a only a few non-zero masses, the curve away from the polynomial limit has the coefficients replaced with polynomials of trigonometric functions of the bare masses. A convenient basis for these functions are the set of T_{2n} defined by taking

$$\begin{aligned} t_8 &= \prod_{i=1}^{8} \sin m_i, \qquad T_{16} = (t_8)^2 , \qquad T_{2n} = G_{2n} - T_{2n+2} \qquad 1 < n < 8 , \\ &\text{where} \quad G_{2n} = \sum_{i_1<..i_n} \sin^2 m_{i_1} ... \sin^2 m_{i_n} . \end{aligned} \tag{4.1}$$

One then finds

$$T_2 = 2 - 2 \prod_{i=1}^{8} \cos m_i .$$

We also define the parameter

$$\widetilde{T}_4 = \frac{1}{4} T_2^2 - T_4 = T_2 - G_2 . \tag{4.2}$$

The instanton expansion is computed as in the previous section, but now instead of showing that a curve gives a character expansion, we assume that the character expansion exists and use this Ansatz to determine the curve for 8 arbitrary masses. Of course, the character expansions for E_8 become large and unwieldy, even for the smaller representations, so it is not practical to explicitly check the instanton coefficients term by term to see if they fit into characters.

However, an important fact is that the maximal subgroup of E_8 is actually $Spin(16)$. This means that only the representations that are in the same conjugacy class as the adjoint or one of the spinor representations of $SO(16)$ appear in the instanton expansion. The terms in the character expansions have the form $exp(2i\vec{m} \cdot \vec{\Lambda})$ where $\vec{\Lambda}$ is a point on the weight lattice. The absence of vector representations and one of the spinor reps, along with their conjugacy classes, simply means that the instanton expansion is invariant under the Z_2 transformation $m_i \to m_i + \pi/2$. In terms of the expressions in (4.1), the Z_2 transformation is

$$\begin{aligned} G_{2n} &\to \sum_{m=0}^{n} \begin{pmatrix} 8-m \\ 8-n \end{pmatrix} (-1)^m G_{2m} \\ t_8 &\to 1 - T_2/2 \qquad\qquad T_2 \to 2 - 2t_8. \end{aligned} \tag{4.3}$$

Surprisingly, insuring that the instanton expansion is invariant under this transformation is sufficient to determine the complete E_8 curve. Furthermore, not many extra terms appear away from the polynomial limit. The extra piece that should be added to the conformal curve in the appendix is

$$\begin{aligned} & x^2(u^2 + 2t_8\widetilde{T}_4) + x\Big(2T_2t_8u^2 + (2t_8T_8 + t_8\widetilde{T}_4^2/2 \\ & + 4t_8^2 - T_{12}\widetilde{T}_4)u + 4t_8^2T_6 - 2t_8T_{10}\widetilde{T}_4 + 4t_8^2T_2\widetilde{T}_4 + T_{14}\widetilde{T}_4^2\Big) \\ & - 8T_8u^4 - (4t_8T_6 + 8T_{14})u^3 - (4T_{14}T_6 + 8t_8T_{12} - 2t_8\widetilde{T}_4T_8 + 8t_8^2\widetilde{T}_4 - t_8^2T_2^2)u^2 \\ & + \big(4t_8^3T_2 - t_8^2(8T_{10} - 2T_2T_8 + 2T_6\widetilde{T}_4 - (T_2\widetilde{T}_4^2)/2) - t_8(4T_{12}T_6 + 4T_{14}\widetilde{T}_4 + T_{12}T_2\widetilde{T}_4 + T_{10}\widetilde{T}_4^2) \\ & + 2T_{14}T_8\widetilde{T}_4\big)u \\ & + 4t_8^4 + t_8^3(4T_2T_6 - 4T_8 + 2T_2^2\widetilde{T}_4 - \widetilde{T}_4^2) - t_8^2(4T_{10}T_6 - T_8^2 + 2T_{12}\widetilde{T}_4 + 2T_{10}T_2\widetilde{T}_4 \\ & - (T_8\widetilde{T}_4^2)/2 - \widetilde{T}_4^4/16) + t_8(T_{12}T_8\widetilde{T}_4 + T_{14}T_2\widetilde{T}_4^2 + (T_{12}\widetilde{T}_4^3)/4) + (T_{12}^2\widetilde{T}_4^2)/4 - T_{10}T_{14}\widetilde{T}_4^2 \end{aligned} \tag{4.4}$$

In fact this term reduces to x^2u^2 if at least three of the masses are zero. In order to fully determine (4.4) it was necessary to compute the fifth instanton in the expansion. We will discuss instanton expansions in more detail in the next section. In the conformal limit, the variables have dimensions $[x] = 10$, $[u] = 6$ and $[T_n] = n$, so that all terms in (B.1) have dimension 30. Using these conformal dimensions, we see that all terms in (4.4) have dimension 32. This of course does not mean that all possible dimension 32 terms appear, in fact the majority of them do not.

4.2. The curves for the other E_n

With the full E_8 curve one can derive the curves for the smaller E_n. To compute the E_7 curve, take $m_7 = i\Lambda + \mu$ and $m_8 = -i\Lambda + \mu$ and take the limit $\Lambda \to \infty$ (which corresponds to a large mass for the five dimensional gauge theory). In this limit $\sin m_7 \approx ie^{\Lambda - i\mu}/2$ and $\sin m_8 \approx -ie^{\Lambda + i\mu}/2$. The T_n parameters scale as

$$\begin{aligned} T_2 &= (e^{2\Lambda}/4)(T_{2,6} - 2) \qquad\qquad \widetilde{T}_4 = (e^{2\Lambda}/4)(T_{2,6} - 4\sin^2\mu) \\ T_6 &= (e^{4\Lambda}/16)(T_{2,6} - T_{2,6}^2/4) \qquad\qquad t_8 = (e^{2\Lambda}/4)t_6 \\ T_{2n} &= (e^{4\Lambda}/16)T_{2n-4,6} \qquad 4 \le n \le 7, \end{aligned} \tag{4.5}$$

where $T_{n,6}$ are the T_n variables for the remaining six masses. Rescaling the u and x variables

$$u \to \frac{1}{4}e^{2\Lambda}u \qquad\qquad x \to \frac{1}{16}e^{4\Lambda}(x+t_6T_2) \tag{4.6}$$

and keeping only the leading terms in e^{Λ} one finds the E_7 curve given in Appendix B.

The E_6 curve can be derived by scaling three masses, with $m_6 = i\Lambda_1 + 2\lambda/3$, $m_7 = i\Lambda_2 + 2\lambda/3$, and $m_8 = -i(\Lambda_1+\Lambda_2) + 2\lambda/3$. Taking the limit $\Lambda_i \to \infty$ the T_n scale as

$$\begin{aligned}
&T_2 = (e^{2(\Lambda_1+\Lambda_2)}/8)(T_{2,5}-2) \qquad\qquad \widetilde{T}_4 = (e^{2(\Lambda_1+\Lambda_2)}/8)(T_{2,5}-2+2e^{-2i\lambda})\\
&T_6 = -(e^{4(\Lambda_1+\Lambda_2)}/64)(1-T_{2,5}+T_{2,5}^2/4) \qquad\qquad T_8 = (e^{4(\Lambda_1+\Lambda_2)}/64)(-T_{2,5}+T_{2,5}^2/4)\\
&t_8 = i(e^{2(\Lambda_1+\Lambda_2)}/8)t_5 \qquad\qquad T_{2n} = -(e^{4(\Lambda_1+\Lambda_2)}/64)T_{2n-6,6} \qquad 5\le n\le 7
\end{aligned} \tag{4.7}$$

After rescaling u and x as

$$u \to ie^{2(\Lambda_1+\Lambda_2)}e^{-i\lambda}u/8\ , \qquad x \to -e^{4(\Lambda_1+\Lambda_2)}\left(e^{-2i\lambda}x + \frac{i}{2}T_2ue^{-i\lambda} + 2it_5 - iT_2t_5\right)/64, \tag{4.8}$$

the E_8 curve reduces to the E_6 curve in the appendix. Note that even for the massless E_6 case, there are imaginary coefficients for the curve. One consequence of this is that the E_6 character expansion will be not be symmetric under complex conjugation.

The smaller E_n can be derived in a similar fashion. The masses satisfy

$$m_i = i\Lambda_i + \frac{2}{9-n}\lambda \qquad n\le i<8, \qquad\qquad m_8 = -i\sum_{i=n}^{7}\Lambda_i + \frac{2}{9-n}\lambda. \tag{4.9}$$

The u and x variables are then scaled as

$$u \to \left(\frac{i}{2}\right)^{9-n} e^{2i\sum\Lambda_i}e^{-i\lambda}u \qquad\qquad x \to \left(\frac{i}{2}\right)^{18-2n} e^{4i\sum\Lambda_i}e^{-2i\lambda}x \tag{4.10}$$

It is also convenient to shift the x variable in order to have a more compact expression. The particular shift depends on which E_n theory is being considered. The curves along with the scaling details for these smaller E_n are given in the appendix.

5. Character Expansions and Holomorphic Curves

5.1. Rational curves in B_n

The contributions of the characters of the form (3.7) to the instanton expansion are given in tables 1, 2 and 3 for the E_n groups. We have expressed the characters in terms of the Weyl orbits instead of the E_n representations. As was shown in [11], this is a natural way to classify the holomorphic curves on the various del Pezzo surfaces.

The B_n del Pezzo surfaces are constructed by blowing up n points on $\mathbb{CP}_2$. The anti-canonical divisor is given as

$$\mathcal{K} = 3l - \sum_{i=1}^{n} e_i \tag{5.1}$$

where l is the anti-canonical divisor on $\mathbb{CP}_2$, in other words, it is a generic line, and the e_i are the n exceptional divisors of the blow-up points. The intersection matrix is generated by $l^2 = 1$, $e_i^2 = -1$ and $e_i \cdot e_j = 0$ if $i \neq j$. A curve in the homology class $a_0 l - \sum a_i e_i$ then has degree

$$d = a \cdot \mu = 3a_0 - \sum a_i, \qquad\qquad \mu = (3, 1, 1, 1, 1, 1, 1, 1, 1). \tag{5.2}$$

The arithmetic genus of this curve is given by

$$g_a = \frac{1}{2}(a_0 - 1)(a_0 - 2) - \frac{1}{2}\sum a_i(a_i - 1), \tag{5.3}$$

which counts the number of double points on $\mathbb{CP}_2$ that are not on the blow-up points.

The holomorphic curves can then be grouped into $U(n)$ Weyl orbits and for $n = 6, 7, 8$, these multiplets can be further combined into E_6, E_7 and E_8 multiplets. The weight length squared for a given curve is

$$L^2 = -a_0^2 + \sum_i a_i^2 + \frac{d^2}{9-n} \tag{5.4}$$

or in terms of the degree and the arithmetic genus is

$$L^2 = \frac{1}{9-n} d^2 + 2(1 - g_a). \tag{5.5}$$

The a_i are non-negative integers and a_0 is positive, except when the curve is one of the exceptional divisors. In the latter circumstance one has $a_0 = a_j = 0$, $j \neq i$ and $a_i = -1$ for the e_i divisor.

Finally, not all combinations of a_0 and a_i are allowed. Obviously, we cannot have a curve with arithmetic genus less than zero. We also cannot have curves where $a_0 < a_i + a_j$ for any i and j. Otherwise, it would be possible to have a line intersecting a curve of degree a_0 in $\mathbb{CP}_2$ more than a_0 times, which violates Bezout's theorem. Other constraints are

$$\begin{aligned}
2a_0 &\geq a_1 + a_2 + a_3 + a_4 + a_5, \\
3a_0 &\geq 2a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 \\
4a_0 &\geq 2a_1 + 2a_2 + 2a_3 + a_4 + a_5 + a_6 + a_7 + a_8 \\
5a_0 &\geq 2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + a_7 + a_8 \\
6a_0 &\geq 3a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + 2a_7 + 2a_8.
\end{aligned} \tag{5.6}$$

One consequence of these constraints is that $a_i \leq d$ for B_6 and B_7 and $a_i \leq 2d$, $|a_i - a_j| \leq d$ for B_8.

		d	1	2	3	4	5
L^2	dim						
0	1		12	−132	4068	−224688	17720400
2	240		1	−20	927	−66912	6381850
4	2160			−2	180	−18496	2207400
6	6720				27	−4656	729000
8	17280				3	−1056	228890
8	240					−976	226100
10	30240					−200	67325
12	60480					−32	18540
14	69120					−4	4725
14	13440						4325
16	138240						1025
16	2160						1100
18	181440						205
18	240						
20	241920						35
20	30240						
22	181440						
22	138240						5
	Total		252	−9252	848628	−114265008	18958064400

Table 1: Coefficients for E_8 Weyl orbits in instanton expansion. The bottom line is the coefficient when all $m_i = 0$.

		d	1	2	3	4	5	6
L^2	dim							
0	1			-20		-976		-179028
3/2	56		1		27		4325	
2	126			-2		-200		-54894
7/2	576				3		1025	
4	756					-32		-15624
11/2	1512						205	
6	2016					-4		-4140
6	56							-3780
15/2	4032						35	
8	4032							-936
8	126							-1020
19/2	4032						5	
19/2	1500							
10	7560							-198
12	10080							-36
12	1312							
14	4032							-6
14	12096							
14	576							-6
	Total		56	-272	3240	-58432	1303840	-33255216

Table 2: Coefficients for E_7 Weyl orbits in instanton expansion.

	d	1	2	3	4	5	6
L^2	dim						
0	1			27			−3780
4/3	27	1			−32		
4/3	27		−2			205	
2	72			3			−936
10/3	216				−4		
10/3	216					35	
4	270						−198
16/3	432						
16/3	432					5	
6	720						−36
8	432						−6
8	72						−6
	Total	27	−54	243	−1728	15255	−153576

Table 3: Coefficients for E_6 Weyl orbits in instanton expansion. The bottom line is the coefficient when all $m_i = 0$.

For instanton number d, one finds that characters with weight lengths squared up to $\frac{1}{9-n}d^2 - d + 2$ contribute to the pre-potential. These maximal weight characters correspond to the holomorphic curves of degree d and arithmetic genus 0. The shorter lengths correspond to curves with non-zero arithmetic genus.

5.2. Flowing from E_8 to E_n

The data given in Tables 1–4 was generated by working with the individual curves for E_n, $5 \leq n \leq 8$. Upon inspection one notices many smiliarities in the orbit degeneracies: namely for a given degree, any number that appears in the E_6 or E_7 table, also appears in the E_8 table (although the reverse is not true). In retrospect, this should not have been a surpise given that we obtained instanton expansions like (3.6) in terms of functions of the form (3.7).

One of the beautiful features of the functional form of (3.7) is that one can easily use it to study the flows down the chain of E_n del Pezzo surfaces. If one thinks of (a purely imaginary) m_i as representing the scale of a del Pezzo 2-cycle, or as representing a mass of a hypermultiplet, then by taking $m_i \to i\infty$ one decouples the corresponding states from the non-critical string, and decouples the corresponding hypermultiplets from the field theory. Thus the scaling procedure given in the last section for getting the E_n curves from the E_8 curve should produce the proper instanton expansions for the E_n theory. This is indeed what we find.

To be more specific, to get the E_n curves, not only were $9-n$ of the masses given large imaginary values, $i\Lambda_i$, but u was rescaled as well. Recall that to leading order, $u = exp(-2\pi i\phi)$, so the rescaling in u corresponds to a shift in ϕ, and hence a rescaling of q. Indeed, for the scaling in (4.10) one finds $q \to e^{-2\sum\Lambda_i}q$. The mass shifts generate a scaling in $e^{2i\vec{v}\cdot\vec{m}} \to e^{\sum_i v_i\Lambda_i}e^{2i\vec{v}\cdot\vec{m}}$. Thus, depending upon which of these scalings wins out, there are three possibilities for the function $q^d e^{2i\vec{v}\cdot\vec{m}}/(1-q^d e^{2i\vec{v}\cdot\vec{m}})$ in (3.7): (i) it vanishes, (ii) it is independent of the Λ_i, or (iii) it goes to -1. If the last possibility is realized then it generates a contribution to the constant term at the front of the instanton expansion. To be consistent with the anomaly computation in the five-dimensional field theory [7], this constant must change from -1 in the E_8 theory to $(n-9)$ in the E_n theory.

To find out what happens to the contributions of the various vectors, $\vec{v}$, under the rescaling, it is most convenient to work in the basis described in section (5.1). The inner product between any two vectors a_1 and a_2 that correspond to rational curves is

$$(a_1, a_2) = -(a_1 - d_1\mu)\cdot(a_2 - d_2\mu) = d_1 d_2 - a_1\cdot a_2, \tag{5.7}$$

where d_1 and d_2 are the degrees for a_1 and a_2 respectively. In this basis, it is clear that the mass shift vector can be chosen to be

$$i\Lambda = \sum_{i=1}^{8-n}\Lambda_i e_{n+i} \tag{5.8}$$

Hence, the inner product of a vector a with $i\Lambda$ is

$$(a, i\Lambda) = d\sum_i^{8-n}\Lambda_i - \sum_i^{8-n}\Lambda_i a_{n+i} \tag{5.9}$$

Assuming that $\Lambda_i > 0$, we see that $(a, i\Lambda) > d\sum_i\Lambda_i$ only if some of the a_i are negative. But this is possible only for the e_i divisors, with $i \geq n$. Hence, the contribution of these vectors to the rescaled instanton sum is $n-8$, exactly as required. For all other a, the a_i components are non-negative, so the inner product is an equality only if $a_i = 0$ for $i \geq n$. Hence, the characters from these vectors will flow to E_n characters and the coefficients in front of the characters remain the same. If any of these a_i are positive, then the corresponding contribution to the E_8 character flows to zero.

We can also flow to the B_0 surface, which is $\mathbb{CP}_2$, with the mass shift $i\Lambda = \sum_{i=1}^{8} \Lambda_i e_i$. In the $SO(16)$ basis $e_1 = \frac{1}{2}(1,1,1,1,1,1,1,1,1)$. In this case, one finds that the instanton expansion only has contributions when $d = 0 \bmod 3$. Presumably, this instanton expansion is giving us information about rational curves on $\mathbb{CP}_2$.

The fact that this works so simply, and is completely consistent with the results coming from the Calabi-Yau compactifications [4], and from the field theory [7], gives even more support to the contention that the effective action is faithfully capturing the structure of the non-critical string.

5.3. Reducing E_8 with real values of m_i

We have jsut seen that by tuning m_7 and m_8 to large imaginary values, we could flow from the E_8 curve to the E_n curve. These values of the masses correspond to Wilson lines along the sixth dimension. By T duality, we would expect a similar result for Wilson lines along the fifth dimension.

In particular, consider what happens to the instanton expansion when $m_7 = m_8 = \pi/2$, with all other $m_i = 0$. A straightforward calculation gives for the Yukawa coupling

$$\partial_\phi^2 \phi_D = -1 + 28\frac{q}{1-q} - 136\frac{8q^2}{1-q^2} + 1620\frac{27q^3}{1-q^3} - 29216\frac{64q^4}{1-q^4} + ... \tag{5.10}$$

This is the massless E_7 instanton expansion, up to a factor of two. If $m_7 = m_8 = \pi$, then we get back the original massless E_8 expansion.

Likewise, when $m_6 = m_7 = m_8 = \pi/3$ and all other $m_i = 0$, then the E_8 characters lead to the expansion

$$\partial_\phi^2 \phi_D = -1 + 9\frac{q}{1-q} - 18\frac{8q^2}{1-q^2} + 81\frac{27q^3}{1-q^3} - 5085\frac{64q^4}{1-q^4} + ... \tag{5.11}$$

which is the E_6 expansion, up to a factor of three.

5.4. General Structure of the Instanton Expansion

Rational curves of degree d in B_n are not isolated for $d \geq 2$: they have moduli spaces of dimension $d-1$. In [12,11] the curve counting was stablized by requiring that the curves pass through $d-1$ points in general position. If the curve has arithmetic genus equal to zero, then according to [11] this imposes an additional $d-1$ linear constraints. This suggests that the moduli space of curves of arithmetic genus zero is $\mathbb{CP}_{d-1}$. For the curves of small degree one can easily check this explicitly. For instance, consider conics going through p of the blow-up points on $\mathbb{CP}_2$. The degree of such a curve is $6-p$. A general conic on $\mathbb{CP}_2$ has the form

$$0 = a_1 X^2 + a_2 Y^2 + a_3 Z^2 + a_4 XY + a_5 XZ + a_6 YZ \ . \tag{5.12}$$

Since conics are automatically rational, there are no constraints on the a_i. Hence the moduli space for conics on $\mathbb{CP}_2$ is $\mathbb{CP}_5$. Requiring that the conic pass through p points leads to p linear constraints on the a_i and reduces the moduli space to $\mathbb{CP}_{5-p}$.

The usual expectation from using mirror symmetry to count rational curves is that if there is a non-trivial space of moduli, then the the "number" of such curves is the Euler characteristic of the moduli space. On a more physical level the Euler characteristic should be thought of as a "net number" after some deformation has broken the continuous degeneracy of the space of rational curves. At any rate, since the Euler characteristic of $\mathbb{CP}_{d-1}$ is d, and the degeneracy of rational curves of arithmetic genus zero is indeed $d(-1)^{d+1}$ (where the sign is due to the embedding of the holomorphic curve), we seem to have some agreement with what one expects from mirror symmetry. However, this only "explains" the counting of curves of arithmetic genus zero.

The computation of the character expansions within the instanton expansion gives us the ability to a take a family of curves of a given degree and separate out curves of different arithmetic genus: for a given degree, the length-squared of the vector on the root lattice decreases with the arithmetic genus. Moreover, it is possible to have more than one Weyl orbit of vectors of a particular length, and starting at arithmetic genus three, these different orbits can come with different non-zero coefficients in the instanton expansion. As a result we should be able to make a finer distinction between the various parts of the moduli spaces that contribute to the entire moduli space of curves, and somehow see this reflected in the computation of the Euler characteristic. It has been suggested that what we are seing is the Euler characteristic of different "stratifications of the moduli space" [13].

In order to try to get some control over the large amount of data that we have gathered, we now try to extract some universal information. As we have seen, the longest weights at a given degree d have degeneracy d. We also note that the second longest weights also fit into a pattern. Except for the $d = 1$ case, this coefficient appears to be $12d - d^2$. These are the characters that correspond to the curves with arithmetic genus 1.

We have also found patterns for the curves with $g_a \leq 4$, and indeed have found polynomials that fit the Weyl orbit degeneracies. In order to find these polynomials, it is necessary to compute the instanton expansions at least up to order 10. This is impractical for E_6 and higher but is possible for $E_5 = SO(10)$. The $SO(10)$ case will not contain all of the coefficients, but it contains enough to at least study the curves for $g_a \leq 4$. Table 4 contains the $SO(10)$ coefficients for degrees 7 through 10. Using these numbers and the lower numbers obtained from the E_8, E_7 and E_6 curves, we can construct Table 5.

As with $g_a = 1$, the the first number in each row actually violates the polynomial rule. Given this, the reader might be surprised to note that we were able to derive the second fifth order polynomial for the $g_a = 4$ case with just two data points (the first point is assumed to violate the rule). However, we actually have more information, since we assumed that the unknown coefficients were positive integers for $d < 15$. If we also use the Ansatz that the polynomial contains the product of two quadratic polynomials, then there is a unique result.

An interesting fact about the instanton expansion is that not all Weyl orbits appear in the expansion at a given instanton number d, even if other orbits of equal or greater length appear. For instance, for $d = 3$ in the E_8 case, the 240 of $L^2 = 8$ does not appear. As it so happens these holomorphic curves do not exist, since they violate the constraints in (5.6). Another interesting observation about these holomorphic curves is that curves with arithmetic genus zero appear at all degrees, but seem to have an upper degree limit for $g_a \neq 0$. For instance, for $g_a = 1$, there are no curves with $d > 9$. The $g_a = 1$ curves have the coefficients $12d - d^2$, hence this number never changes sign. We expect a similar result for higher values of g_a.

		d 7	8	9	10
L^2	dim				
0	1		-9604		
5/4	16			25758	
1	10				-181550
5/4	16	812			
2	40		-2752		
13/4	80			7992	
3	80				-61700
13/4	80	182			
4	10		-768		
4	80		-672		
5	16				-17770
5	16				-20000
5	80				-20750
21/4	160			2106	
21/4	160	35			
6	240		-160		
7	320				-5700
29/4	80			630	
29/4	160			531	
29/4	80	7			
8	40		-32		
9	10				-2250
9	240				-1550
37/4	320			135	
10	80		-8		
11	240				-500
11	160				-400
45/4	16			27	
13	80				-110
13	80				-110
53/4	80			9	
17	80				-10

Table 4: Coefficients for $SO(10)$ Weyl orbits in the instanton expansion. We have only included those Weyl orbits that contribute up to $d = 10$.

g_a \ d	1	2	3	4	5	6	7	8	9	10	Polynomial
0	1	2	3	4	5	6	7	8	9	10	$d/0!$
1	12	20	27	32	35	36	35	32	27		$(12-d)d/1!$
2		132	180	200	205	198	182	160	135	110	$(d^2-27d+192)d/2!$
3			927	1056	1025	936	812	672	531	400	$(d^2-30d+248)(15-d)d/3!$
3				976	1100	1020	?	768	630	500	$(d^2-29d+240)(16-d)d/3!$
4				4656	4725	4140	?	2752	2106	1550	$(d^2-31d+270)(d^2-35d+312)d/4!$
4					4325	3780	?	?	?	2250	$(d^2-19d+198)(15-d)(20-d)d/4!$

Table 5: Coefficients for a given degree and arithmetic genus. The question marks indicate values that are non-zero but which we did not determine. The Polynomial gives the d dependence for the coefficients. For non-zero g_a, the first term in each row violates the polynomial rule.

We should stress that except for the $d = 1$ or $g_a = 0$ curves, the coefficients in Table 5 are not the Euler numbers for the moduli spaces of the relevant curves. For instance the coefficient for the $d = 2,\ g_a = 1$ curves is -20, but the Euler number for the moduli space of these curves is -4. One can derive this as follows. Let $F(x, y, z) = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$ be a pencil of cubics that intersects seven points on $\mathbb{CP}_2$. To reduce this pencil to the space of *rational* cubics, there must be a double point, in other words a point where $\partial_x F = \partial_y F = \partial_z F = 0$. This has a solution for a set $\{\lambda_1, \lambda_2, \lambda_3\}$ if the determinant $\Delta = |\partial_i f_j|$ is zero for some point on $\mathbb{CP}_2$. The determimant Δ is a sixth order polynomial on $\mathbb{CP}_2$, which naively is a genus 10 Riemann surface. However, Δ has a double point at each of the original seven points, hence the genus is 3 and the Euler number is -4.

Even though the instanton coefficients are not the Euler numbers, we believe that the polynomials in Table 5 contain information about the topological structure of the spaces of moduli of the curves. For example, the coefficients for the terms linear in d are very interesting numbers: they appear in the work of [12,11], and for a given g_a represent the number of rational curves of degree d through $d-1$ specified points in general position in $\mathbb{CP}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. The leading term in the polynomial seems to have the universal form $d^{g_a+1}/(g_a)!$ and presumably reflects some combinatorial factor. It is amusing to conjecture that the d^p coefficients of the polynomials are related to the Euler characteristic of the $p-1$ dimensional space of rational curves of degree d passing through $d-p$ points.

As regards the "errors" for $g_a = 1$ in the polynomials in table 5, it is tempting to try to associate this with the fact that we are looking at a degenerated torus compactification of the non-critical string. For example, there are actually 12 curves with $d = 1, g_a = 1$, rather than the 11 predicted by the polynomial formula. These rational curves can be found explictly and correspond to the singular fibers on the 2-fold in (x, y, u) defined by (2.4). One of these fibers is at infinity in the u plane, leaving 11 at finite u.

While the mathematical interpretation of these degeneracies is as yet unclear, it seems very likely that one will ultimately be able to find the proper interpretation. However, we feel that this is not the right way to understand the issue. There should be a simple physical characterization of these degeneracies, and the mathematical interpretation will then amount to a magical property of this partition function of the non-critical string.

From the physics perspective, the most important fact about the polynomials in Table 5. are that they are universal: that is, they represent degeneracies for the E_n string for any n. One should recall that in terms of the compactified string, the degree d represents a winding number and momentum state of a string on a circle. The belief is that the d-wound state is, in fact, a bound state, and so these degeneracy polynomials are fundamental, group theory independent, properties of the bound states spectrum.

5.5. Counting states via BPS geodesics

Our approach to the E_n string has been based upon a IIB compactification on a Calabi-Yau manifold in which one has integrated out two dimensions to obtain a torus. As mentioned earlier, this should enable us to see the string rather explicitly, and as in [8,14,15,16,17], count BPS states by counting indecomposable geodesics on the torus with metric $ds^2 = |\lambda_{\rm SW}|^2$. The existence of BPS geodesics can be rather subtle in that the curves of minimal length with a given set of winding numbers could be decomposable into concatenations closed geodesics of other winding numbers. In such circumstances, the corresponding BPS state will be either marginally stable or unstable. Thus, at strong coupling, some purely electric states can become unstable (like the W-boson in $N = 2$ supersymmetric gauge theory). However, if one is interested in the purely electric bound states of the E_n string, one might hope that if the string tension is high enough, then all such states would be stable against decay into magnetically charged states. Thus all the fundamental electrically charged excitations of the E_n non-critical strings should be counted by looking at all the homotopy classes of strings on the torus with winding numbers $(1, 0)$ about the (A, B)-cycles. We will therefore see to what extent this approach replicates the state counting that we have already done.

To count the geodesics properly, one must of course, keep track of the hypermultiplet charges, and this is done by turning on all of the mass parameters and keeping track of the winding numbers around the simple poles of $\lambda_{\rm SW}$ whose residues are linear combinations of the m_i. We will refer to such poles as *relevant* poles. We will ignore both the multi-sheeting of the torus induced by the logarithm, and the (*irrelevant*) poles of residue $2\pi iN/R_5$, since as we discussed before, this structure is related to Kaluza-Klein momenta. The obvious hypothesis for the set of stable, fundamental electric BPS states is the set of curves that pass once around the A-cycle of the torus, passing between relevant poles of $\lambda_{\rm SW}$. The number of such states is equal to the number of relevant poles of $\lambda_{\rm SW}$.

So far in this paper we have not needed $\lambda_{\rm SW}$ for the massive theory in order to do the BPS state counting. Now we need it, and we need to exploit an ambiguity in its definition. As discussed in [10,6], the Seiberg-Witten differential is defined by finding $\lambda_{\rm SW}$ so that $\Omega_{(2)} \equiv \omega \wedge du = d\lambda_{\rm SW}$, where ω is the holomorphic differential on the torus. The problem is that $\Omega_{(2)}$ is not exact – it has non-trivial integrals over the 2-cycles in the surface that is defined by (x, y, u). To define $\lambda_{\rm SW}$ one must first excise these 2-cycles, and for $\lambda_{\rm SW}$ to be meromorphic, once must excise these curves holomorphically. One also wants to preserve the proper discrete symmetries, so one must make excisions in an appropriately Weyl invariant manner. Thus one is to excise Weyl orbits of rational curves: but there is the choice of the degree of these curves. One usually excises lines, but this is for the sake of simplicity and convenience: As was evident in [6] one could equally well excise quadratics or cubics, or even higher degree curves. As described earlier, rational curves are labelled by weight vectors, and the length of the weight vector increases with degree. By Bezout's theorem, such a curve generically intersects the Seiberg-Witten torus (defined by $u = constant$) d times. Thus excising such a curve introduces in $\lambda_{\rm SW}$, d simple poles each with the residue $\vec{v} \cdot \vec{m}$, where $\vec{v}$ is the weight label, and the components of $\vec{m}$ are the mass parameters.

Thus excising a Weyl orbit of rational curves of degree d gives rise to d poles for each vector in the orbit, and hence the indecomposable BPS states come with an additional degeneracy factor of d. Thus we have another understanding of the degeneracy of the curves of arithmetic genus 0 – the multiplicity comes from the intersections of each curve with the Seiberg-Witten torus.

One can also begin to see how the degeneracies might work for curves of non-zero arithmetic genus. As we stressed in the previous section, the degeneracies are not simply the Euler number for the moduli space of rational curves, but probably some combination of Euler numbers of various pieces of the moduli space. Here we propose a slightly more precise physical description: the degeneracy is computed by counting intersections of curves of degree d with the planes $u = u_0$, where u_0 is a constant (*i.e.* intersections with the

Seiberg-Witten torus). As described above, each such curve intersects the torus d-times, and so each degeneracy polynomial must have a factor of d. For higher arithmetic genus the necessary refinement is that the presence of double points may require that we look at tori at specific values of u_0, and then sum over choices of u_0. For example, one of the points in the moduli space of the curves with arithmetic genus 1 is the torus itself, *i.e.* the curve $u = u_0$. For this curve to be rational, it must have a double point, and so the discriminant must vanish at $u = u_0$. For E_8 the number of such zeroes is 12, and for E_n it is $n + 3$. For E_n, curves of arithmetic genus 1 first appear at degree $d = 9 - n$, and since the number of moduli is $d - 1$, the degeneracy for E_8 is simply 12 (there is a singular fiber at $u = \infty$). For E_n, $n < 8$ each singular torus belongs to a family of rational curves of the same degree. A generic member of this family intersects $u = u_0$ at d points. Thus, counting the self-intersections of the singular Seiberg-Witten tori gives us $d(12 - d)$ which is the correct degeneracy for $g_a = 1$ rational curves. We also see more clearly that the "error" for $d = 1$ is associated with the singular fiber at $u = \infty$.

The foregoing discussion of curves is far from rigorous, but very suggestive. It would thus be very interesting to revisit the IIB string compactifications that lead to these models and see how the choice of the degree of the excised curves arises in the construction, and how the BPS geodesic methods are to be modified so as to properly incorporate the curves of higher arithmetic genus. This might lead to a simple physical understanding of the degeneracy polynomials and their relationship to the topology of moduli spaces.

6. Toroidal compactification of the non-critical string

Thus far we have considered the non-critical string compactified on a degenerate torus with $R_5/R_6 = \infty$. We now briefly consider the corresponding story with R_5/R_6 finite.

From the form of the trigometric tori in the appendices and in section 2, it is fairly obvious how to restore the lost modulus of the torus, and the corresponding double periodicity of the complexified Wilson line parameters. For the curves with up to two non-zero masses one simply replaces the sine functions by the corresponding Jacobi elliptic functions, $sn(u, k)$, where $k = \vartheta_2^4(0|\tau)/\vartheta_3^4(0|\tau)$ is the elliptic modulus, and τ is the usual Teichmuller parameter of the torus. The surface (2.6) becomes

$$\begin{aligned} y^2 \ &= \ x^3 \ + \ (1+k^2)u^2x^2 \ + \ k^2 \ u^4 \ x \\ &\qquad - \ 2u \ (u^2 \ + \ sn^2(m_+)x) \ (u^2 \ + \ sn^2(m_-)x) \ . \end{aligned} \tag{6.1}$$

It is shown in Appendix A that this is indeed the proper form for the curve. For the curves with more than two non-zero masses the situation becomes considerably more complicated (essentially because there are many natural elliptic functions that reduce to unity in the trigonometric limit). However, the curves can be obtained using the approach of [9].

In appendix A we also construct the Seiberg-Witten differential associated with this surface, and even for two masses the explicit expressions are extremely complicated. The important point for the discussion here is that the logarithms in (2.2) are replaced by inverse elliptic functions:

$$\int_0^{sn(m)} \frac{dt}{(1-t^2)(1-k^2t^2)} \; = \; m \; . \tag{6.2}$$

This is necessary to make the residues of the Seiberg-Witten differential linear in the masses, while having the curve itself doubly periodic in its dependence on the masses. Moreover, the non-critical string compactified on a torus must have Kaluza-Klein excitations of mass $2\pi i(N_1/R_5 + N_2/R_6)$, for all integers N_1 and N_2. One sees that this is properly encoded in the differential if it has a prefactor of $1/R_5$ as in section 2, but is now the inverse of an elliptic function with $Im(\tau) = R_5/R_6$.

We can now push the construction of [5] backwards so as to reconstruct a non-compact Calabi-Yau manifold with two moduli that are the complexified versions of $k_D + k_E$ and k_E. To resolve an ambiguity in how to do this we need a slightly more explicit form of the Seiberg-Witten differential. The construction in Appendix A generically involves the following indefinite integral:

$$\int_0^1 \frac{dv}{\sqrt{x^3 + (1+k^2)u^2v^2x^2 + k^2u^4v^4x + f(x,u;m_i)}} \; , \tag{6.3}$$

where

$$f(x,u;m_i) \;\equiv\; y^2 - (x^3 + (1+k^2)u^2x^2 + k^2u^4x) \tag{6.4}$$

represents the perturbation of the curve away from the massless point. In particular, note that f is independent of the integration variable, v. Reversing the calculation of [5] this suggests that we should interpret v as one of the Calabi-Yau coordinates, and the point $v = 1$ should correspond to a limit of integration that is set (as in [5]) by the integration over the third Calabi-Yau coordinate. Thus, one can easily arrive at the following expression for the non-compact Calabi-Yau manifold (without mass parameters [4]):

$$w^2 \ = \ z_1^3 \ + \ z_2^6 \ + \ z_3^6 \ - \ \frac{1}{z_4^6} \ + \ \psi^2(1+k^2)(z_1 z_2 z_3 z_4)^2 \ + \ k^2\psi^4 z_1 (z_2 z_3 z_4)^4 \ . \qquad (6.5)$$

Given the identification of the new parameter in terms of the torus compactification of the non-critical string, one can use the work of [4] and [5] to relate it to modulus of the Calabi-Yau in the IIA compactification. Indeed it is the complexification, t_E, of the Kähler modulus k_E.

Thus we propose that this non-compact Calabi-Yau manifold (or the corresponding torus) captures the sector of the non-critical string that is defined by the closed sub-monodromy problem described in [5] that associated with the two parameters t_S and t_E. In practical terms, this means that we should be able to pass between phase I ($t_S < t_E$) and phase II ($t_S > t_E$), perhaps seeing the phase transition or some curve of marginal stability. We should also be able to generate the full instanton counting of [4], with independent d_E and d_D independent. Work is continuing along these lines, and preliminary calculations indicate that one should be able to find expressions for the degeneracies explicitly in terms of modular functions of τ. The precise computation is rather complicated as one needs to carefully evaluate the "constants of integration," in ϕ, and these "constants" can, in principle, be very complicated functions of τ.

7. Conclusions

We have shown that the effective action of a non-critical string does indeed capture much of the information about the BPS structure of the theory. We have shown how the formulation of the massive effective action is closely parallel to the corresponding object in field theory, and yet it contains information that is appropriate to the string compactification. In particular we used this effective action to count BPS states replete with the full set of E_n character parameters. The fact that this computation works and provides answers that are consistent with results from field theory and from Calabi-Yau manifolds already represents a remarkable number of consistency checks on the overall approach. Combined with the internal self-consistency of the character expansions, and

[4] The construction of the Calabi-Yau manifold with mass parameters cannot be done by such a simple procedure: if one tries the naive approach one obtains a Calabi-Yau manifold with too few independent moduli.

the proper behaviour of the flows from E_8 to E_n, we feel that we have made a compelling case, not only for the correctness but also for the utility of the approach.

We have focussed on a particular sub-sector of the non-critical string: One that corresponds to the string wrapped d times around a circle and in a phase where all such states are becoming massless. The general belief is that such multiply wrapped strings should be bound states. By extracting the degeneracy polynomials in Table 5, we have obtained the first predictions for the universal (E_n independent) degeneracies of such bound states. As we remarked in section 5, there is almost certainly a beautiful mathematical characterization of these degeneracies. However, we believe that there should also be some simple physical description of the degeneracies.

Based upon the picture of non-critical strings in terms of membranes stretching between a 9-brane and a 5-brane, a natural suggestion for the spectrum is a represenation of E_8 current algebra at level one (inherited from the 9-brane) multiplied by some eta-functions (associated with the 5-brane). This is indeed what one finds [4] for the lowest level of the non-critical string ($d_E = 1$). The most natural first guess for the compactified and multiply wound string is the same current algebra representation, but with more complicated structure, perhaps through level matching, coming from the 5-brane degrees of freedom. This possibility is ruled out by our data: The level one representations generally have every Weyl orbit occurring once. One might be able to get around this by multiplying by some eta-functions, or other modular functions and then doing some exotic level matching, but our data shows that distinct Weyl orbits of vectors of the *same length* usually come with different degeneracies. This cannot be realized by a simple level matching of level one E_n characters with other modular functions. A less naive suggestion is that the bound states may involve E_n current algebras at level d for the string wrapped d times. Preliminary calculations for $d = 2$ indicate that this does not appear to work either. Thus, in spite of all this data, a simple physical characterization of the spectrum of the E_n string still eludes us.

Another potentially useful physical application of our degeneracy polynomials is in entropy calculations where one need to estimate of the growth of the number of BPS states [18]. For example, the degeneracy polynomials strongly suggest that the number of E_n Weyl orbits corresponding to curves of degree d and arithmetic genus g_a grows as $d^{g_a+1}/g_a!$. Summing over g_a for a given d one sees that the number of Weyl orbits must grow as $e^d \sim e^L$, where L is the length of the vectors in the corresponding Weyl orbit. Since the number of weight vectors of E_n of a given length grows as L^n, and so this does not modify the exponential growth. A similar result has been obtained from a numerical fit using the mirror map on the Calabi-Yau manifold [19].

Finally, we believe that the ideas described in section 6 will lead to far more complete characters for the states of the E_n non-critical strings. In particular, we expect to obtain explicit modular functions for some of the non-critical string degenaracies. This should not only shed some light on the modular structure of the spectrum, but should also enable some sharp estimates of the growth of the number of BPS states.

Acknowledgements

We would like to thank S. Katz, W. Lerche, P. Mayr and C. Vafa for valuable discussions. N.W. is also grateful to the ITP in Santa Barbara, and the Institute for Advanced Study in Princeton for hospitality while this work was being done. This work is supported in part by funds provided by the DOE under grant number DE-FG03-84ER-40168, and by the National Science Foundation under grant No. PHY94-07194.

Appendix A. Derivation of the E_8 curve with up to two Wilson lines.

A.1. Introducing one mass

To construct the curves with the mass parameters, we follow the methods developed in [6]. Before starting it is convenient to make the shift $x \to x - \frac{1}{4}u^2$ in (2.4), and rescale to obtain the curve

$$y^2 \;=\; x^3 - \frac{1}{48}u^4 x + \frac{1}{864}u^6 + u^5. \tag{A.1}$$

We now add one mass parameter, which breaks the E_8 symmetry down to $SO(14)$. The general form of the curve consistent with $SO(14)$ symmetry has the form

$$y^2 \;=\; x^3 - (\frac{1}{48}u^4 + bu^3 + 3\lambda u^2)\; x \;+\; (\frac{1}{864}u^6 + \beta u^5 \gamma u^4 + 2\lambda u^3)\;. \tag{A.2}$$

Here $b, \beta,\ \gamma$ and λ are constants that we will determine bellow. The discriminant of (A.2) is given by

$$\Delta \;=\; 4\,(\frac{1}{48}u^4 + bu^3 + 3\lambda u^2)^3 - 27\,(\frac{1}{864}u^6 + \beta u^5 \gamma u^4 + 2\lambda u^3)^2. \tag{A.3}$$

Since the global symmetry of the pertubed curve is $SO(14)$ the discriminant has to be of order u^9 as $u \to 0$. This fixes: $\gamma = b\lambda$ and $\beta \;=\; b^2/12\lambda$. The rest of the constants can be fixed by finding the appropriate lines. We will assume that the lines have the same form as the ones in the polynomial curve of [6]. Therefore we look for lines of the form

$$x \;=\; \mu^2 u^2 \;+\; \nu u\;. \tag{A.4}$$

With $\nu \;= -\lambda$ we have the spinor line. The adjoint line is obtained by setting $\nu \;=\; 2\lambda$. First consider the spinor line. If we set $\mu \;=\; t^2 - \frac{1}{6}$ it is easy to verify that the line (A.4) gives rise to a perfect square, and one obtains

$$y \;=\; iu^3(t^2 - \tfrac{1}{4})t\;. \tag{A.5}$$

For our later analysis it is convenient to shift back $x \to x + \frac{1}{12}u^2 - \lambda u$. The curve with one mass now has the form

$$y^2 \;=\; x^3 + (\tfrac{1}{4}u^2 - \lambda u)x^2 + 6\lambda(t^2 - \tfrac{1}{4})u^3 x \;-\; 3\lambda(t^2 - \tfrac{1}{4})^2 u^5. \tag{A.6}$$

Now it is also easy to verify that adjoint line is

$$x \;= -(t - \frac{1}{4t})^2 u^2 - 3\lambda u \tag{A.7}$$

with

$$y \;=\; -\,i\Big(\tfrac{1}{8}(t-\frac{1}{4t})^2(t+\frac{1}{4t})^2u^3-\tfrac{9}{2}\lambda^2\;u(t-\frac{1}{12t})\Big)\;. \tag{A.8}$$

Note here we have written the adjoint line for the curve (A.6).

One can easily recover to the polynomial limit of this curve as $\lambda \to 0$. From (A.6) we see that t has to diverge in order to have a finite u^5 term. With $t \approx \Lambda/m$ and $\lambda \;=\; -\frac{1}{3}\Lambda^2 m^4$ it is easy to see that when $m \to 0$ we reduce to the polynomial limit.

Next we construct the Seiberg-Witten differential λ_{SW} for the foregoing curve. By definition one must have

$$\frac{d\lambda_{SW}}{du} \;=\; \frac{dx}{y}+\frac{d}{dx}(\ldots)\;. \tag{A.9}$$

One starts by considering the following differential:

$$\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{x}\;, \tag{A.10}$$

where

$$y^2 \;=\; x^3+(\tfrac{1}{4}u^2-3\lambda)x^2+6\lambda\xi u^3 x-3\lambda\xi \tag{A.11}$$

and $\xi \;=\; (t^2-1/4)$. The derivative of (A.10) with respect to u gives

$$\frac{d}{du}\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{x} \;=\; \frac{x^3-\frac{3}{2}\lambda ux^2-3\lambda\xi u^3x+\frac{9}{2}\lambda\xi^2u^5}{y^2-\frac{1}{4}u^2x^2}\;\frac{dx}{y}\;. \tag{A.12}$$

This shows that (A.10) will not do the job and we need additional terms. First consider differential

$$\frac{1}{u}\frac{d}{dx}\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)dx \;=\; \frac{-\frac{1}{2}x^3+3\lambda\xi u^3x-3\lambda\xi^2u^5}{y^2-\frac{1}{4}u^2x^2}\frac{dx}{y}\;. \tag{A.13}$$

It is clear from (A.13) that (A.12) is not enough to cancel off the dominator. To this end consider

$$\log\left(\frac{y+\frac{1}{2}u\;x}{y-\frac{1}{2}u\;x}\right)\frac{dx}{l_s}\;, \tag{A.14}$$

where $l_s \;=\; x-\xi u^2$ is the spinor line. With this definition we can rewrite our curve in the form

$$\begin{aligned} y^2 \;&=\; l_s\;q+r^2\;,\qquad r_s \;=\; u^3t\xi \\ q \;&=\; x^2+(t^2u^2-3\lambda u)x+u^2\xi(u^2t^2+3\lambda u)\;. \end{aligned} \tag{A.15}$$

The derivatives of (A.14) are given by

$$\Big(\frac{d}{du}-\frac{d}{dx}\frac{dl}{du}\Big)\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{l} \;=\; \frac{-\frac{1}{2}\{l,q\}u\;x+xq-uq\frac{dl}{dq}-2r^2}{y^2-\frac{1}{4}u^2x^2}\frac{dx}{y}\;, \tag{A.16}$$

where $\{l,q\} = \frac{dl}{dx}\frac{dq}{du} - \frac{dl}{du}\frac{dq}{dx}$. Combining equations (A.12), (A.13) and (A.16) we can construct a Seiberg-Witten differential

$$\lambda_{SW}^{(s)} = \log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{x} - \tfrac{2}{3}\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{x-\xi u^2} \tag{A.17}$$

that satisfies

$$\frac{d}{du}\lambda_{SW}^{(s)} = \tfrac{1}{6}\frac{dx}{y} - \tfrac{1}{3}\frac{d}{dx}\left(\log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{u}\right) - \frac{d}{dx}2\xi u\ \log\left(\frac{y+\frac{1}{2}ux}{y-\frac{1}{2}ux}\right)\frac{dx}{x-\xi u^2}\ . \tag{A.18}$$

In the this construction we have only used the spinor line, but the full Seiberg-Witten differential should also contain a contribution from the adjoint line

$$l_a = x-(t-\frac{1}{4t})^2u^2 - 3\lambda u \quad r_a = \tfrac{1}{8}(t-\frac{1}{4t})^2(t+\frac{1}{4t})u^3 - \tfrac{9}{2}\lambda^2\ u(t-\frac{1}{12t})\ . \tag{A.19}$$

In order to include the adjoint line l_a we have to proceed in a slightly different fashion. Note the argument of $\log\left(\frac{y+\frac{1}{2}(ul_s+\alpha_s r_s)}{y-\frac{1}{2}(ul_s+\alpha_s r_s)}\right)$ is different from $\log\left(\frac{y+\frac{1}{2}(ul_a+\alpha_s r_a)}{y-\frac{1}{2}(ul_a+\alpha_s r_a)}\right)$. This means if one takes derivatives of these functions with respect to u the dominators will generically be different, and so these terms cannot combine to form a Seiberg-Witten differential. However if we choose α_s and α_a so that

$$-x_s u + \alpha_s r_s = -x_a u + \alpha_a r_a \tag{A.20}$$

then the dominators will be the same and we can construct a Seiberg-Witten differential. We can solve equation (A.20) for α_s and α_s by comparing the terms of order u^2 and u^3. After simple algebra we have

$$\alpha_s = -\alpha_a = \frac{2}{3(t-\frac{1}{12t})}\ . \tag{A.21}$$

It is easy to see that the Seiberg-Witten differential with the adjoint line will be of the form

$$\lambda_{SW}^{(a)} = \log\left(\frac{y+\frac{1}{2}(ul_a+\alpha_a r_a)}{y-\frac{1}{2}(ul_a+\alpha_a r_a)}\right)\left(\frac{dx}{l_a+\alpha\frac{r_a}{u}} + b_1\ \frac{dx}{l_a} + b_2\frac{dx}{l_s}\right)\ , \tag{A.22}$$

where the constants b_1 and b_2 are determined by demanding that $\lambda_{SW}^{(a)}$ satisfies the following condition

$$\frac{d}{du}\lambda_{SW}^{(a)} + \frac{d}{dx}(\cdots) = k\frac{dx}{y}\ . \tag{A.23}$$

A simple substitution allows as to solve for the unknown coefficients to obtain $b_1 = -\frac{1}{2}$, $b_2 = -1$ and $k = 0$. Surprisingly, we find $k = 0$, and so we have a *null* contribution to the Seiberg-Witten differential. It is clear that the full Seiberg-Witten will be a linear combination of $\lambda^{(s)}$ and $\lambda^{(a)}$

$$\lambda_{SW} \;=\; C_1 \lambda_{SW}^{(s)} \;+\; C_2 \lambda_{SW}^{(a)} \;. \tag{A.24}$$

We fix the coefficients C_1 and C_2 by comparing this with the superconformal or polynomial limit. We described above how to make this limit for the curve, and applying the same procedure to the Seiberg-Witten differential, we find that the leading term is given by

$$(\frac{C_1}{3} - \frac{C_2}{2})u\frac{dx}{y} - (\tfrac{2}{3}C_1 + \tfrac{2}{3})\frac{r_s}{t}\frac{dx}{y l_s} + \tfrac{1}{3}C_2 \frac{r_a}{t}\frac{dx}{y\; l_a} \;. \tag{A.25}$$

The Seiberg-Witten differential in the polynomial limit with one mass is given by

$$\frac{1}{2\sqrt{2}\pi}(60u\;\frac{dx}{y} - 64i\frac{\frac{m}{2}r_s}{x - x_s}\frac{dx}{y} - 14i\frac{m r_a}{x - x_a}\frac{dx}{y} + 42u\frac{dx}{y}) \;. \tag{A.26}$$

They are consistent if we set $C_1 = \frac{180}{2\sqrt{2}\pi}$ and $C_2 = \frac{-84}{2\sqrt{2}\pi}$ and then we have

$$\frac{d}{du}(C_1 \lambda_{SW}^{(s)} \;+\; C_2 \lambda_{SW}^{(a)}) \;=\; \frac{30}{2\sqrt{2}\pi}\frac{dx}{y} \tag{A.27}$$

Where the coefficient, 30, is the E_8 dual Coexeter number, as it should be.

To obtain a physical interpretation of the parameter t one calculates the residue both for the spinor and adjoint line. The residue of the spinor line is $\log\frac{t+\frac{1}{2}}{t-\frac{1}{2}}$ and the residue of the adjoint is $2\log\frac{t+\frac{1}{2}}{t-\frac{1}{2}}$. Since the adjoint has a residue that is twice the residue of the spinor we can identify the residue with the mass term

$$\log\frac{t+\frac{1}{2}}{t-\frac{1}{2}} \;=\; m \;. \tag{A.28}$$

Before generalizing this result to two masses it is useful to summarize the final form of our curve with one mass. We have found that curve with one mass is given by

$$y^2 \;=\; x^3 + \frac{u^2}{4}x^2 - 2\Lambda^6 u(4\sin^2\frac{m}{4\Lambda}x + u^2)^2 \;. \tag{A.29}$$

Here we have set $\lambda = \frac{32}{3}\Lambda^6 \sin^4\frac{m}{4\Lambda}$.

A.2. The curve and differential with two masses

We start the case of two masses by first studying the polynomial limit

$$y^2 \;=\; x^3 - (T_2 u^3 + \frac{\tilde{T}_4^2 u^2}{12})x - (2u^5 + u^4 \frac{T_2 \tilde{T}_4}{6} + \frac{\tilde{T}_4^3}{108}) \;, \tag{A.30}$$

where $\tilde{T}_4 \;=\; \frac{1}{4}T_2^2 - T_4$ and $T_2 \;=\; m_1^2 + m_2^2$, $T_4 \;=\; m_1^2 m_2^2$. After shifting $x \to x - \frac{1}{6}\tilde{T}_4 u$ the curve becomes

$$y^2 \;=\; x^3 - 2\; ux^2 (\frac{m_1+m_2}{2})^2 (\frac{m_1-m_2}{2})^2 - 2u^3 x((\frac{m_1+m_2}{2})^2 + (\frac{m_1-m_2}{2})^2) - 2\; u^5 \tag{A.31}$$

In the shifted form the spinor line has a very simple form

$$x \;=\; \frac{4}{(m_1 \pm m_2)^2}\; u^2 \tag{A.32}$$

with

$$y \;=\; \frac{8i}{(m_1 \pm m_2)^2} u^3 \tag{A.33}$$

The adjoint lines, $l_\pm$, have the form

$$x \;=\; -\frac{1}{(m_+ \pm m_-)^2} u^2 + 2m_+^2 m_-^2 \;, \tag{A.34}$$

with

$$y \;=\; i(\frac{u^3}{(m_+ \pm m_-)^3} \pm \frac{2\; m_+ m_-}{m_+ \pm m_-}(m_+^2 \pm m_+ m_- + m_-^2)\; u^2) \;. \tag{A.35}$$

Here we have also introduced the notation $m_\pm \;=\; \frac{m_1 \pm m_2}{2}$.

To construct the trigonometric curve we make the Ansatz

$$y^2 \;=\; x^3 + \tfrac{1}{4}u^2 x^2 + \tfrac{1}{2}T_4' u x^2 - 2\; u(u^2 + \tfrac{1}{4}T_2' x) \;. \tag{A.36}$$

Recall that with one mass the curve was obtained by replacing m^2 with $16\sin^2\frac{m}{4}$. This suggest then that the curve with two masses is obtained in a very similar fashion, namely replacing $m_\pm$ with $4\sin^2\frac{m_\pm}{4}$. This then leads us to the curve

$$y^2 \;=\; x^3 + \tfrac{1}{4}u^2\; x^2 - 2u(4x\; \sin^2\; \tfrac{m_+}{2} + u^2)(4x\; \sin^2\; \tfrac{m_-}{2} + u^2) \;. \tag{A.37}$$

Again we look for lines, starting with the spinor line. The generalization of equation (A.32) is straightforward and we have

$$x \;=\; -\frac{u^2}{4\sin^2\frac{m_\pm}{2}} \;, \tag{A.38}$$

with

$$y \;=\; i\frac{\cos\; m_\pm/2}{8\sin^3 m_\pm}\;. \tag{A.39}$$

Next we consider the adjoint line that has the form

$$x \;=\; -\frac{u^2}{4\sin^2\frac{m_+ + m_-}{2}} + 32\sin^2\frac{m_+}{2}\sin^2\frac{m_-}{2}\;u\;, \tag{A.40}$$

with

$$\begin{aligned} y \;=\; & \frac{i\;u^3\cos\frac{m_+ + m_-}{2}}{8\;\sin^3\frac{m_+ + m_-}{2}} \pm \frac{16\;i\;u^2\sin\frac{m_+}{2}\sin\frac{m_-}{2}}{\sin\frac{m_+ + m_-}{2}}. \\ & (\sin^2\frac{m_+}{2} + \sin^2\frac{m_+}{2} - \sin^2\frac{m_+}{2}\sin^2\frac{m_-}{2} \pm \sin\frac{m_+}{2}\sin\frac{m_-}{2}\cos\frac{m_+}{2}\cos\frac{m_-}{2})\;. \end{aligned} \tag{A.41}$$

As before, in order to construct the Seiberg-Witten differential we start with the spinor residue and rewrite our curve using the spinor line l_s in the form $y^2 \;=\; l_s\;q + r_s^2$ where

$$\begin{aligned} l_s \;&=\; x + \frac{u^2}{4\sin\frac{m_\pm}{2}}\;,\quad r_s^2 \;=\; -\frac{\cos^2\frac{m_\pm}{2}}{64\sin^6\frac{m_\pm}{2}}u^6 \\ q \;&=\; x^2 - \frac{\cos\frac{m_\pm}{2}x}{4\sin^2\frac{m_\pm}{2}}u^2 + \frac{\cos^2\frac{m_\pm}{2}}{16\sin^4\frac{m_\pm}{2}}u^4 - 8usin^2\frac{m_\pm}{2}(u^2 + 4\sin^2\frac{m_\pm}{2}x)\;. \end{aligned} \tag{A.42}$$

Consider the differential

$$\log\left(\frac{y + \frac{1}{2}(ul + \alpha r)}{y + \frac{1}{2}(ul + \alpha r)}\right)\frac{dx}{l}\;. \tag{A.43}$$

If we set $\alpha = \alpha_\pm = \frac{2i\sin\frac{m_\pm}{2}}{\cos\frac{m_\pm}{2}}$ it is easy to see that the residue $im_\pm$. Proceeding in the same fashion as in the one mass case one can construct a Seiberg-Witten differential using the spinor line:

$$\frac{d}{du}\left(\log\left(\frac{y + \frac{1}{2}ux}{y - \frac{1}{2}ux}\right)\left(\frac{dx}{x} - \frac{1}{3}\frac{dx}{l_+} - \frac{1}{3}\frac{dx}{l_-}\right)\right) \;=\; \frac{1}{6}\frac{dx}{y} + \frac{d}{dx}(\ldots)\;. \tag{A.44}$$

With only one mass we saw that the full Seiberg-Witten differential requires the existence of a null differential. It is not surprising that this will also be also for two masses. Again before solving for these null differentials it is instructive to study the polynomial limit. For two masses we have are six different poles: $m_+,\; m_-,\; m_+ + m_-,\; m_+ - m_-,\; m_{a_+} = 2m_+$ and $m_{a_-} = 2m_-$. As we have seen, the spinor lines correspond to residues $m_\pm$ and the adjoint lines corresponds to the residue $m_+ \pm m_-$. The residues $2m_\pm$ can be identified with a second set of adjoint lines $l_{a_\pm}$

$$x \;=\; \frac{-u^2}{4m_\pm^2} + u(3m_\pm^2 - m_\mp^2)m_\pm^2 - m_\pm^6(m_+^2 - m_-^2)^2\;, \tag{A.45}$$

with

$$y^2 \;=\; -(\frac{u^3}{8m_\pm^3}+6u^2(5m_\pm^6-m_\pm^4 m_\mp^2)-4um_\pm^8(9m_\pm^2-5m_\mp^2)(m_\pm^2-m_\mp^2)+8m_\pm^{12}(m_\pm^2-m_\mp^2)^3)^2 \; . \tag{A.46}$$

It turns out that one can make a consistent Seiberg-Witten differential out of any two lines l_a and l_b. The following Ansatz leads to a Seiberg-Witten differential:

$$\begin{aligned} &\frac{d}{du} i m_a r_a \frac{dx}{l_a y} - \frac{dl_a}{du}\frac{d}{dx} i m_a r_a \frac{dx}{l_a y} + A_b\Big(\frac{d}{du} - \frac{dl_b}{du}\frac{d}{dx}\Big) i m_b r_b \frac{dx}{l_b y} + \frac{d}{du}\alpha(U+T)\frac{dx}{y} \\ &+\beta\frac{d}{dx}(x+S)\frac{dx}{y} \;=\; k\frac{dx}{y} \; . \end{aligned} \tag{A.47}$$

For example if we choose the lines l_+ and l_- we find that the unknown coefficients are given by $A=-1$, $k=-\frac{1}{2}$, $T=0$, $\beta=1$ and $S = 2u^2(m_+^2+m_-^2)/m_+^2 m_-^2$

We can now use this this result find the final set of adjoint lines. The missing lines are the generalization of the adjoint lines $l_{a_\pm}$ of the polynomial curve. From our experience with the one mass case one should be able to construct a null Seiberg-Witten differential using the lines l_+ and $l_{a_\pm}$. If this is the case then we have to satisfy the following condition

$$\log\frac{y+\frac{1}{2}(ul_+ + \alpha_+ r_+ + l_+ C)}{y-\frac{1}{2}(ul_+ + \alpha_+ r_+ + l_+ C)} \;=\; \log\frac{y+\frac{1}{2}(ul_{a_+} + \alpha_{a_+} r_{a_+} + l_{a_+} C)}{y-\frac{1}{2}(ul_{a_+} + \alpha_{a_+} r_{a_+} + l_{a_+} C)} \; . \tag{A.48}$$

For $l_+ = x + u^2/4\sin^2\frac{m_+}{2}$, $r_+ = iu^3/8\sin^3\frac{m_+}{2}$ and for $l_{a_+} = x + \frac{1}{4}\frac{u^2}{\sin\frac{m_+}{2}} - b_1 u - \lambda^2$ and $r_{a_+} \;=\; \frac{i\cos m_+}{8\sin^3 m_+}u^3 + a_2 u^2 a_1 u\lambda^3$. Substituting these into (A.48), and demanding the l_+ is line on our curve, gives, after some algebra:

$$\begin{aligned} \lambda &= -32i\frac{\sin^3\frac{m_+}{2}}{\cos\frac{m_+}{2}}(\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2}) \\ b_1 &= \frac{16\sin^2\frac{m_+}{2}}{\cos^2\frac{m_+}{2}}\Big(\tfrac{1}{2}\sin^2 m_+ \sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2}\Big) \\ a_1 &= \frac{-256i\sin^5\frac{m_+}{2}}{\cos^3\frac{m_+}{2}}(\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2})\Big(3(\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2}) + 2\cos^2\tfrac{m_+}{2}(3\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2})\Big) \\ a_2 &= 8i(\cos\; m_+ + 2)\frac{\sin^3\frac{m_+}{2}}{\cos\frac{m_+}{2}} + 6i\frac{\sin\frac{m_+}{2}}{\cos^3\frac{m_+}{2}}(\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2}) \\ C &= -128\frac{\sin^4\frac{m_+}{2}}{\cos\; m_+ + 2}(\sin^2\tfrac{m_+}{2} - \sin^2\tfrac{m_-}{2}) \; . \end{aligned} \tag{A.49}$$

Having found the second adjoint line we are now ready write down the null Seiberg-Witten differentials. After some algebra we find the following null-differentials:

$$
\begin{aligned}
\lambda_1^{(n)} &= \log\Big(\frac{y+\frac{1}{2}(ul_1+\alpha_1 r_1)}{y-\frac{1}{2}(ul_1+\alpha_1 r_1)}\Big)\Big(\frac{dx}{l_1+\alpha_1 r_1/u} - \tfrac{1}{2}\frac{dx}{l_1} - \tfrac{1}{2}\frac{dx}{l_+} - \tfrac{1}{2}\frac{dx}{l_-}\Big) \\
\lambda_2^{(n)} &= \log\Big(\frac{y+\frac{1}{2}(ul_2+\alpha_2 r_2)}{y-\frac{1}{2}(ul_2+\alpha_2 r_2)}\Big)\Big(\frac{dx}{l_2+\alpha_2 r_2/u} - \tfrac{1}{2}\frac{dx}{l_2} - \tfrac{1}{2}\frac{dx}{l_+} - \tfrac{1}{2}\frac{dx}{l_-}\Big) \\
\lambda_+^{(n)} &= \log\Big(\frac{y+\frac{1}{2}(ul_{a_+}+\alpha_{a_+} r_{a_+}+l_{a_+}C_+)}{y-\frac{1}{2}(ul_{a_+}+\alpha_{a_+} r_{a_+}+l_{a_+}C_+)}\Big)\Big(\frac{dx}{l_{a_+}+\alpha_{a_+} r_{a_+}/u} - \tfrac{1}{2}\frac{dx}{l_{a_+}} - \tfrac{1}{2}\frac{dx}{l_{a_+}} - \tfrac{1}{2}\frac{dx}{l_+}\Big) \\
\lambda_-^{(n)} &= \log\Big(\frac{y+\frac{1}{2}(ul_{a_-}+\alpha_{a_-} r_{a_-}+l_{a_-}C_-)}{y-\frac{1}{2}(ul_{a_-}+\alpha_{a_-} r_{a_-}+l_{a_-}C_-)}\Big)\Big(\frac{dx}{l_{a_-}+\alpha_{a_-} r_{a_-}/u} - \tfrac{1}{2}\frac{dx}{l_{a_-}} - \tfrac{1}{2}\frac{dx}{l_{a_-}} - \tfrac{1}{2}\frac{dx}{l_-}\Big)\ .
\end{aligned}
\tag{A.50}
$$

The full Seiberg-Witten differential is a linear combination of the spinor Seiberg-Witten differential and the null Seiberg-Witten differentials constructed above:

$$
C_s\lambda_s + C_1\lambda_1^{(n)} + C_2\lambda_2^{(n)} + C_{a_+}\lambda_{a_+}^{(n)} + C_{a_-}\lambda_{a_-}^{(n)}\ , \tag{A.51}
$$

where $C_s = -\frac{180}{2\sqrt{2}\pi}$. This is the same constant that appeared in the one mass case. There are 12 l_1 and 12 l_2 lines, each have a residue $m_+ + m_-$ and $m_+ - m_-$ and

$$
C_1 = \frac{-24(m_+ + m_-)i}{2\pi\sqrt{2}\log\big(\frac{1+\alpha_1/2}{1-\alpha_1/2}\big)}\ , \qquad C_2 = \frac{-24(m_+ - m_-)i}{2\pi\sqrt{2}\log\big(\frac{1+\alpha_2/2}{1-\alpha_2/2}\big)}\ . \tag{A.52}
$$

There is only one l_{a_+} line and one l_{a_-} line with residues $2m_+$ and $2m_-$

$$
C_{a_+} = \frac{-2m_+ i}{2\pi\sqrt{2}\log\big(\frac{1+\alpha_{a_+}/2}{1-\alpha_{a_+}/2}\big)}\ , \quad C_{a_-} = \frac{-2m_+ i}{2\pi\sqrt{2}\log\big(\frac{1+\alpha_{a_-}/2}{1-\alpha_{a_-}/2}\big)}\ . \tag{A.53}
$$

The total residue of the lines $l_\pm$ is $32m_\pm$, as expected since there are 32 such lines on top of each other.

A.3. The curve with elliptic parameters

Next we want to generalize the curve to the elliptic case. We will start with two masses and take as our Ansatz

$$
y^2 = x^3 + \gamma x^2 u^2 - 2u\mu(u^2 + sn^2 m_+ x)(u^2 + sn^2 m_- x) + \beta x u^2\ , \tag{A.54}
$$

where γ, μ and β are constants to be determined. Here we have chosen to write the curve in terms of the Jacobi elliptic functions

$$sn(u) = \frac{\vartheta_3(0)}{\vartheta_2(0)}\frac{\vartheta_1(u)}{\vartheta_4(u)} , \quad cn(u) = \frac{\vartheta_4(0)}{\vartheta_2(0)}\frac{\vartheta_2(u)}{\vartheta_4(u)} , \quad dn(u) = \frac{\vartheta_4(0)}{\vartheta_3(0)}\frac{\vartheta_3(u)}{\vartheta_4(u)} , \tag{A.55}$$

with

$$k = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)} . \tag{A.56}$$

The reason for using the Jacobi functions, rather than Weierstrass or theta functions, is that some of our results above can be easily generalized by just replacing the trigonometric functions with Jacobi elliptic functions. We would like to draw the readers attention to the new term xu^4 in (A.54) since this will give rise to elliptic functions in the Seiberg-Witten differential. It is easy to verify that he trigonometric curve is obtained in the limit of $k \to 0$. To fix the unknown coefficients we look for lines. As before we start with the spinor line, which is given by

$$x = \frac{-u^2}{sn^2 m_\pm} . \tag{A.57}$$

If we set $\gamma = 1 + k^2$ and $\beta = k^2$ it is easy to see that equation (A.57) gives rise to a perfect square

$$y = \frac{-iu^3 cn(m_\pm) dn(m_\pm)}{sn^3(m_\pm)} . \tag{A.58}$$

Next we consider the adjoint line of the form

$$x = \frac{-1}{sn^2 m_+ \pm m_-} u^2 + bu \quad , \tag{A.59}$$

where b is a constant that we determine by substituting the line into our curve and demanding that y is a perfect square. Upon substitution, the lowest power of u in y^2 is u^3, which cannot be part of a perfect square, and so setting this to zero gives:

$$b = -2\mu sn^2(m_+)\ sn^2(m_-) . \tag{A.60}$$

With the adjoint line we can fix the remaining constant in the curve. After a rather lengthy but straightforward calculation we find that if $\mu = 1$, y is a perfect square given by

$$y = \frac{icn(m_+ \pm m_-) dn(m_+ \pm m_-)}{cn^3(m_+ \pm m_-)} u^3 . \tag{A.61}$$

So far we have found the spinor line l_s and the adjoint line $l_\pm$. From our previous analysis we know that we have a second adjoint line $l_{a_\pm}$. To find this line we use the same trick as we used in the trigonometric case, namely set

$$ul_+ + Cl_+\alpha_+ + r_+ = ul_{a_+} + Cl_{a_+} + \alpha_{a_+} r_{a_+} , \tag{A.62}$$

where

$$l_+ \; = \; x \; + \frac{u}{snm_+} \;, \qquad r_+ \; = \; \frac{iu^3 cnm_+ dnm_+}{sn^3 m_+} \;. \tag{A.63}$$

We look for a solution that has the form

$$l_{a_\pm} \; = x - (b_2 u^2 + b_1 u + \lambda^2) \;, \qquad r \; = \; a_3 u^3 + a_2 u^2 + a_1 u + \lambda^3 \;. \tag{A.64}$$

Matching the coefficient of different powers of u in equation (A.62) we find after some algebra

$$\begin{aligned}
a_1 &= 3sn^2(m_+)(sn^2(m_+) - \frac{sn^2(m_-)}{3} + \frac{sn^2(m_+) - sn^2(m_-)}{cn(2m_+)dn(m_+)} - i\frac{sn^3(m_+)(sn^2(m_+) - sn^2(m_-))}{cn^2(m_+)dn^2(m_+)} \\
a_2 &= i\;\frac{cn(2m_+)\;dn(2m_+)}{sn^3(2m_+)} \\
a_3 &= \frac{i\Delta(cn^2(m_+) - sn^2(m_+)dn^2(m_+))(dn^2(m_+) - k^2 sn^2(m_+)cn^2(m_+))}{sn^3(m_+)cn^2(m_+)dn^2(m_+)} \\
b_1 &= 2sn^2(m_+)\Big(sn^2(m_+) + \frac{sn^2(m_+) - sn^2(m_-)}{cn(2m_+)\;dn(2m_+)}\Big) \\
b_2 &= -\frac{\Delta^2}{4sn^2(m_+)cn^2(m_+)dn^2(m_+)} \\
\lambda &= -i\frac{sn^3(m_+)(sn^2(m_+) - sn^2(m_-)}{cn(m_+)\;dn(m_+)} \\
\Delta &= 1 - k^2 sn^4(m_+)\;.
\end{aligned} \tag{A.65}$$

A.4. Seiberg-Witten differential for the elliptic case

Having found all the lines we are ready to construct the Seiberg-Witten differential. However, it is first instructive to reconsider the trigonometric problem. Recall that the Seiberg-Witten differential has a piece that is of the form

$$\tfrac{1}{2}\log\left(\frac{y + ul + \alpha r}{y - (ul + \alpha r)}\right)\frac{u\;dx}{ul + \alpha r} \tag{A.66}$$

The generalization to the elliptic case relies on the integral representation of the log-function

$$\int_0^1 \frac{u\;dt}{(y^2 - (ul + \alpha r)^2 + (ul + \alpha r)^2 t^2)^{\frac{1}{2}}} \tag{A.67}$$

Similarly the differential $\log \frac{y+ul+\alpha r}{y-ul-\alpha r}$ has the following integral representation

$$\int_0^1 \frac{ul+\alpha r}{l} \frac{dt}{(y^2-(ul+\alpha r)^2+(ul+\alpha r)^2 t^2)^{\frac{1}{2}}} \ . \tag{A.68}$$

The generalization of (A.68) to elliptic case is now straightforward

$$\int_0^1 \frac{ul+\alpha r}{l} \frac{dt}{(y^2-\gamma(ul+\alpha r)^2+\gamma(ul+\alpha r)^2 t^2+k^2u^3(ul+\alpha r)+k^2u^3(ul+\alpha r)t^4)^{\frac{1}{2}}} \ . \tag{A.69}$$

First we can check that that residue at line $l = x + u^2/sn^2(m)$ with $r(u) = iu^3 sn(m) dn(m)/sn^3(m)$ and $\alpha = isn(m)/sn(m)dn(m)$ is indeed m. From (A.69) it follows that the residue at the pole is proportional to

$$\alpha r \int_0^{sn(m)} \frac{dt}{u^3(1-\gamma t^2+k^2t^4)^{\frac{1}{2}}} = -im \ . \tag{A.70}$$

The last equality follows from the definition of the inverse of the elliptic function. To construct the Seiberg-Witten differential we will use the same combination of derivatives that lead to the Seiberg-Witten differential in the trigonometric case. Here we want to be left a rational function after performing the integral. This will restrict the allowed form of the integrand. It is easy to see that the integral

$$\int_0^1 \frac{dt}{(a_0+a_2t^2+a_4t^4)^{\frac{1}{2}}} \tag{A.71}$$

is not rational, but the following integral is:

$$\int_0^1 \frac{dt(a_0-a_4t^4)}{(a_0+a_2t^2+a_4t^4)^{\frac{1}{2}}} \ . \tag{A.72}$$

We can use this since for us $a_0 = y^2-\gamma(ul+\alpha r)-k^2u^3(ul+\alpha r)$, $a_2 = \gamma(ul+\alpha r)^2$, $a_4 = k^2u^3(ul+\alpha r)$, so that $a_0+a_2+a_4 = y^2$. This means that if we can arrange that the numerators is proportional to $y^2-\gamma(ul+\alpha r)-k^2u^3(ul+\alpha r)-k^2u^3(ul+\alpha rt^4)$, then after integrating (A.72) we get the holomorphic differential.

As before act with the $\frac{d}{du}-\frac{dl}{du}\frac{d}{dx}$ on the integral (A.69) to give

$$L_1 = (\frac{d}{du}-\frac{dl}{du}\frac{d}{dx})\int_0^1 \frac{ul+\alpha r}{l} \frac{dt}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{1}{2}}} =$$
$$\int_0^1 dt \frac{x^3+4u^5+2u^3+snm_-^2+u^3xsnm_+^2-snm_+^2snm_-^2ux^2)-t^4k^2u^4x}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{3}{2}}} \ . \tag{A.73}$$

Above we have used the fact that for a spinor line $u\frac{dr}{du} = 3r$. Next consider

$$
\begin{aligned}
L_u \;&=\; \frac{d}{du}(u\int_0^1 dt\frac{1}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{1}{2}}}) \\
&\int_0^1 \frac{x^3+3u^5+u^3x(snm_+^2+snm_-)-ux^2snm_+^2snm_- - k^2t^4u^4x}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{3}{2}}}\ ,
\end{aligned} \tag{A.74}
$$

and similarly

$$
\begin{aligned}
L_x \;&=\; \frac{d}{dx}(x\int_0^1 dt\frac{1}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{1}{2}}}) \\
&\int_0^1 \frac{-\frac{1}{2}x^3-2u^5-u^3x(snm_+^2+snm_-)+\frac{1}{2}ux^2snm_+^2snm_- - k^2t^4u^4x}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{3}{2}}}\ .
\end{aligned} \tag{A.75}
$$

Consider the following combination

$$
\begin{aligned}
L_1+L_2-3L_u-L_x \;&=\; -\tfrac{1}{2}\int_0^1 dt(\frac{y^2-\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4-t^4k^2u^4x}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{3}{2}}} \\
&= -\tfrac{1}{2}\left.\frac{t}{(y^2-(ul+\alpha r)^2+\gamma(ul+\alpha r)^2t^2+k^2u^3(ul+\alpha r)t^4)^{\frac{1}{2}}}\right|_0^1 \\
&=\; -\tfrac{1}{2}y\ .
\end{aligned} \tag{A.76}
$$

Hence this combination leads a Seiberg-Witten differential. The construction of the null differential is very similar to the trigonometric case. Recall that we had

$$
\begin{aligned}
&\Big(\frac{d}{du}-\frac{d}{dx}\frac{dl_s}{du}\Big)\log\left(\frac{y+(ul+\alpha r)}{y-(ul+\alpha r)}\right)\frac{dx}{l_s}+\tfrac{1}{2}\Big(\frac{d}{du}-\frac{d}{dx}\frac{dl_a}{du}\Big)\log\left(\frac{y+(ul+\alpha r)}{y-(ul+\alpha r)}\right)\frac{dx}{l_a} \\
&-\Big(\frac{d}{du}u-\frac{d}{dx}\frac{dul_a+\alpha r}{du}\Big)\log\left(\frac{y+(ul+\alpha r)}{y-(ul+\alpha r)}\right)dx \;=\; 0\ .
\end{aligned} \tag{A.77}
$$

This can now easily generalized to the elliptic case. All we have to do is to replace the log terms with the appropriate integrals. The full Seiberg-Witten differential is again given by a linear combination of null-differentials and the one constructed out of the spinor line.

Appendix B. General E_n curves

The polynomial E_8 curve is given as

$$\begin{aligned}
y^2 =& x^3 - x^2\big(u\widetilde{T}_4/2 + T_{10} - t_8 T_2\big) \\
&- x\Big(T_2 u^3 + (14t_8 + T_8)u^2 + u(8T_{14} - T_{12}T_2 + 8t_8 T_6 - T_{10}\widetilde{T}_4 + 4t_8 T_2 \widetilde{T}_4) + 2t_8 T_{12} \\
&+ 4T_{14}T_6 - T_{12}T_8 + 2t_8^2\widetilde{T}_4 + 2T_{14}T_2\widetilde{T}_4 - t_8 T_8 \widetilde{T}_4 + (T_{12}\widetilde{T}_4^2)/4 + (t_8\widetilde{T}_4^3)/4\Big) \\
&- 2u^5 - T_6 u^4 + u^3(4T_{12} - 2t_8 T_2^2 - 5t_8\widetilde{T}_4 + (T_8\widetilde{T}_4)/2) \\
&+ u^2\big(16t_8 T_{10} - 8t_8^2 T_2 - T_{14}T_2^2 + 2T_{12}T_6 - 4t_8 T_2 T_8 - 4T_{14}\widetilde{T}_4 + (T_{12}T_2\widetilde{T}_4)/2 \\
&- 2t_8 T_6 \widetilde{T}_4 - (T_{10}\widetilde{T}_4^2)/4 - (t_8 T_2 \widetilde{T}_4^2)/4\big) \\
&+ u\big(-8t_8^3 - 2T_{12}^2 + 8T_{10}T_{14} - 4t_8 T_{14}T_2 + 8t_8 T_{10}T_6 - 8t_8^2 T_2 T_6 + 8t_8^2 T_8 - 2T_{14}T_2 T_8 \\
&- 2t_8 T_8^2 - 3t_8 T_{12}\widetilde{T}_4 + 4t_8 T_{10}T_2\widetilde{T}_4 - 4t_8^2 T_2^2 \widetilde{T}_4 - 2T_{14}T_6\widetilde{T}_4 - (T_{12}T_8\widetilde{T}_4)/2 - 3t_8^2\widetilde{T}_4^2 \\
&- (T_{14}T_2\widetilde{T}_4^2)/2 + (t_8 T_8 \widetilde{T}_4^2)/2 + (T_{12}\widetilde{T}_4^3)/8\big) \\
&- 4t_8^2 T_{14} - T_{12}^2 T_6 + 4T_{10}T_{14}T_6 - 4t_8 T_{14}T_2 T_6 + 4t_8 T_{14}T_8 - T_{14}T_8^2 - (T_{12}^2 T_2 \widetilde{T}_4)/2 \\
&+ 2T_{10}T_{14}T_2\widetilde{T}_4 - 2t_8 T_{14}T_2^2\widetilde{T}_4 - 2t_8 T_{12}T_6\widetilde{T}_4 - t_8 T_{14}\widetilde{T}_4^2 - t_8 T_{12}T_2\widetilde{T}_4^2 - t_8^2 T_6 \widetilde{T}_4^2 \\
&+ (T_{14}T_8\widetilde{T}_4^2)/2 - (t_8^2 T_2 \widetilde{T}_4^3)/2 - (T_{14}\widetilde{T}_4^4)/16
\end{aligned} \tag{B.1}$$

For the polynomial limit the T_{2n} satisfy

$$T_{2n} = \sum_{i_i<i_2..<i_n}^{8} m_{i_1}^2 ... m_{i_n}^2, \qquad t_8 = \prod_i^8 m_i, \qquad \widetilde{T}_4 = T_2^2/4 - T_4.$$

The expression differs slightly from the curve in [6] since we have shifted the x variable.

The lower E_n curves are derived using the scaling described in the text. The E_7 curve, in terms of $SO(12) \times SU(2)$ variables is given by

$$
\begin{aligned}
y^2 = x^3 &+ \Big(u^2 - (u\widetilde{T}_2)/2 - 2t_6 - 2t_6T_2 - T_6 + 2t_6\widetilde{T}_2\Big)x^2 \\
&+ \Big(u^3(2 - T_2) - u^2(4t_6 + T_4) + u\big(2t_6T_4 - 8t_6T_2 + 2t_6T_2^2 - 2T_8 + T_2T_8 + 8t_6\widetilde{T}_2 \\
&- 3t_6T_2\widetilde{T}_2 + T_6\widetilde{T}_2 - T_8\widetilde{T}_2 + t_6\widetilde{T}_2^2/2\big) \\
&+ 8t_6^2T_2 - 4T_{10}T_2 + T_{10}T_2^2 + 2t_6T_2T_6 + T_4T_8 - 8t_6^2\widetilde{T}_2 \\
&+ 4T_{10}\widetilde{T}_2 - 2T_{10}T_2\widetilde{T}_2 + t_6T_4\widetilde{T}_2 - 2t_6T_6\widetilde{T}_2 + T_{10}\widetilde{T}_2^2 - (T_8\widetilde{T}_2^2)/4 - t_6\widetilde{T}_2^3/4 - 8t_6\widetilde{T}_2 \\
&- 3t_6T_2\widetilde{T}_2 + T_6\widetilde{T}_2 - T_8\widetilde{T}_2 + t_6\widetilde{T}_2^2/2\Big)x \\
&+ u^4(T_2^2/4 - T_2) + u^3(2t_6T_2 - 8t_6 + T_4\widetilde{T}_2/2) + u^2\big(4t_6^2 - 4T_{10} + 8t_6T_4 - 3t_6T_2T_4 \\
&+ 2T_2T_8 - T_2^2T_8/2 - 2t_6T_2\widetilde{T}_2 + t_6T_2^2\widetilde{T}_2/2 + 2t_6T_4\widetilde{T}_2 - T_8\widetilde{T}_2 \\
&+ (T_2T_8\widetilde{T}_2)/2 + t_6\widetilde{T}_2^2/2 - t_6T_2\widetilde{T}_2^2/4 - (T_6\widetilde{T}_2^2)/4\big) \\
&+ u\Big(16t_6^2T_2 - 4t_6^2T_2^2 - 4t_6^2T_4 + 4T_{10}T_4 - 2T_{10}T_2T_4 - 2t_6T_4^2 + 8t_6T_2T_6 - 2t_6T_2^2T_6 - 2t_6T_2T_8 \\
&- 16t_6^2\widetilde{T}_2 + 6t_6^2T_2\widetilde{T}_2 - 2T_{10}T_2\widetilde{T}_2 + T_{10}T_2^2\widetilde{T}_2/2 + 2T_{10}T_4\widetilde{T}_2 - 8t_6T_6\widetilde{T}_2 + 3t_6T_2T_6\widetilde{T}_2 \\
&+ 2t_6T_8\widetilde{T}_2 - T_4T_8\widetilde{T}_2/2 - t_6^2\widetilde{T}_2^2 + T_{10}\widetilde{T}_2^2 - T_{10}T_2\widetilde{T}_2^2/2 + t_6T_4\widetilde{T}_2^2/2 - t_6T_6\widetilde{T}_2^2 + (T_8\widetilde{T}_2^3)/8\Big) \\
&- 8t_6^3T_2 + 8t_6T_{10}T_2 - 2t_6T_{10}T_2^2 + t_6^2T_4^2 - T_{10}T_4^2 - 4t_6^2T_2T_6 + 4T_{10}T_2T_6 - T_{10}T_2^2T_6 \\
&- t_6T_2T_4T_8 - T_2T_8^2 + T_2^2T_8^2/4 + 8t_6^3\widetilde{T}_2 - 8t_6T_{10}\widetilde{T}_2 + 4t_6T_{10}T_2\widetilde{T}_2 \\
&- t_6^2T_2T_4\widetilde{T}_2 + 4t_6^2T_6\widetilde{T}_2 - 4T_{10}T_6\widetilde{T}_2 + 2T_{10}T_2T_6\widetilde{T}_2 - 2t_6T_2T_8\widetilde{T}_2 + t_6T_2^2T_8\widetilde{T}_2/2 \\
&+ t_6T_4T_8\widetilde{T}_2 + T_8^2\widetilde{T}_2 - T_2T_8^2\widetilde{T}_2/2 - 2t_6T_{10}\widetilde{T}_2^2 - t_6^2T_2\widetilde{T}_2^2 + t_6^2T_2^2\widetilde{T}_2^2/4 + t_6^2T_4\widetilde{T}_2^2/2 \\
&+ T_{10}T_4\widetilde{T}_2^2/2 - T_{10}T_6\widetilde{T}_2^2 + 2t_6T_8\widetilde{T}_2^2 - 3t_6T_2T_8\widetilde{T}_2^2/4 + T_8^2\widetilde{T}_2^2/4 + t_6^2\widetilde{T}_2^3 - t_6^2T_2\widetilde{T}_2^3/4 \\
&+ t_6T_8\widetilde{T}_2^3/4 + t_6^2\widetilde{T}_2^4/16 - T_{10}\widetilde{T}_2^4/16
\end{aligned} \tag{B.2}
$$

The T_n variables have the same form as in the E_8 case, with $T_{2n} = -T_{2n+2} + G_{2n}$ for $n > 1$ and where

$$G_{2n} = \sum_{i_1<..i_n}^{6} \sin^2 m_{i_1} ... sin^2 m_{i_n}$$

We also have that

$$T_2 = 2\left(1 - \prod_{i=1}^{6} \cos m_i\right) \qquad\qquad t_6 = \prod_{i=1}^{6} \sin m_i$$

and

$$\widetilde{T}_2 = T_2 - 4\sin^2(\mu/2).$$

In deriving this curve, x was shifted by t_6T_2. In terms of the E_7 dimensions, where $[x]=6$, $[u]=4$ and $[T_n]=n$, we see that all terms in the E_7 curve are either dimension 18 or 20. The dimension 18 terms are what remain in the Kodaira limit.

The E_6 curve is

$$\begin{aligned}
y^2 =& x^3 + \Big(u^2 + u(2\sin\lambda - iT_2e^{i\lambda}) + 4it_5 - T_4e^{2i\lambda}\Big)x^2 \\
&- e^{i\lambda}\Big(2iu^3 + u^2\big(2i\sin\lambda T_2 + T_2e^{i\lambda}\big) + u\big(4\sin\lambda T_4e^{i\lambda} - 8t_5 - 2iT_6 + 4i\sin 2\lambda t_5\big) \\
&\quad + 16\sin^3\lambda t_5 - 6\sin\lambda T_2t_5 + 2\sin\lambda T_2t_5e^{2i\lambda} + 4\sin\lambda^2 T_6e^{i\lambda} - T_2T_6e^{i\lambda} \\
&\quad + 4T_8e^{-i\lambda} - it_5(T_2^2 - 4T_4)e^{i\lambda}\Big)x \\
&- e^{2i\lambda}\Big(u^4 + 2\sin\lambda T_2u^3 - u^2\big(2T_6 - 4\sin^2\lambda T_4 - 4\sin 2\lambda t_5\big) \\
&\quad - u\big(8\sin\lambda T_8 - 8\sin^3\lambda T_6 + 2\sin\lambda T_2T_6 - 4\sin 2\lambda\sin\lambda T_2t_5 + 2\cos\lambda t_5(T_2^2 - 4T_4)\big) \\
&\quad + 4\sin^2 2\lambda t_5^2 + 4\sin 2\lambda t_5T_6 + T_8(T_2^2 - 4T_4) + T_6^2 + 16\sin^4\lambda T_8 - 8\sin^2\lambda T_2T_8\Big).
\end{aligned} \tag{B.3}$$

with

$$t_5 = \prod_i^5 \sin m_i$$

The shift used on x is $-it_5(2-T_2)e^{2i\lambda} + iT_2ue^{i\lambda}/2$.

The $E_5 = SO(10)$ curve is

$$\begin{aligned}
y^2 =& x^3 + \big(u^2 + u(T_2e^{i\lambda} - 4\cos\lambda) - T_2e^{2i\lambda} - 8t_4 + T_2^2e^{2i\lambda}/4\big)x^2 \\
&+ \big(u^2(4-2T_2) + u(4T_2e^{i\lambda} - 8t_4e^{-i\lambda} - T_2^2e^{i\lambda} - 4T_4e^{i\lambda}) \\
&+ 8iT_4\sin\lambda e^{i\lambda} + 16it_4\sin\lambda(e^{i\lambda} - 2e^{-i\lambda}) + 12t_4T_2 + 16T_6 - 2T_2T_4e^{2i\lambda}\big)x \\
&+ u^2(T_2^2 - 4T_2) + u(32it_4\sin\lambda + 8t_4T_2e^{-i\lambda} - 16iT_4\sin\lambda + 4T_2T_4e^{i\lambda}) \\
&- 16t_4T_4 - 16T_2T_6 + 16t_4^2e^{-2i\lambda} + 64T_6\sin^2\lambda + 4T_4^2e^{2i\lambda}
\end{aligned} \tag{B.4}$$

In deriving this curve we shifted x by $(T_2-2)(ue^{i\lambda}/2 + t_4e^{2i\lambda})$. If we set $T_n = t_4 = \lambda = 0$ then the curve in (B.4) reduces to

$$y^2 = x^3 + (u^2 - 4u)x^2 + 4u^2x$$

The discriminant of this curve is $128u^7 - 16u^8$, which describes an $SO(10)$ singularity. We could have also derived a curve where the $SO(10)$ symmetry is manifest by taking the curve in (B.3) and sending λ to $i\infty$ and then rescaling.

The $E_4 = SU(5)$ curve is

$$\begin{aligned} y^2 =& x^3 + \big(u^2 + u(iT_2 e^{i\lambda} - 2i(e^{i\lambda} + 2e^{-i\lambda})) - (1 - T_2/2)^2 e^{2i\lambda} - 6T_2 - 16it_3 + 12\big)x^2 \\ &+ \big(u(16iT_2 + 32t_3 - 32i - 8ie^{2i\lambda})e^{-i\lambda} - 64T_4 + 16iT_2t_3 + 8T_2^2 - 32it_3 \\ &+ 4T_2(e^{2i\lambda} - 8) + 8(6 - e^{2i\lambda})\big) - 64iue^{-i\lambda} - 256t_3^2 e^{-2i\lambda} + 256iT_2t_3e^{-2i\lambda} - 256T_4e^{-2i\lambda} + 128it_3 - 3 \end{aligned} \tag{B.5}$$

The shift in x is $(T_2 - 2)(it_3e^{2i\lambda} - 2 + iue^{i\lambda}/2)$. In the massless case, (B.5) reduces to

$$y^2 = x^3 + (u^2 - 6iu + 11)x^2 + (40 - 40iu)x + 48 - 64iu$$

and the discriminant is $-256iu^5(u^2 - iu + 1)$ which describes an $SU(5)$ singularity.

The $E_3 = SU(3) \times SU(2) \times U(1)$ curve is

$$\begin{aligned} y^2 =& x^3 + \big(u^2 + 2u(e^{i\lambda} + 4e^{-i\lambda} - T_2e^{i\lambda}) + 12T_2 + 32t_2 - 24 + \widetilde{T}_4e^{2i\lambda}\big)x^2 \\ & 64\big(u(T_2 - 2t_2 - 2)e^{-i\lambda} - 16t_2e^{-2i\lambda} - 2\widetilde{T}_4 + 2t_2T_2 - 2t_2 + 4\big) \\ &+ 4096(t_2^2 - T_2t_2 + \widetilde{T}_4 + 2t_2 - 1)e^{-2i\lambda} \end{aligned} \tag{B.6}$$

where $\widetilde{T}_4 = (1 - T_2/2)^2$. The shift in x is $(T_2 - 2)(-t_2e^{2i\lambda} + 1 - ue^{i\lambda}/2)$. In the massless case, the curve reduces to

$$x^3 + (u^2 + 10u - 23)x^2 + 128(1 - u)x$$

and the discriminant is $-16384(1+u)^3(1-u)^2(17+u)$. Hence this has an $SU(3)\times SU(2)\times U(1)$ singularity structure.

The $E_2 = SU(2) \times U(1)$ curve is

$$\begin{aligned} y^2 =& x^3 + \big(u^2 + iu(16e^{-i\lambda} + 2e^{i\lambda} + t_1^2e^{i\lambda}) - \widetilde{T}_4e^{2i\lambda} - 24t_1^2 + 64it_1 + 48\big) \\ &+ 256\big(u(2i + 2t_1 - t_1^2)e^{-i\lambda} - 2\widetilde{T}_4 + 2it_1(1 + 16e^{-2i\lambda}) - it_1^3\big)x \\ &+ 65536(t_1^2 - \widetilde{T}_4 - it_1^3 + 2it_1)e^{-2i\lambda} \end{aligned} \tag{B.7}$$

where $t_1 = \sin m_1$ and $\widetilde{T}_4 = (1 - sin^2 m_1/2)^2$. The shift in x is $(t_1^2 - 2)(-it_1e^{2i\lambda} - iue^{i\lambda}/2 - 8)$. The massless curve is

$$x^3 + (u^2 + 18iu + 47)x^2 + 512(iu - 1)x - 65536$$

and its discriminant is $4194394(5i + u)^2(-iu^3 + 23u^2 - 117iu - 565)$.

The $E_1 = SU(2)$ curve is

$$y^2 = x^3 + \big(u^2 - 2u(e^{i\lambda} + 16e^{-i\lambda}) + e^{2i\lambda} - 224\big)x^2 - 65536e^{-2i\lambda}x \tag{B.8}$$

The shift in x is $-2e^{2i\lambda} - ue^{i\lambda} - 32$. In the massless case, the discriminant is up to a numerical factor $(u - 17)^2(15 + u)(u - 49)$.

The $\widetilde{E}_1 = U(1)$ curve is found by letting $m_8 = +i\sum \Lambda_i - 2\lambda/8$, that is it has the opposite sign as in the E_1 case. This curve is given by

$$y^2 = x^3 + \big(u^2 - 2u(e^{i\lambda} + 16e^{-i\lambda}) + 32 + e^{2i\lambda}\big)x^2 + 4096\big(ue^{-i\lambda} - 16e^{-2i\lambda} - 1\big)x + 4194304e^{-2i\lambda} \tag{B.9}$$

References

[1] O. Ganor and A. Hanany, Nucl. Phys. B474 (1996) 122, hep-th/9602120; N. Seiberg and E. Witten, Nucl. Phys. B471 (1996) 121, hep-th/9603003; M. Duff, H. Lu and C.N. Pope, Phys. Lett. B378 (1996) 101, hep-th/9603037; M.R. Douglas, S. Katz, C. Vafa, *Small Instantons, del Pezzo Surfaces and Type I' theory,* hep-th/9609071; E. Witten, Mod. Phys. Let. A11 (1996) 2649, hep-th/9609159.

[2] E. Witten, Nucl. Phys. B471 (1996) 195, hep-th/9603150.

[3] O. Ganor, Nucl. Phys. B479 (1996) 197, hep-th/9607020; Nucl. Phys. B488 (1997) 223, hep-th/9608109.

[4] A. Klemm, P. Mayr and C. Vafa, *BPS states of exceptional non-critical strings,* CERN-TH-96-184, hep-th/9607139.

[5] W. Lerche, P. Mayr and N.P. Warner, *Non-Critical Strings, Del Pezzo Singularities and Seiberg-Witten Curves,* CERN-TH/96-326,USC-96/026, hep-th/9612085.

[6] J.A. Minahan and D. Nemeschansky, Nucl. Phys. B464 (1996) 3, hep-th/9507032; Nucl. Phys. B468 (1996) 72, hep-th/9601059.

[7] N. Seiberg, Phys. Lett. B388 (1996) 753, hep-th/9608111; D.R. Morrison and N. Seiberg, Nucl. Phys. B483 (1997) 229, hep-th/9609070.

[8] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner,Nucl. Phys. B477 (1996) 746, hep-th/9604034.

[9] O. Ganor, D. Morrison and N. Seiberg, Nucl. Phys. B487 (1997) 93, hep-th/9610251.

[10] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[11] C. Itzykson, Int. J. Mod. Phys. AB8 (1994) 1994.

[12] M. Kontsevich and Yu. Manin. Comm. Math. Phys. 164 (1994) 525, hep-th/9402147.

[13] D.R. Morrison, private communication.

[14] W. Lerche, *Introduction to Seiberg-Witten Theory and its Stringy Origin*, hep-th/9611190.

[15] A. Brandhuber and S. Stieberger, *Self-dual Strings and Stability of BPS States in $N = 2$ $SU(2)$ Gauge Theories,* hep-th/9610053.

[16] J. Rabin, *Geodesics and BPS StatesGEODESICS in $N = 2$ Supersymmetric QCD*, hep-th/9703145.

[17] J. Schulze and N.P. Warner, *BPS Geodesics in $N = 2$ Supersymmetric Yang-Mills Theory,* preprint USC-97-001, to appear in *Nuclear Physics B*, hep-th/9702012.

[18] C. Vafa, private communication.

[19] P. Mayr, private communication.